Optimal Pants Decompositions and Shortest Homotopic Cycles on an Orientable Surface*

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Abstract. A *pants decomposition* of a compact orientable surface $\mathcal{M}$ is a set of disjoint simple cycles which cuts $\mathcal{M}$ into *pairs of pants*, i.e., spheres with three boundaries. Assuming $\mathcal{M}$ is a polyhedral surface, with weighted vertex-edge graph G, we consider combinatorial pants decompositions: the cycles are closed walks in G that may overlap but do not cross.

We give an algorithm which, given a pants decomposition, computes a homotopic pants decomposition in which each cycle is a shortest cycle in its homotopy class. In particular, the resulting decomposition is *optimal* (as short as possible among all homotopic pants decompositions), and any optimal pants decomposition is made of shortest homotopic cycles. Our algorithm is polynomial in the complexity of the input and in the longest-to-shortest edge ratio of G. The same algorithm can be applied, given a simple cycle C, to compute a shortest cycle homotopic to C which is itself simple.

1 Introduction

Let $\mathcal{M}$ be a connected, compact, orientable surface. A *pants decomposition* of $\mathcal{M}$ is a set of disjoint simple cycles in $\mathcal{M}$ which cuts $\mathcal{M}$ into *pairs of pants*, i.e., spheres with three boundaries. See Fig. 1A.

We shall consider pants decompositions on a polyhedral surface (a surface obtained by assembling simple polygons). In our combinatorial setting, the cycles are closed walks on the weighted vertex-edge graph G of $\mathcal{M}$; the cycles may share edges and vertices of G, provided that they can be spread apart with a thin space so that they become simple and disjoint.

We describe a conceptually simple, iterative scheme which takes a given pants decomposition and outputs a shorter homotopically equivalent pants decomposition. We prove that, at the end of the process, each cycle is a shortest cycle in its homotopy class. In particular, the resulting decomposition is *optimal* in

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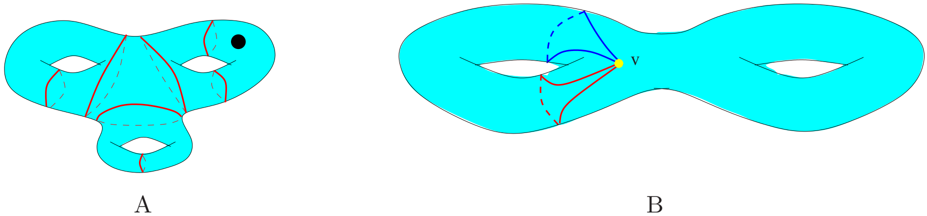

Fig. 1. A: A pants decomposition of a genus three surface with one boundary. B: The two cycles on this double-torus are freely homotopic, though non-homotopic when considered as loops with endpoint v.

the sense that it is as short as possible among all homotopic pants decompositions, and any optimal pants decomposition is made of shortest homotopic cycles. Furthermore, this scheme can be implemented, leading to an algorithm which is polynomial in the complexity of the surface and of the input pants decomposition, and in the longest-to-shortest edge ratio of G.

Let γ be a non-contractible simple cycle on $\mathcal{M}$, and let Γ be the set of all shortest cycles homotopic to γ. We can compute an element of Γ which is simple: the idea is to extend γ into a pants decomposition of $\mathcal{M}$; after optimization, this pants decomposition contains such a cycle. Even the existence of a simple cycle in Γ is non-obvious, and the fact that this optimization problem has polynomial complexity was previously unknown.

The problem of shortening a pants decomposition of a combinatorial surface was raised in the conclusion of [4]; to our knowledge, we present the first algorithm for this purpose. Concerning the optimization of a single cycle, our result extends [6] to more general surfaces in the case of simple cycles. The present work is also a natural extension of our former paper [2] where we treat the case of optimal simple loops in a given class of homotopy with fixed basepoint.

This paper is organized as follows. In Sect. 2, we review elementary topological notions, and present the framework and our main theorem. Its proof is given in the next three sections. Finally, we discuss the computational issues and give the complexity of our algorithm.

2 Framework and Result

2.1 Homotopy and Pants Decompositions

We begin with some useful definitions.

Let $\mathcal{M}$ be a connected, compact, orientable surface, possibly with boundary. A *path* is a continuous mapping $p : [0, 1] \to \mathcal{M}$; its *endpoints* are $p(0)$ and $p(1)$. A *closed path*, or *loop*, is a path whose endpoints coincide. A *cycle* is a continuous mapping $\gamma : \mathbb{R} \to \mathcal{M}$, such that $\gamma(x) = \gamma(x + 1)$ for all $x \in \mathbb{R}$. A path is *simple* if it is one-to-one; a cycle is *simple* if its restriction to $[0, 1)$ is one-to-one.

Two paths p and q, both with endpoints a and b, are *homotopic* if there is a continuous family of paths with endpoints a and b which joins p and q. More formally, a *homotopy* between p and q is a continuous mapping $h : [0,1] \times [0,1] \to \mathcal{M}$ such that $h(0,.) = p$, $h(1,.) = q$, $h(.,0) = a$, and $h(.,1) = b$. A closed path is *contractible* if it is homotopic to the constant path. Two cycles γ and γ' are *homotopic* if there is a continuous family of cycles joining γ to γ'. Equivalently, if p and p' denote the restrictions of γ and γ' to $[0,1]$, there exists a path β joining $p(0)$ to $p'(0)$ such that the loop $\beta^{-1}.p.\beta.p'^{-1}$ is contractible ("." denotes paths concatenation).

Homotopy of cycles (also called free homotopy) and homotopy of loops (also called homotopy with basepoint) are two different equivalence relations (Fig. 1B).

A *pants decomposition* of $\mathcal{M}$ is an ordered set of simple, pairwise disjoint cycles which split $\mathcal{M}$ into pairs of pants (see [5]). Every compact orientable surface, except the sphere, disk, cylinder, and torus, admits a pants decomposition, obtained for example by cutting the surface iteratively along an essential cycle (a simple cycle which does not bound a disk nor a cylinder). Although pants decompositions do not exist for the torus and the cylinder, this paper applies to these surfaces as well (with minor changes) if we allow a pants decomposition to decompose the surface into pairs of pants *and/or cylinders*. If $\mathcal{M}$ has genus g and b boundary cycles, a pants decomposition is made of $3g + b - 3$ cycles.

We can augment any pants decomposition, s, of $\mathcal{M}$ to form a *doubled pants decomposition*. Just add to s a copy of each of its cycles and a copy of each of the boundaries of $\mathcal{M}$, slightly translated, in the same homotopy class, such that s is still a set of pairwise disjoint simple cycles. A doubled pants decomposition $s = (s_1, \ldots, s_N)$ is thus made of $N = 6g + 3b - 6$ cycles. A cycle of s and its translated copy, or a boundary of $\mathcal{M}$ and its translated copy, are called *twins*. For a cycle s_j in s, the closure of the component of $\mathcal{M} \setminus \{s \setminus s_j\}$ that contains s_j is a pair of pants.

2.2 Length of Cycles

$\mathcal{M}$ is assumed to be a polyhedral surface, whose edges have positive weights. Let G be the (weighted) vertex-edge graph of $\mathcal{M}$, and G^* be its dual graph embedded into $\mathcal{M}$.[1] We are interested in sets of piecewise linear (PL) curves (paths and cycles) drawn on $\mathcal{M}$ which are *regular* with respect to G^*. More precisely, the set of (self-)intersection points between the curves (resp. between the curves and G^*) is finite and, at such points, exactly two curve parts (resp. exactly one curve part and one edge of G^*) meet and actually cross. *Regularity is always assumed throughout this paper, although omitted in all statements.* If a curve c crosses the edges $e_1^*, \ldots, e_k^*$ of G^*, its *length* $|c|$ is defined to be the sum of the weights of $e_1, \ldots, e_k$, counting multiplicities.

Any set of simple, pairwise disjoint, PL cycles on $\mathcal{M}$ can be retracted onto closed walks on G without changing their homotopy classes, see Fig. 2AB; and

[1] This means that there is a vertex of G^* in each face of G and an edge of G^* crossing each interior edge of G. Furthermore, for each edge of G on the boundary of $\mathcal{M}$, we put a vertex of G^* on this edge and link it with an edge of G^* to the vertex of G^* in the incident face of G.

the length of a cycle is the length of the corresponding closed walk. The resulting walks can fail to be simple or disjoint as they can travel several times through a same vertex or edge of G; however, it is always possible to perturb them to get disjoint, simple cycles. This observation motivates the definition of length of a cycle: in this paper, we are interested in combinatorial sets of cycles in G. From an algorithmic point of view, it will be sufficient to work with cycles stored as closed walks on G, with the additional information, if several edges of these walks go along a same edge e of G, of their ordering, from left to right, along e.

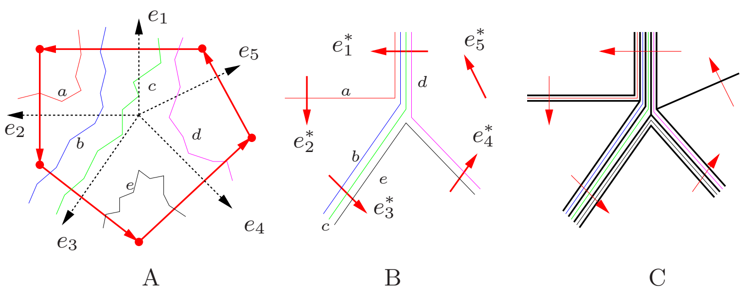

Fig. 2. A, B: A retraction of the set of simple, pairwise disjoint cycles (a, b, c, d, e) onto G, in the neighborhood of a vertex whose incident edges are $e_1, \ldots, e_5$. C: The construction of the graph $G(\mathcal{P}_j)$, represented in bold lines, for the computation of an Elementary Step (described in Sect. 6).

2.3 Our Result

Definition 1. *Let s be a doubled pants decomposition of $\mathcal{M}$. An* Elementary Step *$f_j(s)$ consists in replacing the jth cycle s_j by a shortest simple homotopic cycle in the pair of pants of $\mathcal{M} \setminus (s \setminus s_j)$ containing s_j. A* Main Step *$f(s)$ is the application of $f = f_N \circ f_{N-1} \circ \ldots \circ f_2 \circ f_1$ to s. These operations transform a doubled pants decomposition into another one, keeping the homotopy class of the decomposition.*

Here is our main theorem:

Theorem 2. *Let s^0 be a doubled pants decomposition of $\mathcal{M}$, and let $s^{n+1} = f(s^n)$. For some $m \in \mathbb{N}$, s^m and s^{m+1} have the same length and, in this situation, s^m is a doubled pants decomposition homotopic to s^0 made of simple cycles which are individually as short as possible among all cycles in their (free) homotopy class. In particular, s^m is an optimal doubled pants decomposition of $\mathcal{M}$, and contains an optimal pants decomposition.*

Since any non-contractible simple cycle can be extended to a doubled pants decomposition of $\mathcal{M}$, we obtain:

Corollary 3. *Let γ be a non-contractible simple cycle in $\mathcal{M}$. There exists a* simple *cycle γ' homotopic to γ which is as short as possible among all cycles homotopic to γ.*

3 Crossing Words

In this section, we will introduce the main ingredient of this paper: the crossing word between a set of disjoint, simple paths or cycles, and a given path or cycle.

3.1 Universal Cover and Lifts

We refer the reader to any textbook in algebraic topology (e.g. [7]) for the details.

The *universal cover* of $\mathcal{M}$ is a simply connected surface, $\tilde{\mathcal{M}}$, (i.e., each closed path is contractible) together with a continuous *projection* π from $\tilde{\mathcal{M}}$ onto $\mathcal{M}$ satisfying: each point x of $\mathcal{M}$ has an open, arcwise connected neighborhood U so that $\pi^{-1}(U)$ is a union of disjoint open sets $(U_i)_{i\in I}$ and $\pi|_{U_i} : U_i \to U$ is a homeomorphism. It is known that the universal cover of a surface is unique up to isomorphism. A *translation* τ of $\tilde{\mathcal{M}}$ is a projection-preserving homeomorphism: $\pi \circ \tau = \pi$. A *lift* of a path p is a path $\tilde{p}$ on $\tilde{\mathcal{M}}$ such that $\pi \circ \tilde{p} = p$. Analogously, a *lift* of a cycle γ is a continuous mapping $\tilde{\gamma} : \mathbb{R} \to \tilde{\mathcal{M}}$ such that $\pi \circ \tilde{\gamma} = \gamma$. The main properties of $\tilde{\mathcal{M}}$ used in this paper are:

- the *lift property*: let p be a path in $\mathcal{M}$ with source point y; let $x \in \tilde{\mathcal{M}}$ be such that $\pi(x) = y$. Then there is a unique path $\tilde{p}$ in $\tilde{\mathcal{M}}$, starting at x, such that $\pi \circ \tilde{p} = p$;
- the *homotopy property*: two paths p_1 and p_2 with the same endpoints are homotopic in $\mathcal{M}$ if and only if they have two lifts $\tilde{p}_1$ and $\tilde{p}_2$ with the same endpoints in $\tilde{\mathcal{M}}$;
- the *intersection property*: a path p in $\mathcal{M}$ self-intersects if and only if either a lift of p self-intersects, or two lifts of p intersect.

If $\tilde{\gamma}$ is a lift of a cycle γ, and $k \in \mathbb{Z}$, note that $\tilde{\gamma}'(.) := \tilde{\gamma}(k + .)$ and $\tilde{\gamma}$ are identical as points sets. We therefore define the *geometric lifts* of a cycle γ to be all lifts of γ, identifying two lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ whenever there exists $k \in \mathbb{Z}$ such that $\tilde{\gamma}'(.) = \tilde{\gamma}(k + .)$. If γ is simple, the geometric lifts of γ correspond precisely to the connected components of $\pi^{-1}(\gamma)$. By definition, for paths, the sets of lifts and of geometric lifts coincide.

A *lifted set* $\mathcal{C}$ is a set of simple, pairwise disjoint curves on $\mathcal{M}$ which are:

- either non-contractible cycles in the interior of $\mathcal{M}$,
- or paths in $\mathcal{M}$, whose intersections with the boundary of $\mathcal{M}$ are precisely their endpoints,

together with the data, for each curve c in $\mathcal{C}$, of an enumeration $(c^\alpha)^{\alpha\in\mathbb{N}}$ of its geometric lifts in $\tilde{\mathcal{M}}$.

In the rest of this paper, $\mathcal{C}$ is a lifted set. The proof of the following lemma is a consequence of the Jordan–Schönflies theorem (see [1, p. 417]).

Lemma 4. *Each geometric lift of a curve in $\mathcal{C}$ separates $\tilde{\mathcal{M}}$ in two connected components.*

3.2 Crossing Words for Paths

We consider words on the alphabet made of letters of the form c^α and $\bar{c}^\alpha$, where $c \in \mathcal{C}$ and $\alpha \in \mathbb{N}$. Let w be a word. If w contains a subword $c^\alpha \bar{c}^\alpha$ or $\bar{c}^\alpha c^\alpha$, let w' be the word resulting from removing this subword from w; we say that w' is deduced from w by an *elementary c-reduction*. An *elementary reduction* is an elementary c-reduction for some c. A word w is *(c-)irreducible* if it can be applied no elementary (c-)reduction, it *(c-)reduces* to w' if w' can be obtained from w by successive elementary (c-)reductions.

Let $\tilde{p}$ be a path in $\tilde{\mathcal{M}}$. Walk along $\tilde{p}$ and, at each crossing encountered with a geometric lift c^α in $\mathcal{C}$, write down the symbol c^α or $\bar{c}^\alpha$, according to the orientation of the crossing (with respect to a fixed orientation of $\tilde{\mathcal{M}}$ – recall that $\tilde{p}$ and the elements of $\mathcal{C}$ are oriented). The word we obtain is called the *crossing word* of $\tilde{p}$ with $\mathcal{C}$, and denoted by $\mathcal{C}/\tilde{p}$.

In all this paper, the following situation will often occur: $\tilde{p}$ is a lift of a path p, and an elementary reduction is possible on $\mathcal{C}/\tilde{p}$. This reduction corresponds to two intersection points, a and a', of a lift $\tilde{c}$ of $\mathcal{C}$ with $\tilde{p}$. The subpaths associated with this possible elementary reduction are the parts $\tilde{c}_1$ and $\tilde{p}_1$ of $\tilde{c}$ and $\tilde{p}$ which are between a and a'. We will often remove these two crossings, by replacing $\tilde{p}_1$ by a path with the same endpoints, going along $\tilde{c}$, and obtaining by projection onto $\mathcal{M}$ a new path p' which crosses $\mathcal{C}$ twice less than p. Obviously, $\mathcal{C}/\tilde{p}'$ is deduced from $\mathcal{C}/\tilde{p}$ by proceeding to the elementary reduction; and the new path p' is homotopic to p. See Fig. 3.

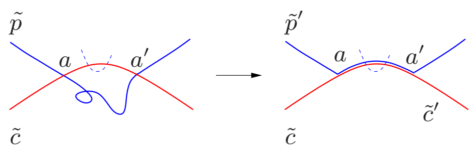

Fig. 3. The fundamental operation of uncrossing the parts of two curves c and p corresponding to an elementary reduction on $\mathcal{C}/\tilde{p}$. $\tilde{p}_1$ and $\tilde{c}_1$ are the parts of $\tilde{p}$ and $\tilde{c}$ between a and a'; $\tilde{p}_1$ is not necessarily simple, and $\tilde{c}_1$ can cross other pieces of $\tilde{p}$.

Lemma 5. *Any word w reduces to exactly one irreducible word.*

Of interest are the words which are *parenthesized*, i.e., those reducing to the empty word ε. The proof of the following lemma relies on Lemma 4.

Lemma 6. *Let $p : [0,1] \to \mathcal{M}$ be a contractible closed path, and let $\tilde{p}$ be a lift of p. Then, $\mathcal{C}/\tilde{p}$ is parenthesized.*

3.3 Crossing Words Sets for Cycles

The goal of this section is to define the analogue of the crossing word between $\mathcal{C}$ and a geometric lift $\tilde{\gamma}$ of a non-contractible cycle γ. A *lifted period* of $\tilde{\gamma}$ is a path which is the restriction of (an element of) $\tilde{\gamma}$ to $[a, a+1]$ for some $a \in \mathbb{R}$ (reparameterized over $[0,1]$). The *crossing words set* of $\tilde{\gamma}$ with $\mathcal{C}$, denoted by $[\mathcal{C}/\tilde{\gamma}]$, is the set of crossing words $\mathcal{C}/\tilde{p}$, over all lifted periods $\tilde{p}$ of $\tilde{\gamma}$ whose endpoints are not on lifts of $\mathcal{C}$. Our first task will be to show that the crossing words set $[\mathcal{C}/\tilde{\gamma}]$ is entirely determined once we know one of its elements.

We note that $\tilde{\gamma}$ induces a translation $\tau_{\tilde{\gamma}}$ in $\tilde{\mathcal{M}}$, as follows. Let $v \in \tilde{\mathcal{M}}$. Let $\tilde{p}$ be a lifted period of $\tilde{\gamma}$; consider a path β^0 joining $\tilde{p}(0)$ to v and call β^1 the lift of $\pi(\beta^0)$ starting at $\tilde{p}(1)$. The target v' of β^1 satisfies $\pi(v) = \pi(v')$; intuitively, $\tilde{\gamma}$ translates v to v'. It is readily seen that v' does not depend on the choice of β^0 and $\tilde{p}$. We therefore define $\tau_{\tilde{\gamma}}(v) := v'$. In particular, $\tau_{\tilde{\gamma}}$ sends a geometric lift of a curve $c \in \mathcal{C}$ to another geometric lift of c.

Define a permutation $\phi_{\tilde{\gamma}}$ over the set of words by: $\phi_{\tilde{\gamma}}(c^\alpha.w) = w.\tau_{\tilde{\gamma}}(c^\alpha)$, $\phi_{\tilde{\gamma}}(\bar{c}^\alpha.w) = w.\overline{\tau_{\tilde{\gamma}}(c^\alpha)}$, and $\phi_{\tilde{\gamma}}(\varepsilon) = \varepsilon$ (w is any word, and "." denotes concatenation).

Lemma 7. *For any word w in $[\mathcal{C}/\tilde{\gamma}]$, we have: $[\mathcal{C}/\tilde{\gamma}] = \{\phi^n_{\tilde{\gamma}}(w), n \in \mathbb{Z}\}$.*

If w is a word, we define $[w]_{\tilde{\gamma}}$ to be the set $\{\phi^n_{\tilde{\gamma}}(w), n \in \mathbb{Z}\}$. The sets of words having this form are called the $\tilde{\gamma}$*-words sets*. Note that $\phi_{\tilde{\gamma}}$ does not affect the length of a word, so that the *length* of a $\tilde{\gamma}$-words set is well-defined. Let W be a $\tilde{\gamma}$-words set. If there exists $w \in W$ containing a subword $c^\alpha\bar{c}^\alpha$ or $\bar{c}^\alpha c^\alpha$, denote by w' the word resulting from removing this subword from w; we say that W (which equals $[w]_{\tilde{\gamma}}$) *elementarily c-reduces* to $[w']_{\tilde{\gamma}}$.

Lemma and Definition 8. *Any $\tilde{\gamma}$-words set W c-reduces (resp. reduces) to exactly one c-irreducible (resp. irreducible) $\tilde{\gamma}$-words set. We define $g^{\tilde{\gamma}}_c(W)$ (resp. $g^{\tilde{\gamma}}(W)$) to be this $\tilde{\gamma}$-words set.*

When an elementary c-reduction is possible on $[\mathcal{C}/\tilde{\gamma}]$, exactly the same phenomenon occurs as in Fig. 3 (with $\tilde{\gamma}$ instead of $\tilde{p}$), and we may also proceed to the reduction by modifying γ, removing the two crossings.

Proposition 9. *Let γ be a cycle homotopic in $\mathcal{M}$ to some cycle γ' disjoint from $\mathcal{C}$. Let $\tilde{\gamma}$ be a geometric lift of γ. Then $g^{\tilde{\gamma}}([\mathcal{C}/\tilde{\gamma}]) = [\varepsilon]_{\tilde{\gamma}}$.*

Proof. Let p and p' be the restrictions of γ and γ' to $[0,1]$. There exists a path β joining $p(0)$ to $p'(0)$ such that the path $q := \beta^{-1}.p.\beta.p'^{-1}$ is contractible in $\mathcal{M}$. Let $\tilde{q}$ be a lift of q, concatenation of the inverse of β^0, $\tilde{p}$, β^1, and the inverse of $\tilde{p}'$ (respectively lifts of β, p, β, and p'). We choose $\tilde{q}$ so that $\tilde{p}$ is a lifted period of $\tilde{\gamma}$.

Since p' is disjoint from $\mathcal{C}$, $w := \mathcal{C}/\tilde{q}$ is the concatenation of $\mathcal{C}/(\beta^0)^{-1}$, $\mathcal{C}/\tilde{p}$, and $\mathcal{C}/\beta^1$. Furthermore, $\tau_{\tilde{\gamma}}(\beta^0)$ is equal to β^1; hence, if the kth symbol of $\mathcal{C}/\beta^0$ is equal to c^α (resp. $\bar{c}^\alpha$), then the kth symbol of $\mathcal{C}/\beta^1$ is equal to $\tau_{\tilde{\gamma}}(c^\alpha)$ (resp. $\overline{\tau_{\tilde{\gamma}}(c^\alpha)}$). It follows that $[w]_{\tilde{\gamma}}$ reduces to $[\mathcal{C}/\tilde{\gamma}]$. Now, by Lemma 6, w is parenthesized, so that $[w]_{\tilde{\gamma}}$ also reduces to $[\varepsilon]_{\tilde{\gamma}}$. Lemma 8 concludes. □

So far, we have introduced the notations $\mathcal{C}/\tilde{p}$ and $[\mathcal{C}/\tilde{\gamma}]$, where $\tilde{p}$ and $\tilde{\gamma}$ are lifts a path p and a cycle γ of $\mathcal{M}$, respectively. In the rest of this paper, when no risk of confusion arises, we will also use the notations $\mathcal{C}/p$ or $[\mathcal{C}/\gamma]$, meaning that we consider in fact $\mathcal{C}/\tilde{p}$ or $[\mathcal{C}/\tilde{\gamma}]$, where $\tilde{p}$ and $\tilde{\gamma}$ are *any* fixed geometric lifts of p or γ. Furthermore, the notation $[p/\gamma]$ will mean that we consider the crossing words set of γ with the geometric lifts of $\mathcal{C}$, this lifted set being made of only one path p in $\mathcal{M}$ with an arbitrary enumeration of its geometric lifts.

4 Curves on Pairs of Pants

In this section, we use crossing words to prove some basic facts regarding curves on pairs of pants.

Proposition 10. *Let K be a cylinder or a pair of pants, and γ be a cycle homotopic to a boundary of K. There exists a* simple *cycle homotopic to and not longer than γ.*

Proof. We will only give a proof when K is a pair of pants; the proof of the case where K is a cylinder is simpler. Let p be a shortest path between the two boundaries of K which are not homotopic to γ; let C be the cylinder obtained when cutting K along p; and let p' be a shortest path in C between its two boundaries.

$[p/\gamma]$ reduces to $[\varepsilon]_{\tilde{\gamma}}$ by Proposition 9; if it is not empty, let $\tilde{\gamma}_1$ and $\tilde{p}_1$ be the subpaths of lifts of γ and p corresponding to an elementary reduction. Since p is a shortest path, $|\tilde{p}_1| \leq |\tilde{\gamma}_1|$, and we can, like in Fig. 3, proceed to the elementary reduction by changing γ to another cycle, which is homotopic to and not longer than γ, and has two crossings less than γ with p. By induction, we obtain that there exists a cycle γ', homotopic to and not longer than γ, which does not cross p. Using similar techniques, we prove that there exists a cycle γ'', homotopic to and not longer than γ, which does not cross p and crosses p' only once, say at some point a.

Cutting C along p', we obtain two copies a' and a'' of a, and γ is transformed into a path between a' and a''. Hence, a shortest path between a' and a'' leads to a cycle in C which is simple, not longer than γ, and homotopic to γ in K. □

Proposition 11. *Let K be a cylinder or a pair of pants, and γ be one boundary of K. Assume γ is a shortest cycle among the simple cycles homotopic to γ. Let q be a path in K whose endpoints are on γ and which is homotopic to a path whose range (set of values) is included in the range of γ. Then the shortest path on γ homotopic to q is not longer than q.*

The proof is omitted and relies on similar ideas as the proof of Proposition 10.

Proposition 12. *Let s be a (doubled) pants decomposition of $\mathcal{M}$. Assume that a cycle γ is inside one component K of $\mathcal{M} \setminus s$ (a cylinder or a pair of pants), and homotopic in $\mathcal{M}$ to a cycle s_k. Then γ is homotopic,* in K, *to one boundary of K.*

Lemma 13. *Any cycle inside K which is contractible (in $\mathcal{M}$) is also contractible in K.*

Proof (of Proposition 12). Again, we assume that K is a pair of pants. Let S be a lifted set whose curves are the cycles in s, with an arbitrary enumeration of the geometric lifts. Let γ' be a simple cycle, disjoint from all cycles of s, and homotopic in $\mathcal{M}\backslash(s\backslash s_k)$ to s_k. Let p and p' be the restrictions of γ and γ' to $[0,1]$. There exists a path β joining $p(0)$ to $p'(0)$ such that the path $q := \beta^{-1}.p.\beta.p'^{-1}$ is contractible in $\mathcal{M}$. Without loss of generality, assume that S/β is irreducible. If this crossing word is empty, then q is contractible in K by Lemma 13, hence γ and γ' are homotopic in K; so are γ and s_k, and the proof is complete. Assume this crossing word is non-empty.

Since p and p' do not cross s, S/q is the concatenation of $S/(\beta^0)^{-1}$ and S/β^1, where β^0 is a lift of β, and $\beta^1 = \tau_{\tilde{\gamma}}(\beta^0)$. Because S/β is irreducible and S/q can be elementarily reduced, the first lifts of S crossed by β^0 and β^1 must be the same, say s_j^α. Let β' be the beginning of β before its first crossing with s; we get that $\beta'^{-1}.p.\beta'$ is homotopic to a power of s_j in $\mathcal{M}$, hence also in K by Lemma 13. By [3, Theorem 4.2], the nth power of s_j is homotopic to no simple cycle if $|n| \geq 2$. Hence γ is homotopic, in K, to s_j or its reverse. □

5 Proof of Theorem 2

Let b denote the set of boundary cycles of $\mathcal{M}$. As for a lifted set, we will assume that the geometric lifts of the cycles in b are enumerated. Consider now a doubled pants decomposition s of $\mathcal{M}$. We define a lifted set S whose curves in $\mathcal{M}$ are the curves in s.

Note that a cycle in s and its twin (in s or in b) are homotopic disjoint cycles, hence bound a cylinder by [3, Lemma 2.4]; the lifts of this cylinder in $\tilde{\mathcal{M}}$ are disjoint infinite strips which contain no geometric lift of s or b in their interior. Let $k \in [1,N]$, and let $s_{k'}$ or $b_{k'}$ be the twin of s_k (depending on whether it is an element of s or b). We can choose the enumeration of the geometric lifts of s_k so that, for each α, s_k^α and $s_{k'}^\alpha$ (or $b_{k'}^\alpha$) bound a strip which contains no lift of s and b in its interior.

Fix $j \in [1,N]$; let $r = f_j(s)$. We consider the lifted set R whose curves are the cycles in r; the enumeration of the geometric lifts of r is as follows. If $k \neq j$, then $r_k^\alpha = s_k^\alpha$ for any $k \in \mathbb{N}$. For the enumeration of the geometric lifts of r_j, we note that r_j and its twin, $r_{j'}$ (or $b_{j'}$), bound a cylinder in $\mathcal{M} \setminus r$; as above, we choose r_j^α so that r_j^α and $r_{j'}^\alpha = s_{j'}^\alpha$ (or $b_{j'}^\alpha$) bound an infinite strip containing no lift of r or b in its interior.

Finally, fix $i \in [1,N]$; let t_i be a shortest cycle among all cycles homotopic to s_i, and $\tilde{t}_i$ be a geometric lift of t_i. Henceforth, the words on the lifted sets R and S will be written differently as above, by omitting the "r" and the "s" (for example, we shall write ${}^3_1\,\overline{{}^7_5}\,\overline{{}^4_2}$ instead of $s_1^3\,\overline{s_5^7}\,\overline{s_2^4}$). This allows to say, for example, that $[R/\tilde{t}_i] = [S/\tilde{t}_i]$ if t_i does not cross r_j nor s_j. Let $\mathcal{P}_j$ be the pair of pants bounded by $s \setminus s_j$ in which s_j and r_j are.

Proposition 14. $g_j^{\tilde{t}_i}([R/\tilde{t}_i]) = g_j^{\tilde{t}_i}([S/\tilde{t}_i])$.

We note S_j the lifted set made of the cycle s_j, with the enumeration of its geometric lifts induced by S; the same holds for R_j.

Lemma 15. *Let p be a path in $\mathcal{P}_j$ whose endpoints are on the boundary of $\mathcal{P}_j$, and $\tilde{p}$ be a lift of p. Then $S_j/\tilde{p}$ and $R_j/\tilde{p}$ reduce to the same irreducible word.*

Proof (of Proposition 14). Assume first that t_i is contained in $\mathcal{P}_j$. By Proposition 9, we have $g^{\tilde{t}_i}([R/\tilde{t}_i]) = [\varepsilon]_{\tilde{t}_i} = g^{\tilde{t}_i}([S/\tilde{t}_i])$. But this also equals $g_j^{\tilde{t}_i}([R/\tilde{t}_i])$ and $g_j^{\tilde{t}_i}([S/\tilde{t}_i])$, and this concludes the proof. If t_i is not entirely contained in $\mathcal{P}_j$, then let t'_i be a maximal subpath of t_i which is inside $\mathcal{P}_j$, and $\tilde{t}'_i$ be a lift of t'_i; it is sufficient to prove that $R_j/\tilde{t}'_i$ and $S_j/\tilde{t}'_i$ reduce to the same irreducible word; but this follows from Lemma 15. □

Proposition 16. *There exists a cycle t'_i, homotopic to and not longer than t_i, and a geometric lift $\tilde{t}'_i$ of t'_i, such that $\tau_{\tilde{t}_i} = \tau_{\tilde{t}'_i}$ and $[R/\tilde{t}'_i] = g_j^{\tilde{t}_i}([S/\tilde{t}_i])$.*

Proof. By Proposition 14, $[R/\tilde{t}_i]$ j-reduces to $g_j^{\tilde{t}_i}([S/\tilde{t}_i])$. If $[R/\tilde{t}_i]$ is j-irreducible, there is nothing to show. Otherwise, an elementary j-reduction is possible on $[R/\tilde{t}_i]$. We can apply Proposition 11 to the subpath of t_i corresponding to this j-reduction, and apply the uncrossing operation to t_i. We obtain a geometric lift $\tilde{t}'_i$ of a cycle t'_i which is homotopic to and not longer than t_i. Clearly, $\tau_{\tilde{t}_i} = \tau_{\tilde{t}'_i}$ (which implies that $g^{\tilde{t}_i} = g^{\tilde{t}'_i}$). Furthermore, $[R/\tilde{t}'_i]$ results from $[R/\tilde{t}_i]$ by this elementary j-reduction. By induction, we obtain the desired t'_i. □

Proposition 17. *Assume t_i is disjoint from s, and that t_i and s_k are homotopic in the cylinder or pair of pants of $\mathcal{M} \setminus s$ containing t_i. Then, there exists a cycle t'_i, homotopic to and not longer than t_i, which is disjoint from r, and which is homotopic to r_k in the cylinder or pair of pants of $\mathcal{M} \setminus r$ containing t'_i.*

The (omitted) proof relies on Propositions 10 and 11. We now conclude the proof of our main theorem.

Proof (of Theorem 2). We consider a lifted set S^0 whose curves are s^0, the enumeration of the geometric lifts being chosen as described at the beginning of this section. By induction on $n \in \mathbb{N}$, we construct a lifted set S^n whose set of curves is s^n, with the enumeration of the lifts being chosen also as in the beginning of the section.

Let $\tilde{t}_i^0$ be a lift of a shortest cycle t_i^0 homotopic to s_i^0. By Proposition 9, $[S^0/\tilde{t}_i^0]$ reduces to $[\varepsilon]_{\tilde{t}_i^0}$. By Proposition 16, we can construct a sequence $(\tilde{t}_i^n)$ of lifts of shortest homotopic cycles such that the length of $[S^n/\tilde{t}_i^n]$ strictly decreases until it becomes empty at some stage n. By Proposition 12, t_i^n and a cycle s_k^n are homotopic in the cylinder or pair of pants of s^n containing t_i^n. By $k-1$ applications of Proposition 17, and then using Proposition 10, $|s_k^{n+1}| = |t_i^n|$. The cycle s_k^{n+1} is either s_i^{n+1} or its twin; in the latter case, since s_i^{n+1} and s_k^{n+1} bound a cylinder, $|s_i^{n+2}| = |s_k^{n+2}| = |t_i^n|$. From this discussion, it follows that the length of $(s_i^n)_{n\in\mathbb{N}}$ becomes stationary. It remains to prove that all lengths remain unchanged once s^n and s^{n+1} have the same lengths. The proof of this fact uses the same tools as above, and is omitted from this version.[2] □

6 Computational Issues

We describe here the combinatorial framework (which is similar to the one described in [2]) used to perform the optimization process.

6.1 Edge-Ordered Set of Cycles

We (temporarily) view G, the vertex-edge graph of $\mathcal{M}$, as a directed graph: each edge of G is replaced by two opposite directed edges. An *edge-ordered set of cycles* (EOSC for short) S on the graph G is a set of closed walks (without basepoint) in G, with the data, for each oriented edge e of G, of an order $\preceq_e$ over all edges of the walks in S corresponding to e or $-e$ (edge e with opposite orientation). These orders should be consistent in the following sense: $a \preceq_e b$ if and only if $b \preceq_{-e} a$. Intuitively, $a \preceq_e b$ if and only if a is on the left of b on e.

Let v be a vertex of G, and $e_1, \ldots, e_n$ be the clockwise-ordered list of oriented edges of G whose source is v. We define a cyclic order $\preceq_v$ over the edges of the walks in S meeting at v, by enumerating its elements in this order: first, the edges of the walks in S on e_1 or $-e_1$, in $\preceq_{e_1}$-order; then the edges of the walks in S on e_2 or $-e_2$, in $\preceq_{e_2}$-order; and so on. We say that two subpaths of length two, a_1, a_2 and b_1, b_2, of walks in S *cross* if the targets of a_1 and b_1 are the same vertex, v, and if, in the cyclic order $\preceq_v$, a_1 and a_2 separate b_1 and b_2. The EOSC S is *simple* if no crossing occurs in S.

Clearly, a set of disjoint simple cycles s can be retracted onto G to get a simple EOSC $S = \rho(s)$ (see Fig. 2AB), this retraction preserving the lengths (with the appropriate definitions) and homotopy classes of the cycles. The converse is also true: the closed walks of a simple EOSC S can be expanded along the edges to get a set of disjoint, simple cycles $s \in \rho^{-1}(S)$.

6.2 Computation of Shortest Paths

By the proof of Proposition 10, we can proceed to an Elementary Step f_j as follows: find the pair of pants $\mathcal{P}_j$ of $s \setminus s_j$ that contains s_j; find a shortest path p between the two boundaries of $\mathcal{P}_j$ which are not homotopic to s_j, and a shortest path p' between the two boundaries of $C := \mathcal{P}_j \setminus p$. Cutting C along p' yields a topological disk D, where points of p' on C correspond to pairs of points on D. The solution is found by considering the shortest paths between all such pairs of points and taking the shortest of these shortest paths.

[2] Note that this fact is unnecessary if we modify the algorithm as follows: if $f_i(s)_i$ has the same length as s_i, we choose $f_i(s)_i$ to be equal to s_i.

From an algorithmic point of view, let $S = \rho(s)$ be a simple EOSC. Define $G(\mathcal{P}_j)$ to be the weighted graph whose vertices are the components of $\mathcal{P}_j \setminus G^*$ and whose edges join two vertices separated by (a piece of) an edge e^* of G^*; such an edge has the same weight as e. This graph represents the vertex-edge graph of the surface $\mathcal{M}$ after cutting along the cycles of S. Its construction is easy and skipped in this abstract (see Fig. 2C). We can perform the computation of a shortest homotopic cycle in $\mathcal{P}_j$ by translating the operations of the previous paragraph to this combinatorial framework.

6.3 Complexity Analysis

Let R be a simple EOSC. For each vertex v of $\mathcal{M}$, consider the parenthesized word formed by the pairs of consecutive edges of R meeting at v. The *multiplicity* of vertex v (w.r.t. R) is the maximal number of nested parentheses in (any cyclic permutation of) this expression. Thus any 2-path a_1, a_2 crosses R a number of times which is at most the multiplicity of R at the target of a_1.

Let n be the complexity of $\mathcal{M}$ (total number of vertices, edges, and faces), g its genus and b its number of boundaries. Let α be the longest-to-shortest edge ratio of $\mathcal{M}$. Let s be a doubled pants decomposition of $\mathcal{M}$ composed of $O(g+b)$ cycles, and $S = \rho(s)$. Let μ be the maximum, over $j \in [1, N]$ and the vertices v of $\mathcal{M}$, of the multiplicity of S_j at v.

Bounding the number of Elementary Steps in the algorithm reduces to bounding the maximum, over i, of the minimal number of crossings between s_i and a shortest homotopic loop t_i. Doing so, we obtain (using Dijsktra's algorithm for the computation of shortest paths):

Theorem 18. *This algorithm computes an optimal pants decomposition homotopic to s in $O((g+b)^2\alpha^3\mu^4n^3\log(\alpha\mu n))$ time.*

Finally, assuming $\mathcal{M}$ is triangulated, and given an EOSC made of a single simple cycle γ, with multiplicity μ, we can compute a doubled pants decomposition containing γ which has multiplicity $O(\mu)$ (details omitted). This implies that computing a shortest cycle homotopic to γ is possible in time $O((g+b)^2\alpha^3\mu^4n^3\log(\alpha\mu n))$.

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