On the Convergence Rate of Hermite-Fejér Interpolation

Shuhuang Xiang and Guo He

1 Introduction

For an arbitrarily given system of points

$$\{x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\}_{n=1}^{\infty}, \tag{1}$$

Faber [3] in 1914 showed that there exists a continuous function $f(x)$ in $[-1, 1]$ for which the Lagrange interpolation sequence $L_n[f]$ $(n = 1, 2, \ldots)$ is not uniformly convergent to f in $[-1, 1]$, where $\omega_n(x) = (x - x_1^{(n)})(x - x_2^{(n)}) \cdots (x - x_n^{(n)})$

$$L_n[f](x) = \sum_{k=1}^{n} f(x_k^{(n)})\ell_k^{(n)}(x),\ \ell_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})}. \tag{2}$$

Whereas, based on the Chebyshev pointsystem

$$x_k^{(n)} = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, \tag{3}$$

S. Xiang
Department of Mathematics, Central South University, Changsha, China
e-mail: xiangsh@mail.csu.edu.cn

G. He (✉)
Department of Mathematics, Jinan University, Guangzhou, Guangdong, China

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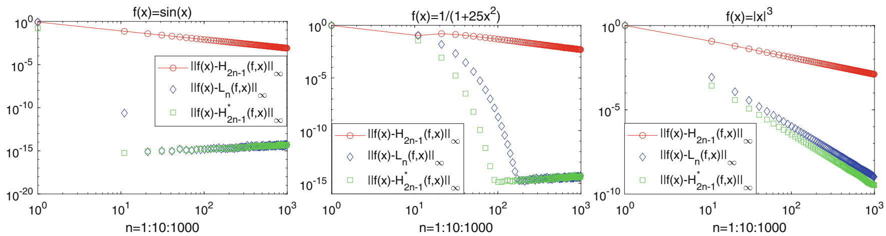

Fig. 1 $\|H_{2n-1}(f,x)-f(x)\|_\infty$, $\|L_n(f,x)-f(x)\|_\infty$ and $\|H^*_{2n-1}(f,x)-f(x)\|_\infty$at $x=-1:0.001:1$ by using Chebyshev pointsystem (3) for $f(x)=\sin(x)$, $f(x)=\frac{1}{1+25x^2}$ and $f(x)=|x|^3$, respectively

Fejér [4] in 1916 proved that if $f\in C[-1,1]$, then there is a unique polynomial $H_{2n-1}(f,x)$ of degree at most $2n-1$ such that $\lim_{n\to\infty}\|H_{2n-1}(f)-f\|_\infty=0$, where $H_{2n-1}(f,x)$ is determined by

$$H_{2n-1}(f,x_k^{(n)})=f(x_k^{(n)}),\quad H'_{2n-1}(f,x_k^{(n)})=0,\quad k=1,2,\ldots,n. \tag{4}$$

This polynomial is known as the Hermite-Fejér interpolation polynomial.

It is of particular notice that the above Hermite-Fejér interpolation polynomial converges much slower compared with the corresponding Lagrange interpolation polynomial at the Chebyshev pointsystem (3) (see Fig. 1).

To get fast convergence, the following Hermite-Fejér interpolation of $f(x)$ at nodes (1) is considered [6, 7]:

$$H^*_{2n-1}(f,x)=\sum_{k=1}^{n}f(x_k^{(n)})h_k^{(n)}(x)+\sum_{k=1}^{n}f'(x_k^{(n)})b_k^{(n)}(x), \tag{5}$$

where $h_k^{(n)}(x)=v_k^{(n)}(x)\left(\ell_k^{(n)}(x)\right)^2$, $b_k^{(n)}(x)=(x-x_k^{(n)})\left(\ell_k^{(n)}(x)\right)^2$ and $v_k^{(n)}(x)=1-(x-x_k^{(n)})\frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}$.

Fejér [5] and Grünwald [7] also showed that the convergence of the Hermite-Fejér interpolation of $f(x)$ also depends on the choice of the nodes. The pointsystem (1) is called normal if for all n

$$v_k^{(n)}(x)\ge 0,\quad k=1,2,\ldots,n,\quad x\in[-1,1], \tag{6}$$

while the pointsystem (1) is called strongly normal if for all n

$$v_k^{(n)}(x)\ge c>0,\quad k=1,2,\ldots,n,\quad x\in[-1,1] \tag{7}$$

for some positive constant c.

Fejér [5] (also see Szegö [12, pp 339]) showed that for the zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n ($\alpha > -1, \beta > -1$)

$$v_k^{(n)}(x) \geq \min\{-\alpha, -\beta\} \quad \text{for } -1 < \alpha \leq 0, -1 < \beta \leq 0, k = 1, 2, \ldots, n \text{ and } x \in [-1, 1].$$

For (strongly) normal pointsystems, Grünwald [7] showed that for every $f \in C^1(-1, 1)$, $\lim_{n\to\infty} \|H_{2n-1}^*(f) - f\|_\infty = 0$ if $\{x_k^{(n)}\}$ is strongly normal satisfying (7) and $\{f'(x_k^{(n)})\}$ satisfies

$$|f'(x_k^{(n)})| < n^{c-\delta} \quad \text{for some given positive number } \delta, \quad k = 1, 2, \ldots, \quad n = 1, 2, \ldots,$$

while $\lim_{n\to\infty} \|H_{2n-1}^*(f) - f\|_\infty = 0$ in $[-1+\epsilon, 1-\epsilon]$ for each fixed $0 < \epsilon < 1$ if $\{x_k^{(n)}\}$ is normal and $\{f'(x_k^{(n)})\}$ is uniformly bounded for $n = 1, 2, \ldots$.[1]

Moreover, Szabados [11] showed the convergence of the Hermite-Fejér interpolation (5) at the Chebyshev pointsystem (3) satisfies

$$\|f - H_{2n-1}^*(f)\|_\infty = O(1)\|f - p^*\|_{C^1[-1,1]} \tag{8}$$

where p^* is the best approximation polynomial of f with degree at most $2n-1$ and $\|f - p^*\|_{C^1[-1,1]} = \max_{0\leq j\leq 1} \|f^{(j)} - p^{*(j)}\|_\infty$.

Hermite-Fejér interpolation has plenty of use in computer geometry aided geometric design with boundary conditions including derivative information. The convergence rate under the infinity norm has been extensively studied in [5–7, 11, 14]. The efficient algorithm on the fast implementation of Hermite-Fejér interpolation at zeros of Jacobi polynomial can be found in [17].

In this paper, the following convergence rates of Hermite-Fejér interpolation $H_{2n-1}^*(f, x)$ at Gauss-Jacobi pointsystems are considered.

- If f is analytic in $\mathcal{E}_\rho$ with $|f(z)| \leq M$, then

$$\|f(x) - H_{2n-1}^*(f, x)\|_\infty = \begin{cases} O\left(\dfrac{4\tau_n M[2n\rho^2 + (1-2n)\rho]}{(\rho-1)^2\rho^{2n}}\right), & \gamma \leq 0, \\ O\left(\dfrac{n^{2+2\gamma}[2n\rho^2 + (1-2n)\rho]}{(\rho-1)^2\rho^{2n}}\right), & \gamma > 0 \end{cases}, \quad \gamma = \max\{\alpha, \beta\} \tag{9}$$

[1] In fact, Grünwald in [7] considered more general cases with any vector $\{d_k^{(n)}\}$ instead of $\{f'(x_k^{(n)})\}$.

where

$$\tau_n = \begin{cases} O(n^{-1.5-\min\{\alpha,\beta\}}\log n), & \text{if } -1 < \min\{\alpha,\beta\} \le \gamma \le -\frac{1}{2} \\ O(n^{2\gamma-\min\{\alpha,\beta\}-\frac{1}{2}}), & \text{if } -1 < \min\{\alpha,\beta\} \le -\frac{1}{2} < \gamma \le 0 \\ O(n^{2\gamma}), & \text{if } -\frac{1}{2} < \min\{\alpha,\beta\} \le \gamma \end{cases}. \tag{10}$$

- If $f(x)$ has an absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ on $[-1,1]$ for an integer $r \ge 3$, and a rth derivative $f^{(r)}$ of bounded variation $V_r = \mathrm{Var}(f^{(r)}) < \infty$, then

$$\|f(x) - H^*_{2n-1}(f,x)\|_\infty = \begin{cases} O\left(n^{-r}\log n\right), & \gamma \le -\frac{1}{2}, \\ O\left(n^{2\gamma-r+1}\right), & \gamma > -\frac{1}{2}, \end{cases} \tag{11}$$

while if $f(x)$ is differentiable and $f'(x)$ is bounded on $[-1,1]$, then

$$\|f(x) - H^*_{2n-1}(f,x)\|_\infty = \begin{cases} O\left(n^{-1}\log n\right), & \gamma \le -\frac{1}{2}, \\ O\left(n^{2\gamma}\right), & \gamma > -\frac{1}{2}. \end{cases}$$

Comparing these results with

$$f(x) - H_{2n-1}(f,x) = \begin{cases} O\left(n^{-1}\log n\right), & \text{if } \gamma \le -\frac{1}{2} \\ O(n^{2\gamma}), & \text{if } \gamma > -\frac{1}{2} \end{cases}, \quad \text{(Vértesi [14])},$$

which is sharp and attainable (see Fig. 2), we see that $H^*_{2n-1}(f,x)$ converges much faster than $H_{2n-1}(f,x)$ for analytic functions or functions of higher regularities (see Fig. 1). Particularly, $H_{2n-1}(f,x)$ diverges at Gauss-Jacobi pointsystems with $\gamma \ge 0$, whereas, $H^*_{2n-1}(f,x)$ converges for functions analytic in the Bernstein ellipse or of finite limited regularity.

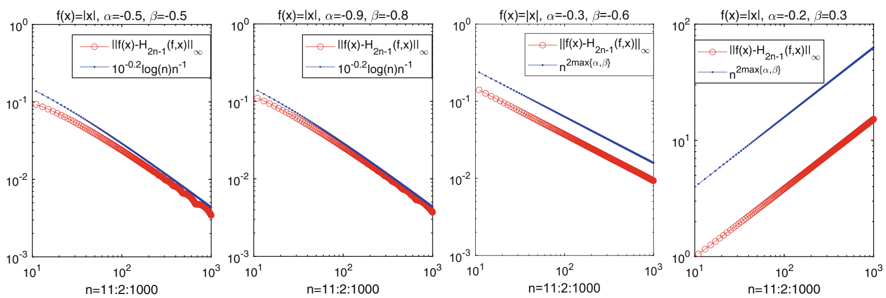

Fig. 2 $\|H_{2n-1}(f,x) - f(x)\|_\infty$ at $x = -1 : 0.001 : 1$ by using Gauss-Jacobi pointsystem for $f(x) = |x|$ with different α and β, respectively

For simplicity, in the following we abbreviate $x_k^{(n)}$ as x_k, $\ell_k^{(n)}(x)$ as $\ell_k(x)$, $h_k^{(n)}(x)$ as $h_k(x)$, and $b_k^{(n)}(x)$ as $b_k(x)$. $A \sim B$ denotes there exist two positive constants c_1 and c_2 such that $c_1 \le |A|/|B| \le c_2$.

2 Main Results

Suppose $f(x)$ satisfies a Dini-Lipschitz condition on $[-1, 1]$, then it has the following absolutely and uniformly convergent Chebyshev series expansion

$$f(x) = {\sum_{j=0}^{\infty}}' c_j T_j(x), \quad c_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}} dx, \quad j = 0, 1, \ldots. \tag{12}$$

where the prime denotes summation whose first term is halved, $T_j(x) = \cos(j \cos^{-1} x)$ denotes the Chebyshev polynomial of degree j.

Lemma 1

(i) (Bernstein [2]) If f is analytic with $|f(z)| \le M$ in the region bounded by the ellipse $\mathcal{E}_\rho$ with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $j \ge 0$,

$$|c_j| \le \frac{2M}{\rho^j}. \tag{13}$$

(ii) (Trefethen [13]) For an integer $r \ge 1$, if $f(x)$ has an absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ on $[-1, 1]$ and a rth derivative $f^{(r)}$ of bounded variation $V_r = \mathrm{Var}(f^{(r)}) < \infty$, then for each $j \ge r + 1$,

$$|c_j| \le \frac{2V_r}{\pi j(j-1)\cdots(j-r)}. \tag{14}$$

Suppose $-1 < x_n < x_{n-1} < \cdots < x_1 < 1$ in decreasing order are the roots of $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$, and $\{w_j\}_{j=1}^n$ are the corresponding weights in the Gauss-Jacobi quadrature.

Lemma 2 *For $j = 1, 2, \ldots, n$, it follows*

$$(x - x_j)\ell_j(x) = \sigma_n(-1)^j \frac{\sqrt{(1-x_j^2)w_j}}{2^{(\alpha+\beta+1)/2}} \sqrt{\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}} P_n^{(\alpha,\beta)}(x), \tag{15}$$

where $\sigma_n = +1$ for even n and $\sigma_n = -1$ for odd n.

Proof Let $z_n = \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta[P_n^{(\alpha,\beta)}(x)]^2dx$ and K_n the leading coefficient of $P_n^{(\alpha,\beta)}(x)$. From Abramowitz and Stegun [1], we have

$$z_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\cdot\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}, \quad K_n = \frac{1}{2^n}\frac{\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}.$$

Furthermore, by Szegö [12, (15.3.1)] (also see Wang et al. [15]), we obtain

$$\begin{aligned}(x-x_j)\ell_j(x) = \frac{1}{\omega_n'(x_j)}\omega_n(x) &= \sigma_n(-1)^j\sqrt{\frac{K_n^2 2n(1-x_j^2)w_j}{2n(2n+\alpha+\beta+1)z_n}}\omega_n(x)\\ &= \sigma_n(-1)^j\sqrt{\frac{(1-x_j^2)w_j}{z_n(2n+\alpha+\beta+1)}}P_n^{(\alpha,\beta)}(x),\end{aligned}$$

which implies the desired result (15). □

Lemma 3 *For $j = 1, 2, \ldots, n$, it follows*

$$(1-x_j^2)w_j = O\left(n^{-1}\right). \tag{16}$$

Proof From $w_j = O\left(\frac{2^{\alpha+\beta+1}\pi}{n}\left(\sin\frac{\theta_j}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta_j}{2}\right)^{2\beta+1}\right)$ Szegö [12, (15.3.10)], we see for $x_j = \cos\theta_j$ that $(1-x_j^2)w_j = O\left(\frac{2^{\alpha+\beta+3}\pi}{n}\left(\sin\frac{\theta_j}{2}\right)^{2\alpha+3}\left(\cos\frac{\theta_j}{2}\right)^{2\beta+3}\right)$, which derives the desired result. □

Lemma 4 ([10, 16]) *For $t \in [-1, 1]$, let x_m be the root of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ which is closest to t. Then for $k = 1, 2, \ldots, n$, we have*

$$\ell_k(t) = \begin{cases} O\left(|k-m|^{-1} + |k-m|^{\gamma-\frac{1}{2}}\right), & k \neq m\\ O(1) & k = m\end{cases}, \quad \gamma = \max\{\alpha, \beta\}. \tag{17}$$

Lemma 5 (Szegö [12, Theorem 8.1.2]) *Let α, β be real but not necessarily greater than -1 and $x_k = \cos\theta_k$. Then for each fixed k, it follows*

$$\lim_{n\to\infty} n\theta_k = j_k, \tag{18}$$

where j_k is the kth positive zero of Bessel function J_α.

Lemma 6 *For $k = 1, 2, \ldots, n$, it follows*

$$v_k(x) = 1 - (x-x_k)\frac{\omega_n''(x_k)}{\omega_n'(x_k)} = O(n^2). \tag{19}$$

Proof Note that $P_n^{(\alpha,\beta)}(x)$ satisfies the second order linear homogeneous Sturm-Liouville differential equation [12, (4.2.1)]

$$(1-x^2)y'' + (\beta-\alpha-(\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0.$$

By $\omega_n(x) = \frac{P_n^{(\alpha,\beta)}(x)}{K_n}$, we get

$$\frac{\omega_n''(x_j)}{\omega_n'(x_j)} = -\frac{\beta-\alpha-(\alpha+\beta+2)x_j}{1-x_j^2} \quad ([12, (14.5.1)]). \tag{20}$$

In addition, by Lemma 5 with $x_j = \cos\theta_j$, we see that $\theta_1 \sim \frac{1}{n}$. Similarly, by $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ we have $\theta_n \sim \frac{1}{n}$. These together yield

$$\frac{1}{1-x_1^2} = O(n^2), \quad \frac{1}{1-x_n^2} = O(n^2), \quad \frac{1}{1-x_j^2} \le \max\left(\frac{1}{1-x_1^2}, \frac{1}{1-x_n^2}\right) = O(n^2)$$

and then by (20) it deduces the desired result. □

Theorem 1 *Suppose $\{x_j\}_{j=1}^n$ are the roots of $P_n^{(\alpha,\beta)}(x)$ with $\alpha, \beta > -1$, then the Hermite-Fejér interpolation (5) for f analytic in $\mathcal{E}_\rho$ with $|f(z)| \le M$ at $\{x_j\}_{j=1}^n$ has the convergence rate (9).*

Proof Since the Chebyshev series expansion of $f(x)$ is uniformly convergent under the assumptions, and the error of Hermite-Fejér interpolation (5) on Chebyshev polynomials satisfies $|E(T_j, x)| = |T_j(x) - H_{2n-1}^*(T_j, x)| = 0$ for $j = 0, 1, \ldots, 2n-1$, then it yields

$$|E(f,x)| = |f(x) - H_{2n-1}^*(f,x)| = |\sum_{j=0}^{\infty} c_j E(T_j, x)| \le \sum_{j=2n}^{\infty} |c_j||E(T_j, x)|. \tag{21}$$

Furthermore, $|E(T_j, x)| = |T_j(x) - \sum_{i=1}^n T_j(x_i)h_i(x) - \sum_{i=1}^n T_j'(x_i)b_i(x)|$. In the following, we will focus on estimates of $|E(T_j, x)|$ for $j \ge 2n$.

In the case $\gamma \le 0$: Notice that the pointsystem is normal which implies $h_i(x) \ge 0$ for all $i = 1, 2, \ldots, n$ and for all $x \in [-1, 1]$,

$$1 \equiv \sum_{i=1}^n h_i(x) = \sum_{i=1}^n v_i(x)\ell_i^2(x).$$

Then we have

$$|\sum_{i=1}^n T_j(x_i)h_i(x)| \le \sum_{i=1}^n h_i(x) = 1, \quad j = 0, 1, \ldots. \tag{22}$$

Additionally, by Lemma 2, it obtains for $j = 2n, 2n+1, \ldots$ that

$$\begin{aligned} &|\textstyle\sum_{i=1}^n T_j'(x_i)b_i(x)| \\ &= j|\textstyle\sum_{i=1}^n U_{j-1}(x_i)(x-x_i)\ell_i^2(x)| \\ &= \tfrac{j}{2^{(\alpha+\beta+1)/2}}\sqrt{\tfrac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}|P_n^{(\alpha,\beta)}(x)\textstyle\sum_{i=1}^n U_{j-1}(x_i)\sqrt{(1-x_i^2)w_i}\ell_i(x)| \\ &= \tfrac{j}{2^{(\alpha+\beta+1)/2}}\sqrt{\tfrac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}|P_n^{(\alpha,\beta)}(x)\textstyle\sum_{i=1}^n \sin((j-1)\arccos(x_i))\sqrt{w_i}\ell_i(x)| \\ &= jO\left(|P_n^{(\alpha,\beta)}(x)|\sqrt{\|\{w_i\}_{i=1}^n\|_\infty}\Lambda_n\right) \end{aligned}$$

(U_{j-1} is the second kind of Chebyshev polynomial of degree $j-1$) since $\sqrt{\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}$ is uniformly bounded in n for $\alpha, \beta > -1$ due to

$$\frac{(n+1)!\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+2)\Gamma(n+\beta+2)} = \left(1 - \frac{\alpha\beta}{(n+1)^2+(\alpha+\beta)(n+1)+\alpha\beta}\right) \times \frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)},$$

which implies $\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$ is uniformly bounded in n and then $\sqrt{\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}$ is uniformly bounded. Here $\Lambda_n = \max_{x\in[-1,1]}\sum_{i=1}^n |\ell_i(x)|$ is the Lebesgue constant. Then from

$$P_n^{(\alpha,\beta)}(x) = \begin{cases} O(n^{-\frac{1}{2}}), & \text{if } \max\{\alpha,\beta\} \le -\frac{1}{2} \\ O(n^{\max\{\alpha,\beta\}}), & \text{if } \max\{\alpha,\beta\} > -\frac{1}{2} \end{cases},$$

$$w_i = \begin{cases} O(n^{-2-2\min\{\alpha,\beta\}}), & \text{if } \min\{\alpha,\beta\} \le -\frac{1}{2} \\ O(n^{-1}), & \text{if } \min\{\alpha,\beta\} > -\frac{1}{2} \end{cases}$$

(see Szegö [12, pp 168, 354]) and

$$\Lambda_n = \begin{cases} O(\log n), & \text{if } \max\{\alpha,\beta\} \le -\frac{1}{2} \\ O(n^{\max\{\alpha,\beta\}+\frac{1}{2}}), & \text{if } \max\{\alpha,\beta\} > -\frac{1}{2} \end{cases} \quad ([12, \text{pp } 338]),$$

we have

$$|\sum_{i=1}^n T_j'(x_i)b_i(x)| = j\tau_n. \tag{23}$$

Then by (22) and (23), we find $|E(T_j, x)| \le 2 + j\tau_n < 2j\tau_n$ for $j \ge 2n$, and consequently

$$|E(f, x)| = |f(x) - H^*_{2n-1}(f, x)| \le \sum_{j=2n}^{\infty} |c_j||E(T_j, x)| = 2\tau_n \sum_{j=2n}^{\infty} j|c_j|,$$

which, directly following [18], leads to the desired result.

In the case $\gamma > 0$: From $|E(T_j, x)| = |T_j(x) - \sum_{i=1}^{n} T_j(x_i)h_i(x) - \sum_{i=1}^{n} T_j'(x_i)b_i(x)|$, by Lemmas 3 and 6 we obtain

$$\sum_{i=1}^{n} |v_i(x)|\ell_i^2(x) = O\left(n^2 \int_1^n t^{2\gamma-1} dt\right) = O(n^{2+2\gamma}),$$

and

$$T_j(x) - \sum_{i=1}^{n} T_j(x_i)h_i(x) = T_j(x) - \sum_{i=1}^{n} T_j(x_i)v_i(x)\ell_i^2(x) = O\left(n^{2+2\gamma}\right).$$

These together with

$$\begin{aligned} &|\textstyle\sum_{i=1}^{n} T_j'(x_i)b_i(x)| \\ &= \tfrac{j}{2^{(\alpha+\beta+1)/2}}\sqrt{\tfrac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}|P_n^{(\alpha,\beta)}(x)\textstyle\sum_{i=1}^{n} \sin((j-1)\arccos(x_i))\sqrt{w_i}\ell_i(x)| \\ &= j\tau_n \end{aligned}$$

and then $|E(T_j, x)| = O\left(j^{2+2\gamma}\right)$ for $j \ge 2n$, similar to the above proof in the case of $\gamma \le 0$, implies the desired result. □

From the definition of τ_n, we see that when $\alpha = \beta = -\frac{1}{2}$ the convergence order on n is the lowest. In addition, if f is of limited regularity, we have

Lemma 7 (Vértesi [14]) *Suppose $\{x_j\}_{j=1}^n$ are the roots of $P_n^{(\alpha,\beta)}(x)$, for every continuous function $f(x)$ we have*

$$|H_{2n-1}(f, x) - f(x)| = O(1)\sum_{j=1}^{n}\left[w\left(f; \frac{j\sqrt{1-x^2}}{n}\right) + w\left(f; \frac{j^2|x|}{n^2}\right)\right] j^{2\bar{\gamma}-1}, \tag{24}$$

where $w(f; t) = w(t)$ is the modulus of continuity of $f(x)$, and $\bar{\gamma} = \max\left(\alpha, \beta, -\frac{1}{2}\right)$.

Theorem 2 *Suppose $\{x_j\}_{j=1}^n$ are the roots of $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$, and $f(x)$ has an absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ on $[-1, 1]$ for some $r \ge 3$,* *and a rth derivative $f^{(r)}$ of bounded variation $V_r < \infty$, then the Hermite-Fejér interpolation (5) at $\{x_j\}_{j=1}^n$ has the convergence rate (11).*

Proof Consider the special functional $L(g) = E_n(g, x)$, where $E_n(g, x)$ is defined for $\forall g \in C^1([-1, 1])$ by

$$E_n(g, x) = g(x) - \sum_{j=1}^{n} g(x_j)v_j(x)\ell_j^2(x) - \sum_{j=1}^{n} g'(x_j)(x - x_j)\ell_j^2(x). \tag{25}$$

By the Peano kernel theorem for $n \geq r$ (see Peano [9] or Kowalewski [8]), $E_n(f, x)$ can be represented as

$$E_n(f, x) = \int_{-1}^{1} f^{(r)}(t)K_r(t)dt \tag{26}$$

with $K_r(t) = \frac{1}{(r-1)!}L\left((x-t)_+^{r-1}\right)$ for $r = 3, 4, \cdots$, that is

$$K_r(t) = \frac{1}{(r-1)!}(x-t)_+^{r-1} - \frac{1}{(r-1)!}\sum_{j=1}^{n}(x_j - t)_+^{r-1}v_j(x)\ell_j^2(x)$$
$$-\frac{1}{(r-1)!}\sum_{j=1}^{n}(x_j - t)_+^{r-2}(x - x_j)\ell_j^2(x),$$

where

$$(x-t)_+^{k-1} = \begin{cases} (x-t)^{k-1}, & x \geq t; \\ 0, & x < t. \end{cases} \quad (k \geq 2), \qquad (x-t)_+^{0} = \begin{cases} 1, & x \geq t; \\ 0, & x < t. \end{cases} \quad (k = 1).$$

Moreover, noting that

$$\frac{1}{(k-2)!}(x-u)_+^{k-2} = \int_u^1 \frac{1}{(k-3)!}(x-t)_+^{k-3}(t)dt, \quad k = 3, 4, \cdots,$$

we get the following identity

$$K_{s-1}(u) = \int_u^1 K_{s-2}(t)dt, \quad s = 4, 5, \cdots,$$

where $K_2(t)$ is defined by

$$K_2(t) = (x-t)_+^1 - \sum_{j=1}^{n}(x_j - t)_+^1 v_j(x)\ell_j^2(x) - \sum_{j=1}^{n}(x_j - t)_+^0(x - x_j)\ell_j^2(x).$$

In addition, it can be easily verified that $K_s(-1) = K_s(1) = 0$ for $s = 2, 3, \ldots$.

Since $f^{(r)}$ is of bounded variation, directly applying the similar skills of Theorem 2 and Lemma 4 in [16], we get

$$\|E_n(f,x)\|_\infty \leq V_r \|K_{r+1}\|_\infty, \tag{27}$$

and

$$\|K_{s+1}\|_\infty \leq \frac{\pi}{2n-s} \sup_{-1\leq t\leq 1} |K_s(t)|, \quad \text{for } s = 2, 3, \cdots, \tag{28}$$

respectively. Then from (27) and (28), we can obtain that

$$\|E_n(f,x)\|_\infty \leq \frac{\pi^{r-1} V_r}{(2n-2)(2n-3)\cdots(2n-r))} \|K_2\|_\infty. \tag{29}$$

In addition, by Lemma 7, we have

$$\|(x-t)_+^1 - \sum_{j=1}^{n} (x_j - t)_+^1 v_j(x) \ell_j^2(x)\|_\infty = \begin{cases} O\left(\frac{\log n}{n}\right), \ \gamma \leq -\frac{1}{2} \\ O\left(n^{2\gamma}\right), \ \ \gamma > -\frac{1}{2}, \end{cases} \tag{30}$$

while by Lemmas 2–3, we get

$$|\sum_{j=1}^{n} (x_j - t)_+^0 (x - x_j) \ell_j^2(x)| \leq \sum_{j=1}^{n} |(x - x_j) \ell_j^2(x)| = \begin{cases} O\left(\frac{\log n}{n}\right), \ \gamma \leq -\frac{1}{2} \\ O\left(n^{2\gamma}\right), \ \ \gamma > -\frac{1}{2}. \end{cases} \tag{31}$$

Together (30) and (31), we can obtain the desired results by using

$$K_2(t) = \begin{cases} O\left(\frac{\log n}{n}\right), \ \gamma \leq -\frac{1}{2} \\ O\left(n^{2\gamma}\right), \ \ \gamma > -\frac{1}{2}. \end{cases}.$$

Finally, We use a function of analytic $f(x) = \frac{1}{1+25x^2}$ and a function of limited regularity $f(x) = |x|^5$ to show that the convergence rate of $\|f(x) - H_{2n-1}^*(f,x)\|_\infty$ is dependent on α and β in Fig. 3.

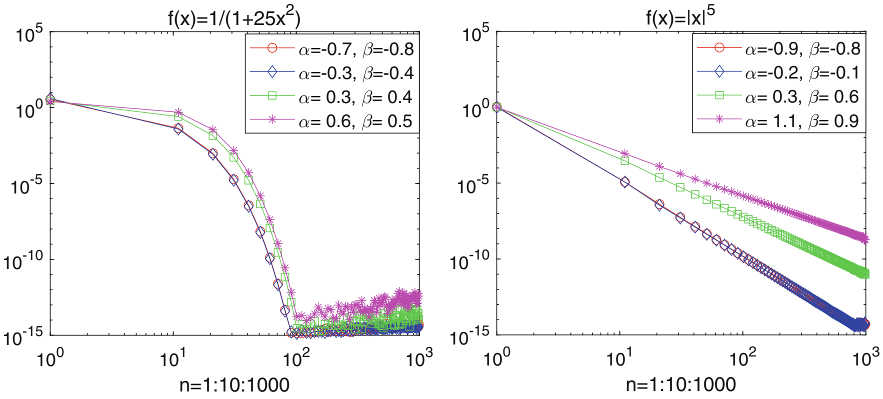

Fig. 3 $\|H^*_{2n-1}(f,x) - f(x)\|_\infty$ at $x = -1 : 0.001 : 1$ by using Gauss-Jacobi pointsystem for $f(x) = \frac{1}{1+25x^2}$ and $f(x) = |x|^5$ with different α and β, respectively

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