# Connecting Sequent Calculi with Lorenzen-Style Dialogue Games

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**Abstract** Lorenzen has introduced his dialogical approach to the foundations of logic in the late 1950s to justify intuitionistic logic with respect to first principles about constructive reasoning. In the decades that have passed since, Lorenzen-style dialogue games turned out to be an inspiration for a more pluralistic approach to logical reasoning that covers a wide array of nonclassical logics. In particular, the close connection between (single-sided) sequent calculi and dialogue games is an invitation to look at substructural logics from a dialogical point of view. Focusing on intuitionistic linear logic, we illustrate that intuitions about resource-conscious reasoning are well served by translating sequent calculi into Lorenzen-style dialogue games. We suggest that these dialogue games may be understood as games of information extraction, where a sequent corresponds to the claim that a certain information package can be systematically extracted from a given bundle of such packages of logically structured information. As we will indicate, this opens the field for exploring new logical connectives arising by consideration of further forms of storing and structuring information.

## 1 Introduction

In the preface of their recent monograph *Linking Game-Theoretical Approaches with Constructive Type Theory*, Clerbout and Rahman (2015) identify the talk entitled 'Logik und Agon',[1] presented by Paul Lorenzen in 1958 to the World Congress of Philosophy in Venice, as inaugurating event for a broad and steady stream of research connecting logic, games, and epistemology. While the corresponding research paradigm may look back to conceptions of logic

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[1] Published as Lorenzen 1960 and included in the collection Lorenzen and Lorenz 1978.



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in antiquity and medieval philosophy, Clerbout and Rahman also view it as providing foundations to recent developments in the fields of theoretical computer science, computational linguistics, artificial intelligence, social sciences, and legal reasoning. Indeed, Lorenzen's dialogical approach to logic is often mentioned as an important early contribution to what Johan van Benthem and others call the 'dynamic turn' of contemporary logic (van Benthem 2014; Peregrin 2003). More specifically, the field of substructural logics, which takes Gentzen's sequent calculus as its starting point and, most prominently, includes Girard's (1987) Linear Logic, engendered a host of work on so-called game semantics. This field has been initiated by Blass (1992), who duly cites Lorenzen as a source of inspiration. However, it is also evident that the current understanding of the paradigm that interprets formulas as games and logical connectives as operators for constructing new games out of given ones is quite distant from Lorenzen's original concerns about a dialogical foundation for constructive reasoning. In fact, the 'games' that interpret formulas in contemporary game semantics, as exemplified, e.g., by Abramsky and Jagadeesan (1994) are quite abstract entities which relate to computational processes, but are hardly any longer connected to (idealized) dialogues between human agents. At least to some extent, this is at odds with the motivation of the move from classical or intuitionistic logic to substructural logics in terms of resource-conscious reasoning.

An example of resource consciousness consists in dropping the contraction rule from Gentzen's sequent calculus, which can be seen as a means to keep track of the number of times a particular assumption is explicitly used in a proof in order to obtain a certain conclusion. Likewise, deriving a sequent without using the (left) weakening rule amounts to a proof that uses all and only those assumptions listed as formula occurrences on the left side of the end sequent. Similarly, dropping the exchange rule (permutation) means that the use of listed assumptions has to follow a certain order. In any case, the crucial point of substructural logics is that a more fine-grained analysis of proofs along the indicated lines leads to new connectives that arise as different refinements of classical or intuitionistic connectives, when proper attention is paid to the use of formulas in corresponding sequent rules.

The main purpose of this contribution is to introduce and explore an interpretation of substructural sequent calculi that is arguably closer to Lorenzen's original dialogue game for intuitionistic logic than the contemporary forms of game semantics mentioned above. This exploits the fact that, quite generally, the logical inference rules of sequent calculi can be interpreted as modeling an interaction between a Proponent **P** and an opponent **O**. For example, the lower sequent of the rule

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge, \mathrm{r})$$

for introducing conjunction on the right side can be seen as representing a state in a dialogue game where **P** seeks to defend her assertion of $A \wedge B$ in the context of a multiset $\Gamma$ of formulas that have already been granted by **O**. The branching in the rule expresses the fact that **P** can only successfully defend her claim if she is prepared to defend both, $A$ as well as $B$, in the context $\Gamma$. This in turn means that the dialogue game is to be continued either in a state represented by the sequent $\Gamma \vdash A$ or in one represented by $\Gamma \vdash B$. Crucially, it is **O** who decides which of these two states should be chosen. In contrast, if **P** defends an assertion $A \vee B$, then she herself can choose whether the game should continue in a state where she claims that $A$ can be defended if **O** has granted $\Gamma$ or with her claim that $B$ can be defended in the presence of $\Gamma$.

To emphasize aspects of information processing implicitly entailed in interpretations of structural sequent rules, we will explore a variation of this idea by interpreting every (single-conclusion) sequent $\Gamma \vdash F$ as state of a game in which an agent seeks to extract the package of information represented by the formula $F$ from the bundle of information represented by the formulas contained in $\Gamma$.[2] In a sense, our aim is quite modest: we only want to discuss a particular reading of sequents for a certain family of substructural calculi and make no pretensions as to nontrivial mathematical results here. However, in another sense, we actually propose a rather ambitious research program, triggered by our specific interpretation. We argue that the proposed game-based view of sequents as states in games of information extraction leads not only to new perspectives on known calculi, but rather opens a way to new research directions, involving new types of formal systems of reasoning. Some of these systems are quite different from established calculi. For example, we will see that there are cases where cut admissibility is neither to be expected nor even justified on semantic grounds.

There is a potential source of misunderstanding about our endeavor that we should clarify right away. In speaking of 'Lorenzen-style games', we only refer to certain structural features of the framework that Lorenzen has introduced, but not to the philosophical aims that informed his writings on dialogical logic. Lorenzen and Kuno Lorenz, his principal collaborator in this endeavor, aimed not merely at a characterization of intuitionistic logic in terms of a two-player perfect-information game, but explicitly strove to derive their logical system from first principles about correct reasoning in mathematics and beyond. For example, in the introduction to the collection of relevant papers and book excerpts, Lorenzen and Lorenz (1978) emphasize that they not only aim at a *characterization* of intuitionistic and classical logic, but that they believe to have found a *justification* and *explanation* (*Begründung*) for those logics (see also Lorenz 2001). This strong claim has been met with skepticism. For instance, Hodges (2001) explicitly

[2] A first version of this interpretation has been presented in Fermüller and Lang 2017. Here we take a broader, less technical view of the topic. A different, but vaguely related, analysis of sequent rules in terms of a proof search game is offered by Lang, Olarte, Pimentel, and Fermüller 2019.

rejects this claim as untenable on several accounts. The more sympathetic account of Lenk (1982)[3] puts Lorenzen's dialogical approach into the context of a number of earlier attempts to provide *a priori* reasons for singling out certain connectives as 'logical', but arrives at largely negative conclusions as to the possibility of such an endeavor as well. It is probably fair to say that later developments in logic, alluded to by Clerbout and Rahman (2015), as mentioned above, persuaded most logicians to ignore the philosophical concerns of Lorenzen regarding the foundations of mathematical reasoning. In particular, contemporary logic is informed by a pragmatic pluralism regarding logical formalisms. It is shaped by a vast and fast-growing number of nonclassical logics that provide tools for modeling and analyzing many different aspects of formal reasoning, but these logics are hardly seen as competitors in the quest for determining the 'one and only' correct form of inference. In any case, our attempt to connect substructural sequent calculi with a type of game that is inspired by Lorenzen's logical dialogue games is informed by a very pragmatic approach to logic. We do not seek to contribute to the debates about the feasibility of dialogical logic as a foundational program, but rather want to illustrate, by way of a specific, but very flexible, approach to substructural calculi, that Lorenzen's innovations continue to inspire contemporary research in logic, independently of the fate of his own philosophical concerns.

The rest of the paper is organized as follows. Section 2 revisits Gentzen's sequent calculus $\mathsf{LI}$ for intuitionistic logic as a reference point, as well as our main example of a substructural calculus: intuitionistic linear logic ($\mathsf{ILL}$). In Section 3 we first present an information extraction game $\mathcal{G}_{\mathsf{I}}$ motivated by different ways of structuring and accessing information. We then define a resource-conscious variant $\mathcal{G}_{\mathsf{ILL}}$ of $\mathcal{G}_{\mathsf{I}}$. As pointed out in Section 4, winning strategies for player $\mathbf{P}$ in $\mathcal{G}_{\mathsf{ILL}}$ directly correspond to proofs in $\mathsf{ILL}$. As also explained there, the relation between $\mathbf{P}$'s winning strategies in $\mathcal{G}_{\mathsf{I}}$ and $\mathsf{LI}$-proofs is less obvious, but can be established by well-known proof-theoretic transformations. Section 5 clarifies in which sense our information extraction games can be classified as Lorenzen-style. Section 6 provides a selective overview over some other substructural calculi and corresponding information extraction games. The final Section 7 briefly recapitulates the overall story conveyed here and concludes with comments on a number of topics for further research, which are intended to show that connecting concerns about resources in sequent-based inference with Lorenzen-style games constitutes a varied and fruitful research paradigm that largely still remains to be explored.

[3] That essay is an expanded version of a talk originally presented in the context of the author's habilitation procedure at TU Berlin, 1966.

## 2 Some sequent calculi

In his seminal paper on the concept of logical consequence, Gentzen (1935) introduced the sequent calculus for intuitionistic as well as for classical logic as the main tool of investigation. We will only refer to the propositional part of the intuitionistic version LI[4] here, which is presented in Table 1. In fact LI,

**Axioms (initial sequents):** $A \vdash A$

**Structural rules:**

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{w}, \mathrm{l}) \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\mathrm{w}, \mathrm{r}) \qquad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (\mathrm{c})$$

$$\frac{\Gamma, A, B, \Sigma \vdash \Delta}{\Gamma, B, A, \Sigma \vdash \Delta}\ (\mathrm{p}) \qquad \frac{\Gamma \vdash A \quad A, \Pi \vdash \Delta}{\Gamma, \Pi \vdash \Delta}\ (\mathrm{cut})$$

**Logical rules (rules for propositional connectives):**

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \Big/ \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge, \mathrm{l}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge, \mathrm{r})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ (\vee, \mathrm{l}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Delta, A \vee B} \Big/ \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee, \mathrm{r})$$

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta}\ (\to, \mathrm{l}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ (\to, \mathrm{r})$$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash}\ (\neg, \mathrm{l}) \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}\ (\neg, \mathrm{r})$$

**Table 1** An additive version of Gentzen's LI.

as presented in Table 1, constitutes a variant of Gentzen's original calculus, where the logical rules that branch do not split the context of formulas denoted by $\Gamma$. In Girard's terminology (see below), we use the *additive* formulations of rules. This is somewhat unusual for the implication rule $(\to, \mathrm{l})$, which is traditionally presented in the context-splitting *multiplicative* form, as shown in rule $(\multimap, \mathrm{l})$ in Table 2. The reason for this deviation from tradition is that sequent proofs can be more directly interpreted as corresponding to **P**'s winning strategy in a Lorenzen-style dialogue game if purely additive rules are used (see, e.g., Fermüller 2003). The notational conventions there are as usual: $A$, $B$ are arbitrary formulas; $\Gamma$, $\Sigma$, and $\Pi$ denote (possibly empty) sequences of formula; $\Delta$ is either empty or a single formula; '$\vdash$' is called the 'sequent arrow'. The term 'substructural' refers to the structural rules,

[4] In the literature, Gentzen's intuitionistic sequent calculus is often referred to as 'LJ'. This seems to be due to the fact that now-obsolete typographical conventions let the letter following 'L' in Gentzen's original naming of the calculus appear to look more like a 'J' than an 'I' to later readers. But in any case, it is clear that the letter was meant to abbreviate the (German) word *intuitionistisch*, as opposed to the 'K' in 'LK', which abbreviates *klassisch*.

rendered here as (w, l), (w, r), (c), and (p), for ‘weakening left’, ‘weakening right’, ‘contraction’, and ‘permutation’, respectively.[5] The prefix ‘sub’ in ‘substructural’ indicates that one is interested in the effect of dropping, or at least restricting, some or even all of these rules. A host of different logics, with a diverse range of intended applications, arises in this manner. For details, we refer to Paoli 2002 and Restall 2002. Here, we just present the sequent calculus for intuitionistic linear logic ILL (see Table 2). In contrast

**Axioms:** $A \vdash A \qquad \Gamma \vdash \top \qquad \vdash 1 \qquad 0, \Gamma \vdash \Delta$

**Weakening with** $1$**:**

$$\frac{\Gamma \vdash \Delta}{1, \Gamma \vdash \Delta}\ (\mathrm{w}_1)$$

**Rules for propositional connectives:**

$$\frac{A, B, \Gamma \vdash \Delta}{A \otimes B, \Gamma \vdash \Delta}\ (\otimes, \mathrm{l}) \qquad \frac{\Gamma \vdash A \quad \Sigma \vdash B}{\Gamma, \Sigma \vdash A \otimes B}\ (\otimes, \mathrm{r})$$

$$\frac{A, \Gamma \vdash \Delta}{A \& B, \Gamma \vdash \Delta} \Big/ \frac{B, \Gamma \vdash \Delta}{A \& B, \Gamma \vdash \Delta}\ (\&, \mathrm{l}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&, \mathrm{r})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \oplus B, \Gamma \vdash \Delta}\ (\oplus, \mathrm{l}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \Big/ \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus, \mathrm{r})$$

$$\frac{\Gamma \vdash A \quad B, \Sigma \vdash \Delta}{A \multimap B, \Gamma, \Sigma \vdash \Delta}\ (\multimap, \mathrm{l}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap, \mathrm{r})$$

**Rules for the exponential:**

$$\frac{\Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ (\mathrm{w}!) \qquad \frac{!A, !A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ (\mathrm{c}!) \qquad \frac{A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ (!, \mathrm{l}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!, \mathrm{r})$$

**Table 2** Sequent calculus ILL for intuitionistic linear logic.

to LI, the left side of a sequent in ILL is a *multiset*, rather than a sequence, of formulas. This entails that the ‘comma’ separating formula occurrences is now to be interpreted as multiset union rather than as concatenation. Moreover, the permutation rule of LI is redundant in ILL. Unlike in LI, in ILL the right side of a sequent is never empty; i.e., $\Delta$ in Table 2 always denotes a formula. The cut rule is omitted in Table 2; it is the same as for LI (see Table 1).

Linear logic has been introduced by Girard (1987) and has triggered a steady stream of research, due to its tight connections with computational paradigms. Note that ILL features two forms of conjunction: $\otimes$ (‘times’ or multiplicative conjunction) and & (’with’ or additive conjunction). It is not hard to see that the rules for $\otimes$ and & are interderivable if we add (unrestricted) weakening and contraction to the calculus. In that sense, $\otimes$ and &

[5] The rule (cut), too, is structural. However, in contrast to the other structural rules, its admissibility and, in fact, stepwise eliminability (cut elimination, as expressed in Gentzen’s *Hauptsatz*) is usually deemed essential also for substructural sequent calculi.

are different refinements of ordinary conjunction. The connective ! ('bang' or 'of course') is referred to as 'exponential'. It is used to restrict applications of weakening and contraction to specific formulas. The rules for the exponential, (w!), (c!), (!, l), and (!, r), are called 'weakening', 'contraction', 'dereliction', and 'promotion', respectively. In the promotion rule, $!\Gamma$ denotes the multiset of formulas arising from prefixing each formula in $\Gamma$ by the exponential.

As already indicated in the introduction, substructural logics and in particular also ILL are intended to model resource-conscious reasoning. We recall a frequently cited example presented by Girard:

> In linear logic, two conjunctions $\otimes$ (*times*) and & (*with*) coexist. They correspond to two radically different uses of the word "and". Both conjunctions express the availability of two actions; but in the case of $\otimes$, both will be done, whereas in the case of &, only one of them will be performed (but we shall decide which one). To understand the distinction consider $A$, $B$, $C$:
>
> $A$: to spend \$1,
> $B$: to get a pack of Camels,
> $C$: to get a pack of Marlboro.
>
> An action of type $A$ will be a way of taking \$1 out of one's pocket (there may be several actions of this type since we own several notes). Similarly, there are several packs of Camels at the dealer's, hence there are several actions of type $B$. An action type $A \multimap B$ is a way of replacing any specific dollar by a specific pack of Camels.
>
> Now, given an action of type $A \multimap B$ and an action of type $A \multimap C$, there will be no way of forming an action of type $A \multimap B \otimes C$, since for \$1 you will never get what costs \$2 (there will be an action of type $A \otimes A \multimap B \otimes C$, namely getting two packs for \$2). However, there will be an action of type $A \multimap B \& C$, namely the superimposition of both actions. In order to perform this action, we have first to choose which among the two possible actions we want to perform, and then to do the one selected. (Girard 1995, p. 2)

The idea of interpreting formulas as action types has indeed turned out to be very fruitful. But here we are interested in a particular ambiguity in the above-cited presentation. Girard says that to perform an action of type $B \& C$ *we* have to choose whether to perform $A$ or to perform $B$. One may ask: who is 'we'? At first glance, it might seem nit-picking to insist on an answer to this question. However, in order to interpret a disjunctive formula (written as $B \oplus C$ in ILL), one has to speak of a choice as well. But presumably it should not be the same agent that is to choose between $B$ and $C$ when faced with $B \oplus C$ or with $B \& C$, respectively, when these formulas occur on the same side of a sequent.

At this point a look at Lorenzen's dialogical logic may help to clarify the semantic intuitions regarding conjunction and disjunction. Recall that in Lorenzen's dialogue game for intuitionistic logic (see Lorenzen and Lorenz 1978) there are two players: a proponent **P** and an opponent **O**. The rules of the game stipulate that, if **P** asserts $B \wedge C$ then **O** is entitled to choose either $B$ or $C$ and **P** has to assert the chosen conjunct. But if **P** asserts $B \vee C$, then, if attacked by **O**, **P** herself gets to choose whether to continue the dialogue with asserting $B$ or with asserting $C$. Revisiting Girard's presentation above, we may say that Lorenzen decided to model conjunction as &, which in his context actually coincides with $\otimes$, since in intuitionistic logic unrestricted (implicit or explicit) contraction and weakening are present. Moreover, from this perspective it becomes apparent that the agent referred to as 'we' by Girard should actually be split into two roles, akin to the proponent and the opponent in Lorenzen's dialogue game. Indeed, game-based models of linear logic, starting with Blass 1992, do exactly that. On the other hand, contemporary game semantics quickly departs from Lorenzen's concept of dialogical logic by interpreting formulas as (abstract) games, rather than as concrete assertions. To bridge the gap, we propose a game that does not interpret formulas as games, but rather as packages of information.

## 3 Two different information extraction games

We suggest to interpret a (single-conclusion) sequent $\Gamma \vdash F$ as a situation in which a client seeks to obtain the information represented by $F$ from the bundle of information $\Gamma$ provided by a server.[6] The purpose of the game is to stepwise reduce claims of the form 'The client can obtain $F$ from a server providing $\Gamma$' to less complex claims of the same form until an obviously true or an obviously false statement appears. The reduction proceeds by an interaction between two players: a proponent **P**, who acts in the interest of the client and thus seeks to reduce the initial claim to an obviously true one; and an opponent **O**, who acts as adversary to **P** (and thus also the client). The rules guiding this interaction refer to the topmost operator used to form a selected *information package* (*ip*) from simpler ips. For now, we just consider three such operators:

- $\mathsf{any\,of}(F_1, \ldots, F_n)$, with the intended meaning that the client can get any of the ips $F_1$ or ... or $F_n$ she likes, if $\mathsf{any\,of}(F_1, \ldots, F_n)$ is among those currently offered by the server;

[6] We do not claim any direct relevance to client–server configurations as studied in contemporary computer science. Moreover, one might object to our use of the term 'information': since we do not care about any specific content, but only about the type of structuring, it may be more appropriate to just speak of '(types of) packages of data'. However, we think that no corresponding confusion will arise in our context and hence prefer to stick with the term 'information packages', introduced in Fermüller and Lang 2017. In the latter paper the players of the game are identified with the client and the server, respectively. Here, we will speak of a 'proponent' and an 'opponent', instead, which refer to client–server situations of the indicated type. This shift in terminology is arguably more in line with a simple and coherent interpretation of the game and the underlying sequent calculus. Another advantage is its obvious relation to Lorenzen's scenario for dialogical logic.

- $\mathsf{some\ of}(F_1, \ldots, F_n)$, where the client also gets one of $F_1, \ldots, F_n$, but she (respectively player **P**, acting in her interest) has no control over which of those ips gets picked;
- $F_1\,\mathsf{given}\,F_2$, where the client can get $F_1$ under the condition that $F_2$ can be obtained as well from whatever is currently offered by the server.

Among the *atomic* ips, which admit no further reduction, there is the special ip $*$, which may be viewed as a 'wildcard': if $*$ is offered by the server then, since it can stand in for any ip that the client may want to obtain, **P** wins the game.

At first glance, it may seem that we have lost the connection with formulas of intuitionistic and linear logic by introducing the above operators. We immediately restore this connection, by first observing that it is sufficient to consider binary versions of $\mathsf{any\ of}$ and $\mathsf{some\ of}$. Clearly the indicated intended meaning justifies the following corresponding identities:

$$\mathsf{any\ of}(F) = F, \qquad \mathsf{some\ of}(F) = F,$$

$$\mathsf{any\ of}(F_1, \ldots, F_n) \;=\; \mathsf{any\ of}\Big(F_1,\, \mathsf{any\ of}\big(F_2, \ldots, \mathsf{any\ of}(F_{n-1}, F_n) \ldots\big)\Big),$$

$$\mathsf{some\ of}(F_1, \ldots, F_n) \;=\; \mathsf{some\ of}\Big(F_1,\, \mathsf{some\ of}\big(F_2, \ldots, \mathsf{some\ of}(F_{n-1}, F_n) \ldots\big)\Big).$$

Secondly, from now on, we write $\mathsf{any\ of}(F_1, F_2)$ as $F_1 \wedge F_2$, $\mathsf{some\ of}(F_1, F_2)$ as $F_1 \vee F_2$, $F_1\,\mathsf{given}\,F_2$ as $F_2 \to F_1$, and $*$ as $\bot$.

So far, we have only alluded to the intended meaning of the operators for the case where they occur among ips provided by the server. But we also want to consider states where the ip the client seeks to obtain is structured likewise. In particular, if the client wants to get $\bot$, the only way to succeed is to find $\bot$ also on the server's side. To fully fix the meaning of the operators, we have to specify additional reduction rules that refer to compound ips on the client's side.

More precisely, states of the game $\mathcal{G}_\mathsf{I}$ are denoted as $\Gamma \triangleright F$, where $F$ is an ip and $\Gamma$ is a multiset of ips. $\{G\} \cup \Gamma$ is denoted as $G, \Gamma$. The game proceeds in *rounds*. Each round is initiated by **P** choosing an occurrence of a compound ip, called the 'selected ip', in the current state. We will indicate the selected ip by underlining it. The following rules regulate the transition from the exhibited current state to its successor state.

$(\wedge_l^{\mathcal{G}})$ $\underline{F \wedge G}, \Gamma \triangleright H$:
: **P** chooses whether to extract $F$ or $G$ from the ips provided by the server; i.e., **P** decides whether the game continues in state $F, F \wedge G, \Gamma \triangleright H$ or in state $G, F \wedge G, \Gamma \triangleright H$.

$(\vee_l^{\mathcal{G}})$ $\underline{F \vee G}, \Gamma \triangleright H$:
: **O** chooses whether the server provides $F$ or provides $G$; accordingly, the game continues in state $F, F \vee G, \Gamma \triangleright H$ or in state $G, F \vee G, \Gamma \triangleright H$.

$(\to_l^{\mathcal{G}})$ $\underline{F \to G}, \Gamma \triangleright H$:
: **O** chooses whether the server simply provides $G$ or whether the next

state should correspond to the claim that $G$ can be obtained from $F$ together with other ips stored on the server. Accordingly, the game continues in state $G, F \to G, \Gamma \rhd H$ or in state $F \to G, \Gamma \rhd F$.

$(\wedge_r^{\mathcal{G}})$ $\Gamma \rhd \underline{F \wedge G}$:
**O** chooses whether the claim reduces to one about obtaining $F$ or about obtaining $G$; i.e., **O** decides whether the game continues in state $\Gamma \rhd F$ or in state $\Gamma \rhd G$.

$(\vee_r^{\mathcal{G}})$ $\Gamma \rhd \underline{F \vee G}$:
**P** chooses whether the claim reduces to one about obtaining $F$ or about obtaining $G$; i.e., **P** decides whether the game continues in state $\Gamma \rhd F$ or in state $\Gamma \rhd G$.

$(\to_r^{\mathcal{G}})$ $\Gamma \rhd \underline{F \to G}$:
No choice is involved; the game continues in state $F, \Gamma \rhd G$.

Any state of the form $H, \Gamma \rhd H$ or $\bot, \Gamma \rhd H$ is a *winning state* for **P**.

A *play* or *run* of the game is a (finite or infinite) sequence of states where each successor state arises from the previous one according to one of the above rules. **P** *wins* the play if the last state is a winning state for **P**. A *strategy* (*for* **P**) $\sigma$ for some *initial state* $S$ is given by a tree of states rooted in $S$. The branches of $\sigma$ consist in runs of the game starting with $S$. If $S'$ is a node (state) in $\sigma$, then it has a single successor node if, according to the rules, it is on **P** to choose the succeeding state in the play. If, however, the rules require **O** to choose between two possible successor states, then $\sigma$ branches at $S'$ with those two states as child nodes. If every branch of $\sigma$ ends in a leaf node that is a winning state for **P**, then $\sigma$ is a *winning strategy* (for **P**).

Let us make some observations regarding the structure and rules of the game $\mathcal{G}_{\mathsf{I}}$ from a semantic perspective:

- Since **P** selects the ip to focus on next, we may call **P** the 'scheduler' of the game. We will retain this feature when, in later games, we decide to assign even more 'scheduling power' to **P**.
- We did not specify winning states for **O**. The role assigned to **O** is to prevent **P** from reducing the game to a state where she (**P**) wins. Since the rules of $\mathcal{G}_{\mathsf{I}}$ entail that all decomposed ips provided by the server are retained, **P** may force the game to loop ad infinitum, if at least one compound ip is provided initially. But such infinite plays are never won by **P** and may thus be classified as won by **O**.
- Notice that the rules $(\wedge_l^{\mathcal{G}})$ and $(\vee_l^{\mathcal{G}})$ directly capture the intended meaning of any of and some of, respectively.
- The rules $(\wedge_r^{\mathcal{G}})$ and $(\vee_r^{\mathcal{G}})$ are strictly dual to $(\wedge_l^{\mathcal{G}})$ and $(\vee_l^{\mathcal{G}})$, respectively: a choice of a subformula by one of the players implies a corresponding choice by the other player, if the selected formula occurrence is on the other side. This furthermore entails that the rules for $\wedge$ arise from those for $\vee$ (and vice versa) by exchanging the roles of **P** and **O**.
- While the rule $(\to_r^{\mathcal{G}})$ is rather self-explanatory, it is less obvious that $(\to_l^{\mathcal{G}})$ corresponds to the indicated meaning of the conditional packag-

ing operator 'given'. To capture the intended semantics more directly, it seems more appropriate to apply a rule $(\rightarrow^{\mathcal{G}}_{10})$ which stipulates that the state $\underline{F \rightarrow G}, \Gamma \triangleright H$ is succeeded by the state $G, \Gamma \triangleright H$, provided that $F$ occurs in $\Gamma$. Indeed, $(\rightarrow^{\mathcal{G}}_{10})$ is sound, but redundant in the presence of $(\rightarrow^{\mathcal{G}}_{1})$. This is easy to see, as follows. Up to a copy of $F$, the states $F, F \rightarrow G, \Gamma \triangleright H$ and $F \rightarrow G, \Gamma \triangleright H$ are identical if $F \in \Gamma$. This means that $(\rightarrow^{\mathcal{G}}_{10})$ amounts to an instance of $(\rightarrow^{\mathcal{G}}_{1})$ when $H = G$, because in that case the other possible successor state, according to $(\rightarrow^{\mathcal{G}}_{1})$, is $F \rightarrow G, \Gamma \triangleright G$. Since the latter is a winning state for **P**, we may assume without loss of generality that **O** never chooses this option.

Seen in this light, $(\rightarrow^{\mathcal{G}}_{1})$ arises, since the fact that $G$ can be obtained from $F \rightarrow G$ in the presence of $F$ does not entail the following more general claim: if $F \rightarrow G$ is provided, the client may obtain $H$ from $G$ (and the other provided ips) if she has a way to obtain $F$ from the ips provided by the server. The two involved sub-claims (about obtaining $H$ from $G$, and about obtaining $F$, respectively) directly correspond to the two possible successor states according to rule $(\rightarrow^{\mathcal{G}}_{1})$. Since it is **O**'s role to point out, if possible, that at least one of the two claims is wrong, the rule lets **O** choose in which of the two states the game should continue.

In Section 4, we will show that **P** has a winning strategy for $\Gamma \triangleright F$ in $\mathcal{G}_{\mathsf{I}}$ if and only if $\Gamma \vdash F$ is provable in $\mathsf{LI}$. More important for our topic is the fact that $\mathcal{G}_{\mathsf{I}}$ invites certain modifications that are motivated by a resource-conscious reading of the claim that the client can obtain certain information from the server. Winning strategies in those variations of the game can be shown to correspond to proofs in appropriate substructural sequent calculi.

We conclude this section with the game $\mathcal{G}_{\mathsf{ILL}}$ for intuitionistic linear logic. The overall format of the game is as in $\mathcal{G}_{\mathsf{I}}$. The language is as in $\mathsf{ILL}$, i.e., ips are built up from simpler ips using the operators &, $\otimes$, $\oplus$, and $\multimap$. Among the atomic ips are the following special ones: $0$, $1$, and $\top$. The rules for reducing selected compound information packages are as follows.

$(\otimes^{\mathcal{G}}_{1})$ $\underline{F \otimes G}, \Gamma \triangleright H$:
The game continues in state $F, G, \Gamma \triangleright H$.

$(\&^{\mathcal{G}}_{1})$ $\underline{F \& G}, \Gamma \triangleright H$:
**P** decides whether the game continues in state $F, \Gamma \triangleright H$ or in state $G, \Gamma \triangleright H$.

$(\oplus^{\mathcal{G}}_{1})$ $\underline{F \oplus G}, \Gamma \triangleright H$:
**O** decides whether the game continues in state $F, \Gamma \triangleright H$ or in state $G, \Gamma \triangleright H$.

$(\multimap^{\mathcal{G}}_{1})$ $\underline{F \multimap G}, \Gamma \triangleright H$:
**P** chooses a partition of $\Gamma$ into disjoint multisets $\Gamma_1$ and $\Gamma_2$; then **O** decides whether the game continues in state $\Gamma_1 \triangleright F$ or in state $G, \Gamma_2 \triangleright H$.

$(!^{\mathcal{G}}_{\mathrm{l}})$ $\underline{!F}, \Gamma \rhd H$:
**P** has three different options: she may convert the selected ip $!F$ into $F$ or add a copy of it or dismiss it. Accordingly, **P** decides whether the game continues in state $F, \Gamma \rhd H$, in state $!F, !F, \Gamma \rhd H$ or in state $\Gamma \rhd H$.

$(\otimes^{\mathcal{G}}_{\mathrm{r}})$ $\Gamma \rhd \underline{F \otimes G}$:
**P** chooses a partition of $\Gamma$ into disjoint multisets $\Gamma_1$ and $\Gamma_2$; then **O** decides whether the game continues in state $\Gamma_1 \rhd F$ or in state $\Gamma_2 \rhd G$.

$(\&^{\mathcal{G}}_{\mathrm{r}})$ $\Gamma \rhd \underline{F \& G}$:
**O** decides whether the game continues in state $\Gamma \rhd F$ or in state $\Gamma \rhd G$.

$(\oplus^{\mathcal{G}}_{\mathrm{r}})$ $\Gamma \rhd \underline{F \oplus G}$:
**P** decides whether the game continues in state $\Gamma \rhd F$ or in state $\Gamma \rhd G$.

$(\multimap^{\mathcal{G}}_{\mathrm{r}})$ $\Gamma \rhd \underline{F \multimap G}$:
The game continues in state $F, \Gamma \rhd G$.

$(!^{\mathcal{G}}_{\mathrm{r}})$ $!\Gamma \rhd \underline{!H}$:
The game continues in state $!\Gamma \rhd H$. Note that **P** can only select $!H$ on the client's (right) side if all ips on the server's (left) side have ! as their outermost operator.

$(1^{\mathcal{G}})$ $\underline{1}, \Gamma \rhd H$:
The game continues in state $\Gamma \rhd H$.

States of the form $H \rhd H$, $\Gamma \rhd \top$, $\rhd 1$, or $0, \Gamma \rhd H$ are *winning states* for **P**. The definition of plays and winning strategies is as for $\mathcal{G}_{\mathsf{I}}$.

Like for the game $\mathcal{G}_{\mathsf{I}}$ above, we offer remarks on the structure and rules of $\mathcal{G}_{\mathsf{ILL}}$ that refer to the intended interpretation of the game.

- We still want to read a state $\Gamma \rhd H$ as expressing the claim that a client can obtain the ip $H$ from the multiset $\Gamma$ of ips offered by a server. However, we add the qualification 'exactly'; in other words, $\Gamma \rhd H$ asserts that to get the ip $H$ no less *and no more* than the resources $\Gamma$ are needed.
- Disregarding the different language for forming ips, arguably the most essential difference between $\mathcal{G}_{\mathsf{I}}$ and $\mathcal{G}_{\mathsf{ILL}}$ is the fact that the selected ips on the server's (left) side now disappear when reduced. We view this as a hallmark of resource consciousness: compound information packages can be 'unpacked' upon request of **P**, but only the corresponding parts (depending on the type of packaging operator) remain available for further reduction or immediate use by the client.
- Except for the just-mentioned removal of the selected ip on the server's side, the rules for & and $\oplus$ are the same as for $\wedge$ and $\vee$, respectively. This means that we suggest to read & as a resource-conscious (binary) version of any of, and $\oplus$ as a corresponding version of some of.
- The informal reading of $F \otimes G$ suggested by $(\otimes^{\mathcal{G}}_{\mathrm{l}})$ is that of a package containing both $F$ and $G$. Rule $(\otimes^{\mathcal{G}}_{\mathrm{l}})$ signifies explicit resource consciousness: in order to reduce the claim that the client can obtain $F \otimes G$ from the bundle of information $\Gamma$ provided by the server, **P** has to

declare which part of $\Gamma$ is needed to get $F$ and which (disjoint) part of $\Gamma$ is needed to obtain $G$.

- Rule $\left(\multimap_{\mathrm{r}}^{\mathcal{G}}\right)$ is exactly as $\left(\to_{\mathrm{r}}^{\mathcal{G}}\right)$ of $\mathcal{G}_{\mathsf{I}}$. The explanation that we gave for $\left(\to_{\mathrm{l}}^{\mathcal{G}}\right)$ above also applies to $\left(\multimap_{\mathrm{l}}^{\mathcal{G}}\right)$ but has to be augmented by attention to the same type of resource consciousness that we just pointed out for the case of $\left(\otimes_{\mathrm{l}}^{\mathcal{G}}\right)$: unpacking $F \multimap G$ entails that **P** has to declare which part of the provided resources is used to get $G$ from $F$ and which part is used to get $F$.
- The special operator ! is used to 'protect' ips from removal from the server if **P** wishes to retain them. Roughly, ! corresponds to 'arbitrarily often'.[7] Considered as a packaging operator, the reader is invited to think of $!A$ as a special 'protected' resource that, upon request, can be converted into an ordinary instance of $A$, but that also can be copied (and hence protected) or removed if desired. Note that the option of removal is significant because (disregarding special atomic ips) it is **P**'s aim to maneuver the game into a final state where only the ip that the client wants to obtain is provided by the server. The rule $\left(!_{\mathrm{l}}^{\mathcal{G}_{\mathsf{ILL}}}\right)$ directly reflects the indicated understanding of protection. The rule $\left(!_{\mathrm{r}}^{\mathcal{G}_{\mathsf{ILL}}}\right)$ expresses the stipulation that the client can obtain $!A$ itself from resources offered by the server if those resources are all likewise protected and if $A$ can be obtained from them.
- In $\mathcal{G}_{\mathsf{I}}$ the only atomic ip with a fixed meaning is the 'wildcard' $*$, written as $\perp$. In $\mathcal{G}_{\mathsf{ILL}}$ we have three special atomic ips:
  - The intended meaning of 1 is that of an empty resource (piece of information). Nothing has to be and nothing should be provided by the server in order for the client to obtain it. This explains why $\rhd 1$ is a winning state for **P**. But it also explains why, according to rule $\left(1^{\mathcal{G}}\right)$, it may be removed from the ips provided by the server whenever **P** wishes to do so.[8]
  - The ip 0 is a kind of joker for the client. As soon as **P** finds it among the ips provided by the server, she wins (on behalf of the client). On the other hand, the claim that the client can get hold of this joker, i.e., that **P** can win in state $\Gamma \rhd 0$, can only be verified by pointing out that 0 itself is provided by the server. (Obviously, 0 is closely related to the 'wildcard' $\perp$ of $\mathcal{G}_{\mathsf{I}}$.)
  - $\top$ serves as a specific kind of winning signal for the client: a state in which $\top$ appears on the client's (right) side is winning for **P**, independently of what is currently provided by the server.

[7] As pointed out by Fermüller and Lang (2017), the reading of $!A$ as 'arbitrarily many copies of $A$' is somewhat problematic. We consider our information extraction games as a tool for connecting the informal semantics of substructural logics with formal semantics in a manner that sheds light on some of the subtleties and ambiguities involved in the informal reading.

[8] Note that $\left(1^{\mathcal{G}}\right)$ corresponds to the weakening rule for 1 in $\mathsf{ILL}$.

While we are not concerned here with the systematic search for **P**'s winning strategies in $\mathcal{G}_{\mathsf{I}}$ and $\mathcal{G}_{\mathsf{ILL}}$, one may still point out a correlation between the technique of 'focusing' in proof search and the pattern of **P–O** interaction emerging in the above rules. Focusing has spawned quite a stream of literature; we refer to Andreoli 2001 and Liang and Miller 2009 for detailed presentations. To see the connection, note that for some of the above game rules, e.g., $(\otimes_{\mathrm{l}}^{\mathcal{G}})$, $(\&_{\mathrm{r}}^{\mathcal{G}})$, and $(\multimap_{\mathrm{r}}^{\mathcal{G}})$, no choice of **P** is involved. However, other rules, like $(\otimes_{\mathrm{r}}^{\mathcal{G}})$ and $(\multimap_{\mathrm{l}}^{\mathcal{G}})$, require **P** to partition the context $\Gamma$ of the current state. The first type of rules corresponds to invertible, the second type to non-invertible sequent rules. Focusing recognizes that applications of invertible and non-invertible rules, respectively, are best bundled into different phases of proof search. Whereas, in a 'negative phase', invertible rules can be applied exhaustively in any order, non-invertible rules are applied in a chain-like fashion, in the 'positive phase'. In terms of our games, these phases relate to the respective necessity or lack of choices by **P** that may have to be withdrawn to guarantee a successful play for **P**. We refer to Sticht 2018 for a presentation of multi-agent games for intuitionistic and modal logics that makes the connection to focusing in corresponding sequent calculi very explicit.

## 4 Relating games and calculi

Notice that the winning states (for **P**) in game $\mathcal{G}_{\mathsf{ILL}}$ turn into the axioms of the sequent calculus ILL if we replace '$\triangleright$' by '$\vdash$'. Moreover, the rules of $\mathcal{G}_{\mathsf{ILL}}$ also directly correspond to the rules of ILL. Except for the three sequent rules referring to the exponential on the left side, which we have mapped into a single game rule, there is a one-to-one correspondence between ILL-rules and $\mathcal{G}_{\mathsf{ILL}}$-rules. Branching sequent rules are interpreted as game rules calling for a corresponding choice of **O**. Choices between different applicable rules or different instances of the same rule are left to **P**. Since we have defined winning strategies for **P** as trees of game states, the outlined correspondence generalizes to the level of proofs and winning strategies, respectively.

**Theorem 1** *Every cut-free* ILL*-proof of a sequent $\Gamma \vdash F$ directly corresponds to a winning strategy for* **P** *in $\mathcal{G}_{\mathsf{ILL}}$ with initial state $\Gamma \triangleright F$, and vice versa.*

Mathematically minded readers may be disappointed about the 'shallowness' of Theorem 1. Indeed, the immediacy of the relation between ILL-proofs and winning strategies in $\mathcal{G}_{\mathsf{ILL}}$ just invites a particular reading or viewpoint of proofs, without involving any deeper insights into the properties of ILL. We consider this to be a virtue rather than a defect. After all, our aim here is to show that the game-based interpretation sheds light on the informal, resource-conscious semantics of the (unmodified) sequent calculus. But we readily admit that this interpretation only obtains genuine profile if we take it as a principle for the analysis of 'substructurality' in general, rather than as an isolated form of attaching meaning to ILL-derivations. To this aim, we will look in Section 6 at different ways to variate and generalize $\mathcal{G}_{\mathsf{ILL}}$ that are triggered by semantic intuitions rather than by the goal to find alternative characterizations of known sequent calculi. In the rest of the current section, we want to cast light on the relation between intuitionistic logic and (intuitionistic) linear logic from the point of view of information extraction.

In contrast to the case for ILL, the relation between LI-proofs and **P**'s winning strategies in $\mathcal{G}_{\mathsf{I}}$ is not immediately obvious. To establish a one-to-one correspondence between these strategies and proofs in a sequent calculus that is sound and complete for intuitionistic logic, we precede by stepwise transforming Gentzen's LI into its variant LIp.

1. Gentzen introduced the sequent calculus as a kind of meta-calculus, dealing with inferential forms (*Schlußweisen*) rather than directly representing informal mathematical proofs.[9] He found it natural to organize the assumptions, i.e., the formulas on the left side[10] of sequents, as sequences of formulas. The presence of the permutation rule $(\mathrm{p})$ of LI corresponds to the irrelevance of the order in which assumptions are presented. In principle, we could model Gentzen's approach in a corresponding game by allowing **P**, in her capacity as scheduler, to re-order the ips on the server's side whenever she wants. However, we prefer to follow the by-now-usual format of dispensing with permutation and instead declaring the left side of sequents to be a multiset of formulas (like in ILL).
2. To get rid of the weakening rule $(\mathrm{w})$, we generalize the axiom sequents by allowing arbitrary 'side formulas' to additionally occur on the left side. This corresponds to shifting weakenings to the axioms.
3. The contraction rule $(\mathrm{c})$ also becomes redundant when we keep a copy of the principal formula (the one exhibited) in the upper sequent(s) of the rules $(\wedge,\mathrm{l})$, $(\vee,\mathrm{l})$, and $(\to,\mathrm{l})$.
4. Finally, note that in $\mathcal{G}_{\mathsf{I}}$ there is no operator corresponding to negation and that in any state there is some ip that the client wants to obtain. The latter feature corresponds to requiring that the right side of sequents is never empty. Since $(\neg,\mathrm{r})$ is the only rule where—read from bottom to top—a formula disappears from the right side, the two mentioned features are intimately related. LI-sequents can be brought in line with game states by declaring $\neg F$ to be an abbreviation of $F \to \bot$ and adding an axiom referring to the occurrence of the new constant $\bot$ on the left side of sequents.

[9] The latter task was assigned to the calculus of Natural Deduction (*natürliches Schließen*).

[10] In the classical sequent calculus LK, also the formulas on the right side of a sequent, representing alternative conclusions, appear as a sequence.

The sequent calculus $\mathsf{LIp}$ resulting from these modifications is exhibited in Table 3. One may add to $\mathsf{LIp}$ a cut rule, which is exactly as for $\mathsf{LI}$; but, like

**Axioms (initial sequents):** $A, \Gamma \vdash A \qquad \bot, \Gamma \vdash A$

**Logical rules (rules for propositional connectives):**

$$\frac{A,\, B,\, A{\wedge}B,\, \Gamma \vdash C}{A{\wedge}B,\, \Gamma \vdash C}\ (\wedge,\mathrm{l}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A{\wedge}B}\ (\wedge,\mathrm{r})$$

$$\frac{A,\, A{\vee}B,\, \Gamma \vdash C \quad B,\, A{\vee}B,\, \Gamma \vdash C}{A{\vee}B,\, \Gamma \vdash C}\ (\vee,\mathrm{l}) \quad \frac{\Gamma \vdash A}{\Gamma \vdash A{\vee}B} \Big/ \frac{\Gamma \vdash B}{\Gamma \vdash A{\vee}B}\ (\vee,\mathrm{r})$$

$$\frac{A{\rightarrow}B, \Gamma \vdash A \quad B,\, A{\rightarrow}B, \Gamma \vdash C}{A{\rightarrow}B, \Gamma \vdash C}\ ({\rightarrow},\mathrm{l}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A{\rightarrow}B}\ ({\rightarrow},\mathrm{r})$$

**Table 3** The 'proof-search-friendly' variant $\mathsf{LIp}$ of Gentzen's $\mathsf{LI}$.

for $\mathsf{LI}$ itself, it is redundant.

Actually, $\mathsf{LIp}$ is neither new nor specific to concerns about a game-based interpretation. The textbook by Troelstra and Schwichtenberg (2000) presents a closely related variant under the name $\mathsf{G3i}$ and establishes the equivalence with $\mathsf{LI}$ and thus soundness and completeness with respect to intuitionistic logic. While the rules that absorb contraction may seem awkward if read in the usual top-to-bottom fashion, $\mathsf{LIp}$ amounts to a natural way of presentation if viewed from the point of view of goal-directed (bottom-up) proof search. Incorporating contraction into the logical rules corresponds to the fact that we do not have to decide during proof search how many copies of identical formulas may eventually be needed. Similarly, shifting weakening to the axioms means that we do not have to decide in advance which formulas will actually be needed to establish the proof. Anyway, for our current purpose it suffices to rely on the fact that $\mathsf{LIp}$, just like $\mathsf{LI}$, is sound and (cut-free) complete for intuitionistic logic. This entails that the following statement expresses that the information extraction game $\mathcal{G}_{\mathsf{I}}$ characterizes intuitionistic logic.

**Theorem 2** *Every cut-free $\mathsf{LIp}$-proof of a sequent $\Gamma \vdash F$ directly corresponds to a winning strategy for $\mathbf{P}$ in $\mathcal{G}_{\mathsf{I}}$ with initial state $\Gamma \triangleright F$, and vice versa.*

In light of the correspondence between *cut-free* sequent proofs and winning strategies expressed in Theorems 1 and 2, one may wonder whether there is a game-based interpretation of the cut rule as well. Indeed, we have the following fact.

**Proposition 1** *For $\mathcal{G}_{\mathsf{I}}$ as well as $\mathcal{G}_{\mathsf{ILL}}$: whenever $\mathbf{P}$ has winning strategies for $\Gamma \triangleright F$ and for $F, \Pi \triangleright H$, $\mathbf{P}$ has a winning strategy for $\Gamma, \Pi \triangleright H$ as well.*

Note that, for $\mathcal{G}_{\mathsf{ILL}}$, Proposition 1 is a direct consequence of Theorem 1 and the well-known fact that the cut rule is admissible in $\mathsf{ILL}$. In fact, the cut rule is not just admissible, but constructively eliminable. That is, any given ILL-proof can be stepwise transformed into a cut-free one (see, e.g., Troelstra 1992). The case for $\mathcal{G}_{\mathsf{I}}$ is analogous, except that in appealing to Theorem 2, we rely on the fact that LI- and LIp-proofs can be translated into each other (following, e.g., Troelstra and Schwichtenberg 2000). Thus, while not adding any technical insights to well-established proof-theoretical facts, Proposition 1 suggests a reading of cut-elimination in terms of information extraction. If $H$ can be extracted from some bunch of ips augmented by the ip $F$, and $F$ in turn can be extracted from yet another multiset of ips, then $H$ can be extracted from the union of the two indicated multisets of ips. Thus, in a sense, the proposition serves as a sanity check for the games.[11] Actually, as is well known, cut is not only redundant in standard sequent calculi, but applications of the cut rule can be eliminated from given proofs by algorithmic, stepwise transformations.[12] For corresponding games this means that given winning strategies can be combined *effectively* in the indicated fashion.

## 5 Lorenzen-style games?

At least at a first glance, the format of our information extraction games seems to be very different from that of Lorenzen's dialogue game for intuitionistic logic. We nevertheless prefer to speak of 'Lorenzen-style games', which calls for some explanation. Let us start by presenting Lorenzen's game in a manner close to his own, in particular also followed by his student and collaborator Kuno Lorenz.[13]

While Lorenzen always also had quantifiers and interpreted statements (e.g., of arithmetic) in view, we confine attention to formal—as opposed to material—dialogue systems for purely logical, constructive reasoning at the propositional level. There are several versions of formal dialogue systems that can be traced back to Lorenzen's original contributions. For an overview and historical remarks, we refer to Lorenzen and Lorenz 1978, Krabbe 1985, Barth and Krabbe 1982, and Lorenz 2001. Here, we sketch a version that retains the overall structure and essential features of Lorenzen's game for intuitionistic logic, but that will also allow us to highlight the close relation to the information extraction game $\mathcal{G}_{\mathsf{I}}$ presented in Section 3.

A *dialogue* (run of the game) starts with the assertion of a formula by the proponent **P**, who seeks to defend his assertion against systematic attacks by the opponent **O**. The dialogue proceeds by strictly alternating moves

[11] Note, however, that the indicated combination of strategies depends on the organization of the information offered by the server as multisets of ips. As we will see in Section 6, organizing this information in the form of, e.g., a stack may well prevent strategies from being composable in this manner.

[12] In the case of LI, this is established in the proof of the *Hauptsatz* of Gentzen 1935.

[13] The book Lorenzen and Lorenz 1978 collects corresponding papers and book excerpts.

between **P** and **O**, in accordance with the rules presented in Table 4. The

| assertion by **X** | attack by **Y** | defense by **X** |
|---|---|---|
| $A \wedge B$ | $?_l$ or $?_r$ (**Y** chooses) | $A$ or $B$, accordingly |
| $A \vee B$ | ? | $A$ or $B$ (**X** chooses) |
| $A \rightarrow B$ | $A$ | $B$ |
| $\neg A$ | $A$ | (no defense possible) |

If **X** is **P**, then **Y** is **O**; if **X** is **O**, then **Y** is **P**.

**Table 4** Lorenzen's dialogue rules for propositional connectives.

special symbols ? and $?_l/?_r$ are used to denote attack moves on disjunctions and conjunctions, respectively. Observe that attacking an implication or a negated formula consists in asserting a corresponding (sub)formula. Therefore, in general, both players must attack the other player's assertions as well as defend their own assertions during a dialogue. There are different ways to define what it means to win a dialogue. If one wants to stick with the idea that a dialogue has been won by **P** if and only if **O** has no legitimate way to continue it, then one has to impose special rules that prevent **O** from forcing the game to cycle indefinitely around the same assertion(s). This is achieved by artificially setting, at the beginning of each dialogue, a limit on the number of times that **O** may attack each assertion of **P** or defend each of his own assertions. Here, we prefer to declare **P** the winner of the dialogue if **O** attacks an assertion of **P** that **O** himself has already asserted previously. Following Barth and Krabbe (1982), one may speak of an *ipse dixisti* rule ('you said it yourself').

Another, rather inconsequential, matter concerns negation. Rather than viewing $\neg$ as a primitive logical symbol, we may follow the route already taken in Section 4 for the transition from $\mathsf{LI}$ to $\mathsf{LIp}$ and declare $\neg A$ to be an abbreviation of $A \rightarrow \bot$. The atomic statement $\bot$ is indefensible, of course. Hence any state in which **O** has asserted $\bot$ is a winning state for **P** too.

The above rules and conventions are not yet sufficient to characterize intuitionistic validity via the existence of a winning strategy for **P**. We have to impose further *structural rules* (*Rahmenregeln*, in Lorenzen's diction) to achieve this. Again, this can be done in various ways, for which we refer the interested reader to Krabbe 1985 and Felscher 1986. For our current purpose it suffices to just highlight a rule that regulates the progress of dialogues in a manner that is central for capturing intuitionistic rather than classical logic:

Intuitionistic Round Closure Rule: Player **X** can only defend against the *last* attack of the other player **Y**.

Traditionally, dialogues are presented in a table with two columns, where the left column records the moves of **O** and the right column those of **P** and where defense moves are placed in the same row as a corresponding attack move by the other player. This convention lets states of the dialogue game appear to be quite different from the sequent-like states of the game $\mathcal{G}_{\mathsf{I}}$. However, already Barth and Krabbe (1982) had introduced the notion of a *dialogue sequent* for recording a state of a dialogue game. These dialogue sequents contain more information than an ordinary sequent. This is necessary since dialogue games are usually not organized in rounds as in $\mathcal{G}_{\mathsf{I}}$, where the formula (ip) selected by **P** has to be reduced immediately. These reduction steps correspond to (possibly implicit) attacks by **O**, followed by an immediate reply (defense) by **P**. Certainly, a comparison between Table 4 and the rules of $\mathcal{G}_{\mathsf{I}}$ reveals a close connection in the treatment of conjunction, disjunction, and implication, which carries over to negation modulo the definition $\neg A = A \to \bot$.

Concerning the more liberal interaction between moves by **P** and **O** in dialogue games, compared to our information extraction games, let us point out that already Felscher (1985) had introduced a structural rule ('*E*-rule') that requires **O** to always react to the immediately preceding move by **P**. This rule brings dialogue strategies closer to sequent derivations, or equivalently, to analytic tableaux, in the case of Felscher 1985. An essential part of Felscher's proof of the adequateness of Lorenzen's dialogue game for intuitionistic logic consists in showing that the *E*-rule is actually redundant. However, adding just the *E*-rule to Lorenzen's original dialogue system does not yet enforce that dialogues proceed in rounds akin to those of the game $\mathcal{G}_{\mathsf{I}}$. To achieve a direct correspondence between plays of $\mathcal{G}_{\mathsf{I}}$ and formal dialogues one would have to impose an additional structural rule that requires also **P** to immediately reply to any attack by her opponent **O**.

We hope that the above remarks suffice to make clear that the game $\mathcal{G}_{\mathsf{I}}$, despite its quite different terminology and motivation, may be seen as a variant of Lorenzen's dialogue game, where a stricter regime on the succession of attack and defense moves is enforced, but where logical connectives are treated in a very similar fashion. In any case, we invite the reader to compare $\mathcal{G}_{\mathsf{I}}$ and its substructural cousin $\mathcal{G}_{\mathsf{ILL}}$ with the game-based semantics of (fragments of) linear logic, presented by, e. g., Blass (1992) and Abramsky and Jagadeesan (1994). The central paradigm underlying the latter type of game semantics is to interpret *formulas as games* and *logical connectives as operators on games*. This view has certainly turned out to be very fruitful for providing compositional, abstract semantics to various formalisms, including functional programming languages, but it involves a major *structural*—and not just motivational—deviation from Lorenzen's dialogical logic. In Lorenzen's dialogues, like in $\mathcal{G}_{\mathsf{I}}$ and $\mathcal{G}_{\mathsf{ILL}}$, one cannot simply identify (sub)formulas with (sub)games: states in those games are inherently more complex than single formulas, but they can be denoted as (possibly augmented) sequents.

## 6 Game variants for other substructural calculi

In the last section, we have pointed out that Lorenzen's dialogical approach to the foundations of constructive reasoning and our characterization of intuitionistic logic in terms of interactive information extraction are closely related. On the other hand, we have already seen in Section 3 that a few straightforward modifications turn the information extraction game $\mathcal{G}_{\mathsf{I}}$ into the game $\mathcal{G}_{\mathsf{ILL}}$ for intuitionistic linear logic. In this section we want to emphasize that the two just-mentioned games are only examples of a whole class of related games that allow one to model a wide range of substructural calculi as specific systems of information extraction.

Recall that we have interpreted a state $\Gamma \triangleright H$ in $\mathcal{G}_{\mathsf{ILL}}$ as the claim that *no less and no more* than the resources (ips) $\Gamma$ are needed to obtain the ip $H$. A particularly simple and natural variation $\mathcal{G}_{\mathsf{aILL}}$ of $\mathcal{G}_{\mathsf{ILL}}$ arises if we want to interpret $\Gamma \triangleright H$ as 'The resources (ips) $\Gamma$ *suffice* to obtain $H$', instead.[14] That is, like in the game $\mathcal{G}_{\mathsf{I}}$ for intuitionistic logic, **P** is not required to use *every* resource provided by the server when showing how the client can obtain the desired information package. On the other hand, like in $\mathcal{G}_{\mathsf{ILL}}$, but unlike in $\mathcal{G}_{\mathsf{I}}$, we want to remain resource-conscious in all other respects. In particular, we maintain that in 'unpacking' a selected ip, only (one of) its sub-ips are (is) retained, unless the selected ip is 'protected' by the exponential (!). In accordance with the modified interpretation, we define winning states in $\mathcal{G}_{\mathsf{aILL}}$ as in $\mathcal{G}_{\mathsf{I}}$, rather than as in $\mathcal{G}_{\mathsf{ILL}}$: **P** wins in any state of the form $H, \Gamma \triangleright H$ or $\bot, \Gamma \triangleright H$. We also retain $\Gamma \triangleright \top$ as winning state, but the connectives 0 and 1 are dropped from the language, since our modifications render them redundant. Moreover, we modify rule $(!_r^{\mathcal{G}})$ of $\mathcal{G}_{\mathsf{ILL}}$ as follows. For selecting $!H$ on the client's side (right side of $\triangleright$) we no longer require that ips on the server's (left) side are protected by !; but we stipulate that all unprotected ips (i.e., those not preceded by !) are dropped from the server (left side of $\triangleright$) when reducing the client's desired $!H$ to $H$. Readers familiar with the corresponding terminology will recognize that the modified game $\mathcal{G}_{\mathsf{aILL}}$ directly corresponds to the sequent calculus $\mathsf{aILL}$ for *affine* linear logic.

Paoli (2002) bases his take on substructural logics on the sequent calculus $\mathsf{LL}$ that corresponds to linear logic without exponentials. In light of Theorem 1 of Section 4, it is quite obvious that a single-conclusion (i.e., intuitionistic) version of Paoli's $\mathsf{LL}$ corresponds to the game arising from $\mathcal{G}_{\mathsf{ILL}}$ by simply dropping the rules for !. Analogously, we may model (partially) contraction-free versions of intuitionistic logic by composing a game where we select rules for particular connectives from either $\mathcal{G}_{\mathsf{I}}$ or $\mathcal{G}_{\mathsf{ILL}}$, as appropriate. Different options for dropping contraction—and thus for introducing some limited form of resource consciousness—in intuitionistic logic thus arguably receive a more tangible meaning. For example, let us consider conjunction $(A \land B)$, which in $\mathcal{G}_{\mathsf{I}}$ we interpreted as $\mathsf{any\ of}(A, B)$, indicating

[14] This interpretation was already considered by Fermüller and Lang (2017).

that the client can get either $A$ or $B$ from this ip, if offered by the server. Clearly, it makes a difference whether the ip $\mathsf{any\,of}(A, B)$ remains available for further extraction after one of its sub-ips has been extracted, as in the rule $(\wedge_1^{\mathcal{G}})$ of $\mathcal{G}_{\mathsf{I}}$, or whether it is removed upon such an extraction, as in the rule $(\&_1^{\mathcal{G}})$ for $\mathcal{G}_{\mathsf{ILL}}$. In the first case, both conjuncts can be obtained, even more than once if desired, whereas in the second case, only one of the conjuncts is available for the client in any given play. Replacing $(\wedge_1^{\mathcal{G}})$ with $(\&_1^{\mathcal{G}})$, where & is identified with $\wedge$, but leaving the rest of the rules of $\mathcal{G}_{\mathsf{I}}$ unchanged, is indeed adequate for a version of intuitionistic logic that is contraction-free for conjunctions. Yet other versions of contraction-free intuitionistic logics can be obtained from $\mathcal{G}_{\mathsf{aILL}}$ by letting different combinations of left and right rules for $\otimes$ and & refer to a single conjunction operator. As discussed, e.g., by Troelstra (1992), not all such combinations result in contraction-free calculi where the cut rule is admissible. But in any case, we may use information extraction games to illustrate the intended meaning of these calculi in terms of resource-conscious reasoning.

An arguably more interesting variation of a substructural sequent calculus arises from the fact (pointed out, e.g., in Danos, Joinet, and Schellinx 1993) that the exponentials (! and ?) of linear logics are not uniquely determined by the corresponding logical rules. For the single-conclusion sequent setting this means that we may have different, non-equivalent variants of !, called 'subexponentials', in the calculus. For example, we may enrich ILL by replacing the rules $(\mathrm{w}!)$, $(\mathrm{c}!)$, $(!, \mathrm{l})$ with different copies of them that refer to corresponding variants, say $!^a$, of the original exponential !, where $a$ is an element of some finite set of labels. Moreover, we impose some partial order $\preceq$ on the labels and introduce the following variant of the rule $(!, \mathrm{r})$:

$$\frac{!^{a_1}A_1, \ldots, !^{a_n}A_n \vdash B}{!^{a_1}A_1, \ldots, !^{a_n}A_n \vdash !^a B}\ (!^a, \mathrm{r}) \quad \text{provided that } a \preceq a_i \text{ for } 1 \leq i \leq n.$$

For the rich and versatile setting of subexponential calculi arising by this kind of modification, we refer to, e.g., Nigam, Olarte, and Pimentel 2017. Here, we just want to emphasize that information extraction games provide a flexible tool for capturing certain semantic intuitions that motivate those modifications. For a concrete example, we refer to Fermüller and Lang 2017, where an information extraction game is introduced for a set of subexpontials in which the corresponding labels are linearly ordered and interpreted as 'safety levels'. The underlying idea is that the access to information packages might be limited according to some hierarchical scheme. In particular, the client may extract an ip $!^a H$ from the server only if the information packages stored there are marked by labels that indicate that they can be used 'safely' with respect to level $a$.

Yet another, more daring variant of information extraction games has been explored, at least in a preliminary fashion, in Fermüller and Lang 2017. So far, we have confined attention to the case where the collection of ips offered by the server is represented by a multiset of ips, entailing that the order of access to these resources is unrestricted, even if the number of copies of identical ips might matter. It seems natural to ask what happens if we confine access by organizing the server's ips using a more restrictive data structure. For this reason we have introduced a stack-based game and corresponding calculus in Fermüller and Lang 2017. The calculus features the following rules for a new connective ';', where the intended reading of $(A;B)$ is 'first $A$, then $B$':

$$\frac{\Gamma, B, A \vdash C}{\Gamma, (A;B) \vdash C}\ (;,\mathrm{l}) \qquad \frac{\Gamma_1 \vdash B \quad \Gamma_2 \vdash A}{\Gamma \vdash (A;B)}\ (;,\mathrm{r})$$

For the intended interpretation of these rules it is important to keep in mind that the right side of a sequent $\Gamma \vdash F$ is understood as a stack rather than as a multiset of formulas: only the top element of the stack, denoted by the right-most formula in the list $\Gamma$ of formulas, can be accessed. Moreover, $\Gamma_1, \Gamma_2$ denotes a stack which results from putting the elements of $\Gamma_2$ on top of those of $\Gamma_1$.[15] Correspondingly, in the information extraction game, **P** can only select the right-most of the ips offered by the server, i.e., of those ips that are listed to the left of $\triangleright$. The rules for the new connective can be formulated as follows.

$(;^{\mathcal{G}}_{\mathrm{l}})$ $\Gamma, \underline{(F;G)} \triangleright H$:
The game continues in state $\Gamma, G, F \triangleright H$.

$(;^{\mathcal{G}}_{\mathrm{r}})$ $\Gamma \triangleright \underline{(F;G)}$:
**P** splits the stack $\Gamma$ into a lower part $\Gamma_1$ and an upper part $\Gamma_2$; **O** decides whether the game continues in state $\Gamma_2 \triangleright F$ or in state $\Gamma_1 \triangleright G$.

Note that in $(;^{\mathcal{G}}_{\mathrm{l}})$, $\Gamma, G, F$ denotes the stack that results from stack $\Gamma$ by first pushing $G$ and then $F$ on its top. This means that in $\Gamma, G, F$, one can only access *first* $F$ and *then* $G$, as intended. In $(;^{\mathcal{G}}_{\mathrm{r}})$, **O**'s choice between the two indicated successor states corresponds to the fact that **P** has to be prepared to establish two claims: (1) $F$ can be obtained from some number of ips stored in $\Gamma$ that have to be accessed in the intended top-first fashion and hence constitute an upper part $\Gamma_2$ of $\Gamma$; (2) $G$ can be obtained from the rest of the stack $\Gamma_1$, i.e., from the stack resulting from $\Gamma$ when the 'higher-up' elements contained in $\Gamma_2$ have been consumed and thus only the lower part of the stack remains.

[15] In contrast to the usual sequent calculi, the stack-based calculus does not admit a cut rule. However, in light of the intended interpretation this is neither surprising nor should it be seen as a defect. It simply corresponds to the fact that, in general, one cannot compose strategies for extracting information from *stacks* in the same direct manner as for *multisets* or (permutable) *lists*.

## 7 Conclusion – an extended research agenda

Resource consciousness is routinely cited as a motivation for substructural logics. However, the reference to resources is usually kept informal, like in Girard's cigarette example. Game semantics for fragments and variants of linear logics (see, e.g., Abramsky and Jagadeesan 1994; Blass 1992) redress this situation by identifying formulas with games, and connectives with operators on games. While useful in the context of providing abstract semantics for programming languages, this paradigm amounts to a rather radical departure from the oldest type of game semantics, namely Lorenzen's dialogue games. On the other hand, Lorenzen attempted to justify constructive (and classical) mathematical reasoning from first principles and did not consider resource consciousness in the sense of substructural logics. To connect Lorenzen-style games with substructural calculi, we introduced the concept of information extraction games. While motivated very differently, these latter games share important structural features with Lorenzen's game for intuitionistic logic, as pointed out in Section 5.

We have already indicated in Section 6 that information extraction games are not just a tool for characterizing provability in some specific well-known sequent calculi, but reach beyond established formats of calculi and corresponding logics. In fact, we prefer to think of information extraction games as a paradigm that opens up a varied landscape of new research questions related to, but not necessarily limited by, the field of substructural logics. Therefore we want to conclude with a list of topics for further research.

- It should be clear that multisets, permutable lists, and stacks (underlying $\mathsf{ILL}$, $\mathsf{LI}$, and the stack-based calculus of Section 6, respectively) constitute only three particular ways to organize access to formulas, interpreted as packages of information. The organization of the server's information could be extended to cover also (versions of) other substructural calculi, like the one of Lambek (1958). More ambitiously, one may devise a general approach to this type of interpretation that is guided by a systematic account of the many different data structures that are available for data storage and retrieval.
- We have interpreted logical connectives as operators for forming structured information packages. Certainly various further forms of packaging data are conceivable. In particular, there remains the challenge of interpreting, not just propositional connectives, but also quantifiers in this fashion.
- One of the distinguishing features of a game-based approach to logics is that it brings into focus notions that remain marginal in alternative frameworks. For example, the central notion of a play (run of a game) corresponds to a single branch in a derivation or, more generally, a branch of a proof search tree. Considering variations, restrictions, and extensions on that level may have interesting repercussions also for

sequent-based approaches to logical inference.

- So far, we have analyzed information extraction games only from the proponent's (**P**'s) perspective. The reason for this is clear: **P**'s winning strategies correspond to proofs. However, from a game-theoretic point of view, it seems natural to investigate **O**'s strategies as well. A winning strategy for **O** establishes that there is no guarantee that the agent can extract the desired information from the server, which, in logical terms, shows that a particular formula does *not* follow from given premises. Thus there is a close relation between **O**'s winning strategies and counterexamples, which might be exploited to devise new forms of semantics.
- It seems natural to consider various restrictions of strategies. The requirement that they should be computable seems obvious and thus does not amount to any real restriction in the games considered so far. However, one might want to further restrict strategies with respect to their complexity, uniformity, specific representability, etc.
- In the presented information extraction games, **P** always acts as the *scheduler*, meaning that **P** can always freely select the (compound) ip that is to be reduced in the next round. It might be interesting to consider game variants with other forms of scheduling, including random selections, assigning scheduling rights to **O**, or some mixed form of scheduling.
- As they stand, our games are perfect-information games: both players are aware of each of the other player's moves and thus always know the current state. Considering a variant where **P** or **O** may not always be fully informed about the other player's choice amounts to a game of imperfect information. In general, such games are not determined, i.e., neither **P** nor **O** may have a winning strategy. In the context of Hintikka's evaluation game for classical first-order logic, admitting imperfect information leads to so-called independence-friendly (IF) logic (see, e.g., Mann, Sandu, and Sevenster 2011). For the corresponding kind of games, one can define an *equilibrium semantics*, which refers to equilibrium values for randomized strategies, where one identifies winning with pay-off 1 and losing with pay-off 0. The concept can also be transferred to other logical games, as demonstrated by Fermüller and Majer (2015) for the case of Giles's game for Łukasiewicz logic (Giles 1974; Giles 1977). It would be interesting to see whether information extraction games with imperfect information can be treated analogously.
- Fermüller and Metcalfe (2009) describe a tight connection between a "hypersequent" calculus for Łukasiewicz logic and the above-mentioned game of Giles 1974. Another interpretation of hypersequent calculi for intermediary logics in terms of *parallel* versions of Lorenzen-style games has been presented by Fermüller 2003. Analogously, one may consider parallel runs of information extraction games.

This may lead to alternative interpretations of various known, but also new, hypersequent systems via different options for the exchange of information between parallel runs of the game.

The above list is not exhaustive, of course. In any case, it is offered here to bear witness to the fact that Lorenzen's dialogical approach to methodical reasoning continues to inspire contemporary research.

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