Chapter 9
"Necklaces": A Didactic Sequence for Missing-Value Proportionality Problems

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9.1 Introduction: The Little Math Problem Factory

Multiplication or division problems that seemingly establish a relationship among three values are, in fact, "missing-value"[1] proportionality problems where a fourth value becomes involved (Vergnaud, 1988, 1990). For example, "If a pencil costs 3 pesos, how much would 5 pencils cost?"

Pencils	Pesos
1	3
5	x

Three problems (one multiplication and two divisions) can be obtained, based on these four values, by shifting the position of the unknown value.

Multiplication	
Pencils	Pesos
1	3
5	x

Partitive Division	
Pencils	Pesos
1	x
5	15

Quotative Division	
Pencils	Pesos
1	3
x	15

Problems solved through division present different relationships between numbers: in partitive division, 15 pesos are distributed equally among 5 pencils; quotative division consists of finding out how many 3 pesos groups can be made

[1] Also known in the old ratio and proportion theory as "fourth proportional" problems.

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M. Isoda, R. Olfos (eds.), *Teaching Multiplication with Lesson Study*,
https://doi.org/10.1007/978-3-030-28561-6_9

from 15 pesos. Various studies have shown that young children see considerable differences between each type of division. When unaware of which problems require division, children approach these problems in very different ways (Nesher, 1988; Martínez & Moreno, 1996, and others).

The three problems above show the unit value (the price of one pencil), as a given value (for multiplication or partitive division) or as the object of a question (quotative division). There is a fourth type of problem where the unit value is neither requested nor provided, as shown in the table below.

Division/multiplication	
Pencils	Pesos
3	15
7	x

This is a typical missing-value proportionality problem. There are several ways to solve it, as will be shown later on, but all methods imply one division and one multiplication. In general, these problems are more complex than the previous ones.

Primary school students are expected to learn to solve all four types of problems, which, along with other types of problems, belong to the conceptual field of multiplicative structures.[2] While some circumstances may require students to approach these problems separately, simultaneous learning is convenient in other circumstances. The "necklaces" sequence presented in this article explores this possibility. Next is a description of the sequence followed by the results of its application with a 4th grade primary school group (9 and 10 years old).

9.2 "Necklaces": a Didactic Sequence

The following sequence is an adaptation of Guy Brousseau's original idea as developed in B. Mopondi's doctoral thesis[3] (1986).

(a) The setting

The setting is a factory that produces necklaces based on an initial "sample necklace." Each sample necklace has a certain number of different-colored beads. The samples vary depending on the number of beads of each color. For example (see Fig. 9.1), 1 necklace has 2 blue beads, 1 red bead, 4 green beads, and 3 yellow beads.

Before making n necklaces from a given sample, the factory requires a purchase order listing the exact number of each type of bead. Both the number of beads used in the sample necklace and the ones in the order can be organized into tables such as the following one.

[2] The conceptual field of multiplicative structures is made of "situations that can be analyzed as simple or multiple proportion problems and that usually require multiplication or division" (Vergnaud, 1988).

[3] In Mexico, an adaptation of this situation has been published in Block, Martínez & Moreno (2013).

Fig. 9.1 Sample necklace

1 necklace	15 necklaces
4 blue	60 blue
9 red	135 red
7 green	105 green

(b) The type of problem

In missing-value problems, every element in what we will now call the "initial" set is matched to an element in the "final" set. For example, a given number of pencils corresponds to a certain amount of money—to a cost. In the necklace problem, each element or each number of necklaces in the set is matched with the numbers in the final set that represent the number of each color of bead required for that number of necklaces. For example, 15 necklaces require 60 blue beads, 135 red beads, and 105 green beads. Meanwhile, the unit value for this necklace is composed of several values: 1 necklace → (4 blue, 9 red, 7 green). We call this a "one-to-many" relationship.

One-to-one relationship	
Initial set pencils	Final set pesos
3	15
7	x

One-to-many relationship	
Initial set necklaces	Final set beads
1	(*4 blue, 9 red, 7 green*)
15	(x, y, z)

A known example of this kind of relationship is the typical school problem where a certain number of students are matched to certain amounts of ingredients in a cooking recipe.[4] However, necklaces are more tangible and familiar to students than recipes and provide empirical ways of verifying results.

As we discuss later in this chapter, one-to-many relationships lead to a greater wealth of relationships among elements than problems with "one-to-one" relationships.

(c) Didactic variables and situation sequences

When modified, didactic variables may increase the difficulty or trigger changes in the strategy or procedure used to solve a situation (Chevallard, Bosch, & Gascón, 1997, p. 216). In the "necklaces" situation sequence, the main variable is the presence or absence of unit values (beads in a necklace). During the first stage, situations provide the unit value, while situations in the second stage do not. Other variables are as follows: (1) if unit values are absent, the relationship between two numbers

[4] A proportional relationship between a number of boxes and a number of objects, when all boxes have the same quantities, is probably the first relationship that students recognize and use in primary school. For more on the role of contexts or settings in identifying proportionality, see Burgermeister and Coray (2008).

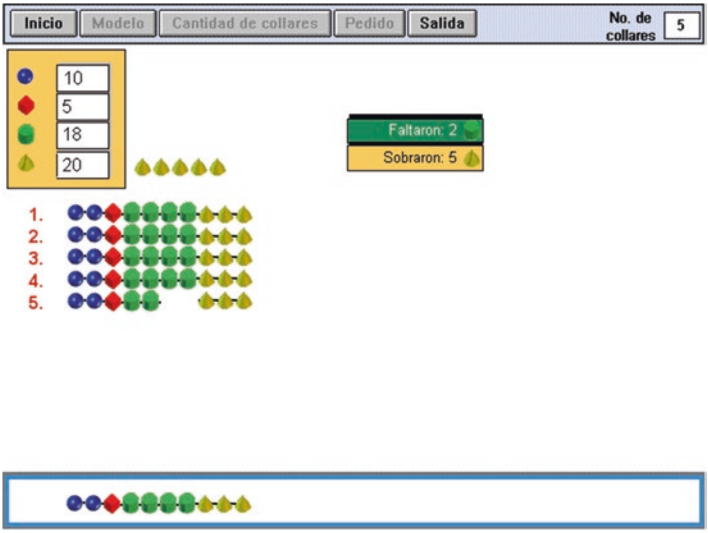

Fig. 9.2 An example of a mistaken resolution displayed in the computer software

of necklaces could be whole or not; and (2) the number of beads could be small or relatively large. The table provided in the Annex offers examples of both types of problems and their characteristics.

(d) Verifying results using computer software

These didactic situations are designed to give students means of empirically verifying their results while assisting them in finding errors and ways to correct them.[5] At the end of each exercise, students enter the following information into a computer program[6] specifically designed for this purpose: the number of each color of bead in the sample necklace, the number of necklaces they are making, and, finally, the number of each color of bead in the order. The computer then displays the sample necklace, all completed necklaces, and any leftover or missing beads (see Fig. 9.2).

(e) The purposes of the sequence

The sequence has two intended objectives: (1) to improve knowledge of division and multiplication with natural numbers by placing them in problems implying distribution (division), multiplicative comparison between quantities (division), and addition of given quantities several times (multiplication); and (2) to enable students to develop different procedures for calculating missing values in relationships between proportional quantities, specifically the unit value procedure.[7]

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[5] For more on validation procedures in didactic situations theory, see Brousseau (1998).

[6] The software was designed by a team from General Academic Computing Services (Dirección General de Servicios y Cómputo Académico (DGSCA)) at the National Autonomous University of Mexico (Universidad Nacional Autónoma de México (UNAM)).

[7] Unit value procedures are those where a value corresponding to a single unit is calculated in order to get the value that corresponds to any number of units.

9.3 Applying the Sequence

9.3.1 Methodology

(a) Theoretical framework

This study is based on a methodology known as “didactic engineering” (Artigue, 1995; Chevallard, et al. 1997, p. 213), which belongs to didactic situations theory (DST). Didactic engineering is known for its experiment structure “based on conceiving, creating, observing, and analyzing sequences in teaching” and also for its way of achieving results by confronting data obtained through experimentation with previously formulated hypotheses. Artigue (1995, p. 36–37) calls it “internal validation.”

(b) Conditions

School and Students We worked with a 35 students group (aged 9 and 10) of 4th grade primary school, in Mexico City. According to official Mexican curricula, year 3 and year 4 mathematics is focused on learning multiplication and division. An expected outcome throughout the sequence was for students to strengthen their grasp on multiplication and division. On the other hand, during the second stage of this research, students were asked to calculate intermediate values, which can be significantly more complex than simple multiplication and division. According to official school programs (Secretaría de Educación Pública, 1993), these problems are meant for Fifth grade curriculum. However, given the outcomes of the first stage of this research and given the context (based on a factory where necklaces are made with beads), it was decided that Fourth grade students would be able to approach these problems. As we will learn in the following pages, this was not the case for all students.

Duration The experiment took place during nine 60- to 90-minute sessions over a period of 2 months. The project leader guided the activities in the classroom. Five observers were assigned to log each session and to follow more or less seven participating pairs of students as well as group activities.

Development Some aspects were consistent in every situation: the teacher read the instructions, students worked in pairs for between 15 and 30 minutes, then results were verified followed by a group discussion. Result verification took place in two stages: (1) after completing their tables, students would review their partners’ results; and (2) any remaining doubts could be verified by entering the quantities into the computer.

The Available Technology Two portable computers and one projector for all sessions.[8]

[8] Two computers were insufficient for the number of students we worked with. Despite implementing measures to help the process, time was lost while students waited for the two computers in the class to turn on.

9.3.2 Results

These results have been reported in Block (2001) and in Reséndiz (2005).

This section is divided into three parts: first, the students' answers to problems where the relationship was 1 necklace to *n* necklaces; second, solutions to problems where the relationship was *n* to *m* necklaces (where *n* and *m* were greater than 1); and, finally, a brief discussion of the feedback from each situation, which turned out to be both problematic and interesting at the same time.

9.3.2.1 Problems Where the Relationship Is Between the Number of Beads in One Necklace and the Number of Beads in *n* Necklaces (the Unit Value Is Given or Present in a Question)

During sessions 1 and 2, students were asked to solve basic tables, which allowed most of them to understand the problem and become familiar with table formats and computer software.

Table 1-B	
1 necklace	Order for ___ necklaces
4 blue	16 blue
6 red	24 red
5 green	20 green

Table 2-A	
1 necklace	Order for ___ necklaces
3 blue	___ blue
___red	48 red
___green	56 green

We will focus on situations solved during sessions 3–5, where six tables were presented and each table led to three problems: one partitive division, one quotative division, and one multiplication.

Table 3-A	
1 necklace	__ necklaces
3 blue	15 blue
6 red	__ red
__ green	30 green

What to Calculate First?

When solving this variant, the first challenge is knowing where to start calculating the number of necklaces in the order. While most students figured this out on their own, some took longer than others, as seen in the following example.

Table 3-B	
1 necklace	___ necklaces
3 blue	24 blue
9 red	___ red
___green	8 green

Silvia: “Do we have to put 3 beads in each necklace? I don’t understand how the sample is made.”
(The observer drew the sample with 3 blue beads and 9 red beads, and told Silvia that they still didn’t know the number of green beads needed.)
Silvia: “So each necklace has 3 [blue beads]?” (They drew necklaces with 3 beads until they reached 24 beads. They answered that there were 8 necklaces in the order.)

Finding the Number of Necklaces: Different Approaches to Quotative Division

Students attempted a range of procedures for finding the number of necklaces, from iterated necklace drawings to using the conventional division algorithm.

Iterating the Blue Beads from the Sample Necklace with Drawings Claudia and Yadira iterated a sample necklace that had 3 blue beads. They did it over and over while counting in their minds until all 15 blue beads stated in the order had been used. They reached the result by counting the number of times they iterated 3 blue beads. This procedure was common during the first tables in the exercise but later disappeared.

Repetitively Adding the Number of Blue Beads in the Sample This procedure took a step further than the previous one because it occurred on a numerical level: knowing that the sample had 4 blue beads and the order had 36, students added the number 4 several times until they reached 36 (see Fig. 9.3), then they obtained the number of necklaces by counting the number of addends.

Juan Carlos or Alejandro (one of them): “If we multiply 4 times 10, we get 36.”
(The observer asked them to try again and pointed to each 4 as they added them up. She stopped when they reached 36, and they counted the 4s they had used.)
Students: “9.”

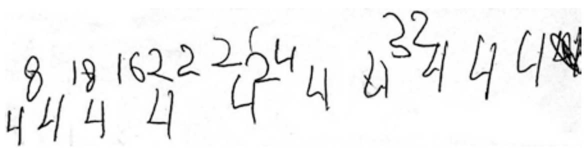

Fig. 9.3 Repetitively adding 4 beads to reach 32

Finding the Unknown Factor in Multiplication This procedure became more and more frequent. Some students recognized that the problem required division. Students used different approaches to find the unknown factor, depending on the size of the quantity in question: successive approaches, multiplication tables, decomposition of known factors, and others. As we will see in the examples, these forms of division were also an opportunity to practice multiplication.

- Successive approaches

Table 3-B	
1 necklace	__ necklaces
3 blue	24 blue
9 red	__ red
__ green	8 green

Francisco (doing "times 3" multiplications out loud): "3 times 5 is 15; 3 times 6, 18; 3 times 7, 21; you can make 7 necklaces and have beads left over."

[. . .]

Observer: "Remember that you should have the least number of beads left over or preferably none at all."

Francisco: "Let's see, 3 on each necklace? We can make 7 and have 3 leftover beads. Oh, no, we can make 1 more, so it's 8 necklaces." (He wrote down "8 necklaces.")

This procedure was frequently used in written form by students completing the last two tables, which was likely due to an increase in the number of beads used in these tables. For example:

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	__ red
__ green	126 green

(Fig. 9.4 shows what the students wrote on the back of their Table 3-E and 3-F worksheets.):

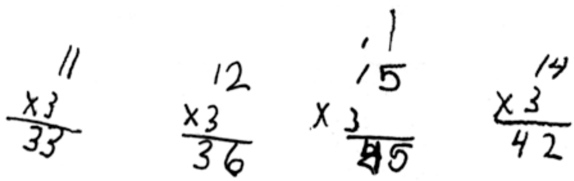

Fig. 9.4 Successive multiplications to find the unknown factor

Some students started by multiplying the number of blue beads in the necklace 10 times. Below is another example.

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	__ red
__ green	126 green

[. . .]

Roberto: "First I did 3 times 10 and got 30, then 3 times 11, and then 3 times 12 until I got to 14; 14 times 3 is 42."

- Using multiplication charts

Looking up quotients on multiplication charts is not a simple task. It requires an understanding of connections between quantities. The following example shows how Francisco and Adriana found the number of necklaces in Table 3-C by searching for a number that when multiplied by 5 would equal 30.

Table 3-C	
Sample	__ necklaces
5 blue	30 blue
3 red	__ red
__ green	24 green

Francisco and Adriana: "It's 6 necklaces because 5 times 6 equals 30." (They showed the observer their multiplication chart and said they used it to find the result.)
Observer: "How did you look for the results on the chart?"
Adriana: "On the '5' chart," (she ran her finger along the 5 row until she reached 30 then moved up along the corresponding column and showed the observer the 6) "so it's 5 times 6 equals 30 . . . then 3 times 6 is 18. We can see it on the '3' chart," (she ran her finger along the 3 row until she reached 18 then moved up along the corresponding column and showed the observer how it intersected with the number 6) "and for the other one, 4 times 6 is 24."

- Using the conventional division algorithm

Some students recognized that the problem was asking them to divide. As the quotient became larger, some of these students used the conventional division algorithm, though not always successfully (see Fig. 9.5).

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	__ red
__ green	126 green

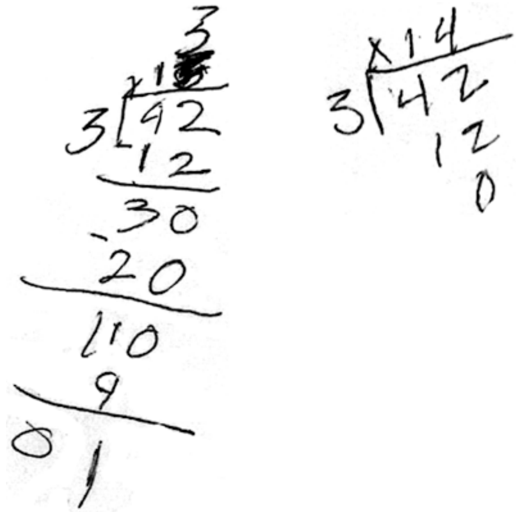

Fig. 9.5 Conventional division algorithm

Finding the Number of Beads in a Necklace from the Numbers in ***n*** Necklaces: Partitive Division

Students' informal procedures reflected subtle differences in comparison to quotative division procedures employed during the previous stage, which confirmed their awareness of differences in relationships between quantities (Martínez & Moreno, 1996). Let's look at an example of cyclical distribution.

Claudia and Yadira's worksheet showed the following procedure (see Fig. 9.6).

Table 3-A	
1 necklace	__ necklaces
3 blue	15 blue
6 red	__ red
__ green	30 green

The pair started by drawing 5 necklace "strings" (horizontal lines) and distributing the green beads one at a time. When they finished, they counted the green beads on each string. There was a significant difference between this procedure and their procedure for the previous problem where they drew strings with 3 beads each until they used up all 15 beads and then counted the number of strings.

Another observation was that more students attempted to solve these problems (partitive division) using the division algorithm than in previous problems: at least five pairs used it while only two had used it to solve quotative division problems.

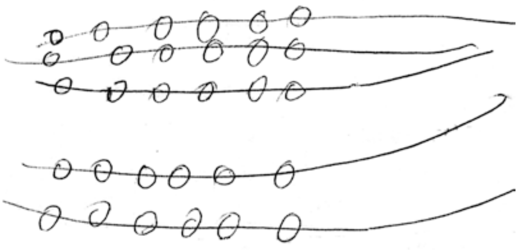

Fig. 9.6 Distributing the green beads one by one

This increase may have been a consequence of elementary schools using more partitive division problems to teach division.

Finding the Number of Beads in *n* Necklaces from the Beads in a Single Necklace: Different Ways of Solving Problems That Require Multiplication

The following problem required simple multiplication. Again, students solved these problems through a variety of methods, from iteration and counting drawings to repeated addition and multiplication. Even without having previously used the algorithm, students could distinguish multiplication problems more often than division problems. These students used the algorithm most, especially when trying to find larger quantities. In this section, we will look specifically at the explicit use of multiplication.

Multiplying the Number of Red Beads in the Sample Necklace by the Number of Necklaces Students used this procedure more than any other. As they progressed toward the final tables, they found themselves needing to write their calculations on paper. Some students used multiplication charts.[9] In the following example, students went as far as decomposing a multiplication factor.

When multiplying 6 by 14, Corelma and Luis implicitly decomposed a multiplication factor as follows: $14 \times 6 = (10 + 4) \times 6 = (10 \times 6) + (4 \times 6) = 60 + 24$.

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	___ red
___green	126 green

$$\begin{array}{r} 60 \\ \underline{+24} \\ 84 \end{array}$$

[9] Also known as a "Table of Pythagoras."

Incorrect Procedures Only a few of the students who correctly calculated the number of necklaces displayed procedural errors when attempting to solve the second problem. Errors were apparently the result of not being able to discern what quantities were represented in the table. The following example shows what happened when David and Karina attempted to solve Table 3-B.

Table 3-B	
Sample	__ necklaces
3 blue	24 blue
9 red	__ red
__ green	8 green

David (looking at the worksheet): "It's 8 necklaces because 3 times 8 is 24." (They wrote "8 necklaces" in the table.)
Karina (looking at the worksheet): "8 necklaces? 9 times . . . 9 times 3 is 27, then it's 27, and we need 9 green beads."
Observer: "Are you sure that 8 necklaces require 27 red beads in total?"
David and Karina: "Yes . . . can we check it on the computer?" (They verified their results and saw that their necklaces were incorrect.)

Karina misread the number of necklaces (she said "3" instead of "8") despite David writing down the number of necklaces and despite both of them repeating the number out loud. What happened? She seemed unsure of what each quantity represented. In general, students expressed difficulty adapting to changes in table structure and therefore in the position of the missing value.

However, the results obtained from this set showed that these problems were suitable for 4th grade, as the students were able to approach these problems, develop procedures, and improve them in the process within a context rich in multiplicative relationships.

9.3.2.2 Problems Where the Relationship Is Between the Number of Beads in *m* Necklaces and the Number in *n* Necklaces (the Unit Value Is Neither Presented as a Known Quantity nor Requested)

In the final two sessions, the students solved tables like the ones below:

4 necklaces	7 necklaces
8 blue	
12 red	
20 green	
4 yellow	

4 necklaces	7 necklaces
12 blue	
8 red	
4 green	
20 yellow	

The First Challenge: Understanding that the Number of Beads in the Sample (the Unit Value) Is Not Provided and Is Not Being Requested

During the first of the final two sessions, 11 of 17 pairs—70% of the group—solved the problem as if the four-necklace order (first column) was the sample. They repeated the procedure that had led them to solve previous tables (where the unit value was explicit), which were presented in a very similar format (two columns with missing data).

> “We did the same as usual and multiplied these numbers,” [the numbers in the column under the order for 4 necklaces] “times this number” [7—the number of necklaces in the order on the right side of the table]. “I don’t get it . . . we have to find this order,” [on the right side of the table, 7 necklaces] “with this one?” [on the left side of the table, 4 necklaces].

They were unable to distinguish between the sample and the order. The instructions were repeated using drawings as visual aids. However, some students repeated the same incorrect interpretation the second time. What caused these difficulties? It appears that the problem was not in the instructions but in the notion that both orders came from the same sample, which was unknown and needed to be figured out—or, as was expressed by a student when he finally understood the situation:

> “Oh, I get it! This,” [the 4-necklace order] “is not the sample, and we need the sample. How do we find it if it’s not there anymore? Do we need to find other numbers?”

As a didactic variable, withholding the problem’s unit value considerably increased the degree of difficulty.

Procedures

Once they understood the problem, the students were able to calculate the sample and the order without expressing further difficulties. Informal procedures were common throughout the two sessions: some students used graphic representation for assistance with divisions and, on a lesser scale, for support with multiplication. Most students estimated using multiplication to solve divisions, while very few used the conventional algorithm.

Before analyzing correct procedures, we will look at two incorrect procedures identified during the session.

Incorrectly Reinterpreting the Problem The following procedure was most common during the first attempts.

4 necklaces	7 necklaces
12 blue	84 blue
8 red	56 red
4 green	28 green
20 yellow	140 yellow

Observer (explaining the task): “What do you need to know before making an order for 7 necklaces?”
Pablo: “We must multiply 8 seven times to find the order.”
Observer (speaking to Jorge): “And what are you going to do?”
Jorge: “The same.” (He multiplied mentally and wrote the result in his table.)

Using Additive Constants At least one pair of students used this procedure. At the start of the activity, Jair and Ixami simplified the problem (this error was explained earlier), then looked for an additive constant, and, finally, understood the need to find unit values during the group discussion. Their worksheet was quite smudged (see Fig. 9.7). Below the Fig. 9.7 is the observer’s register.

4 necklaces	7 necklaces		
12 blue	20	24	21
8 red	16	20	14
4 green	12	16	7
20 yellow	28	35	35

Observer’s version
(The numbers in cursive were erased.
The numbers on the right remained
and wereconsidered correct.)

Apparently, to find the number of beads for 7 necklaces, they started by adding 8 beads to each number of beads in the 4-necklace order. Then they added 12 beads to each quantity in the order for 4 necklaces. They became aware of their mistakes when they verified their results on the computer. In the end, they noted the correct results, after the group discussion.

Next, we will review examples of the many correct procedures that appeared in the session.

Using Internal Relationships In Table 4-F, the relationship between the number of necklaces is one to two, or double. At least one student, David, based his procedure on this relationship. With this procedure, he implied that the relationship between the two numbers of necklaces was the same relationship for all pairs of

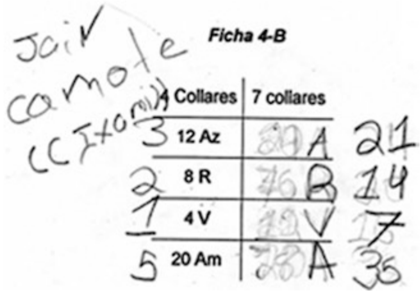

Fig. 9.7 Use of an additive constant

beads of each color in this table. In fact, this relationship presented a proportionality constant. It should also be said that, as expected, this procedure was rare and appeared only in cases where relationships were whole (double).

double

6 necklaces	12 necklaces
18 blue	
60 red	
42 green	
6 yellow	

Obs2: “How did you find it?”
David: “I multiplied. I doubled that (points to the chart where it says ‘6 necklaces’)”

Using the Relationship Between the Number of Necklaces and the Number of Beads of a Certain Color (or the Relationship Between Quantities of Beads) At least two pairs attempted this procedure for Table 4-B by finding the relationship between one quantity of necklaces and one quantity of beads.

Enrique: “Look, it’s just that, I think here,” [in the order for 7 necklaces] “it’s 21, because 4 times 3 is 12 and 7 times 3 is 21.”
Adriana: “I think we need to do 12 times 7.”

Table 4-B

4×3=12 ; 7×3=21

4 necklaces	7 necklaces
12 blue	
8 red	
4 green	
20 yellow	

The use of these factors appeared to be intuitive and probably did not originate from an understanding of the problem. For example, they did not make any connection with the unit value. It also became evident when students who attempted this procedure were asked, “What are the values for the sample necklace?” while verifying results on the computer.[10]

Observer: “What is the sample [in this problem]?”
Adriana: “Mmm, there isn’t one.”
Enrique: “7.”
Observer: “No, you are going to make 7 necklaces. And the sample?” (pointing to the sample on the computer).
Student: “21 blue.”
Adriana: “This is the model” (pointing to the column with 4 necklaces).
Observer: “No, that’s the order for 4 necklaces.”
Enrique: “No, this one,” (pointing to the order for 7 necklaces) “you write in the order.”
Observer: “No, we’re still looking at the sample—look, in the sample.”
Enrique: “Ah!”
Observer: “What should we write here?” (pointing to the space for the sample in the computer). “How did you find these numbers [the order for 7 necklaces]?”
Enrique: “It’s 3 here [blue], 2 here [red], 1 here [green]. Adriana, how many did we say here?” (Both students stopped to think about the number of yellow beads in the order required.) “It’s 5 here.”

At first, Adriana and Enrique tried to answer the sample necklace question by using all the numbers they encountered. It may be worthwhile asking: Did these students realize that the factors they identified (times 3, times 2, etc.) corresponded to the number of beads of each color per necklace? Did the students find the term “sample” (*modelo* in Spanish) confusing (as opposed to using “number of beads per necklace”)? In the end, Enrique seemed to grasp the meaning of the sample and provided the answers that had been requested.

By Recognizing the Need to Know the Sample (the Unit Value) before Obtaining the Quantities in the Order Once the students realized they had to figure out the sample necklace, they completed the implied calculations, dividing then multiplying, using procedures like those mentioned before, both canonical and noncanonical. See the examples below.

- By finding the model through cyclical distribution, with the help of iconic representations

Two teams followed this procedure. Below is an example where the observer helped students by suggesting the pertinence of finding the sample.

Table 4-B

4 necklaces	7 necklaces
12 blue	
8 red	
4 green	
20 yellow	

[10] When entering quantities, students needed to know the values for the sample necklace—that is, the number of beads of each color in one necklace.

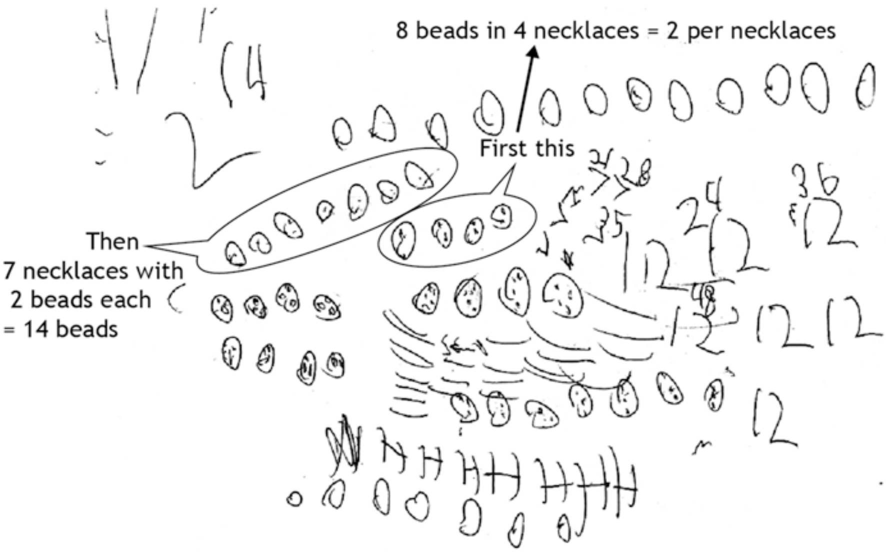

Fig. 9.8 Finding the model through cyclical distribution, with the help of iconic representations

Before this procedure, the pair had simplified the problem.

Observer: “It says here that 12 blue beads were needed to make 4 necklaces. Can you use that information to figure out the number of blue beads in the sample?”
Juan Carlos (drawing 4 small circles then distributing 12 little dots one by one and counting the ones that remained inside one of the circles): “There are 3.”
Observer: “So, then, to make 7 necklaces, how many blue beads do you need?”
Juan Carlos (drawing 7 circles with 3 little dots in each circle; there were 21): “OK, I get it.”

This procedure was apparently used to find several results. The Fig. 9.8 shows an excerpt from the student’s worksheet, demonstrating how he found the answer for the red beads (8 red beads in a 4-neklace order).

- By searching for the unknown factor in a multiplication

Several students decided to calculate the number of beads corresponding to 1 necklace (the sample) by looking for the unknown factors in multiplication where “number of necklaces times ‘×’ beads per necklace = number of beads.” Below is an example that demonstrates the difficulties that even the most skilled students encountered with this task. Also, for Table 4-A, Karina knew she needed to find a sample or, as she called it, “what goes on each necklace.” This is how Karina and Cecilia solved Table 4-A.

Table 4-A	
4 necklaces	7 necklaces
8 blue	
12 red	
20 green	
4 yellow	

Karina: "OK, let's see what goes on each necklace. Since there are 8 [blue beads in 4 necklaces], each necklace needs 2 [blue beads]."

On the multiplication chart (see Fig. 9.9), Karina searched down the 4 column for the number 8 where it intersected with the 2 row; 2 was the quotient of 8 divided by 4 and the number of blue beads per necklace. Then, to find the number of blue beads in the order, she multiplied 2 by 7: she looked for the intersection of the 2 row and the 7 column, which was 14.

Karina started by figuring out the sample, then the order (See Fig. 9.9).

Karina demonstrated her skills using the chart for both dividing and multiplying. She quickly completed the two calculations needed to go from 4 to 7 necklaces by figuring out the quantities required in a single necklace.

Observer: "Karina, what's in the sample? What do we place in the sample?"
Karina: "What do you mean by 'place'?"
Observer: "Remember how the software asks us to create a sample? You wrote 14 [blue], 21 [red], 28 [green], and 28 [yellow], but that is the order for 7 necklaces. If we want to verify this on the computer, we need a sample, right?"
Karina: "Which one is the sample?"
Observer: "How did you find these quantities [for the 7-necklace order]?"
Karina: "Oh! . . . Um, um, that's right. Here, on the blue sample, we need 2 blue beads for each necklace; here [on the red beads] we need 3 per necklace, I think." (They used the multiplication chart to finish the sample.)

Karina's original way of finding the numbers of blue and red beads led, in fact, to finding each unit value. However, she seemed to have forgotten that those values corresponded to the "sample."

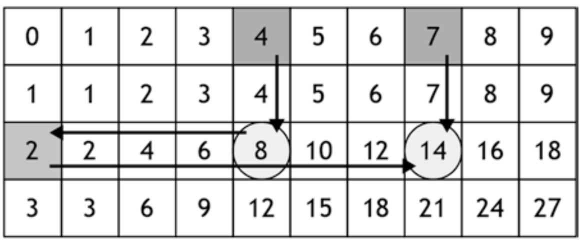

Fig. 9.9 Use of the chart for searching the unknown factor in a multiplication

9.3.2.3 Feedback

During the first stage, where the problems included the unit value (as a given value or as a question), the software fulfilled its purpose of enabling students to understand the problem and empirically verify their results, which saved time and resources. When the problem was solved correctly, the software became a functioning validation tool. When typing errors had been made (miscalculations, for example), the software detected where the error had taken place. Below is a scenario where Irving and Carlos suggested that the order was asking for 3 necklaces (in Table 3) during the group discussion.

The teacher requested the results for Table 3 during the group discussion.

Table 3	
1 necklace	__ necklaces
4 blue	16 blue
6 red	24 red
5 green	20 green

Teacher: "How many necklaces in this order?"
Carlos: "4."
Irving: "3."

The teacher asked each student to explain how they reached their results, but the rest of the class disagreed with their explanations. Then, the teacher suggested verifying their results on the computer.

Teacher: "Irving, maybe we can try something; we can tell the computer that the answer is 3."
Francisco: "There will be leftover beads."

The computer made 3 necklaces, leaving 4 blue beads, 6 red, and 5 green unused. There were enough beads to make another necklace.

Teacher: "See? There were leftover beads."
Francisco: "Enough for another one."
Teacher: "Enough for another necklace, to make the fourth necklace."

During the second stage, where the unit value was not included, verifying results on the computer became considerably more difficult. The most widespread difficulty occurred when students attempted to verify their results on the computer without knowing the sample. They had approached the problem through addition, solved the problem without stopping to find the sample necklace (by identifying regularities), or solved it by other means. When the computer asked students to enter the numbers for the sample, they did not understand what they were meant to do. As a result, some felt bewildered and returned to their seats; others attempted to find a sample for each order, which was a step away from having one sample for both orders. Some students did not realize the pertinence of the sample until the computer requested it.

The program sometimes validated incorrect answers. It happened mainly when students, knowing the numbers for one sample, verified only the values for one of the orders and omitted the second one. Let's analyze an example. Yadira, Claudia, and Alejandro solved Table 4-C using drawings. However, they entered their data incorrectly.

Table 4-C

Sample	5 necklaces	11 necklaces
2	10 blue	22
4	20 red	44
4	15 green	44
1	5 yellow	11

The students drew their sample instead of writing it.

Observer: "What is the sample?"
Yadira: "2 blue, 4 red, 4 green, and 1 yellow."
Observer: "What's in the order?"
Yadira: "22 blue, 44 red, 44 green, and 11 yellow."
Student: "I think the numbers are wrong."
Observer: "Does the sample have the same number of red beads and green beads?"
Students: "Yes."
Observer: "How did you find them, Yadira?"
Yadira: "Where it says 5 necklaces, 1 has 2 blue, so then, where it says 11 necklaces, I placed 2, 2, 2, and got 22," (showing me the drawing on her chart; she had drawn 12 necklace strings and added 2 beads on each of the first strings) "and these [the other numbers in the order]; they found them."
Observer: "OK, let's see if it works, OK?" The necklaces were correct.

The computer accepted the quantities for the sample and the order, and it validated the result without detecting any errors. However, the observer aptly reminded the students that the sample should be the same for both orders.

Observer: "It turned out all right, didn't it? Just remember that the necklaces in this order are identical to the ones in the other [five-necklace] order. . . . Should we test it? . . . Let's use the same sample for the other number of necklaces, and they should be identical. This will tell us if we're right. . . . The sample is the same: 2, 4, 4, and 1, because the necklaces are identical. Can someone read me the numbers in the 5-necklace order?"
Yadira: "10 blue, 20 red, 15 green, and 5 yellow."
Observer: "Let's see if it's true. If they come out the same as these [11 necklaces], then we're fine, but if they don't, there must be some mistake, right?" Three necklaces were made correctly, but the fourth and fifth were incomplete; 5 green beads were missing.

After noticing the missing beads, the students realized they had made a mistake. In this case, however, given the number of relationships at stake, they could not find the error in their procedure.

Other Validation Resources Aside from using the software to validate results empirically, some students were seen verifying results through a range of inherent properties embedded in the problem's relationships. This might have been the case for the student who, seeing the quantities in the following table, stated, "I think the numbers are wrong."

5 necklaces	11 necklaces
10 blue	22 blue
20 red	44 red
15 green	44 green
5 yellow	11 yellow

The student probably thought that if an order presented two equal quantities of different-colored beads, then the number of beads in those two colors should also be the same in other purchase orders.

Others also tried keeping the order: if the number of beads of a certain color was greater than the number of another color in the set (for example, in the sample), that same relationship must carry over to the other set (for example, in the order). See the examples below.

Example 9.1

Table 2	
1 necklace	8-necklace order
3 blue	__ blue
__ red	48 red
__ green	56 green

For the red beads:

Yadira: "6 red beads." (The observer thought she found the answer by testing numbers that when multiplied by 8 would equal 48. Then, she drew 8 lines and added 8 beads only to the first 2 lines.)

For the green beads:

Yadira: "It can't be less than 6." (She realized that now she needed more beads than before and that the number of necklaces was the same. She wrote "7 green" on her worksheet.)

Example 9.2
(That example was worked during a group discussion)

Table 3-A	
Sample	5 necklaces
3 blue	15 blue
6 red	30 red
__ green	30 green

Tea1cher: "Finally, how did you know you needed 6 green beads?"
Student: "Because of 6 times 5."
Luis: "Because if the line above said 6 red becomes 30, the line below is the same because the result is there [in the row for red beads]."

Students favored this type of semantic validation (Brousseau, 1998) in problems where plural relationships were in play. These resources could have been shared more during group sessions in a way that would encourage the students to find and reflect on relationships. For example, if one quantity in the sample was larger than another, the quantities in the order would maintain the same relationship; if the relationship between beads in the sample was "double," the same applied to the order. Meanwhile, additive relationships, where one quantity was "larger than another by 3 beads" would not be reflected as such in the order.

9.4 Final Remarks

Most fourth-grade students were comfortable approaching relationships among three or four quantities of beads on 1 necklace and the corresponding quantities on *n* necklaces, and they could solve all three implied problems (two divisions, one multiplication), usually through noncanonical procedures. It can be said that one-to-many relationships provided richer contexts than those presented in problems with four values.

The relationships between the number of beads in *n* necklaces and *m* necklaces (where *n* and *m* were greater than 1) were significantly more difficult. However, at least 50% of the pairs in this 4th grade group could understand one or more of the problems presented during the last two sessions. It seemed that the greatest challenge was understanding that (1) each set of beads in the table matched a different "order" rather than a single necklace and a single order, and (2) both orders came from the same sample. Once students understood this, they encountered far fewer difficulties calculating the number of beads in the sample based on one of the orders, to then generate the second order. From the perspective of understanding multiplicative relationships, this was a remarkable achievement.

On the other hand, considering their school level, this was probably the first time that most of the students had needed to calculate information that was not directly requested in the problem (the unit value) to solve it. Potentially, more students would be able to approach this variable when they reached 5th grade, which might be a better time to introduce it.

Finally, the computer software, as a “virtual means of interaction” (Mariotti, 2002), supported the students’ understanding of problems and, more specifically, of relationships between the broad range of quantities used in the “necklace factory.” The computer software also provided a way of verifying the results. However, the program showed some limitations. When verifying the relationship between *n* and *m* necklaces, it asked for the sample necklace even when students had not figured out its pertinence for finding the solution. The program did not always enable students to identify the source of their errors and did not invalidate certain types of erroneous procedures. It would be worthwhile to explore other forms of empirical verification for more difficult cases where the relationship in question was between *n* and *m* necklaces. For example, the program could display a graphic representation of the two orders that came from a certain sample. On the other hand, the wealth of relationships in play enabled a variety of semantic forms of verification, which were valuable from a didactic point of view.

Annex

Problem Chart

Variables	Stage 1: unit value provided or requested						Stage 2: unit value not provided			
	Small numbers				Larger numbers		Nonwhole relationships between numbers		Whole relationships between numbers	
	First type of situation		Second type of situation		Third type of situation		Fourth type of situation		Fifth type of situation	
Table example	Table 1		Table 2-A		Table 3-F		Table 4-A		Table 4-F	
	1 necklace	12 necklaces	1 necklace	__ necklaces	1 necklace	__ necklaces	4 necklaces	7 necklaces	6 necklaces	12 necklaces
	__ blue beads	__ blue beads	4 blue beads	12 blue beads	4 blue beads	60 blue beads	8 blue beads	__ blue beads	18 blue beads	__ blue beads
	__ red beads	__ red beads	6 red beads	18 red beads	9 red beads	__ red beads	12 red beads	__ red beads	60 red beads	__ red beads
	__ green beads	__ green beads	5 green beads	15 green beads	__ green beads	105 green beads	20 green beads	__ green beads	42 green beads	__ green beads
	__ yellow beads	__ yellow beads					4 yellow beads	__ yellow beads	6 yellow beads	__ yellow beads
Implied knowledge	Multiplication		Division		Division or multiplication		Division and multiplication Unit value		Division and multiplication Unit value 12C/6C reason	
Number of similar tables applied	One table		Two tables		Six tables		Six tables			
Number of sessions dedicated	One session		One session		Three sessions		Two sessions			

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