Bounds and Methods for *k*-Planar Crossing Numbers

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Abstract. The k-planar crossing number of a graph is the minimum number of crossings of its edges over all possible drawings of the graph in k planes. We propose algorithms and methods for k-planar drawings of general graphs together with lower bound techniques. We give exact results for the k-planar crossing number of $K_{2k+1,q}$, for $k \geq 2$. We prove tight bounds for complete graphs.

1 Introduction

Let $\mathrm{cr}(G)$ denote the standard crossing number of a graph G, i.e. the minimum number of crossings of its edges over all possible drawings of G in the plane. For $k \geq 2$, define the k-planar crossing number as

$$\mathrm{cr}_k(G) = \min\{\mathrm{cr}(G_1) + \mathrm{cr}(G_2) + ... + \mathrm{cr}(G_k)\},$$

where the minimum is taken over all edge disjoint subgraphs $G_i = (V, E_i)$, $i = 1, 2, ..., k$, so that $E = E_1 \cup E_2 \cup ... \cup E_k$.

Motivated by printed circuit boards, Owens [9] introduced the *biplanar crossing number* of a graph G, i.e. the case $k = 2$. He described a biplanar drawing of the complete graph K_n with $\mathrm{cr}_2(K_n) \leq 7n^4/1536 + O(n^3)$. A survey on biplanar crossing numbers is in [5]. Determining $\mathrm{cr}_k(G)$ has application to the design of multilayer VLSI circuits [1].

⋆ This research was supported by the NSF grant CCR9988525.
⋆⋆ Work supported by the EPSRC grant GR/R37395/01.
⋆⋆⋆ This author was visiting the National Center for Biotechnology Information, NLM, NIH, with the support of the Oak Ridge Institute for Science and Education. This research was partially supported by the NSF contract 007 2187.
† Supported partially by the VEGA grant No. 02/3164/23.

G. Liotta (Ed.): GD 2003, LNCS 2912, pp. 37–46, 2004.

Much of this paper extends ideas of the papers [5] and [12] investigating the biplanar crossing number to the k-planar crossing number. Section 2 gives general bounds for the k-planar crossing number and exposes an important extremal problem: how does $\mathrm{cr}_k(G)$ decrease when k increases?

Section 3 yields unexpected exact results for the k-planar crossing number of some complete bipartite graphs. Complete bipartite graphs $K_{p,q}$ are also the best studied graphs with respect to planar crossing numbers. Exact results are known only for $p \leq 6$ and arbitrary q, [6]. Crossing numbers of bipartite graphs drawn on surfaces of higher geni were determined only for $p \leq 3$, and arbitrary q, [10]. Thus our results belong to the same rare class of exact results on crossing numbers (for bipartite graphs), and are direct extensions of the results of [5] for $\mathrm{cr}_2(K_{5,n})$ and $\mathrm{cr}_2(K_{6,n})$. We spell out the results in more details. Recall that the *thickness* $\theta(G)$ of G is the minimum number of planar graphs whose union is G. By definition, $\mathrm{cr}_k(G) = 0$ if and only if $\theta(G) \leq k$. Beineke et al. [4] proved that the thickness of $K_{p,q}$ is given by

$$\theta(K_{p,q}) = \left\lceil \frac{pq}{2(p+q-2)} \right\rceil, \tag{1}$$

except, possibly, when $p \leq q$ are both odd and there exists an integer k such that $\frac{1}{4}(p+5) \leq k \leq \frac{1}{2}(p-3)$ and $q = \lfloor 2k(p-2)/(p-2k) \rfloor$. According to (1) $\mathrm{cr}_k(K_{2k,q}) = 0$, for $k \geq 2$ and any q, so the first interesting bipartite graph is $K_{2k+1,q}$. We prove that for $k \geq 2, q \geq 1$

$$\mathrm{cr}_k(K_{2k+1,q}) = \left\lfloor \frac{q}{2k(2k-1)} \right\rfloor \left(q - k(2k-1) \left\lfloor \frac{q}{2k(2k-1)} \right\rfloor - k(2k-1) \right)$$

and for $k \geq 2$, and $1 \leq q \leq 4k^2$

$$\mathrm{cr}_k(K_{2k+2,q}) = 2 \left\lfloor \frac{q}{2k^2} \right\rfloor \left(q - k^2 \left\lfloor \frac{q}{2k^2} \right\rfloor - k^2 \right).$$

Section 4 improves on the general bounds for the k-planar crossing numbers of complete and complete bipartite graphs. The improvement means constant multiplicative factors.

2 General Bounds

Little is known about lower bounds for the k-planar crossing number in general. Some of the lower bounds for crossing numbers, *mutatis mutandis* apply to k-planar crossing numbers. For example, if $G = (V, E)$, $|V| = n, |E| = m$, then the lower bound resulting from Euler's formula, $\mathrm{cr}(G) \geq m - 3n + 6$ for $n \geq 3$, generalizes to

$$\mathrm{cr}_k(G) \geq m - k(3n-6).$$

There is a strengthening of the lower bound resulting from Euler's formula for graphs G with girth g, $\mathrm{cr}(G) \geq m - g(n-2)/(g-2)$ for $n \geq g$; and we get

$$\mathrm{cr}_k(G) \geq m - \frac{gk}{g-2}(n-2). \tag{2}$$

We state a k-planar version of Leighton's Lemma [7] for crossing numbers (note that we do not go for the best constants here, since the best constant is always getting improved even for the ordinary crossing number).

Lemma 1. *For a simple graph G with n vertices and m edges, we have $m \le 6kn$, or*

$$\mathrm{cr}_k(G) \ge \frac{1}{64} \cdot \frac{m^3}{n^2k^2}. \tag{3}$$

Proof. Recall Leighton's Lemma for the ordinary crossing number: $m \le 4n$ or $\mathrm{cr}(G) \ge m^3/64n^2$. Consider an optimal k-planar drawing of G, such that G_i is the subgraph drawn on the i^{th} plane. Assume that the first x graphs have at most $4n$ edges, while the last $k - x$ graphs have more. We have

$$\mathrm{cr}_k(G) \ge \sum_{i=x+1}^{k} \mathrm{cr}(G_i) \ge \sum_{i=x+1}^{k} \frac{m_i^3}{64n^2} \ge$$

$$\ge \frac{k-x}{64n^2} \cdot \left(\frac{\sum_{i=x+1}^{k} m_i}{k-x} \right)^3 \ge \frac{1}{64n^2} \cdot \frac{(m-4nx)^3}{(k-x)^2} \ge \frac{1}{64} \cdot \frac{m^3}{n^2k^2},$$

where the last inequality holds for $m \ge 6kn$ according to the sign of the derivative. □

Recall that $a(G)$, or arboricity of G, is the minimum number of acyclic subgraphs whose union covers E. By a well known theorem of Nash-Williams [8]

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H) - 1} \right\rceil$$

where the maximum is taken over all subgraphs H of G, with $m(H)$ edges and $n(H)$ vertices. It is easily seen that $a(G) \ge \theta(G)$, moreover, $\theta(G) \ge \lceil a(G)/3 \rceil$, since $m(H) \le 3n - 6$ for any planar graph.

Let $P = \{V_1, V_2, ...V_t\}$ be a partition of V. We denote by E_{ij} the set of edges with one end point in V_i and the other in V_j, hence E_{ii} denotes the set of all edges with both end points in V_i, for $1 \le i \le t$. Let H denote the t vertex graph that is obtained by contracting all vertices in V_i into one single vertex and removing the multiple edges. We call H the *mate* of G with respect to P, or simply the mate of G. Let $T_1, T_2, ...T_{a(H)}$, be a decomposition of the edge set of H into acyclic subgraphs of H. Let $d_i(x)$ denote the degree of $x \in V(H)$ in T_i, $i = 1, 2, ...k$.

Theorem 1. *Let $G = (V, E)$, and let k be a given integer. Let $\{V_1, V_2, ...V_t\}$ be a partition of V and let $H = (V(H), E(H))$ denote the mate of G. If $k \ge a(H)$, then we can construct in polynomial time a k-planar drawing of G with at most*

$$tp^2 + 2pq|E(H)| + \sum_{i=1}^{k} p^2 \sum_{x \in V(H)} d_i^2(x)$$

crossings, where $p = \max\{|E_{ii}|\}$ and $q = \max\{|E_{ij}|\}$, $i, j = 1, 2, ...t$.

Proof Sketch. Consider a drawing of each $T_i, i = 1, 2, ..., k$ in plane i, with no crossings, so that the vertices are placed in the corners of a convex polygon, and each edge is drawn using one straight line segment. Now, replace each vertex $j \in V(H)$ with the set V_j. In particular, place the vertices of V_j in a very small neighborhood around j. Next, draw the edges in E with straight line segments using the drawings of T_i's, to produce a k-planar drawing of G. There will be 3 kinds of crossings:
(a) between edges of E_{ii},
(b) between edges of E_{ii}, and edges of $E_{ij}, i \neq j$, and finally
(c) between edges E_{ij}, where $i, j = 1, 2, ...k$ and $i \neq j$.
The terms in the theorem correspond to these 3 cases. Note that the estimate for (b) is $pq \sum_{i=1}^{k} \sum_{x \in V(H)} d_i(x) = 2pq|E(H)|$. □

Theorem 1 can be used effectively, if the degrees appearing in the last term are small. In fact, in certain cases one can decompose G into a number of (cyclic) outer planar graphs of small maximum degree, and still use the method of Theorem 1 to obtain upper bounds for $\text{cr}_k(G)$. In this paper, we have obtained exact values of $\text{cr}_k(G)$ for certain graphs in this way. Nonetheless, the acyclic decompositions into forests of small maximum degree has been also studied. Let $a_d(G)$ denote the degree bounded arboricity, that is the minimum number of forests that the edges of G can be decomposed to so that the maximum degree of each forest is bounded by d. Truszczýnski [14] conjectured that for every multigraph G and $d \geq 2$,

$$a_d(G) = \begin{cases} \Delta(G)/d \text{ or } \ 1 + \Delta(G)/d \text{ if } a(G) = \Delta(G)/d, \\ \max\left(a(G), \lceil \Delta(G)/d \rceil\right) \text{otherwise.} \end{cases} \quad (4)$$

Truszczýnski actually proved his conjecture for complete and complete bipartite graphs, and also for the case $d \geq \Delta(G) + 1 - a(G)$. Combining Theorem 1 with (4), we immediately obtain

Corollary 1. *For* $n \geq 1$ $\text{cr}_k(K_n) = O(n^4/k^2)$.

However, Corollary 1 also follows from the next theorem:

Theorem 2. *For any graph G on n vertices and m edges,*

$$\text{cr}_k(G) \leq \frac{1}{12k^2}\left(1 - \frac{1}{4k}\right) m^2 + O\left(\frac{m^2}{kn}\right).$$

The corresponding drawing can be found in polynomial time. For any graph G,

$$\text{cr}_k(G) \leq \frac{2\text{cr}(G)}{k^{\log_2 \frac{8}{3}}} = \frac{2\text{cr}(G)}{k^{1.4708...}}.$$

Proof. The first upper bound follows from our paper [11] (Corollary 3.2) and a simple observation that a drawing of a graph G in $2k$ pages gives a drawing of the graph G in k planes. The second upper bound follows by iteration from the inequality $\text{cr}_2(G) \leq \frac{3}{8}\text{cr}(G)$, proved in [5]. □

One challenging question is how cr_k changes from $\text{cr}(G)$ to 0, as k increases from 1 to the thickness of G, $\theta(G)$.

3 Exact Results

Theorem 3. *For $k \geq 2, q \geq 1$*

$$\mathrm{cr}_k(K_{2k+1,q}) = \left\lfloor \frac{q}{2k(2k-1)} \right\rfloor \left(q - k(2k-1) \left(\left\lfloor \frac{q}{2k(2k-1)} \right\rfloor + 1 \right) \right). \quad (5)$$

Proof. *Upper bound.* Beineke [2] proved that the thickness of $K_{2k+1,2k(2k-1)}$ is k by describing a drawing of $K_{2k+1,2k(2k-1)}$ in k planes without crossings. We extend this drawing to a drawing of $K_{2k+1,q}$ in k planes with minimum number of crossings. Let $u_1, u_2, ..., u_{2k+1}$ be the vertices of the first partition. Let $v_1, v_2, ..., v_{2k(2k-1)}$ be the vertices of the second partition. Beineke's drawing possesses the following properties.

1. On every plane, all v_i's lie on the vertices of the regular $2k(2k-1)$-gon.
2. All u_j's lie inside or outside of the polygon.
3. The edges do not cross.
4. For every v_i, its degree on exactly one plane is 3 and 2 on the remaining $(k-1)$ planes. Moreover, on every plane, the vertex v_i has a neighbor inside and a neighbor outside the polygon.

Fig. 1 shows the case $k = 3$, i.e. a drawing of $K_{7,30}$ in 3 planes without crossings. The graphs on the same row are drawn on the same plane. The left (right) graph corresponds to the inside (outside) part of the drawing on a plane.

Now consider $K_{2k+1,q}$ and assume that $q = 2k(2k-1)a + b$, where a, b are integers and $0 \leq b < 2k(2k-1)$. Partition the q-vertices into $2k(2k-1)$ almost equal sets $S_1, S_2, ..., S_{2k(2k-1)}$, where $2k(2k-1) - b$ sets have a vertices, and b sets have $a+1$ elements. On every plane, replace each vertex v_j by the set S_j such that its vertices lie on a very short arc and the arcs do not interfere. Join every vertex S_i to all vertices of S_j on a plane iff u_i was adjacent to v_j on that plane in the Beineke's drawing. Clearly the total number of crossings is

$$\sum_{j=1}^{2k(2k-1)} \binom{|S_j|}{2}.$$

The above sum turns to

$$b\binom{a+1}{2} + (2k(2k-1) - b)\binom{a}{2} = a(b + k(2k-1)(a-1)),$$

which gives the claimed value by substituting $a = \lfloor q/(2k(2k-1)) \rfloor$ and $b = q - 2k(2k-1)a$.

Lower bound. We will proceed by induction on q. The claim is obviously true for $q \leq 2k(2k-1)$. The claim is also true for $2k(2k-1) \leq q \leq 4k(2k-1)$ as the RHS of (5) equals $q - 2k(2k-1)$, which is a lower bound given by (2). Hence assume that the claim is true for some $q \geq 4k(2k-1)$. Using the counting argument with $H = K_{2k+1,q}$, $G = K_{2k+1,q+1}$, i.e. counting the number of crossings

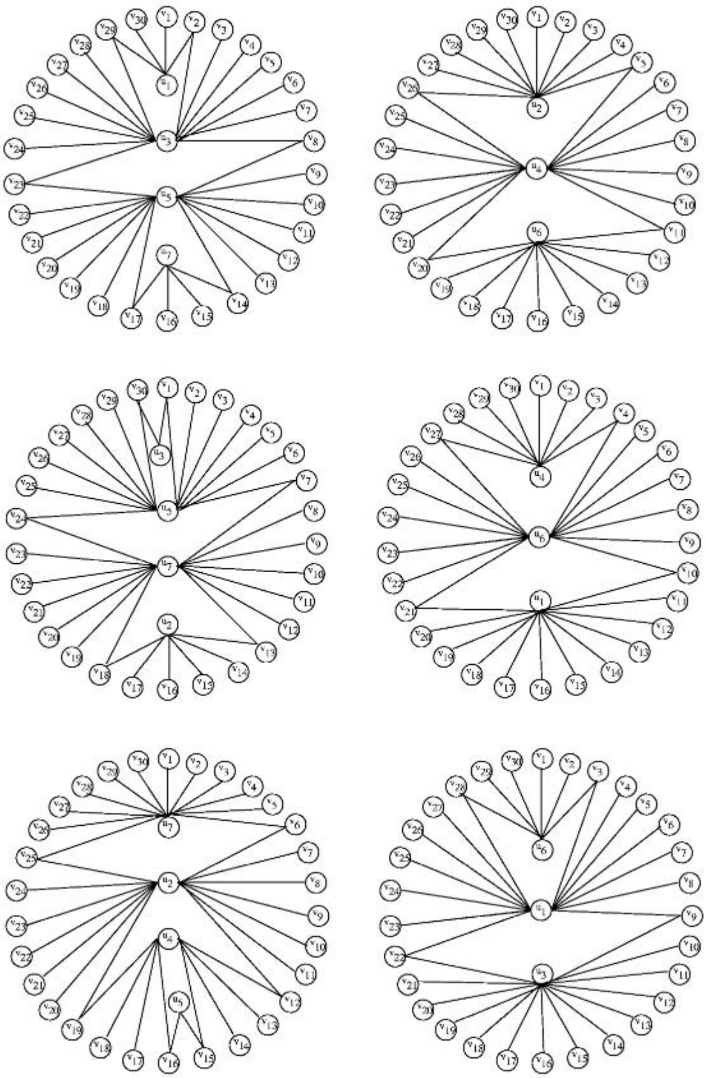

Fig. 1. A drawing of $K_{7,30}$ in 3 planes without crossings.

produced by all occurrencies of H in G and dividing it by the multiplicity of each crossing, we have

$$\mathrm{cr}_k(K_{2k+1,q+1}) - \left\lfloor \frac{q+1}{2k(2k-1)} \right\rfloor \left(q+1-k(2k-1)\left(\left\lfloor \frac{q}{2k(2k-1)} \right\rfloor + 1 \right) \right)$$

$$\geq \left\lceil \frac{\binom{q+1}{q}}{\binom{q-1}{q-2}} \mathrm{cr}_k(K_{2k+1,q}) \right\rceil - \left\lfloor \frac{q+1}{2k(2k-1)} \right\rfloor \left(q+1-k(2k-1)\left(\left\lfloor \frac{q}{2k(2k-1)} \right\rfloor + 1 \right) \right)$$
$$\geq \left\lceil \frac{q+1}{q-1} \left\lfloor \frac{q}{2k(2k-1)} \right\rfloor \left(q-k(2k-1)\left(\left\lfloor \frac{q}{2k(2k-1)} \right\rfloor + 1 \right) \right) \right.$$
$$\left. - \left\lfloor \frac{q+1}{2k(2k-1)} \right\rfloor \left(q+1-k(2k-1)\left(\left\lfloor \frac{q}{2k(2k-1)} \right\rfloor + 1 \right) \right) \right\rceil .$$

To conclude the proof, it is sufficient to show that for $q \geq 4k(2k-1)$ the expression inside the big brackets of the last line is greater than -1. Let $q = 2k(2k-1)a + b$, as above. Distinguish two cases.
If $b < 2k(2k-1) - 1$ then the expression inside the big brackets equals

$$\frac{q+1}{q-1} a(q-k(2k-1)(a+1)) - a(q+1-k(2k-1)(a+1)) = \frac{-a-b}{q-1} > -1.$$

If $b = 2k(2k-1) - 1$ then the expression inside the big brackets equals

$$\frac{q+1}{q-1} a(q-k(2k-1)(a+1)) - (a+1)(q+1-k(2k-1)(a+2)) = 0.$$

□

Theorem 4. *For $k \geq 2$*

$$\mathrm{cr}_k(K_{2k+2,q}) \leq 2 \left\lfloor \frac{q}{2k^2} \right\rfloor \left(q - k^2 \left\lfloor \frac{q}{2k^2} \right\rfloor - k^2 \right). \tag{6}$$

The equality holds for $1 \leq q \leq 4k^2$.

Proof. *Upper bound.* We start with a drawing of $K_{2k+2,2k^2}$ in k planes without crossings and then extend this drawing to a drawing of $K_{2k+2,q}$. Denote the vertices of the first partition class by $u_1, u_2, ..., u_{k+1}$ and $v_1, v_2, ..., v_{k+1}$. Denote the vertices of the second partition class by $a_0, a_2, ..., a_{k^2-1}$ and $b_0, b_1, ..., b_{k^2-1}$.

On the first plane, place the vertices $u_1, u_2, ..., u_{k+1}$ (resp. $v_1, v_2, ..., v_{k+1}$) on the positive (resp. negative) part of the x axis, in this order from the origin. Place the vertices $a_0, a_1, ..., a_{k^2-1}$ (resp. $b_0, b_1, ..., b_{k^2-1}$) on the positive (resp. negative) part of the y axis, in this order from the origin. Join u_i and v_i to $a_{(i-1)(k-1)}, ..., a_{ik-i}$ and $b_{(i-1)(k-1)}, ..., b_{ik-i}$, for all i. On the second plane, the positions of u_i's and v_i's remain unchanged. Shift a_j's (resp. b_j's) cyclically up (down) by k position. Join u_i and v_i to $a_{(i-2)(k-1)}, ..., a_{(i-1)(k-1)}$ and $b_{(i-2)(k-1)}, ..., b_{(i-1)(k-1)}$, where the indices are computed modulo k^2. Continuing in this drawing for all planes we get a drawing of $K_{2k+2,2k^2}$ in k planes without crossings. See Fig. 2 for the case $k = 3$.

Now consider $K_{2k+2,q}$. Partition the q vertices into $2k^2$ almost equal sets, $A_0, A_1, ..., A_{k^2-1}$ and $B_0, B_1, ..., B_{k^2-1}$. Replace every a_j and b_j by A_j and B_j and join u_i and v_i to A_j and B_j on a plane iff u_i and v_i were adjacent to a_j and b_j on that plane. A simple counting shows that the number of crossings is

$$\sum_{j=0}^{k^2-1} 2 \left(\binom{|A_j|}{2} + \binom{|B_j|}{2} \right).$$

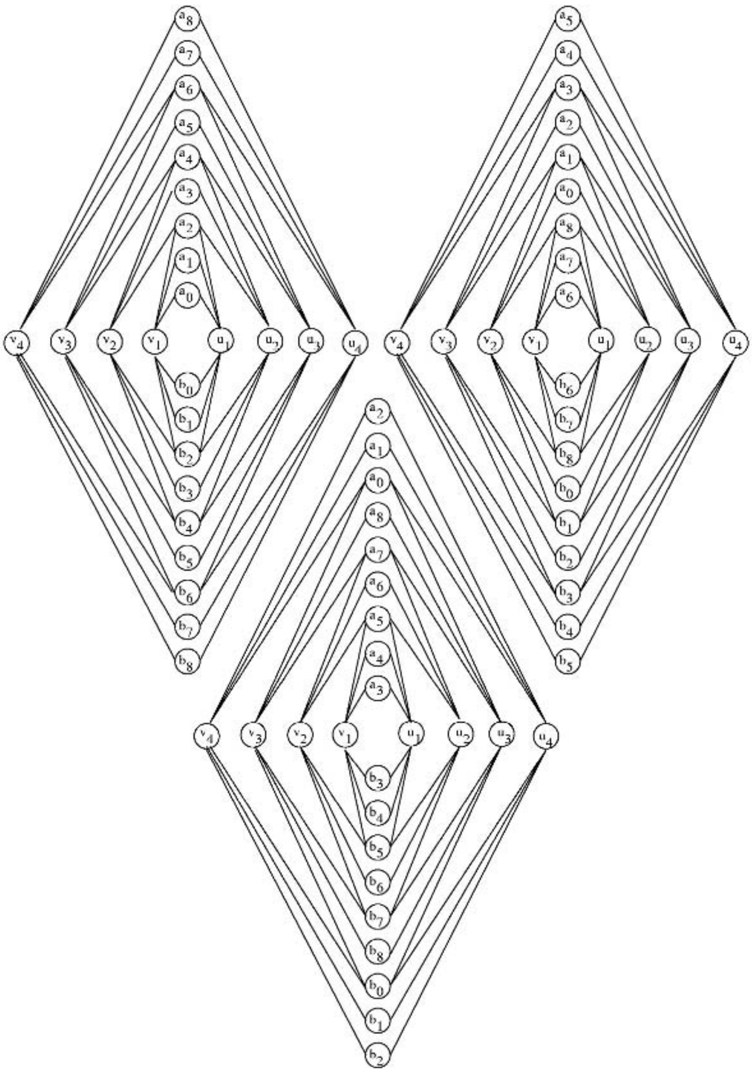

Fig. 2. A drawing of $K_{8,18}$ in 3 planes without crossings.

The rest is similar to the proof of Theorem 3.

Lower bound. As $\theta(K_{2k+2,2k^2}) = k$, from (1), $\mathrm{cr}_k(K_{2k+2,q}) = 0$, for $q \leq 2k^2$. Assume $2k^2 \leq q \leq 4k^2$. In this interval the RHS of (6) equals $2q - 4k^2$, which is the lower bound given by (2). □

4 Improved Bounds on Complete and Complete Bipartite Graphs

4.1 Lower Bounds

For specific graphs we can strengthen the lower bound by the standard counting argument.

Theorem 5. *For $p \geq 6k-1$ and $q \geq \max\{6k-1, 2k^2\}$*

$$\mathrm{cr}_k(K_{p,q}) \geq \frac{1}{3(3k-1)^2}\binom{p}{2}\binom{q}{2}.$$

Proof. The estimation (2) gives

$$\mathrm{cr}_k(K_{6k-1,6k-1}) \geq 12k^2 - 4k + 1.$$

Using the counting argument with $H = K_{6k-1,6k-1}$ and $G = K_{p,q}$ we have

$$\mathrm{cr}_k(K_{p,q}) \geq \frac{\binom{p}{6k-1}\binom{q}{6k-1}}{\binom{p-2}{6k-3}\binom{q-2}{6k-3}}\mathrm{cr}_k(K_{6k-1,6k-1}) > \frac{1}{3(3k-1)^2}\binom{p}{2}\binom{q}{2}.$$

□

Theorem 6. *For $n \geq 2k^2 + 6k - 1$*

$$\mathrm{cr}_k(K_n) \geq \frac{1}{2(3k-1)^2}\binom{n}{4}.$$

Proof. Let $n = p + q$. Combining the counting argument with $H = K_{p,q}$ and $G = K_n$ with the lower bound from Theorem 8 we get the claim. □

4.2 Upper Bounds

For special values of k we can improve on the upper bounds from Corollary 1 and Theorem 2.

Theorem 7. *Let $k-1$ be a power of a prime. For $n \geq (k-1)^2$*

$$\mathrm{cr}_k(K_n) \leq \frac{1}{64}\frac{k}{(k-1)^3}(n+k^2)^4.$$

Proof. To appear in the full verion.

For the complete bipartite graphs and arbitrary k we can extend the construction from the Section 3.

Theorem 8. *For $p \geq 2k+2$ and $q \geq 2k^2$*

$$\mathrm{cr}_k(K_{p,q}) \leq \frac{k^2+k+2}{16k^2(k+1)^2}(p+2k+1)^2(q+2k^2-1)^2.$$

Proof. To appear in the full version.

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