Straight Line Triangle Representations

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Abstract. A straight line triangle representation (SLTR) of a planar graph is a straight line drawing such that all the faces including the outer face have triangular shape. Such a drawing can be viewed as a tiling of a triangle using triangles with the input graph as skeletal structure. In this paper we present a characterization of graphs that have an SLTR that is based on flat angle assignments, i.e., selections of angles of the graph that have size π in the representation. We also provide a second characterization in terms of contact systems of pseudosegments. With the aid of discrete harmonic functions we show that contact systems of pseudosegments that respect certain conditions are stretchable. The stretching procedure is then used to get straight line triangle representations. Since the discrete harmonic function approach is quite flexible it allows further applications, we mention some of them.

The drawback of the characterization of SLTRs is that we are not able to effectively check whether a given graph admits a flat angle assignment that fulfills the conditions. Hence it is still open to decide whether the recognition of graphs that admit straight line triangle representation is polynomially tractable.

1 Introduction

In this paper we study a representation of planar graphs in the classical setting, i.e., vertices are represented by points in the Euclidean plane and edges by non-crossing continuous curves connecting the points. We aim at classifying the class of planar graphs that admit a straight line representation in which all faces are triangles. Haas et al. present a necessary and sufficient condition for a graph to be a pseudo-triangulation [8], however this condition is not sufficient for a graph to have a straight line triangle representation (e.g. see Fig. 2 and [1]). There have been investigations of the problem in the dual setting, i.e., in the setting of side contact representations of planar graphs with triangles. Gansner, Hu and Kobourov show that outerplanar graphs, grid graphs and hexagonal grid graphs are Touching Triangle Graphs (TTGs). They give a linear time algorithm to find the TTG [7]. Alam, Fowler and Kobourov [2] consider proper TTGs, i.e., the union of all triangles of the TTG is a triangle and there are no holes. They give a necessary and a stronger sufficient condition for biconnected outerplanar graphs to be TTG, a characterization, however, is missing. Kobourov, Mondal and Nishat present construction algorithms for proper TTGs of 3-connected

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cubic graphs and some grid graphs. They also present a decision algorithm for testing whether a 3-connected planar graph is proper TTG [10].

Here is the formal introduction of the main character for this paper.

Definition 1. *A plane drawing of a graph such that*

- *all the edges are straight line segments and*
- *all the faces, including the outer face, bound a non-degenerate triangle*

is called a Straight Line Triangle Representation *(SLTR).*

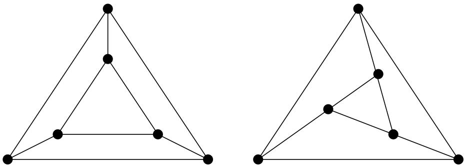

Fig. 1. A graph and one of its SLTRs

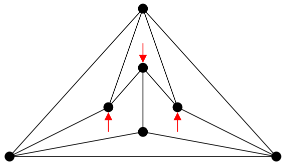

Fig. 2. A Flat Angle Assignment (given by the arrows) that is not an SLTR

Clearly every straight line drawing of a triangulation is an SLTR. So the class of planar graphs admitting an SLTR is rich. On the other hand, graphs admitting an SLTR cannot have a cut vertex. Indeed, as shown below (Prop. 1), graphs admitting an SLTR are well connected. Being well connected, however, is not sufficient as shown e.g. by the cube graph.

To simplify the discussion we assume that the input graph is given with a plane embedding and a selection of three vertices of the outer face that are designated as corner vertices for the outer face. These three vertices are called *suspension vertices*. If needed, an algorithm may try all triples of vertices as suspensions.

If a degree two vertex has an angle of size π in one of its incident faces, then it also has an angle of size π in the face on the other side. Hence, this vertex and its two incident edges can be replaced by a single edge connecting the two neighbors of the vertex. Such an operation is called a *vertex reduction*. The only angles of an SLTR whose size exceeds π are the outer angles at the outer triangle. Therefore, we can use vertex reductions to eliminate all the degree two vertices, except for degree two vertices that are suspensions.

A plane graph G with suspensions s_1, s_2, s_3 is said to be *internally 3-connected* when the addition of a new vertex v_∞ in the outer face, that is made adjacent to the three suspension vertices, yields a 3-connected graph.

Proposition 1. *If a graph G admits an SLTR with s_1, s_2, s_3 as corners of the outer triangle and no vertex reduction is possible, then G is internally 3-connected.*

Proof. Consider an SLTR of G. Suppose there is a separating set U of size 2. It is enough to show that each component of $G \setminus U$ contains a suspension vertex, so that $G + v_\infty$ is not disconnected by U. Since G admits no vertex reduction every degree two vertex is a suspension. Hence, if C is a component and $C \cup U$ induces a path, then there is a suspension in C. Otherwise consider the convex hull of $C \cup U$ in the SLTR. The convex corners of this hull are vertices that expose an angle of size at least π. Two of these large angles may be at vertices of U but there is at least one additional large angle. This large angle must be the outer angle at a vertex that is an outer corner of the SLTR, i.e., a suspension. □

From Prop. 1 it follows that any graph that is not internally 3-connected but does admit an SLTR, is a subdivision of an internally 3-connected graph. Therefore we may assume that the graphs we consider are internally 3-connected.

In Section 2 we present necessary conditions for the existence of an SLTR in terms of what we call a flat angle assignment. A flat angle assignment that fulfills the conditions is shown to induce a partition of the set of edges into a set of pseudosegments. Finally, with the aid of discrete harmonic functions we show that in our case the set of pseudosegments is stretchable. Hence, the necessary conditions are also sufficient. The drawback of the characterization is that we are not aware of an effective way of checking whether a given graph admits a flat angle assignment that fulfills the conditions.

In Section 3 we consider further applications of the stretching approach. First we look at flat angle assignments that yield faces with more than three corners. Then we proceed to prove a more general result about stretchable systems of pseudosegments with our technique. The result is not new, de Fraysseix and Ossona de Mendez have investigated stretchability conditions for systems of pseudosegments in [3,4,5]. The counterpart to Theorem 2 can be found in [5, Theorem 38]. The proof there is based on a long and complicated inductive construction.

2 Necessary and Sufficient Conditions

Consider a plane, internally 3-connected graph $G = (V, E)$ with suspensions given. Suppose that G admits an SLTR. This representation induces a set of *flat angles*, i.e., incident pairs (v, f) such that vertex v has an angle of size π in the face f.

Since G is internally 3-connected every vertex has at most one flat angle. Therefore, the flat angles can be viewed as a partial mapping of vertices to faces. Since the outer angle of suspension vertices exceeds π, suspensions have no flat angle. Since each face f (including the outer face) is a triangle, each face has precisely three angles that are not flat. In other words every face f has $|f|-3$ incident vertices that are assigned to f. This motivates the definition:

Definition 2. *A* flat angle assignment *(FAA) is a mapping from a subset U of the non-suspension vertices to faces such that*
[C_v] *Every vertex of U is assigned to at most one face,*
[C_f] *For every face f, precisely $|f| - 3$ vertices are assigned to f.*

Not every FAA induces an SLTR. An example is given in Fig. 2. Hence, we have to identify another condition. To state this we need a definition. Let H be a connected subgraph of the plane graph G. The *outline cycle* $\gamma(H)$ *of* H is the closed walk corresponding to the outer face of H. An *outline cycle* of G is a closed walk that can be obtained as outer cycle of some connected subgraph of G. Outline cycles may have repeated edges and vertices, see Fig. 3. The interior $\mathsf{int}(\gamma)$ of an outline cycle $\gamma = \gamma(H)$ consists of H together with all vertices, edges and faces of G that are contained in the area enclosed by γ.

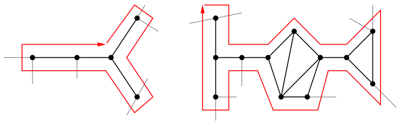

Fig. 3. Examples of outline cycles

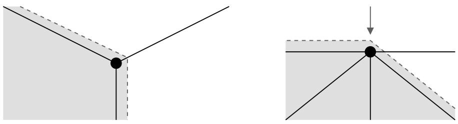

Fig. 4. Combinatorially Convex Corners

Proposition 2. *An SLTR obeys the following condition C_o:*

[C_o] *Every outline cycle that is not the outline cycle of a path, has at least three geometrically convex corners.*

Proofs of Propositions 2 and 3 have been moved to the appendix.

Condition C_o has the disadvantage that it depends on a given SLTR, hence, it is useless for deciding whether a planar graph G admits an SLTR. The following definition allows to replace C_o by a combinatorial condition on an FAA.

Definition 3. *Given an FAA ψ. A vertex v of an outline cycle γ is a* combinatorial convex corner *for γ with respect to ψ if*

- *v is a suspension vertex, or*
- *v is not assigned and there is an edge e incident to v with $e \notin \mathsf{int}(\gamma)$, or*
- *v is assigned to a face f, $f \notin \mathsf{int}(\gamma)$ and there exists an edge e incident to v with $e \notin \mathsf{int}(\gamma)$.*

In Fig. 4 an unassigned and an assigned combinatorially convex corner are shown. The grey area represents the interior of some outline cycle and the arrow represents the assignment of the vertex to the face in which the arrow is drawn.

Proposition 3. *Let G admit an SLTR Γ, that induces the FAA ψ and let H be a connected subgraph of G. If v is a geometrically convex corner of the outline cycle $\gamma(H)$ in Γ, then v is a combinatorially convex corner of $\gamma(H)$ with respect to ψ.*

The proposition enables us to replace the condition on geometrically convex corners w.r.t. an SLTR by a condition on combinatorially convex corners w.r.t. an FAA.

[C_o^*] Every outline cycle that is not the outline cycle of a path, has at least three combinatorially convex corners.

From Prop. 2 and Prop. 3 it follows that this condition is necessary for an FAA that induces an SLTR. In Thm. 1 we prove that if an FAA obeys C_o^* then it induces an SLTR. The proof is constructive. In anticipation of this result we say that an FAA obeying C_o^* is a *good flat angle assignment* and abbreviate it as a *GFAA*.

Next we show that a GFAA induces a contact family of pseudosegments. This family of pseudosegments is later shown to be stretchable, i.e., it is shown to be homeomorphic to a contact system of straight line segments.

Definition 4. *A* contact family of pseudosegments *is a family* $\{c_i\}_i$ *of simple curves* $c_i : [0,1] \to \mathbb{R}^2$, *with different endpoints, i.e.,* $c_i(0) \neq c_i(1)$, *such that any two curves* c_j *and* c_k *(*$j \neq k$*) have at most one point in common. If so, then this point is an endpoint of (at least) one of them.*

A GFAA ψ on a graph G gives rise to a relation ρ on the edges: Two edges, both incident to v and f are in relation ρ if and only if v is assigned to f. The transitive closure of ρ is an equivalence relation on the edges of G.

Proposition 4. *The equivalence classes of edges of* G *defined by* ρ *form a contact family of pseudosegments.*

Proof. Let the equivalence classes of ρ be called arcs.

Condition C_v ensures that every vertex is interior to at most one arc. Hence, the arcs are simple curves and no two arcs cross.

Every arc has two distinct endpoints, otherwise it would be a cycle and its outline cycle has only one combinatorially convex corner. If an arc touched itself, the outline cycle of this equivalence class would have at most one combinatorially convex corner. This again contradicts C_o^*.

If two arcs share two points, the outline cycle has at most two combinatorially convex corners. This again contradicts C_o^*.

We conclude that the family of arcs satisfies the properties of a contact family of pseudosegments. □

Definition 5. *Let* Σ *be a family of pseudosegments and let* S *be a subset of* Σ. *A point* p *of a pseudosegment from* S *is a* free point *for* S *if*

1. *p is an endpoint of a pseudosegment in* S, and
2. *p is not interior to a pseudosegment in* S, and
3. *p is incident to the unbounded region of* S, and
4. *p is incident to the unbounded region of* Σ or
   p is incident to a pseudosegment that is not in S.

With Lem. 1 we prove that the family of pseudosegments Σ that arises from a GFAA has the following property

[C_P] Every subset S of Σ with $|S| \geq 2$ has at least three free points.

Lemma 1. *Let* ψ *a GFAA on a plane, internally 3-connected graph* G. *For every subset* S *of the family of pseudosegments associated with* ψ, *it holds that, if* $|S| \geq 2$ *then* S *has at least 3 free points.*

Proof. Let S be a subset of the contact family of pseudosegments defined by the GFAA (Prop. 4).

Each pseudosegment of S corresponds to a path in G. Let H be the subgraph of G obtained as union of the paths of pseudosegments in S. We assume that H

is connected and leave the discussion of the cases where it is not to the reader. If H itself is not a path, then by C_o^* the outline cycle $\gamma(H)$ must have at least three combinatorially convex corners. Every combinatorially convex corner of $\gamma(H)$ is a free point of S.

If S induces a path, then the two endpoints of this path are free points for S. Moreover, there exists at least one vertex v in this path which is an endpoint for two pseudosegments and not an interior point for any. Now there must be an edge e incident to v, such that $e \notin S$, therefore v is a free point for S. □

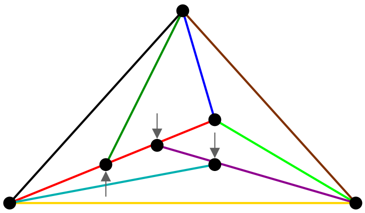

Fig. 5. A stretched representation of a contact family of pseudosegments that arises from a GFAA in the graph of Fig 2

Given an internally 3-connected, plane graph G with a GFAA. To find a corresponding SLTR we aim at representing each of the pseudosegments induced by the FAA as a straight line segment. If this can be done, every assigned vertex will be between its two neighbors that are part of the same pseudosegment. This property can be modeled by requiring that the coordinates $p_v = (x_v, y_v)$ of the vertices of G satisfy a harmonic equation at each assigned vertex.

Indeed if uv and vw are edges belonging to a pseudosegment s, then the coordinates satisfy

$$x_v = \lambda_v x_u + (1 - \lambda_v) x_w \qquad \text{and} \qquad y_v = \lambda_v y_u + (1 - \lambda_v) y_w \tag{1}$$

where the parameter λ_v can be chosen arbitrarily from $(0, 1)$. These are the harmonic equations for v.

In the SLTR every unassigned vertex v is placed in a weighted barycenter of its neighbors. In terms of coordinates this can be written as

$$x_v = \sum_{u \in N(v)} \lambda_{vu} x_u, \qquad y_v = \sum_{u \in N(v)} \lambda_{vu} y_u \,. \tag{2}$$

These are the harmonic equations for an unassigned vertex v. The λ_{vu} can be chosen arbitrarily in the range set by the convexity conditions: $\sum_{u \in N(v)} \lambda_{vu} = 1$ and $\lambda_{vu} > 0$.

Vertices whose coordinates are not restricted by harmonic equations are called *poles*. In our case the suspension vertices are the three poles of the harmonic functions for the x and y-coordinates. The coordinates for the suspension vertices are fixed as the corners of some non-degenerate triangle, this adds six equations to the linear system.

The theory of harmonic functions and applications to (plane) graphs are nicely explained by Lovász [11]. The following proposition is taken from Chapter 3 of [11].

Proposition 5. *For every choice of the parameters λ_v and λ_{vu} complying with the conditions, the system has a unique solution.*

Now we state our main result, it shows that the necessary conditions are also sufficient.

Theorem 1. *Let G be an internally 3-connected, plane graph and Σ a family of pseudosegments associated to an FAA, such that each subset $S \subseteq \Sigma$ has three free points or cardinality at most one. The unique solution of the system of equations that arises from Σ is an SLTR.*

Proof. The proof consists of 7 arguments, which together yield that the drawing induced from the GFAA is a non-degenerate, plane drawing. The proof has been inspired by the proof of Colin de Verdière [6] for convex straight line drawings of plane graphs via spring embeddings.

1. *Pseudosegments become Segments.* Let $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ be the set of edges of a pseudosegment defined by ψ. The harmonic conditions for the coordinates force that v_i is placed between v_{i-1} and v_{i+1} for $i = 2, .., k-1$. Hence all the vertices of the pseudosegment are placed on the segment with endpoints v_1 and v_k.

2. *Convex Outer Face.* The outer face is bounded by three pseudosegments and the suspensions are the endpoints for these three pseudosegments. The coordinates of the suspensions (the poles of the harmonic functions) have been chosen as corners of a non-degenerate triangle and the pseudosegments are straight line segments, therefore the outer face is a triangle and in particular convex.

3. *No Concave Angles.* Every vertex, not a pole, is forced either to be on the line segment between two of its neighbors (if assigned) or in a weighted barycenter of all its neighbors (otherwise). Therefore every non-pole vertex is in the convex hull of its neighbors. This implies that there are no concave angles at non-poles.

4. *No Degenerate Vertex.* A vertex is degenerate if it is placed on a line, together with at least three of its neighbors. Suppose there exists a vertex v, such that v and at least three of its neighbors are placed on a line ℓ. Let S be the connected component of pseudosegments that are aligned with ℓ, such that S contains v. The set S contains at least two pseudosegments. Therefore S must have at least three free points, v_1, v_2, v_3.

By property 4 in the definition of free points, each of the free points is incident to a segment that is not aligned with ℓ. Suppose the free points are not suspension vertices. If v_i is interior to $s_i \in S$, then s_i has an endpoint on each side of ℓ. If v_i is not assigned by the GFAA it is in the strict convex hull of its neighbors, hence, v_i is an endpoint of a segment reaching into each of the two half-planes defined by ℓ.

Now suppose v_1 and v_2 are suspension vertices[1] and consider the third free point, v_3. If v_3 is interior to a pseudosegment not on ℓ, then one endpoint of this pseudosegment lies outside the convex hull of the three suspensions, which is a contradiction. Hence it is not interior to any pseudosegment and at least one of its neighbors does not lie on ℓ, but then v_3 should be in a weighted barycenter of its neighbors, hence again we would find a vertex outside the convex hull of

[1] Not all three suspension vertices lie on one line, hence at least one of the three free points is not a suspension.

the suspension vertices. Therefore at most one of the free points is a suspension and ℓ is incident to at most one of the suspension vertices.

In any of the above cases each of v_1, v_2, v_3 has a neighbor on either side of ℓ.

Let n^+ and $n^- = -n^+$ be two normals for line ℓ and let p^+ and p^- be the two poles, that maximize the inner product with n^+ resp. n^-. Starting from the neighbors of the v_i in the positive halfplane of ℓ we can always move to a neighbor with larger[2] inner product with n^+ until we reach p^+. Hence v_1, v_2, v_3 have paths to p^+ in the upper halfplane of ℓ and paths to p^- in the lower halfplane. Since v_1, v_2, v_3 also have a path to v we can contract all vertices of the upper and lower halfplane of ℓ to p^+ resp. p^- and all inner vertices of these paths to v to produce a $K_{3,3}$ minor of G. This is in contradiction to the planarity of G. Therefore, there is no degenerate vertex.

5. *Preservation of Rotation System.* Let $\theta(v) = \sum_f \theta(v, f)$ denote the sum of the angles around an interior vertex. Here f is a face incident to v and $\theta(v, f)$ is the (smaller!) angle between the two edges incident to v and f in the drawing obtained by solving the harmonic system. If the incident faces are oriented consistently around v, then the angles sum up to 2π, otherwise $\theta(v) > 2\pi$ (see Fig. 6). We do not consider the outer face in the sums so that the b vertices incident to the outer face contribute a total angle of $(b-2)\pi$ to the inner faces.

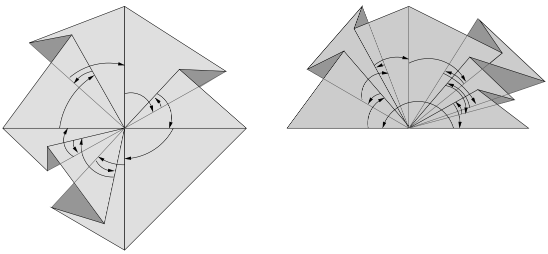

Fig. 6. Vertices with their surrounding faces not oriented consistently

Now consider the sum $\theta(f) = \sum_v \theta(v, f)$ of the angles of a face f. At each vertex incident to f the contribution $\theta(v, f)$ is at most of size π. A closed polygonal chain with k corners, selfintersecting or not, has a sum of inner angles equal to $(k-2)\pi$. Therefore $\theta(f) \leq (|f|-2)\pi$. The sum over all vertices $\sum_v \theta(v)$ and the sum over all faces $\sum_f \theta(f)$ must be equal since they count the same angles in two different ways.

$$(|V|-b)2\pi + (b-2)\pi \leq \sum_v \theta(v) = \sum_f \theta(f) \leq ((2|E|-b) - 2(|F|-1))\,\pi \quad (3)$$

This yields $|V|-|E|+|F| \leq 2$. Since G is planar Euler's formula implies equality. Therefore $\theta(v) = 2\pi$ for every interior vertex v and the faces must be oriented consistently around every vertex, i.e. the rotation system is preserved. Note that the rotation system could have been flipped, between clockwise and counterclockwise but then it is flipped at every vertex.

6. *No Crossings.* Suppose two edges cross. On either side of both of the edges there is a face, therefore there must be a point p in the plane which is covered by at least two faces. Outside of the drawing there is only the unbounded face.

[2] If n^+ is perpendicular to another segment this may not be possible. In this case we can use a slightly perturbed vector n^+_ε to break ties.

Move along a ray, that does not pass through a vertex of the graph, from p to infinity. A change of the cover number, i.e. the number of faces by which the point is covered, can only occur when crossing an edge. But if the cover number changes then the rotation system at a vertex of that edge must be wrong. This would contradict the previous item. Therefore a crossing cannot exist.
7. *No Degeneracy.* Suppose there is an edge of length zero. Since every vertex has a path to each of the three suspensions there has to be a vertex a that is incident to an edge of length zero and an edge ab of non-zero length. Following the direction of forces we can even find such a vertex-edge pair with b contributing to the harmonic equation for the coordinates of a. We now distinguish two cases.

If a is assigned, it is on the segment between b and some b', together with the neighbor of the zero length edge this makes three neighbors of a on a line. Hence, a is a degenerate vertex. A contradiction.

If a is unassigned it is in the convex hull of its neighbors. However, starting from a and using only zero-length edges we eventually reach some vertex a' that is incident to an edge $a'b'$ of non-zero length, such that b' is contributing to the harmonic equation for the coordinates of a'. Vertex a' has the same position as a and is also in the convex hull of its neighbors. This makes a crossing of edges unavoidable. A contradiction. Hence, there are no edges of length zero.

Suppose there is an angle of size zero. Since every vertex is in the convex hull of its neighbors there are no angles of size larger than π. Moreover there are no crossings, hence the face with the angle of size zero is stretching along a line segment with two angles of size zero. Since there are no edges of length zero and all vertices are in the convex hull of their neighbors, all but two vertices of the face must be assigned to this face. Therefore, there are two pseudosegments bounding this face, which have at least two points in common, this contradicts that Σ is a family of pseudosegments. We conclude that there is no degeneracy.

From items 1–7 we conclude that the drawing is plane and thus an SLTR. □

3 Further Applications of the Proof Technique

We have shown that a graph G has an SLTR exactly if it admits an FAA satisfying C_v, C_f and C_o^*. Conditions C_v and C_o^* are necessary for the proof that the system of pseudosegments corresponding to the FAA is stretchable. Condition C_f, however, is only needed to make all the faces triangles. Modifying condition C_f allows for further applications of the stretching technique. Of course we still need at least three corners for every face. Also we have to make sure that all the non-suspension vertices of the outer face are assigned to the outer face. Together this makes the modified face condition:

[C_f^*] For every face f, at most $|f| - 3$ vertices are assigned to f and all non-suspension vertices of the outer face f^o are assigned to f^o.

If we use the empty flat angle assignment, i.e., if the harmonic equations of all non-suspensions are of type (2), then we obtain a drawing such that all non-suspension vertices are in the barycenter of their neighbors. This is the Tutte drawing [12] with asymmetric elastic forces given by the parameters λ_{uv}, see also [11]. Note that in this case the existence of at least three combinatorially convex corners at an outline cycle (condition C_o^*) follows from the internally 3-connectedness of the graph.

The construction of Section 2 also applies when

- the assignment has $|f|-i$ vertices assigned to every inner face f, for $i = 4, 5$ (drawing with only convex 4-gon or only convex 5-gon faces.)
- the assignment has some number c_f of corners at inner face f (drawing with convex faces of prescribed complexity).

The drawback is that again in these cases we do not know how to find an FAA that fulfills C_o^*.

In [9] Kenyon and Sheffield study T-graphs in the context of dimer configurations (weighted perfect matchings). In our terminology T-graphs correspond to straight line representations such that each non-suspension is assigned. In [9] the straight line representations of T-graphs are obtained by analyzing random walks. Cf. [11] for further connections between discrete harmonic functions and Markov chains.

Stretchability of Systems of Pseudosegments. A contact system of pseudosegments is *stretchable* if it is homeomorphic to a contact system of straight line segments. De Fraysseix and Ossona de Mendez characterized stretchable systems of pseudosegments [3,4,5]. They use the notion of an extremal point.

Definition 6. *Let Σ be a family of pseudosegments and let S be a subset of Σ. A point p is an* extremal point *for S if*

1. *p is an endpoint of a pseudosegment in S,* and
2. *p is not interior to a pseudosegment in S,* and
3. *p is incident to the unbounded region of S.*

Theorem 2 (De Fraysseix & Ossona de Mendez [5, Theorem 38]**).** *A contact family Σ of pseudosegments is stretchable if and only if each subset $S \subseteq \Sigma$ of pseudosegments with $|S| \geq 2$, has at least 3 extremal points.*

Our notion of a free point (Def. 5) is more restrictive than the notion of an extremal point. In the following we show that there is no big difference. First in Prop. 6 we show that in the case of families of pseudosegments that live on a plane graph via an FAA, the two notions coincide. Then we continue by reproving Thm. 2 as a corollary of Thm. 1.

Proposition 6. *Let G be an internally 3-connected, plane graph and Σ a family of pseudosegments associated to an FAA, such that each subset $S \subseteq \Sigma$ has three extremal points or cardinality at most one. The unique solution of the system of equations corresponding to Σ, is an SLTR.*

Proof. Note that in the proof of Thm. 1 the notion of free points is only used to show that there is no degenerate vertex. We show how to modify this part of the argument for the case of extremal points:

Consider again the set S of pseudosegments aligned with ℓ. We will show that all extremal points are also free points. Let p an extremal point of S. Assuming that p is not free we can negate item 4. from Def. 5, i.e., all the pseudosegments for which p is an endpoint are in S. By 3-connectivity p is incident to at least three pseudosegments, all of which lie on the line ℓ. Since all regions are bounded by three pseudosegments and p is not interior to a segment of S, all the regions incident to p must lie on ℓ. But then p is not incident to the unbounded region of S, hence p is not an extremal point. Therefore all extremal points of S are also free points of S. Prop. 6 now follows from Thm. 1. □

Proof (of Thm. 2). Let Σ a contact family of pseudosegments which is stretchable. Consider a set $S \subseteq \Sigma$ of cardinality at least two in the stretching, i.e., in the segment representation. Endpoints (of segments) on the boundary of the convex hull of S are extremal points. There are at least three of them unless S lies on a line ℓ. In the latter case, there is a point q on ℓ that is the endpoint of two colinear segments. This is a third extremal point.

Conversely, assume that each subset $S \subseteq \Sigma$ of pseudosegments, with $|S| \geq 2$, has at least 3 extremal points. We aim at applying Prop 6. To this end we construct an extended system Σ^+ of pseudosegments in which every region is bounded by precisely three pseudosegments.

First we take a set Δ of three pseudosegments that intersect like the three sides of a triangle so that Σ is in the interior. The corners of Δ are chosen as suspensions and the sides of Δ are deformed such that they contain all extremal points of the family Σ. Let the new family be Σ'.

Next we add new *protection points*, these points ensure that the pseudosegments of Σ' will be mapped to straight lines. For each inner region R in Σ', for each pseudosegment s in R, we add a protection point for each visible side of s. The protection point is connected to the endpoints of s, with respect to R from the visible side of s.

Now the inner part of R is bounded by an alternating sequence of endpoints of Σ' and protection points. We connect two protection points if they share a neighbor in this sequence. Last we add a *triangulation point* in R and connect it to all protection points of R.

This construction yields a family Σ^+ of pseudosegments such that every region is bounded by precisely three pseudosegments and every subset $S \subseteq \Sigma^+$ has at least 3 extremal points, unless it has cardinality one.

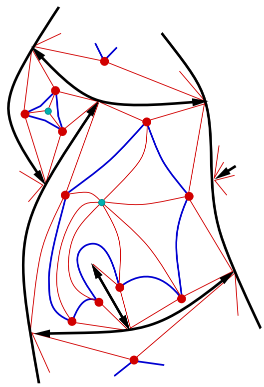

Fig. 7. Protection points in red and the triangulation point in cyan for two faces of some Σ'

Let V be the set of points of Σ^+ and E the set of edges induced by Σ^+. It follows from the construction that $G = (V, E)$ is internally 3-connected.

By Prop. 6 the graph $G = (V, E)$ together with Σ^+ is stretchable to an SLTR. Removing the protection points, triangulation points and their incident edges yields a contact system of straight line segments homeomorphic to Σ. □

4 Conclusion and Open Problems

We have given necessary and sufficient conditions for a 3-connected planar graph to have an SLT Representation. Given an FAA and a set of rational parameters $\{\lambda_i\}_i$, the solution of the harmonic system can be computed in polynomial time. Checking whether a solution is degenerate can also be done in polynomial time. Hence, we can decide in polynomial time whether a given FAA corresponds to an SLTR. In other words, checking whether a given FAA is a GFAA can be done in polynomial time. However, most graphs admit different FAAs of which only some are good. We are not aware of an effective way of finding a GFAA. Therefore we have to leave this problem open: Is the recognition of graphs that have an SLTR (GFAA) in P?

Given a 3-connected planar graph and a GFAA, interesting optimization problems arise, e.g. find the set of parameters $\{\lambda_i\}_i$ such that the smallest angle in the graph is maximized, or the set of parameters such that the length of the shortest edge is maximized.

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