On Orthogonally Convex Drawings of Plane Graphs (Extended Abstract)

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Abstract. We investigate the bend minimization problem with respect to a new drawing style called *orthogonally convex drawing*, which is orthogonal drawing with an additional requirement that each inner face is drawn as an *orthogonally convex polygon*. For the class of bi-connected plane graphs of maximum degree 3, we give a necessary and sufficient condition for the existence of a no-bend orthogonally convex drawing, which in turn, enables a linear time algorithm to check and construct such a drawing if one exists. We also develop a flow network formulation for bend-minimization in orthogonally convex drawings, yielding a polynomial time solution for the problem. An interesting application of our orthogonally convex drawing is to characterize internally triangulated plane graphs that admit floorplans using only orthogonally convex modules subject to certain boundary constraints.

Keywords: Bend minimization, floorplan, orthogonally convex drawing.

1. Introduction

An *orthogonal drawing* of a plane graph is a planar drawing such that each edge is composed of a sequence of horizontal and vertical line segments with no crossings. A classic optimization problem in orthogonal drawing is to minimize the number of *bends*, namely, the *bend-minimization* problem. The problem is NP-complete in the most general setting, i.e., for planar graphs of maximum degree 4 [5]. Subclasses of graphs with bend-minimization of orthogonal drawing tractable include planar graphs of maximum degree 3, series-parallel graphs, and graphs with fixed embeddings [3,12], etc.

Most of the orthogonal drawing algorithms reported in the literature can be roughly divided into two categories, one uses flow or matching to model the problem (e.g., [2,3,12]), whereas the other tackles the problem in a more graph-theoretic way by taking advantage of structure properties of graphs (e.g., [9,10,8]). The former usually solves a more general problem, but requires higher

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time complexity. On the contrary, algorithms in the latter focus on specific kinds of graphs, resulting in linear time complexity in many cases.

In this paper, we introduce a new type of orthogonal drawing called *orthogonally convex drawing*, which requires that each inner face be an *orthogonally convex polygon.* A polygon is *orthogonally convex* if for any horizontal or vertical line, if two points on the line are inside a polygonal region, then the entire line segment between these two points is also inside the polygonal region. The study of this new drawing style is motivated by an attempt to learn more about the geometric aspect of orthogonal drawing, which, in the dual setting, is closely related to *rectangular dual* and *rectilinear dual* which are well-studied in floor-planning and contact graph representations [6,11,14]. Note that if we consider standard convexity instead of orthogonal convexity in the setting of no-bend orthogonal drawing, the problem becomes the "inner rectangular drawing" studied in [7]. There are several recent results on rectilinear duals in cartographic applications, see, e.g., [1]. For other perspectives of orthogonal drawing, the reader is referred to [4] for a survey chapter.

Our contributions include the following:

1. A new drawing style called *orthogonally convex drawing* is introduced, and a necessary and sufficient condition, along with a linear time testing algorithm, is given for a bi-connected plane 3-graph (i.e., of maximum degree 3) to admit a no-bend orthogonally convex drawing.
2. A flow network formulation is devised for the bend-minimization problem of orthogonally convex drawing.
3. By combining the above no-bend orthogonally convex drawing algorithm and the flow network formulation, a polynomial time algorithm (in $O(n^{1.5} \log^3 n)$ time) for constructing a bend-optimal orthogonally convex drawing is presented.
4. We apply our analysis of no-bend orthogonally convex drawing to characterizing internally triangulated graphs that admit floorplans using only orthogonally convex modules that can be embedded into a rectilinear region with its boundary order-equivalent to a given orthogonally convex polygon.

2. Preliminaries

Given a graph $G = (V, E)$, we write $\Delta(G)$ to denote the maximum degree of G. Graph G is called a d-graph if $\Delta(G) \leq d$. A *path* P of G is a sequence of vertices $(v_1, v_2, ..., v_n)$ such that $\forall 1 \leq i \leq n, v_i \in V$ and $\forall 1 \leq i \leq n-1, (v_i, v_{i+1}) \in E$. We write $V(P)$ to denote the set of vertices $\{v_1, ..., v_n\}$, and $E(P)$ to denote the edge set $\{(v_i, v_{i+1}) | 1 \leq i < n\}$ of P. Given two paths P' and P, we write $P' \subseteq P$ if P' is a subsequence of P, and $P' \subset P$ if $P' \subseteq P$ and $P' \neq P$. P is called a *cycle* if $v_1 = v_n$. Unless stated otherwise, paths and cycles are assumed to be *simple* throughout this paper, in the sense that there are no repeated vertices other than the starting and ending vertices. A drawing of a planar graph divides the plane into a set of connected regions, called *faces.* A *contour* of a face F is the cycle formed by vertices and edges along the boundary of F. A cycle that is the boundary, ie., contour, of a face is called a *facial cycle*. The contour of the outer face is denoted as C_O. If G is bi-connected, contours of all the faces are simple cycles.

In our subsequent discussion, we adopt some of the notations and definitions used in [9,10]. A cycle C divides a plane graph G into two regions. The one that is inside (resp., outside) cycle C is called the *interior region* (resp., *outer region*) of C. We use $G(C)$ to denote the subgraph of G that contains exactly C and vertices and edges residing in its interior region. An edge $e = (u, v)$ in the outer region of C is called a *leg* of C if at least one of the two vertices u and v belongs to C. C is *k-legged* if C contains exactly k vertices that are incident to some legs of C. These k vertices are called *legged-vertices* of C. If $\Delta(G) \leq 3$, every legged-vertex v of C is incident to exactly one leg e of C. Note that 3-legged cycles coincide with the so-called *complex triangles* in the dual setting, which play a crucial role in the study of rectilinear duals [11,14].

We call a face or a cycle *inner* if it is not the outer one. If an inner face or inner cycle intersects with the outer one, then we call it *boundary face* or *boundary cycle*. A *contour path* P of a cycle C is a path on C such that P includes exactly two legged-vertices x and y of C, and x and y are the two endpoints of P. Therefore, each k-legged cycle has exactly k contour paths. If a contour path intersects with (i.e., shares some edges with) the outer cycle, we call it *boundary contour path*. In fact, each boundary contour path is a subpath of C_O. Each contour path P of C is incident to exactly one face, denoted as $F_{C,P}$, in the outer region of C. As an illustrating example, consider Figure 1. F_0 is the outer face of G. Consider two cycles $C_1 = (s, t, u, v, s)$ and $C_2 = (x, b, i, a, z, y, c, x)$ (both drawn in bold line). C_1 is a non-boundary 2-legged cycle, of which two legged-vertices are t and v, and two legs are (t, q) and (v, r). C_1 is also a facial cycle, which is the contour of F_1. C_2 is a boundary 3-legged cycle, of which three legged-vertices are x, y, and z. $P_1 = (t, u, v)$ is a contour path of C_1. $P_2 = (z, a, i, b, x)$ is the boundary contour path of C_2. We have $F_{C_1,P_1} = F_2$ and $F_{C_2,P_2} = F_0$.

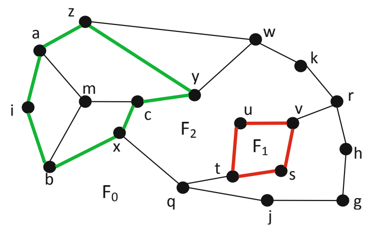

Fig. 1. Illustration of some terms about cycles and paths

Let $D(G)$ be an orthogonal drawing of plane graph G with outer cycle C_O. Given a cycle C, we use $D(C)$ (or equivalently $D(F)$ if C is the contour of a face F) to denote the drawing of C in $D(G)$. $D(C)$ is always a simple polygon as long as C is simple. We call $D(G)$ an *orthogonally convex drawing* of G if $D(F)$ is an *orthogonally convex polygon* for every face F other than the outer one. We use $bc(D(G))$ to denote the bend count, i.e., the total number of bends, of $D(G)$.

In an orthogonal drawing $D(G)$, $ang_C(v)$ denotes the interior angle of v in polygon $D(C)$. We called v a *convex corner*, *non-corner*, and *concave corner* of

C if $ang_C(v)$ is 90°, 180°, and 270°, respectively. A *corner* in the drawing $D(G)$ is either a bend on some edge, or a vertex v of G such that $ang_C(v) \neq 180°$ for some C. If v is a non-corner of C, v is on a side of polygon $D(C)$.

From Section 3 to Section 5, graphs under the name G are assumed to be bi-connected with $\triangle(G) \leq 3$, and may have multi-edges.

3. No-Bend Orthogonally Convex Drawing

Among existing results concerning orthogonal drawings, Rahman et al. [10] gave a necessary and sufficient condition for a bi-connected plane 3-graph to admit a no-bend orthogonal drawing, and they devised an algorithm to test the condition, and subsequently construct the drawing if one exists.

Theorem 1 ([10]). *A bi-connected plane 3-graph G has a no-bend orthogonal drawing iff G satisfies the following three conditions:*

1. *There are four or more 2-vertices (i.e., vertices of degree 2) of G on $C_O(G)$.*
2. *Every 2-legged cycle contains at least two 2-vertices.*
3. *Every 3-legged cycle contains at least one 2-vertex.*

Theorem 1 clearly holds even when G has multi-edges, as such graphs do not have no-bend orthogonal drawings. The no-bend orthogonal drawing algorithm in [10] performs the following steps recursively: (1) reducing the original graph G into a structurally simpler graph G^* by collapsing the so-called "*maximal bad cycles*", (2) drawing G^* in a rectangular fashion, and (3) plugging in the orthogonal drawings of those maximal bad cycles to the rectangular drawing[1] of G^* to yield a no-bend orthogonal drawing of G.

Bad cycles in Step (1) are cycles that are 2-legged or 3-legged if the four designated corner vertices in C_O are considered as legged-vertices. Intuitively, bad cycles are cycles that violate the conditions under which a graph admits a rectangular drawing. For instance, consider the graph in Figure 1. If $\{h, i, j, k\}$ are the 4 designated vertices, then $(w, z, a, i, b, x, c, y, w)$ (a 3-legged cycle as i is considered a legged-vertex) is a bad cycle, whereas $(r, v, u, t, q, j, g, h, r)$ (a 4-legged cycle including legged-vertices h and j) is not a bad cycle. *Maximal bad cycles* are bad cycles that are not contained in $G(C)$ for another bad cycle C. Step (2) involves computing the rectangular drawing of an input graph with four designated corner vertices on $C_O(G)$. It is known that such a graph with four designated vertices admits a rectangular drawing if and only if every 2-legged cycle contains at least two designated vertices, and every 3-legged cycle contains at least one designated vertex [13]. As shown in [10], the G^* (with each of the maximal bad cycles contracted to a single vertex) always meets the condition for the existence of a rectangular drawing. The reader is referred to [10] for more.

Our goal in this section is to give a similar necessary and sufficient condition for graphs to have no-bend orthogonally convex drawings.

[1] A *rectangular drawing* of a graph is a no-bend orthogonal drawing such that each interior face is a rectangle and the boundary of the outer face also forms a rectangle.

Lemma 1. *Consider a no-bend orthogonally convex drawing $D(G)$ of a graph G. For every 2-legged cycle C with legged-vertices x and y and a contour path P of C, the number of convex corners of $D(C)$ in $V(P) \setminus \{x, y\}$ (i.e., the set of vertices along path P excluding x and y) must be at least 1 more than that of concave corners, if either (1) C is a boundary cycle and P is its boundary contour path, or (2) C is non-boundary and P is any of its contour paths.*

We are now in a position to give one of our main results.

Theorem 2. *A bi-connected plane 3-graph G admits a no-bend orthogonally convex drawing if and only if the three conditions in Theorem 1 and the following two additional conditions hold: (1) every non-boundary 2-legged cycle contains at least one 2-vertex on each of its contour paths, and (2) every boundary 2-legged cycle contains at least one 2-vertex on its boundary contour path.*

The necessity of Theorem 2 follows from Lemma 1. A modification to the no-bend orthogonal drawing algorithm described above yields a constructive proof of the sufficiency of Theorem 2. Based on an implementation described in [10], we have the following result.

Theorem 3. *There is a linear time algorithm to construct a no-bend orthogonally convex drawing $D(G)$ if G admits one.*

4. An Alternative Condition

An alternative necessary and sufficient condition is given in this section to characterize bi-connected 3-plane graphs admitting no-bend orthogonally convex drawings, facilitating a min-cost flow formulation for the bend-minimization problem. As Theorem 2 indicates, contour paths along (boundary or non-boundary) 2-legged cycles play a vital role in orthogonally convex drawing. Due to possible overlaps of 2-legged cycles and complex intersections between contour paths, it becomes difficult to capture the amount of convex/concave corners along contour paths in a min-cost flow formulation. To ease this problem, we identify two types of cycles, namely, *proper* and *improper* cycles, which are later used to characterize the presence of orthogonally convex drawings.

Let G^c denote the graph resulting from contracting every 2-vertex of G. Since we require G to be of maximum degree 3, G^c must be 3-regular. A 2-legged cycle of G is called *improper* if its two legs correspond to the same edge in G^c; otherwise, it is called *proper*. See Figure 2 for instance. Due to 3-regularness of G^c and the fact that the two legs of an improper cycle C are the same edge e in G^c, there remains nothing outside $G(C)$ except the leg e. Therefore, improper cycles must be boundary cycles, or conversely, all non-boundary 2-legged cycles are proper. It is also easy to observe that a 2-legged cycle C of G with two leg-vertices x and y is improper iff for the non-boundary contour path P of C, the boundary of $F_{C,P}$ intersects C_O of G in exactly 1 path. Again consider Figure 2. Note that F_1 and F_2 correspond to the $F_{C,P}$ of the 2-legged cycles drawn as bold lines in the left and right figures, respectively. The boundary of F_1 intersects C_O

in exactly one path (x, y), whereas the boundary of F_2 intersects C_O in two paths (z, a) and (y, b, c).

Definition 1. *A path P of G is called* critical *if there is a proper 2-legged cycle C such that: (1) P is a contour path of C, (2) if C is a boundary 2-legged cycle, P is the boundary contour path of C, and (3) P does not edge-intersect with any proper 2-legged cycle other than C that is contained in $G(C)$.*

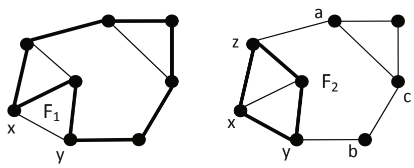

Fig. 2. Proper and improper 2-legged cycles. Left: An improper 2-legged cycle (drawn as a bold line) with leg-vertices x, y. Right: A proper 2-legged cycle (drawn as a bold line) with leg-vertices y, z.

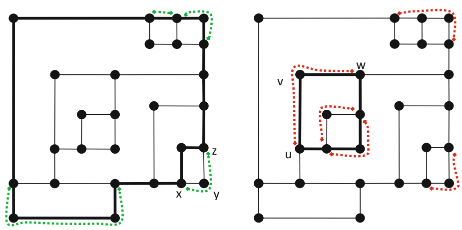

Fig. 3. Critical paths and S_G in a plane graph. Left: Paths in S_G. Right: Critical Paths

To proceed further, we require the following two lemmas.

Lemma 2. *For any bi-connected plane 3-graph G, the critical paths of G are edge-disjoint.*

Lemma 3. *Let P be a path satisfying (1) and (2) in Definition 1. If P is not critical, there must be a critical path P' such that $P' \subset P$.*

Note that the requirement of properness of 2-legged cycles in Definition 1 is essential in the sense that Lemma 2, which is crucial in the subsequent context, is not true if we remove that requirement. Given a path P with endpoints x and y, we write $P_{(x \frown y)}$ to denote the "open" version of P, i.e., excluding x and y. That is, $P_{(x \frown y)}$ consists of $V(P) \setminus \{x, y\}$ and $E(P)$.

Instead of basing on contour paths as in Theorem 2, our new characterization for no-bend orthogonally convex drawing is based upon two kinds of paths defined over proper and improper 2-legged cycles, namely, critical paths defined above for proper 2-legged cycles and a set of paths called S_G associated with improper 2-legged cycles in graph G. S_G is defined to be the set of all paths $C_O \setminus P_{(x \frown y)}$ for every boundary contour path P of an improper 2-legged cycle with two legged-vertices x, y. Note that internal vertices in paths of S_G must have degree 2, and paths in S_G must be in C_O, and hence $P \in S_G$ iff P is a boundary contour path of a facial cycle C that has only one boundary contour path. The following fact summarize the observation.

Fact 1. *Let P be a path of G with two end-vertices x and y. The following three statements are equivalent: (1) P is in S_G; (2) P is the boundary contour path of a facial cycle C that intersects C_O of G in exactly one path; (3) P is $C_O \backslash P'_{(x \frown y)}$, for some boundary contour path P' of an improper 2-legged cycle with two legged-vertices x and y.*

To have better grasp of critical paths and S_G, consider Figure 3 in which a no-bend orthogonally convex drawing of a plane graph G is shown. In the left figure, the four dotted paths are those in S_G, which are edge-disjoint. Let C be the 2-legged cycle drawn as a bold line, and P be its boundary contour path. We have $C_O \backslash P_{(x \frown z)} = (x, y, z)$. In the right figure, the five dotted paths are critical paths, which are edge-disjoint. Let C be the 2-legged cycle drawn as a bold line. We have (1) the path (u, v, w) is one of its contour paths, (2) C is a non-boundary 2-legged cycle, and (3) P does not edge-intersect with any proper 2-legged cycle other than C that is contained in $G(C)$. A path in S_G is either contained in exactly one critical path or intersects with no critical path. The reader is encouraged to verify that the graph satisfies the conditions stated in Theorem 2 and Theorem 4, and the orthogonally convex drawing satisfies the conditions in Lemma 4.

The following theorem enables us to characterize no-bend orthogonally convex drawings in terms of critical paths and S_G.

Theorem 4. *Suppose a bi-connected plane 3-graph G has a no-bend orthogonal drawing. G has a no-bend orthogonally convex drawing iff the following conditions are satisfied:*

1. *Every critical path of G contains at least one 2-vertex.*
2. *$C_O \backslash P$ contains at least one 2-vertex for every $P \in S_G$.*

We note that both S_G and the set of all critical paths can be found in linear time. The algorithm is basically a contour edge-traversal of each face with a mechanism of detecting repeated adjacent faces.

5. Flow Formulation for Bend-Minimization

In this section, we tailor the planar min-cost flow formulation originally designed for orthogonal drawing [12] to coping with orthogonal convexity. To make our subsequent discussion clear, we use *arc* and *node* instead of edge and vertex, respectively, in describing a flow network. A *min-cost flow network* is a directed multi-graph $N = (W, A)$ associated with four functions: *lower bounds* $\lambda : A \to \mathbb{Z}_{\geq 0}$, *capacities* $\mu : A \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, *costs* $c : A \to \mathbb{Z}_{\geq 0}$, *demands* $b : W \to \mathbb{Z}$. A map $f : A \to \mathbb{Z}_{\geq 0}$ is a *flow* if the following constraints are met for each node v and arc a:

$$b(v) + \sum_{(u,v) \in A} f(u, v) - \sum_{(v,u) \in A} f(v, u) = 0, \quad \lambda(a) \leq f(a) \leq \mu(a)$$

The cost of a flow f is $c(f) = \sum_{a \in A} f(a) \times c(a)$. The flow network $N_G = (W_G, A_G)$ associated with a bi-connected plane 3-graph G is

- $W_G = W_V \cup W_F$, where W_V and W_F are the vertex set and face set (including the outer face) of G, respectively, Furthermore, $\forall u_v \in W_V$, $b(u_v) = 2$ if $deg_G(v) = 3$; $b(u_v) = 0$ if $deg_G(v) = 2$. $\forall u_F \in W_F$, $b(u_F) = -4$ if F is an inner face; $b(u_F) = 4$ if F is the outer face.
- $A_G = A_V \cup A_F$, where
  - $A_V = \{(u_v, u_F), (u_F, u_v)|deg(v) = 2\} \cup \{(u_v, u_F)|deg(v) = 3\}$, where $v \in V(G), F \in \text{face}(G), v$ incident to F. $\forall a \in A_V$, $\lambda(a) = 0$, $\mu(a) = 1$, and $c(a) = 0$.
  - $A_F = \{(u_F, u_H)|F, H \in \text{face}(G)$, and F adjacent to $H\}$ is a multi-set of arcs between faces, and the number of (u_F, u_H) in A_F equals the number of shared edges e in contours of F and H. We use $(u_F, u_H)_e$ to indicate the specific arc that corresponds to the shared edges e. $\forall a \in A_F$, $\lambda(a) = 0$, $\mu(a) = \infty$, and $c(a) = 1$.

Although our definition of N_G is slightly different from the original one given in [12], the validity of N_G is apparent as the following explains. Every flow f in N_G corresponds to an orthogonal drawing $D(G)$, and vice versa, such that

- $f(u_v, u_F) - f(u_F, u_v) = -1, 0, 1$ means v is a concave corner, non-corner, convex corner in $D(F)$, respectively,
- $f(u_F, u_H)_e$ is the number of bends on e that are concave corners in $D(F)$ and convex corners in $D(H)$, and
- the total number of bends in $D(G)$ equals $c(f)$.

Fact 2. *Let S_1 (resp., S_2) be any subset of edges (resp., vertices) along the contour of a face F. For any $e \in S_1$, we write F_e to denote the face incident to e other than F. For a flow f in N_G and its corresponding orthogonal drawing D, we must have $\sum_{e \in S_1}[f(u_{F_e}, u_F)_e - f(u_F, u_{F_e})_e] + \sum_{v \in S_2}[f(u_v, u_F) - f(u_F, u_v)]$ equals the difference between the numbers of convex corners and concave corners in S_1 and S_2 of $D(F)$.*

Lemma 4. *A bi-connected plane 3-graph G admits a no-bend orthogonally convex drawing iff there is a no-bend orthogonal drawing (not necessarily orthogonally convex) such that (1) for every critical path P along a contour path of 2-legged cycle C, $\#_{cc}(P_{(x \frown y)}) > \#_{cv}(P_{(x \frown y)})$ in $F_{C,P}$, and (2) for every P in S_G, $\#_{cc}(P_{(x \frown y)}) \leq 3 + \#_{cv}(P_{(x \frown y)})$ in the outer face, where P has endpoints x and y, and $\#_{cv}(P_{(x \frown y)})$ and $\#_{cc}(P_{(x \frown y)})$ represent the numbers of convex and concave corners, respectively, of $P_{(x \frown y)}$.*

In what follows, we show how to construct a flow network N'_G from N_G in such a way that a flow of N'_G corresponds to an orthogonal drawing meeting the conditions stated in Lemma 4. Initially we set $N'_G = N_G$.

- $\forall P \in S_G$ with endpoints x, y and outer face F', add a new node u_P to $W(N'_G)$, and two arcs $(u_{F'}, u_P)$, $(u_P, u_{F'})$ to $A(N'_G)$. We set $b(u_P) = 0$, $\lambda(u_{F'}, u_P) = \lambda(u_P, u_{F'}) = 0$, $\mu(u_{F'}, u_P) = 3$, $\mu(u_P, u_{F'}) = \infty$, and $c(u_{F'}, u_P) = c(u_P, u_{F'}) = 0$. We redirect all the arcs in the current $A(N'_G)$ of the following forms: $(u_{F'}, u_v)$, $(u_v, u_{F'})$, $(u_{F'}, u_F)_e$, $(u_F, u_{F'})_e$ for all

$v \in V(P) \setminus \{x, y\}$, $F \in S_{P,F'}$, $e \in E(P)$ by replacing $u_{F'}$ with u_P. Due to Fact 2, Statement 2 of Lemma 4 holds.

– $\forall$ critical path P with endpoints x, y, C the 2-legged cycle for which P is its contour path, and S the set of faces in $G(C)$ that border P, add a new node u_P to $W(N'_G)$, and a new arc $(u_{F_{C,P}}, u_P)$ to $A(N'_G)$. We set $b(u_P) = 0$, $\lambda(u_{F_{C,P}}, u_P) = 1$, $\mu(u_{F_{C,P}}, u_P) = \infty$, and $c(u_{F_{C,P}}, u_P) = 0$. We redirect all the arcs in the current $A(N'_G)$ of the following forms: $(u_{F_{C,P}}, u_{P'})$, $(u_{P'}, u_{F_{C,P}})$, $(u_{F_{C,P}}, u_v)$, $(u_v, u_{F_{C,P}})$, $(u_{F_{C,P}}, u_F)_e$, $(u_F, u_{F_{C,P}})_e$ for all $P' \in S_G$ such that $P' \subseteq P$, $v \in V(P) \setminus \{x, y\}$, $F \in S$, $e \in E(P)$ by replacing $u_{F_{C,P}}$ with u_P. Due to Fact 2, Statement 1 of Lemma 4 holds.

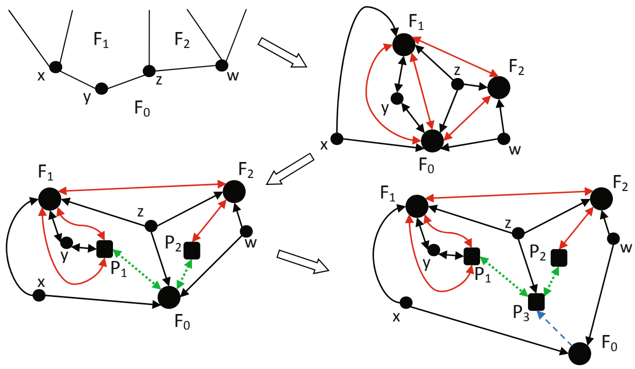

Fig. 4. Illustration of the construction of N'_G: F_0 is the outer face, $P_1 = (x, y, z)$ and $P_2 = (z, w)$ are two paths in S_G, $P_3 = (x, y, z, w)$ is a critical path

For an illustrating example, consider Figure 4 in which the up-left picture is a portion of a graph G with N_G depicted in the up-right. The down-left one illustrates the result of adding two additional nodes representing P_1 and P_2 (the newly added arcs are drawn in dotted line). The down-right one illustrates the result of adding an additional node representing critical path P_3 (the newly added arc is drawn in dashed line).

The validity of the above construction follows from critical paths being mutually edge-disjoint (Lemma 2), and every path in S_G is either a subpath of a critical path or intersects with no critical paths. Note that the number of newly added arcs and nodes is linear in $n = V(G)$, and the maximum possible value of the minimum cost is also $O(n)$. Following an $O(n^{1.5} \log^3 n)$ time algorithm in [2], we have

Theorem 5. *For any bi-connected plane 3-graph G, we can construct a bend-minimized orthogonally convex drawing in $O(n^{1.5} \log^3 n)$ time.*

6. An Application to Floor Planning

In this section, we show an application of orthogonally convex drawing to floor planning. A plane graph is *internally triangulated* if all the inner faces are triangles. For any internally triangulated plane graph $G = (V, E)$, a *rectilinear dual* is a partition of a simple orthogonal polygon (denoted as R) into $|V(G)|$ simple orthogonal regions, one for each vertex, such that two region have a side-contact iff their corresponding vertices adjacent to each other.

Two polygons are said to be *order-equivalent* if they admit the same circular order (in counter-clockwise orientation) of angles. For instance, the following two figures  are order-equivalent. Let Q be an orthogonal polygon, we use *Q-floorplan* to denote a rectilinear dual whose boundary (the R in the definition of rectilinear dual) is order-equivalent to Q. A floorplan is called orthogonally convex if all the boundaries of $|V(G)|$ simple orthogonal regions are orthogonally convex polygons. In this section, graphs under the name G_{dual} are assumed to be simple, connected, internally triangulated plane graph.

Lemma 5. *For any simple, connected, internally triangulated plane graph G_{dual}, there is a unique bi-connected 3-regular plane multi-graph G_{primal} such that G_{dual} is the weak dual*[2] *of G_{primal}, and the following properties are hold: (1) G_{primal} does not have any non-boundary 2-legged cycle, and (2) internal faces (which are orthogonal polygons) of an orthogonal drawing of G_{primal} form a rectilinear dual of G_{dual}.*

We remark that although G_{dual} is required to be simple, G_{primal} may still have multi-edges. Since G_{primal} is bi-connected and $\triangle(G_{primal}) \leq 3$, the results in the previous sections can be applied.

Let $C_O = (v_1, v_2, \ldots, v_s, v_1)$ be the boundary cycle of G_{dual}, which need not be a simple cycle. Then, a triangulated plane multi-graph G' is constructed by adding a new vertex t in the outer face of G_{dual}, and then triangulate the outer face by adding edge (v_i, t) for $1 \leq i \leq s$. Take the dual of G' yields G_{primal}. See Figure 5 for an illustration.

Given an orthogonally convex polygon Q, our goal is to characterize graphs that admit orthogonally convex Q-floorplans, and subsequently realize such floorplans. We use numSide(P) to denote the number of sides of polygon P with non-corner vertices neglected.

Lemma 6. *Let G be a bi-connected plane 3-graph (may have multi-edges) with k boundary critical paths. We have* $\min\{\text{numSide}(D(C_O)) \mid D \text{ is an orthogonally convex drawing of } G\} = \max\{4, 2k-4\}$*. Further, for any orthogonally convex polygon Q of* $\text{numSide}(Q) \geq \max\{4, 2k-4\}$*, there is an orthogonally convex drawing $D(G)$ such that $D(C_O)$ is order-equivalent to Q.*

The concept of critical paths turns out to be pretty clean in the dual setting. We use T_G to denote the block-cutvertex tree of G. We will see in Lemma 7

[2] The weak dual of a plane graph is the subgraph of the dual graph excluding the vertex (and edges) corresponding to the unbounded (i.e., outer) face.

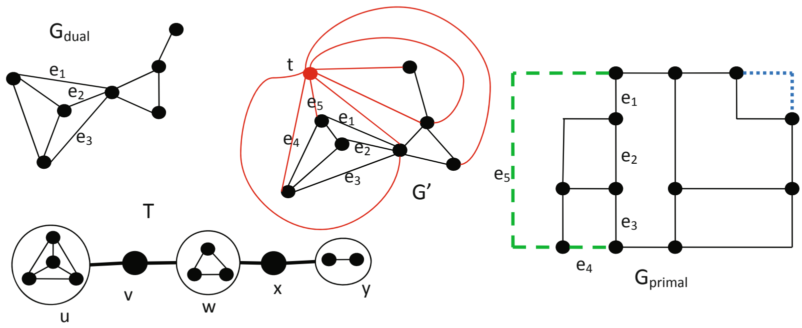

Fig. 5. The construction of G_{primal}

that leaves in $T_{G_{dual}}$ actually have one-to-one correspondence to critical paths in G_{primal}. Let (v, u) be an edge in $E(T_{G_{dual}})$ such that v is a cut-vertex. Now u must be a block. Let $V_{v,u}$ be the vertex set of the component in $G_{dual} \setminus \{v\}$ that contains some vertices in block u, and $F_{v,u}$ denote the corresponding face set in G_{primal}. Since G_{dual} is internally triangulated, the edges in $E(G_{dual})$ that link v to vertices in $V_{v,u}$ must be located consecutively in the circular list of edges incident to v that describes the combinatorial embedding of G_{dual}. We denote the edge set as $E_{v,u}$. According to the definition of duality of plane graphs and the algorithm for constructing G_{primal} from G_{dual}, these edges form a path in G_{primal}. We write $C_{v,u}$ to denote the cycle that is the boundary of union of faces in $F_{v,u}$. For instance, in Figure 5 the set $E_{v,u}$ is $\{e_1, e_2, e_3\}$, which forms the non-boundary contour path with respect to $C_{v,u}=(e_1, e_2, e_3, e_4, e_5)$.

Lemma 7. *$\{$Boundary contour path of $C_{v,u} \mid u$ is a leaf of $T_{G_{dual}}, (v, u) \in E(T_{G_{dual}})\}$ is the set of boundary critical paths in G_{primal}.*

In Figure 5, the boundary contour paths of $C_{v,u}$ and $C_{x,y}$ are the paths drawn in dashed and dotted lines, respectively. These two paths are the boundary critical paths of G_{primal}. Following Lemmas 5, 6, 7 and Theorem 3, we have

Theorem 6. *For any internally triangulated graph G_{dual} and orthogonally convex polygon Q, let k be the number of leaves in the block-cutvertex tree of G_{dual}. G_{dual} admits an orthogonally convex Q-floorplan iff* $\text{numSide}(Q) \geq \max\{4, 2k - 4\}$. *The floorplan can be constructed in linear time.*

7. Conclusion

We studied a new drawing style called orthogonally convex drawing from both combinatorial and algorithmic viewpoints. It would be interesting to see whether results/techniques developed in our work could be extended to other types of convex versions of contact graph representations or floorplans.

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