Teaching and Learning Geometry

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Aims, Themes and Organization of the Topic Study Group

Aims and Themes

This group provided a forum for discussion of the teaching and learning of geometry, with a focus especially on the middle and secondary school and university levels. The focus of the group was on theoretical, empirical, or developmental issues related to

- Curriculum studies of new curriculum implementation, challenges and issues, discussion of specific issues such as place and role of transformations
- An application of geometry on the real world and other subjects,
- The use of instrumentation such as computers in teaching and learning of geometry,
- Explanation, argumentation and proof in geometry education
- Spatial abilities and geometric reasoning
- Teacher preparation in geometry education.

The issues were addressed from the historical and epistemological, cognitive and semiotic, educational points of view related to students' difficulties and related to the design of teaching and curricula.

TSG 10 received 40 submissions. We decided to subdivide the group into 2 subgroups during 3 slots of the group and to organize a poster session during one slot.

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Organization

Each paper was reviewed by two members of the organizing team who gave an evaluation and suggestions for the writing of the full paper. From the reviews and interactions by email among the members of the Organizing Team, an agreement was reached on a final list of presentations and posters, leading to 3 long oral presentations, 17 shorter presentations and 20 posters. Finally, due to cancellations, 3 long oral presentations and 14 presentations took place. Only 4 posters were displayed at the poster session. Most of the poster presenters left their posters in the main poster session of the congress. This turned the poster session of the group into a very interactive and vivid session with a small number of papers, in which each poster was presented by the author(s) and then discussed with all the participants.

The presenters in the group came from 12 different countries of North and South America, Asia and Europe.

Content of the Group

Range of the Themes Addressed in the Group

Several themes dealing with various mathematical contents were addressed in the group (Table 1).

A Multifaceted Approach of Geometry

As visible in the previous table, geometry was approached from various points of view. It should be noted that these points of view are not independent but intertwined. For example, the notion of "geometric transformation" was addressed by several presentations focusing on various themes: curriculum design, students' learning or teachers' knowledge. Some key issues arose from the range of themes addressed by the group:

- the notion of shape and generally of representation in geometry teaching and learning with an extension to the use of Dynamic Geometry environments
- the link between geometry and the real world
- the notion of transformation
- teacher education

The notion of "shape" as a corner stone of school geometry was investigated by Usiskin in his long presentation: "(1) a "figure"—we study many different shapes in geometry; (2) a "type of figure", as in the declaration that an object is triangular-shaped; and (3) a "property of a set of similar figures", as in the statement that two

Table 1 The addressed themes and contents

Theme	Mathematical content	School level
Mathematical analysis of the domain	Shapes and relationships with functions, graphical representations	Secondary, University
Curriculum and textbooks	Plane geometry, transformations	Secondary
Problem solving	Combinatorial problems	Secondary, College, University
Reasoning and proving	3D and 2D configurations	Middle school
Modeling the real world	Mirror and line reflection, trigonometry	Elementary, Middle school, Secondary
Use of tools and technology	Centroids in 2D and 3D geometry, geometrical relationships, tessellations and transformations	Primary, Middle school, Secondary
Introduction to axiomatic system	Geometry of the sphere	College, University
Students' solving strategies	Area of trapezoids	Upper elementary, Early secondary, Secondary, College, University
Students' recognition of shapes	Solids	Primary, Middle school
Reading and writing	3D geometry	Upper secondary
Teacher education	Transformations, measurement	Pre and in-service teacher education

figures are congruent if they have the same size and shape, or two figures are similar if they have the same shape." Usiskin investigated how the notion of shape has been extended in school geometry with four components of present school geometry: coordinate geometry, transformations, applications of geometry, dynamic geometry software environments. An important claim of Usiskin is that whereas geometry is usually considered as studying abstractions of real objects, "geometry studies real figures as well as abstract ones".

This extension of the notion of shape can be linked to the notion of diagram or representation of geometric objects in 2D or 3D. The issue of representation was involved in several contributions.

In 3D, there is a larger variety of representations than in 2D: real models, 2D representations in various perspectives, computer representations. Ludwig and Steinwandel carried out an investigation on 242 10 to 15 year-old students who had to identify the shape of faces and to give the number of faces, edges and vertices of Platonic and Archimedean solids represented by either models, or computer animations or diagrams. In his long presentation, Ludwig showed that students benefit more from real models. The assistance by computer animations and by pictures was not so fruitful in tasks where the students need mental rotation to solve the task. Lavador used the Bruners' classification to design a teacher guide about measurement of solids, starting from enactive representations to move to images and iconic representations that lead then to symbolic representations.

The chosen representations in geometry problem solving (be it in 2D or 3D) may help or hinder a constructive reasoning for 12–15 year old students (Jones, Fujita and Kunimune); for the same problem depending on the diagram students may recognize or not the configuration for applying a known theorem. In his long presentation, Jones showed some examples in 2D and 3D and stressed the existence of prototypical representations that may turn into obstacles for recognizing the same property in other representations. Students' difficulties in interpreting diagrams seem to prevail across the world and are mentioned in contributions from Germany, Japan, and England. Jones concluded that "questions remain about how different mathematical representations influence students' decision making, conjecture production, and proof construction processes in the classroom, and how can such representations can be utilized by teachers to develop students' productive reasoning process." This is exactly the question also addressed in Kageyama's contribution that studies how students recognize analytical and logical properties of figures in construction tasks and use figural properties as justifying tools.

The link between geometry and the real world underlies several contributions and was even the focus of a few presentations. The issue seems to be more complex than expected. In some cases, referring to the real world can be very helpful for students (Ludwig). Whereas for Usiskin, although geometry is usually considered as studying abstractions of real objects, "geometry studies real figures as well as abstract ones", Boehm, Pospiech, Narciss and Körndle claimed that mathematics is an abstract world and they investigated what might be the potential confusions regarding a physical phenomenon after having experienced mathematics and physics lessons on this topic. Their study dealt with a very relevant phenomenon the mirror image in geometrical optics, as very often reflection is introduced in mathematics as modeling the mirror image. Their empirical data showed that we must pay attention to the fact that reality itself is not taught but a model of the reality and we must take into account the role of the used model in the teaching. It may happen that they do not go hand in hand as for reflection and mirror image and students may build inadequate knowledge. The results of the empirical study showed that students learn better when the scientific model is split into different science areas and when they are introduced to a multi-perspective modeling encompassing all model parts.

The link between real objects and theoretical objects of geometry was also viewed from the perspective of physical manipulations: real models for solid geometry (Ludwig, Suarez) but also strings, scissors, geoboard at elementary school (Faggiano). Faggiano stressed the fact that the manipulation by children contributes to the construction of meaning to geometric objects and relations only if they are involved in suitable tasks designed by the teacher.

Representations of geometric objects in Dynamic Geometry Environments are of a new nature and largely extending the range of manipulations and thought operations. Surprisingly a relative small number of contributions addressed this issue. Mammana (Ferrarelo and Pennisi) asked students to generalize properties from 2D to 3D by using two Dynamic Geometry environments (Cabri II plus and Cabri 3D). Their observations showed how the computer environments helped students not only to verify their conjectures but also to prove them. The same idea of combining exploring and generalizing was also investigated by Withney, Kartal and Zawojewsky with collegiate students using Lenart spheres for constructing an axiomatic system of spherical geometry. Faggiano combined the use of dynamic geometry and manipulatives at elementary school and concluded to the benefit of such combination. Lindamann carried out an investigation on the provocative question: "Which learning environment, DGE or traditional one produces a greater learning in a college geometry course?". No significant difference was found between the results of both kinds of learning environments. However as noted by Lindamann, students using technology gained other skills related to technology.

Transformations was a theme addressed by many contributions at least from two perspectives, a curricular perspective and from the perspective of pre- or in-service teacher education. La Ferla et al. compared the Common Core standards in the United States and the Turkish curricula and showed that the teaching of transformations is reinforced by the Common Core standards and becomes more aligned with the Turkish curriculum. Innovative teaching introducing pre-service or in-service teachers not only to transformations, but also to their use in solving geometry problems was reported by several contributions. Saego reported by means of very relevant examples about a professional development and its rich materials guiding teachers to move beyond conceptualizing similarity as a numerical relationship between two discrete figures to instead understand a precise conception of similar figures from a transformations-based perspective. Xhevdet Thaqi compared curricula of Spain and Kosovo and investigated "how do prospective teachers understand, learn and present each component of geometric transformations, if there is any differences between two different countries." The study concluded that of importance among student teachers is the concept image of transformation as displacement and change of place.

Teacher education was part of several presentations, be it the focus of the paper or joint to another issue such as the teaching and learning of transformations. As stressed by Somayajulu, teacher knowledge is especially fragile in geometry as a subject. This is certainly a major motivation for improving teacher education in geometry.

Geometry as a source of problems was illustrated by some contributions: Soifer presented geometry combinatorial problems for advanced students, Manizade and Mason carried out a thorough analysis of possible solving strategies of calculating the area of a trapezoid and showed how solving this task may be done at various Van Hiele levels. Hak Ping Tam and Hsin Han Wang concluded their study about the presentation of Pythagoras theorem in Taiwan textbooks by claiming that this theorem is a good opportunity for making students aware of the fact that multiple proofs can be given for the same theorem.

In conclusion, the various presentations of the group illustrated very well how rich the field of geometry teaching and learning is and how it can be investigated from various points of view with some emerging key issues, namely the nature and the role of representations.