# Chapter 5
# Japanese Lesson Study for Introduction of Multiplication

**Raimundo Olfos and Masami Isoda**

In Chap. 2, we posed questions about the differences in several national curricula, and some of them were related to the definition of multiplication. In Chap. 3, several problematics for defining multiplication were discussed, particularly the unique Japanese definition of multiplication, which is called definition of multiplication by measurement. It can be seen as a kind of definition by a group of groups, if we limit it to whole numbers. In Chap. 4, introduction of multiplication and its extensions in the Japanese curriculum terminology were illustrated to explain how this unique definition is related to further learning. Multiplicand and multiplier are necessary not only for understanding the meaning of multiplication but also for developing the sense to make sense the future learning. The curriculum sequence is established through the extension and integration process in relation to multiplication. In this chapter, two examples of lesson study illustrate how to introduce the definition of multiplication by measurement in a Japanese class. Additionally, how students develop and change their idea of units—that any number can be a unit in multiplication beyond just counting by one—is illustrated by a survey before and after the introduction of multiplication. After the illustration of the Japanese approach, its significance is discussed in comparison with the Chilean curriculum guidebook. Then, the conclusion illustrates the feature of the Japanese approach as being relatively sense making for students who learn mathematics by and for themselves by setting the unit for measurement (McCallum, W. (2018). Making sense of mathematics and making mathematics make sense. *Proceedings of ICMI Study 24 School Mathematics Curriculum Reforms: challenges, changes and Opportunities* (pp. 1–8). Tsukuba, Japan: University of Tsukuba.). A comparison with Chile is

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given in order to demonstrate the sense of it from the teacher's side. In relation to lesson study, this is a good exemplar of how Japanese teachers develop mathematical thinking. It also illustrates the case for being able to see the situation based on the idea of multiplication (Isoda, M. and Katagiri, S. (2012). *Mathematical thinking: How to develop it in the classroom*. Singapore: World Scientific; Rasmussen and Isoda Research in Mathematics Education 21:43–59, 2019), as seen in Figs. 4.2 and 4.3 in Chap. 4 of this book.

## 5.1 Lesson Study for the Introduction of Multiplication

The introduction of multiplication to students does not demand much time. Teaching the meaning of multiplication demands 3 or 4 hours of lessons or sessions[1] of 45 minutes each in the three Japanese textbook series we analyzed (Gakko Tosyo,[2] Tokyo Shoseki,[3] Keirinkan, and PROMETAM[4]) for multiplicative situations.[5] The terms "multiplicand" and "multiplier" are introduced to create the mathematical sentence appropriate for a given situation. Enabling students to see multiplicative

[1] Japanese usually teach mathematics with the whole class and use terminologies in Chap. 4 on the unit plan. A mathematics lesson in Japan corresponds to a session in a subunit of the unit plan. The subunit is usually called a "phase." Another usage of the term "session" refers to one class hour. The term "lesson" refers to the topic addressed by the lesson plan and is sometimes not limited to one class hour. The lesson plan usually refers to a part of the phase in the unit plan, which means a section in the textbook. On the other hand, based on research in mathematics education, sessions usually use the context of the topic sequence. Here we have used the term "session" for one class hour. The lesson plan for lesson study by the group usually has a study theme and the objective of the class with the content topic as the teaching material (see Chap. 1; Isoda, 2015).

[2] The English-translated edition of *Study with Your Friends: Mathematics* (Hitotsumatsu, 2005; Isoda and Murata, 2011; Isoda, Murata, and Yap, 2015; Isoda and Murata, 2020). Thai translated edition (Inprasitha and Isoda, 2010) is from the 2005 edition. Spanish-translated edition (Isoda and Cedillo, 2012) is from the 2005 edition. Indonesian adapted edition (Isoda et al) is from the 2011 edition. Chilean adapted edition (Isoda et al., 2020; Isoda and Estrella, 2020) are the 2005 and 2011 editions.

[3] The English-language edition of *New Mathematics* is used (Hironaka and Sugiyama, 2006).

[4] PROMETAM is the Project for Improving Technical Education in the Area of Mathematics in Honduras, with technical assistance from the Japanese International Cooperation Agency (JICA). The JICA-supported projects PROMESAM in the Dominican Republic, PROMECEM in Nicaragua, GUATEMATICA in Guatemala, and COMPRENDO in El Salvador were also implemented in the period in which the framework for the development of the texts was elaborated.

[5] For analyzing those textbooks, we also referred to the framework of Vergnaud (1990) to describe the concept of multiplication in a situation, the invariant, and the representation. However, we did not explain the lesson using his terminologies because we would not be able to clearly explain the significance of the teaching sequence based on his framework. Actually, the teaching sequence was never discussed on his framework. Instead of using his analytical terminology, we illustrated the real lesson study classroom in Japan and compared the Japanese approach with the Chilean approach to show its significance.

situations with the idea of multiplication is a particular feature of Japanese education.[6] In a later section, it will be compared with the Chilean approach using the terminology "sense making" or "making sense."

### *5.1.1 Lesson Study on the Meaning of Multiplication, by Mr. Natsusaka*

In relation to the subtheme of this book, this section presents an exemplar of lesson study with the lesson plan, implementation of the lesson (an open class), and discussion of the implementation, carried out in June 2008. The open class for the lesson study was implemented by Mr. Satoshi Natsusaka from the Elementary School at the University of Tsukuba. The implementation corresponds to the first of the three lessons that introduce the meaning of multiplication to second-grade students.

#### 5.1.1.1 Description and Plan of the Lesson Being Investigated

The topic to be studied in this lesson was "the meaning of multiplication," developed by Mr. Natsusaka. The goal of the study was to consider lessons that would allow for developing students' competency to use multiplication by linking the situation with multiplication expressions, taking advantage of how students would understand the situation.

*[Lesson plan by Mr. Satoshi Natsusaka]*

1. *Unit name*: Multiplication (1).[7]
2. *Research theme of lesson study*: To develop the eyes to see the situation mathematically.

(a) From "counting" and "discovering" activities to "expressing" activities: When there are a number of groups with the same quantity of elements (a unit of measurement)—say, balls—it is expressed as the "number of balls in a group times

[6] In Japan (as explained in Chap. 1), development of mathematical values, attitudes, ways of thinking, and ideas have been the objectives of mathematics teaching since 1968. Mathematical ideas usually change the way to see the situation. In research on mathematics education, this is sometimes referred to using terms such as "intuition" and "insight" (see van Hiele, 1986).

[7] He is an author of Gakko Tosho textbooks (2005). It has four chapters on multiplication for grade 2. Multiplication (1) provides an introduction and definitions with the meanings of situations. Multiplication (2) covers development of the row of 2, the row of 5, the row of 3, and the row of 4, and learning how to develop the multiplication table. The discussion between rows is used to produce the idea of distribution for extension of the table. Then, Multiplication (3) discusses extension of the multiplication table to include the rows of 6 to 9 and the row of 1. It is expected that students are able to extend it. Multiplication (4) explores the properties of the multiplication table. Finally, the book discusses the making of a project by students (see Fig. 4.2 in Chap. 4, and see Chap. 6). The lesson being analyzed here is his original work. Textbook authors usually try to offer new challenges in their classes to produce innovative ideas for teaching and further revision of textbooks.

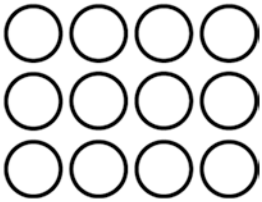


**Fig. 5.1** The way to explain the array diagram such as 4 columns of 3 balls and so on

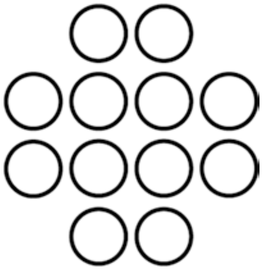


**Fig. 5.2** Grouping the dots for multiplication

the number of the same groups" (in Japanese) which corresponds to the mathematical expression of multiplication. The students express such a situation using phrases like "There are *n* balls in each group (set) and there are *m* groups (set)" even though they do not know the mathematical expression for multiplication (see Fig. 4.3, Fig. 4.8 in Chap. 4). For example, when balls are placed in a box as shown in Fig. 5.1,[8] some students may express this situation by saying, "There are 4 columns of 3 balls." This expression can be considered to identify groups of 3 balls aligned vertically and to show that there are 4 columns with this quantity of balls. There are no lines that separate or encircle groups of 3 balls, but students who use this expression are imagining these lines.

Similarly, some students may observe the same situation from other points of view, such as "3 rows of 4 balls" or "2 groups of 6 balls." In any case, they will try to calculate the total number of balls by identifying groups with the same quantity of elements. If it is understood that there are "4 columns of 3 balls," the total number of balls can be found by making the calculation "3 + 3 + 3 + 3 = 12." It is appropriate to lead the students to the multiplication expression, obtaining the expression from them and confirming what the expression "3 [×] 4" represents.[9]

(b) The competency to see the situation as multiplication:[10] As shown in Fig. 5.1, the quantity of balls in Fig. 5.2 is 12. The students who realized that in Fig. 5.1

8 This was discussed as the array diagram in Chaps. 2 and 3.

9 In this chapter we use the Japanese notation "3 × 4" instead of the English notation "4 × 3" because we quote Japanese textbooks and photos in the classrooms, and we could not change original photos, and so on. Thus, "[×]" is written as "×" from here onward.

10 In Japan (as discussed in Chap. 1), seeing the situation through mathematical ideas has been emphasized as subject matter of teaching to develop mathematical thinking since 1958 (see Isoda and Katagiri, 2012; Rasmussen and Isoda, 2019).

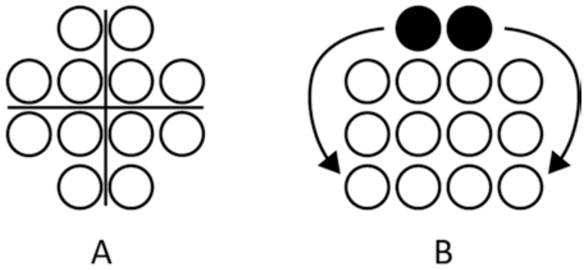



**Fig. 5.3** To see the shape for multiplication: (**a**) 4 corners of 3 balls. (**b**) If we move two balls to appreciate places, it changes to 4 columns of 3 balls

there were 4 groups of 3 balls are asked if they can also see that there are 4 groups of 3 balls. Then, some students may think of separating the balls as shown in Fig. 5.3a or moving the 2 balls placed in the upper part to the corners of the lower part as shown in Fig. 5.3b. As such, the way of placing the balls is changed so that it is the same as in Fig. 5.1.[11]

The custom of observing the figure and determining the quantity of balls per unit or group will increase students' competency to see the situation as a multiplication expression or a model of multiplication. Also, listening to how other students interpret the figure and recognizing the model will allow them to enrich their points of view.

3. *Unit goals*:

(a) To understand the meaning of multiplication through concrete situations.
(b) To be able to formulate the multiplication expression for situations that can be expressed as such.

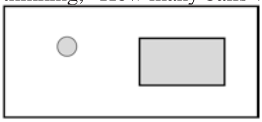
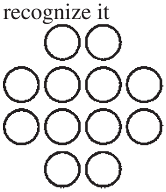
4. *Unit plan* (4 hours):

(a) First phase: The meaning of multiplication (2 hours); this is the first of the 2-hour lesson.
(b) Second phase: Applying multiplication (2 hours).

5. *Lesson outline:*

(a) Goal (objective of the class): Learn to express that "there are *m* groups of *n* quantities" considering groups of the same quantity when the number of elements is counted.
(b) Development of the lesson.

[11] This is the activity that enhances seeing the situation as a multiplicative situation (see Figs. 4.2 and 4.3 in Chap. 4).

| Main learning activity | Considerations |
|---|---|
| *Situation 1*: observing the box and thinking, “How many balls will fit in it?”<br><br>2 + 2 + 2 = 6 (balls)<br>6 balls because 2 balls are placed in 3 columns<br>6 balls because 3 balls are placed in 2 rows<br>*Situation 2*: observing the shape of a box where 12 balls can fit, and thinking, “How many balls will fit in it?”<br>6 sets of 2 balls (2 + 2 + 2 + 2 + 2 + 2)<br>4 sets of 3 balls (3 + 3 + 3 + 3)<br>2 sets of 6 balls (6 + 6)<br>Observing the balls placed as (Fig. 5.4) from the same point of view as in situation 2 if they are moved<br>2 + 4 + 4 + 2 = 12<br>4 sets of 3 balls are seen<br>6 sets of 2 balls are seen | Thinking by considering the role of the rectangular drawing of the box<br>It is desirable for the students to realize that if the number of rows and columns is known, the total number can be determined without putting all the balls in the box<br>Try to get verbal expressions like “there are so many groups of so many balls” from the students or expressions through the additive model<br>It is desirable to take advantage of the point of view of situation 1<br>Make the expression correspond to the words<br>If the numbers are added from the first row downward, it can be expressed using the equation 2 + 4 + 4 + 2 = 12<br>If there are students who try to change the way the balls are placed by moving some, they could also recognize it<br> |

*[End of lesson plan]*

#### 5.1.1.2 A Public Lesson (Open Class) by Mr. Natsusaka

The following is a translation of a transcript of the notes taken during the implementation of the lesson by Mr. Natsusaka with a class of 39 second-grade students from the Tsukuba School in Tokyo on June 19, 2008. These notes were taken in Spanish based on the simultaneous translations from Japanese that were offered to Central American teachers observing the lesson.

The lesson took place in the Elementary School Theater at the University of Tsukuba in Tokyo. Fig. 5.4 shows the arrangement of the desks between the stage and the first row of seats in the theater.

At 9:18 a.m. the students went up to the stage in two lines and received a round of applause from the audience. There were more than 300 people present, the majority of whom were teachers from different parts of Japan. Some of the guardians (parents) participated in recording the class to support Mr. Natsusaka. Without a doubt, the lesson being observed was an important occasion not only for the teachers watching but for the students as well.

After Mr. Natsusaka guided the students in greeting the audience and ceremonially opening the lesson, he flashed on the interactive screen questions about types of triangles and polygons. On various occasions, the students went to the screen and touched a part of it as a way to answer the question asked, such as “Which of these

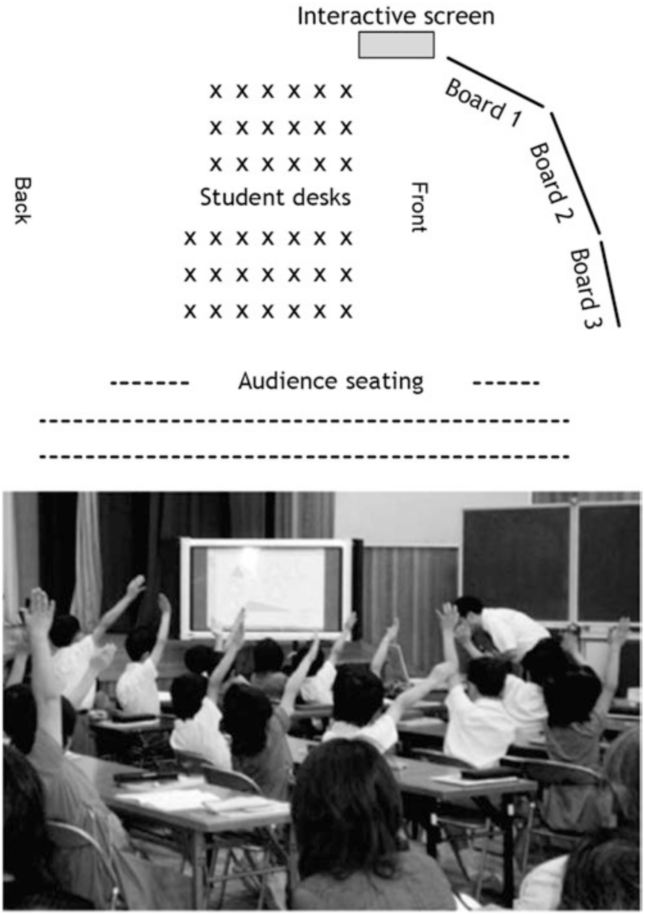



**Fig. 5.4** Open class given by Mr. Natsusaka

figures is a triangle?” and “Which of the triangles is equilateral?” When a student touched the correct answer, the figure was filled in with the color green. Otherwise it was filled in with the color red. The activity let the students recognize 3 triangles, 2 rectangles, 1 pentagon, and 1 hexagon among the figures. Next, Mr. Natsusaka proceeded on to new questions using the interactive software, changing the content flashed on the screen, such that 4 triangles and 6 rectangles appeared. At this point, all the students raised their hands, and it could be seen that they had become comfortable and were involved in the dynamics with the teacher.

9:36 a.m. (Teacher, Mr. Natsusaka, presented on the interactive screen a rectangle with circular pastries in it. He never mentioned that this was an introduction to multiplication. It should be noted that he used the interactive screen (see Fig. 5.5) and not paper or a chalkboard to present the problem situation, as was indicated in the lesson plan.)

**Fig. 5.5** Mr. Natsusaka uses the interactive screen to present the problem situation

Teacher: "How many sweets will fit in the box?" (A rectangle shape and one circular[12] sweet were shown on the screen.) "Make guesses about how many pink sweets will fit in the blue box."

(A student came up to the electronic screen and demonstrated on the screen what he understood. Teacher did not say whether this was good or not; he delayed reacting on purpose to give time for the students to think by themselves.)

9:37 a.m. Teacher: "How many pink sweets will fit?"

9:38 a.m. Teacher: "Open your notebooks and write. How many sweets will fit?" (Teacher observed that some students were not working, then he added the following for those who hadn't thought of it.) "Look at the screen. I would like to know your predictions." (Teacher walked around the classroom from desk to desk and quickly looked at the students' notebooks.) "Now, I'm going to write here," (using the left part of the second board) "some of your answers in your notebooks: 4 . . . , 5 . . . , 6 . . . ; from what I can see, some of you have written '4,' others '5,' and others '6.' One student wrote '8' and another '12.' Which of these answers seem possible to you? Which seem impossible?" (Teacher provided ambiguous situations and let the students fix the necessary conditions by asking questions that made them think. Indeed, the students began to critique.)

Student 1: "It can't be 4; 2 more would fit."

Student 2: "If we look at it, 6 would fit."

9:40 a.m. Student 2: "Can the sweets be placed on top of each other?"

Student 3: "One layer of 6 and another of 6. I don't think that only 6 will fit. If you want it to be a box with 6 on the bottom and 6 on top, it has to be a taller box."

Teacher: "What are the bases for your conjectures?"

Student 4: "I think that 12 will fit: 6 on top and 6 on the bottom." (looking at the box from the top view)

Student 5: "Observing it from above, then it would be 12: 3 layers of 4."

9:45 a.m. Mr. Natsusaka: "We will exclude that case. The box has to have all the pastries visible." (The boxes with layers viewed from the top were excluded.)

[12] They know the circle as a shape; however, they do not know the property of a circle. Thus, it is an ambiguous figure.

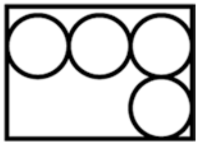


**Fig. 5.6** On the Monitor Screen: Mr. Natsusaka confirmed 3 or 4

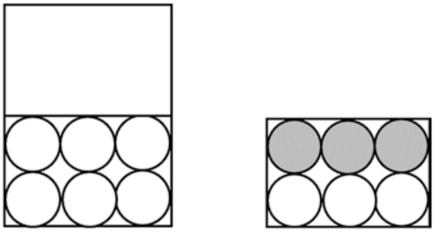


**Fig. 5.7** Show the students ideas on the screen

Student (going to the board and showing with his fingers how the length of the diameter of a circle was contained three times in the length of the rectangle): “3 on the top row and 3 on the bottom row, so 6 fit.”

(Mr. Natsusaka put another box on the screen under the first. With the mouse and software tools, he placed 4 pastries in the box (see Fig. 5.6).)

Teacher: “It’s the same as what you did with cardboard. We have to prove . . . ,” (after drawing) “so 6 fit. There is enough space. If there are 3 in the first row, then . . . 3 fit on top.”

Student 1 (using the software’s copy option to draw another rectangle on the screen and commenting as follows): “Since 6 fit the first time, if the box is tall enough, 6 more will fit.” (In the left part of Fig. 5.7, the 6 balls became a unit, which was the side view of the layer.)

9:50 a.m. Student 2 (speaking from his desk and pointing at the 3 balls in the right part of Fig. 5.7): “The balls are superimposed.”

Teacher (trying to lead them to see it as 6): “Do 6 fit? Raise your hand if you think that 6 fit.” (Several students raised their hands.)

Student 4: “3 fit in 1 row. 3 + 3 is 6.”

Teacher (writing the expression “3 + 3 = 6” on the board): “3 + 3 = 6.”

Student (pressing the software’s buttons, visible on the interactive screen, and drawing as he spoke): “Then there are 6. There is a group of 3 and there is another group of 3.”

Teacher: “How did you divide it?” (on the screen)

Student: “Days ago,” (Mr. Natsusaka did not conduct the class for multiplication, yet) “we made a drawing like this,” (pointing to the ovals drawn on the board (see Fig. 5.8)) “we changed the shape, but is it the same?”

Student: “If we think of 2 groups of 3, there are 6.”

9:59 a.m. Teacher: “First, listen to what your classmate said,” (repeating the student’s idea) “yesterday, someone separated it like that and said there were 2 groups of 3. Could I also say that there are 3 groups of 2?” (See Fig. 5.9.)


**Fig. 5.8** Board Writhing (Bansho), Japanese teacher listen students' idea through questioning and note on the board. See Fig. 1.2, Chap. 1

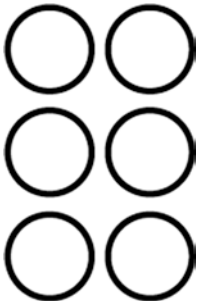


**Fig. 5.9** Read the diagram and explain vertically and horizontal

**Fig. 5.10** Mr. Natsusaka, Teacher, and a student interacting on the board

(Teacher no longer drew with the software on the electronic board; he drew on board 2 with colored chalk, in the upper left-hand portion.)

Teacher: "So . . . there are 3 groups of 2. So, in a row of 2, there are 3 groups. So, 3 groups of 2, there are 3 groups of 2. Do you all see it like that?"

Student: "If we take 3, twice, then it will be 6."

Teacher: "3 + 3 = 6." (See Fig. 5.10.) "So, in this case, how can you express 2 + 2 + 2? If you express it using addition, how can we express it?" (Only half the class raised their hands, and a student asked Mr. Natsusaka a question).

Student: "It's 2, like it's grouped that way. So, is it about groups?"

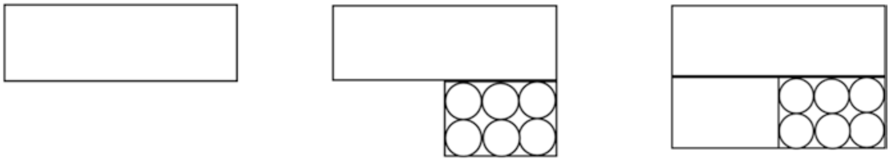


**Fig. 5.11** How many boxes? Here the box is a unit for counting: The inverse idea of splitting (see Chap. 3).

Teacher: "Your classmate asked what this number 2 represents." (Teacher paused and waited for the students to raise their hands, then he spoke to the student who had just spoken): "Could you repeat what you said before?"

Student (returning to the board and explaining it as follows): "A group of 2 repeated 3 times."

Teacher (speaking to another student to evaluate his understanding and focus the discussion): "Can you repeat what your classmate said?"

(The student did not answer, so Teacher did.)

Teacher: "So, 2 represents the number that divides. There are 3 groups of 2. The number that divides $2 \times 3 = 6$ is 2. 3 indicates how many times."

Student: "I can say it more simply: 2 is the number that is going to be multiplied."

Teacher: "He said it in a way that is easier to understand."

Student: "This number, 2, of 2 times 3, leads to 6. 3 shows how many there are."

10:03 a.m. Teacher (drawing 2 pastries inside a circle on the board in blue): "So, 3 indicates 'how many circles.' One way is 2 groups of 3, and another is 3 groups of 2; that is, it can be said in different ways." (He then returned to the interactive screen.) "Now, in this box," (see the left part of Fig. 5.11) "how many will fit?"

Student: "Can you show the previous box again?" (See Fig. 5.6.)

Student: "Can you show both boxes?"

Teacher used the mouse to copy the box, showing both.)

Student: "Can you move one box under the other?"

10:06 a.m. Teacher: "Yes, I can move it." (Since the student came up with arguments, Teacher asked him to go over to the interactive board.) "Come here."

Student: "In this [the box underneath], 6 fit." (He used the software to move the boxes and line them up (see the middle part of Fig. 5.11).)

Teacher: "It looks like it marked it there. Do you know what it's doing?"

Student: "It's covering it up."

Teacher: "He adjusted, marked, and moved. Think, what is Lu's intention?"

(Teacher gave the students 30 seconds to talk in groups of three.)

Teacher: "Are the rectangles below of the same width?"

Teacher (showing that the rectangles had the same width by placing one rectangle beside the other): "So, how many sweets fit in this big box?"

Students (all responding together): "12."

Teacher: "Who thinks that it's not 12?"

Student: "I'm not sure, but it has to be even."


**Fig. 5.12** 6 + 6 = 12 on the third board; is there another way to express the total?

10:12 a.m. Student (in front of the interactive screen, and moving the lower rectangle): “It fits twice, so 12 fit.” (See the right part of Fig. 5.11.)

Student 2: “6 fit in the small box. I marked it there, and the space is equivalent to the box. So, the big box is equivalent to two small boxes. So, the total is obtained by adding two sixes.”

10:15 a.m. Teacher (writing on the board): “You all say that twice 6 is 12.” (See Fig. 5.12.)

$$6+6\rightarrow 12$$

Teacher: “So, 2 groups of 6, 4 groups of 3, 2 groups of 6. Is there another way to express the total?”

Student: “4 + 4 + 4.”

Teacher: “4 and 4 is 8, and 8 and 4 is 12.”

Student: “We can divide 3 times 4 in another way. 4 + 4 + 4 is a new way.”

Teacher: “Are there other ways?” (Teacher then decided to end the lesson.) “I had planned to have you try with stickers, but it’s time to end, so we will have to leave that for the next lesson on Monday. Now we’re going to say goodbye to the teachers visiting us.” (They looked at the visiting teachers.) “I’ll go with you all in a little bit. I’ll catch up.”

(A student asked if they would have another special activity the next day.)

Teacher: “No, you’ll have your normal classes. Tomorrow, there is music class. Don’t forget to bring your pencils, textbooks, and PE uniforms.”

10:19 a.m. (The students left.)

#### 5.1.1.3 Post–Open Class Discussion

Once the students had left the theater in a line, the teacher spoke to the audience to give justifications for his actions according to the goal proposed for the lesson.

Mr. Natsusaka (Teacher): “Thank you; I would like to receive your comments. We just witnessed a second-grade lesson of introducing the meaning of multiplication, in which the students made conjectures about how many pastries fit in a box.

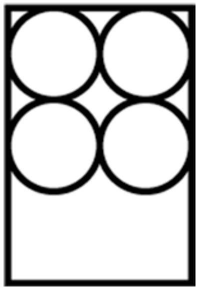


**Fig. 5.13** Estimation of how many balls in the box

My intention was that the students would present at least the number per column. More than half already knew the word ‘multiplication’ although I don’t know if they understand it. But my intention was that the students would learn the meaning of multiplication, so although I heard the word ‘multiplication’ many times or expressions like ‘2 times 3’ I didn’t repeat them because I wanted them to understand. So, I avoided introducing the expression ‘multiplication’ on purpose. I tried to use terms they all know.”

10:31 a.m. Mr. Natsusaka: “In the lesson, the first student said ‘4 + 2 is 6,’ expressing the total as a sum (See Fig. 5.13.). The second student said, writing vertically, that a group of 3 and another group of 3 is 6.” “This introductory part lasted for 10 minutes. There were two expressions that came out of this: ‘2 groups of 3’ and ‘2, 3 times.’ Maybe the students didn’t realize this. But I wanted them to understand. A student said ‘2 + 3’ but this sum cannot be used, so multiplication appeared as something important and necessary. Expressing verbally ‘in 2 groups there are 3’ indicates that there is another way to see it. So, I changed the color from blue to red because it represented something different. I wanted them to learn a new arithmetic operation. So, here,” (pointing to a diagram made during the lesson) “there are 12 units. Then a child explained the situation thinking of figures of 4 objects, 3 groups of 4, separating it in different ways. My intention was for the students to discover different groupings. How many groups could there be? I wanted them to group the objects in different ways before using the term multiplication.”

Visiting teacher from Central America: “Why didn’t you use concrete materials?”

Mr. Natsusaka: “I decided not to use tokens or concrete materials as I had already shown this to the students. Also, we already did that in first grade. That is used in Japan, but this time they didn’t use tokens. Honestly, I was thinking of using a blank piece of paper and different colored stickers to stick on the boxes. In the fourth grade, we study area and dimensions.”

Teacher in the audience: “I am using this program and I see the usefulness of the program. But why didn’t they use the real conjectures in three dimensions and see the height, as maybe it could be shown in different ways? Comparing would be easier for the students with something more real.”

Mr. Natsusaka: “This time I showed 2 vertically and 3 horizontally. What do you suggest?”

Teacher in the audience: “A student showed 2 times 3 vertically. But using a real box would be more efficient. It would be possible to have various boxes and adapt to the students’ answers.”

Mr. Natsusaka: “Before class, I practiced with the software. Maybe more drawings could be included. I didn’t think of using a real box because it could be 6 or 12 that can fit. I didn’t think they would reach that point [three dimensions].”

Teacher in the audience: “I come from an island. You achieved the goal, but I’m lost. You always ask, ‘Why do you think so? Write the reason.’ This kind of behavior was visible and developed reasoning and imagination. But why did the students know how to answer?”

Mr. Natsusaka: “I follow the guidelines of the new program [the national curriculum standards] which places greater emphasis on verbal expression: expressive competency, comprehensive competency, and textual formulations. It is important to ask for the reason or ‘Why can it be written like that?’ Teachers tend to assume what the student expresses. It is important to let the student say why so that the teachers will know how much they have understood and where their limit of understanding is. It is important to know how far they have really understood. The students fail in verbal expression, so they do it with diagrams. The students want to communicate their ideas. There are cases when students have difficulty expressing their ideas verbally, so they just draw diagrams. I understood that one student could not say what he understood, so I asked him to express it in another way or with more words—that is, to paraphrase. It is also important to promote the competency for listening—that they know how to interpret what others are trying to tell them. I intend to listen well to be able to communicate. If I express the ideas ambiguously or unclearly, then more time and situations are needed to communicate. If the students understand, then they ask questions. I asked them to express their ideas in another way by drawing. When the students try it, you have to evaluate what they understand. In the beginning, we played with these shapes.” (He indicated the triangles and rectangles on the interactive screen.) “There is an open polygonal chain. I asked, ‘Why isn’t it a shape, or can it be considered a shape?’ I get the children to think, ‘Why?’ Back to the topic, it is interesting to develop the competency for interpretation. I observe the students’ faces to see if they are listening to me. Looking at their faces, I can see if they don’t understand. As my colleague Mr. Tanaka says, ‘I look at the back of the classroom, and I go to the middle of the room to see if they are listening and understanding.’ It is important to see the students’ faces.”

Teacher in the audience: “As you speak of the meaning of multiplication, you mentioned that in the fourth grade they will study area (dimensions). Regarding area, how would you introduce it? Because now you are drawing pastries (circles) . . .”

Mr. Natsusaka: “It’s a difficult question. Today, for example, I used pastries of the same color; maybe one row could be one color and the row underneath another color. But my intention, which I wanted to develop among the students, was that by looking at the same drawing, they could see various forms. Colors are useful for area; 1 rabbit, 2 ears; 3 rabbits, 6 ears. Such attribute models are in another discussion because it fixes the view to every rabbit. But my intention was that in this lesson, the students would learn to group in different ways by themselves. In the case of rabbits, it’s obvious that they have 2 ears; it cannot be changed.”

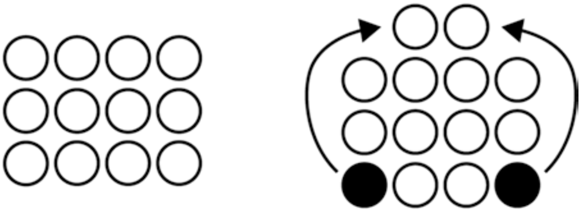


**Fig. 5.14** Establish entangler shape to recognize the situation for multiplication

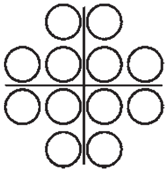


**Fig. 5.15** To find the various unit for multiplication

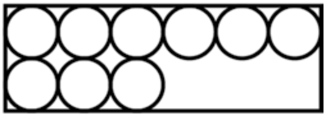


**Fig. 5.16** Change the box for changing the number of balls in it

11:05 a.m. Mr. Natsusaka: "How many circles are there? I thought of expressions like '4 + 4 + 4,' but there are students who thought, '3 in the corners, 4 times.'" (See Figs. 5.14 and 5.15.) "It is important to develop this competency for discovering different groupings. After learning the meaning of multiplication, it can also be applied to this drawing. I moved two circles to give shape to the image." (See Fig. 5.14.) "It's not that they already know, but, rather, that before starting with the multiplication table, they have already learned to group in different ways."

Visiting teacher: "I come from a distant province. You insisted on the competency to group in different ways. But when you said to a student that the big box (pointing to the rectangle drawn on the interactive screen) was the same as the 2 small boxes, he said that it wasn't the same. Maybe he said that 2 boxes of 6 isn't the same as 1 box of 12."

Mr. Natsusaka: "Maybe a student said 3 rectangular boxes in the big box." (See the right part of Fig. 5.16.) "There were students who saw the big box as 2 small boxes. But there were students who saw it as 3 boxes of 3 circles each. Later, in the next lesson, the students can continue with representation. There were a few who thought that there were 3 boxes."

Visiting teacher: "Your lesson gave me ideas for my lesson. Sometimes, comparing with my class, I intervene too much. But what were you trying to do? Also, the board was not used very well."

Mr. Natsusaka: "I asked the students to express their ideas verbally. They didn't do it. I tried to get them to formulate something before coming to the board, as some of them forget when they try to express it verbally. I wrote slowly so that the students could keep up. I also intervened when it was something important. When I posted the four problems, there wasn't enough space on the board for the fourth problem. It depends on the students. Some of them try to economize in their notebooks. The use of the board and the notebooks have a lot to do with each other. There are teachers who insist on writing down the class goals. I don't agree, because the goals are not static. The students' goals, hidden goals, or maybe apparent goals can appear. I think this is wrong. On the other hand, writing down the goals leads the weaker students to understand better. Depending on the nature of the goals, it may be best to write them down or not, as some will be explored and discovered. If I write down the goals and I want them to discover regularities, then the lesson is already over, because if I write it, then they already know that there is regularity; thus, there is no more exploration. There are topics for which the lesson goals cannot be written. They are understood during the development of the lesson. Maybe halfway through the lesson, the intention can be written from the students' perspective."

11:20 a.m. Teacher in the audience: "We were observing the first lesson for understanding multiplication. What will the next lesson be like?"

Mr. Natsusaka: "It continues with the topic on expressing multiplication. For example, '2 × 3 is 6,' and it will show multiplication directly, no longer using 3 + 3 + 3 but, rather, the multiplication expression."

Teacher in the audience: "One student said '1 unit 12 times.' How do you deal with this student?"

Mr. Natsusaka: "I would use the idea of 1 × . . . ; the multiplication table only goes up to 9, but maybe 12 groups of 1 can be expressed, or 1 unit 12 times, even though for now we only express up to 1 × 9. But it can be done."

Teacher in the audience: "For the students to use multiplication, do you think it is important that they see different shapes or groupings?"

Mr. Natsusaka: "The students grouped in different ways, as I have said. In the following lessons, we will use multiplication and the students will learn the multiplication table. To familiarize them with the multiplication table, I use the method of practicing with a written record of their progress in the multiplication table." (He shows notebooks made by the students that they use as a support for memorizing the table (see Fig. 5.17).) "The student learns the multiplication table for each


**Fig. 5.17** Students' homework notebook (journal) for multiplication to show the group as unit and a number of groups

number and is asked to check his progress. Then, the teacher or a family member (parent) signs after checking the memorization of the table. Then the student advances with the multiplication table of 2, of 3, etc. In the following lessons, I have the students write their ideas, then I have them do exercises. Following that, we look at some of the properties of multiplication. I ask them, 'If I add these two rows,' (referring to 2 and 3 in 3 × 2 and 3 × 3) 'is it the same as 3 × 5? How much is it? If there are 6 and 9, then there is 15.' That way, the students in the second grade discover that the results for the row of 5 are the sums of the results of the rows of 2 and 3. Now I cover the part of the multiplication table, and I ask them to say the sum. What I am using is the distributive law. That way, they think of, look for, and discover patterns in the multiplication table. I can cover four numbers at a time. Many things can be learned from the multiplication table, which is why it is good for them to know how to use it well."

11:27 a.m. (Dr. Isoda introduces himself as a professor at the University of Tsukuba.)

Dr. Isoda: "The students learn 'How many more?' but 'How many times?' is something different. Now, the students do not know how to multiply, but through grouping, it is possible that multiplication expressions present themselves. Multiplication, as an arithmetic operation, is important for students to learn how to express relations with a meaning different from that of addition. In multiplication, the first number represents something totally different from the second number. This was not mentioned in the lesson."

Mr. Natsusaka (thanking Dr. Isoda for his contribution and closing the comments and question time): "Thank you for your attention."

(While the audience is leaving the theater, a group of Central American teachers stays in the hall and asks Dr. Isoda some questions.)

Observing teacher from Central America: "How did the teacher carry out the evaluation of the lesson? The students have a tendency to count, and the teacher's intention is that they group."

Dr. Isoda: "Assessment for teaching and rating of students should be distinguished. As a confirmation, the teacher usually assesses students' learning within the lesson, such as observing whether or not the students raise their hands and if they understand. Based on such assessment, teachers make decisions on what is necessary activity and needs to share the ideas or ask students to imagine other's ideas, and so on."

Observing teacher: "Do they all have computers? What was the importance of the use of the interactive screen?"

Dr. Isoda: "In this lesson, only the program with interactive software was used. There is a tendency to use it. Today the software's advantages were not seen well. We can do the similar activity by using cards and so on. It is being experimented with now. It is a good interactive tool as well as other manipulative. There is a tendency to use the interactive screen, to learn Information, Communication and Technological (ICT) tool, not to do something new but, today, it was used rather, as a teaching tool." (See Fig. 5.18.)

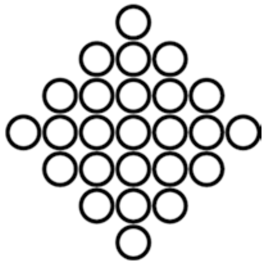
**Fig. 5.18** Dr. Raimundo Olfos (*left*), Dr. Masami Isoda (*middle*), and a translator (*right*)

### *5.1.2 Lesson Plan on Applying the Meaning of Multiplication After Learning the Multiplication Table, by Mr. Tsubota*

In the second grade in Japan, seeing situations in various ways with multiplication are usually learned both in the introduction to the meaning of multiplication and in the application after students have learned the multiplication table. The following lesson plan uses the meaning of multiplication and was developed by Mr. Kozo Tsubota (2007), Vice Principal of the Elementary School at the University of Tsukuba. Please note that the lesson study usually has a research theme. The proposed research theme in this case is "Representing Ideas Using Expressions and Interpreting Expressions" for solving problems using multiplication, which is related to finding the unit for multiplication. However, in this exemplar, students have already learned the multiplication table. This is a good task for students in the next grade. Thus, interpretation between an expression and a situation is the main study theme. It should be noted that in lesson study in Japan, the lesson study theme and the goal/objective of the class should be distinguished (see Chap. 1 and Isoda, 2015a).[13] The study theme is the

[13] Around the world, there have been a number of research on lesson study and some misconceptions about Japanese lesson study. They are related to the research of M. Yoshida (see Fernandez and Yoshida, 2004), who focused on school-based lesson study, which was a very unique activity in the world for professional development 20 years ago. In English, international researchers did not have the opportunity to understand the various meanings of Japanese lesson study (see Chap. 1). School-based lesson study at the elementary school level usually enhances a limited lesson study group as a learning community in the school; this is true. On the other hand, there are several types of lesson study communities in Japan as we mentioned Fig. 1.4 in Chap. 1. A good example is the subject-based lesson study that originated from the Elementary School at the University of Tsukuba in 1873 (see Isoda, Stephens, Ohara and Miyakawa, 2007). It is used for curriculum development too. In subject-based lesson study, the teacher usually focuses a lot on both personal research activity in the research society beyond his or her school and demonstration activity to show his or her practice at several schools as an invited consultant. Indeed, every teacher at the Elementary School at the University of Tsukuba has his or her community of lesson study in his or her subject.

**Fig. 5.19** Various unit for grouping and which is easier to get the number



theme proposed by the teacher who teaches the class and is written as a general issue. The objective of the class is defined for the specific content in the curriculum sequence.

1. *Study theme of lesson study*: Representing ideas using expressions and interpreting expressions.
2. *About the theme*: In this lesson, the students find the number of dots in a collection (arrangement) of dots (see Fig. 5.20), and find ways of counting the number of dots in the arrangement. Some students represent their ways of counting using expressions, and others interpret the meaning of each expression. Through these activities, the students can find unexpected interpretations for their own expressions, and other ways of counting can emerge. We want to use these experiences to encourage students to value learning from each other in studying mathematics. In particular, for each expression presented by a student regarding Fig. 5.19, another student interprets the meaning of the expression. This activity provides an extension of the students' ways of thinking about the expressions.
3. *Goal (objective of the class)*: To understand how to solve problems using multiplication.
4. *Duration of the lesson*: Special 1-hour lesson.
5. *Development of the lesson*:

    (a) Lesson goal: To find ways of counting the total number of dots in a square with 4 dots on each side, represent each way of counting as an expression, and to interpret the meaning of the expressions.
    (b) Development:

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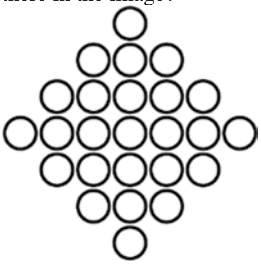
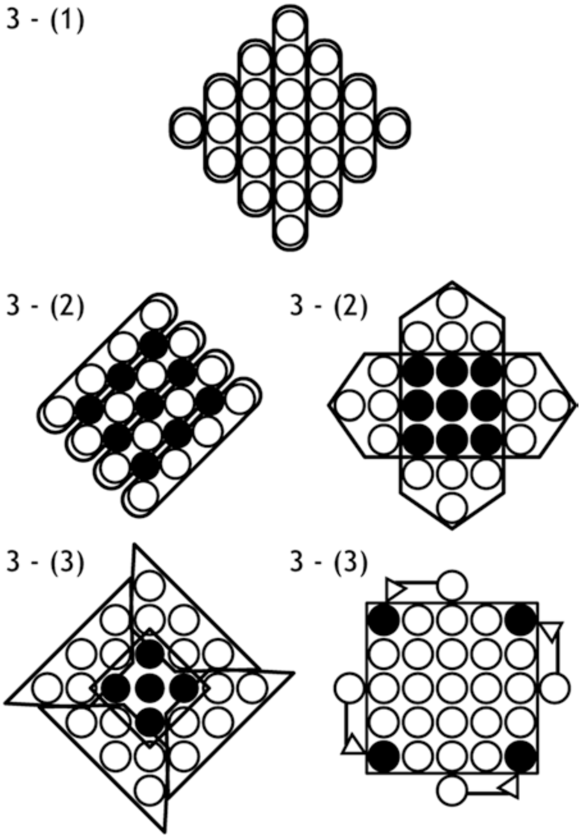
The mathematics study group at the school has its own *Journal of Elementary School Mathematics Education* in Japan. Every mathematics teacher at the school edits at least one issue of the journal in a year in collaboration with his/her study group. At its annual meeting, more than 1000 teachers participate in studying new research issues for lesson study. All of them are professional leading teachers in Japan. The quality of school-based lesson study is maintained by such subject-based lesson study. In this context, if a lesson plan does not have a study theme, it is just preparation for a class and is not for lesson study to show others in Japan. Having only an objective without a study theme, the lesson plan looks like just a preparation of teaching. However, Japanese lesson study theme usually related to develop students who learn mathematics by and for themselves (Isoda and Nakamura, 2010, Isoda 2015a, Isoda 2015b). Thus, the lesson study theme usually focuses on teaching mathematical values, attitudes, ideas, and ways of thinking. Under the same theme, every teacher can develop different exemplars to share what to teach (see Chap. 1).

| Content | Considerations |
|---|---|
| Look at the image below and think, "How many dots are there in the image?"<br><br>Confirm that there are 25 dots and think about how to represent the way of counting them, using an expression<br>The students represent their ways of counting using expressions, and other students interpret the meaning of each expression:<br>$1 + 3 + 5 + 7 + 5 + 3 + 1 = 25$<br>$(3 \times 3) + (4 \times 4) = 25$<br>$5 \times 5 = 25$<br>$6 \times 4 + 1 = 25$<br>$3 \times 8 + 1 = 25$<br>etc.<br>Confirm that there are various ways of counting | Show the image to the students briefly for them to construct a mental image of the arrangement of dots in the image<br>Each student should try to represent his or her own way of counting, using an expression<br>The students look at the expressions made by other students and think about the interpretation of these expressions<br><br><br>Confirm that there are various ways of counting by grouping and that for each expression there are various possible interpretations, as shown in the images |

This activity is developed in the textbook "Item 2" from Shogaku Publishers (2008), including possible student responses, as proposed by Mr. Tsubota during lesson study (see Fig. 5.20)

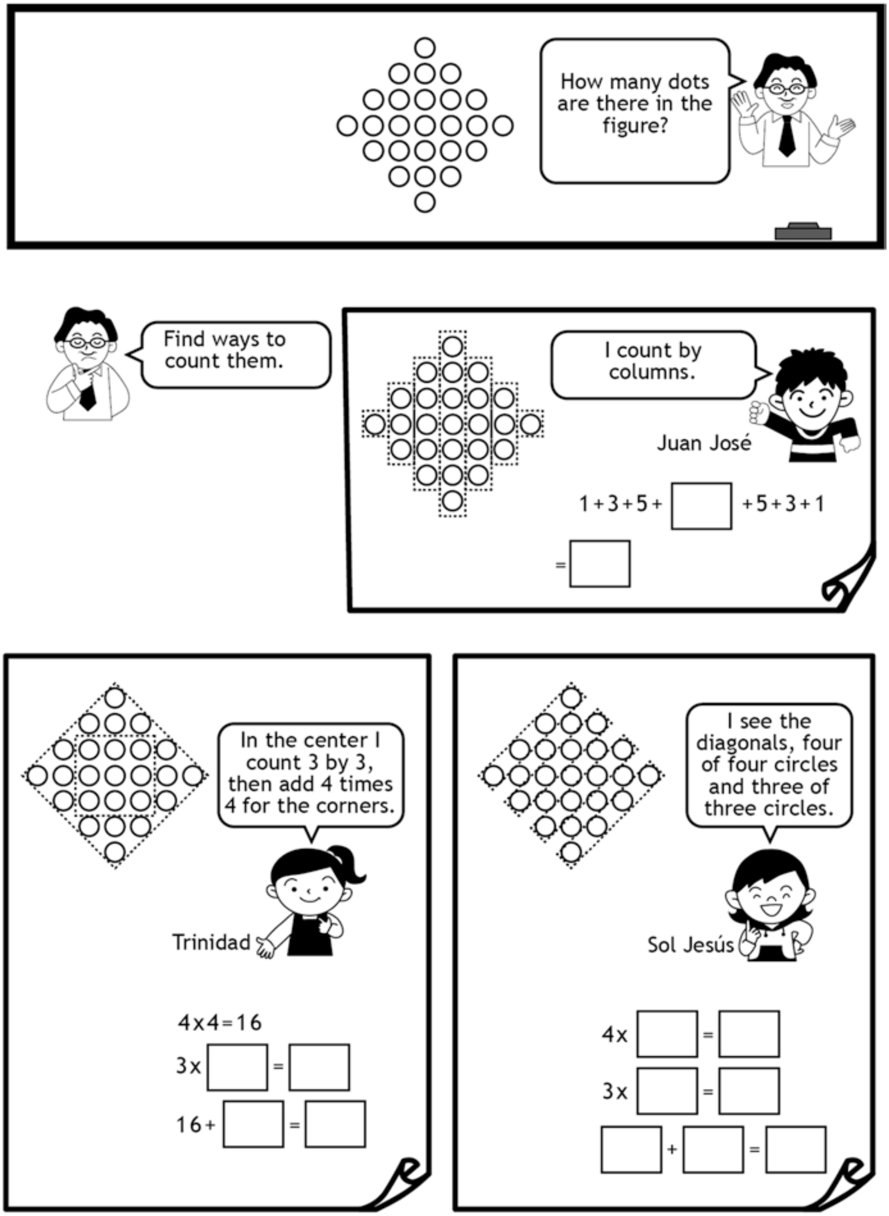



**Fig. 5.20** Possible student responses, as proposed by Mr. Tsubota during lesson study

## 5.2 Evidence to See Any Number as a Counting Unit

The 1989 curriculum standards (Isoda, 2005) reinforced the variety of the types of grouping and that any number can be seen as a unit and every unit is not limited to the base ten (decimal) place value system. The standards are reflected in the second grade at the beginning of the study of multiplication. In Japan, teachers use textbooks, approved by the Ministry of Education, that follow the standards. For example, the number of unit squares in Fig. 5.21 is 27 by counting, by adding 10 + 9 + 8, and by multiplying 9 × 3.[14] The Japanese standards ask teachers to develop students to choose the appropriate unit for counting, depending on what they have learned. This approach has been implemented since 1992 (based on the 1989 curriculum standards).

Isoda and Odajima (1992) researched the development of the cardinal number among students from the viewpoint of grouping strategies. They studied how students' competency for grouping is reorganized, depending on the content of their learning, by comparing the grouping strategies offered by first-, second-, and third-grade students in a survey (see Fig. 5.22).

The results, expressed in percentages, are shown in Table 5.1.

As shown in Fig. 5.23 and Table 5.1, in the first grade, some students use counting or grouping to add. In the second grade, before studying the multiplication table, coins are used in forming groups to add. Some students can use grouping to multiply after their introduction to the meaning of multiplication in the classroom. In the third grade, after all the students have studied the multiplication table and the algorithm with the column method in vertical form, more than half of them use the idea of grouping to add or multiply. This task is not like the one shown in Fig. 5.22. At the time of this survey of student development, the teachers were not yet implementing the new curriculum. Even though it is not easy to find the unit to

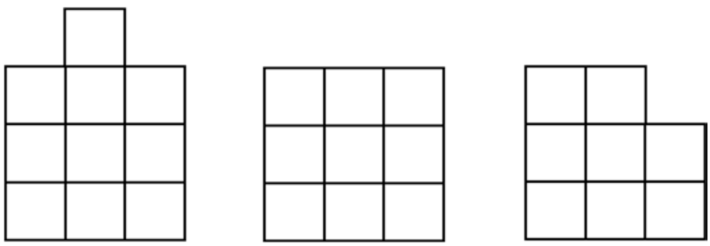


**Fig. 5.21** How many unit squares are there?

[14] Figure 5.21 is used by Prof/Dr. Satoshi Kodo to explain the significance of mathematical ideas in how students change their view to see objects through the learning of mathematics (Personal communication in 1984, see Chap. 1, Mathematical Thinking).

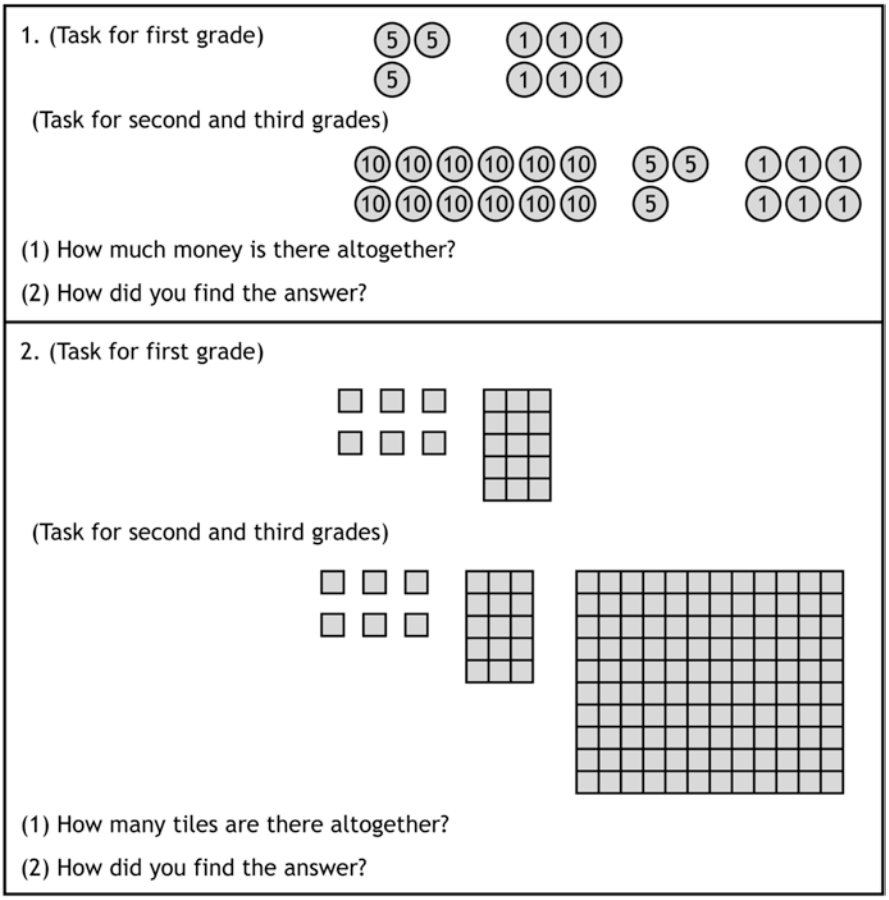



**Fig. 5.22** Grouping strategies offered by first-, second-, and third-grade students in a survey

**Table 5.1** Difference of the ways of counting by setting the various units for counting

| | 197 first-grade students (%) | | 214 second-grade students (%) | | 167 third-grade students (%) | |
|---|---|---|---|---|---|---|
| Method | Coins | Tiles | Coins | Tiles | Coins | Tiles |
| Count one by one | 36 | 47 | 36 | 23 | 10 | 12 |
| Count by 2s or 5s | 3 | 8.6 | 4 | 4 | 1 | 1 |
| Count by 10s | 3 | 5.4 | 10 | 22 | 8 | 8 |
| Simple addition | 10 | 2 | 3 | 4 | 1 | 0 |
| Group to add | 48 | 37 | 38 | 31 | 48 | 43 |
| Group to multiply | 0 | 0 | 9 | 16 | 32 | 36 |

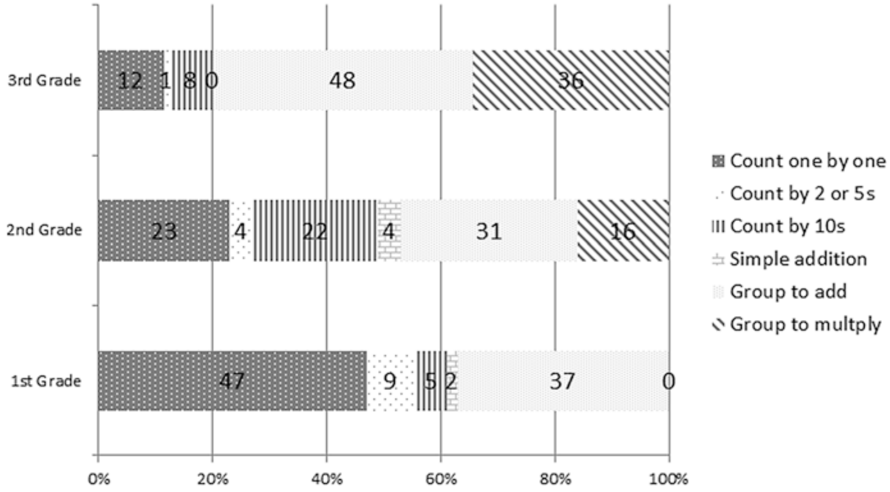



**Fig. 5.23** Students' competency for grouping tiles. (Note: The first data row in Table 5.1 is shown at the bottom row of the graph)

multiply in the tasks and the students are not asked to think about it, they applied multiplication.[15]

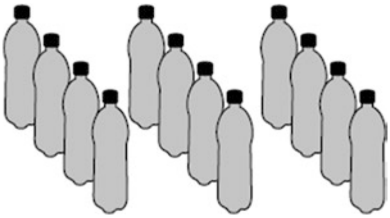
This result shows that learning multiplication develops the idea of grouping. Base ten units such as ones, tens, and hundreds are not the only units used for counting. In the Japanese approach, the students should learn how to choose the appropriate unit, set, or group for counting by using multiplication. In relation to vertical form, students should also fulfill the necessity to reorganize multiplication under the base ten place value system, which will be discussed in Chap. 7. The process of extension and integration is explained in Chap. 1.

[15]This survey included data collected at another school. That school did not follow the national curriculum but used the methods and textbooks proposed by the Toyama group (see Kobayashi, 1986), as explained at Chap. 1. The data showed that the students taught under the Toyama group (AMI) approach did not change their view as in Table 5.1. A limited number of schools preferred the AMI approach at that time. The data (which were taken from two classes) were insufficient to compare the difference with other schools that did follow the national curriculum. However, the data that showed no change represent the critical point for the discussion in the discussion of attribute on the next Sect. 5.3. In that section, the Japanese and Chilean approaches are compared. The Toyama approach is similar to the Chilean approach.

## 5.3 Comparison of the Japanese and Chilean Approaches

This section illustrates the feature of the Japanese introduction of meaning, which was explained in Sect. 5.1, in comparison with Chile. In Chile, multiplication is illustrated by repeated addition with seven sample activities (MINEDUC, 2017, pp. 151–154). If we prefer the activities closest to the Japanese approach, the following sample activities can be quoted:

In activity 1, the students are asked to transform sums in expressions with the word “times” (*veces* in Spanish), asking the following questions in these situations:

| | |
|---|---|
|  | (i) How many times do you repeat the 2 in the case of the number of ears of the 5 children in the image shown on the left? |
|  | (ii) Answer the following questions that relate to the groups of 4 bottles in the image shown on the left: How many times is the row of 4 bottles repeated? How many times is 4 bottles repeated? |

In these examples, it is clear that there are no discussions to set the unit of measurement by students because every pair of ears is fixed to the faces of the students, and counting the number of students corresponds to counting by 2s. Having two ears is an attribute of humans.[16] Thus, instead of counting each ear, we count the number of students. In (ii), the number of 4-bottle sets is asked. Then, the children have to see the set likely to be an attribute of humans. Here, (i) will be a metaphor for (ii) to see the set as an attribute. Thus, the metaphor of the attribute can be seen as a model for the binary operation to introduce multiplication, which is discussed in Chap. 3.

In activity 2, the students are asked:

(i) To draw a situation explaining what they understand about it and answering the question: “I have 5 cats and each one has 4 legs. How many legs are there in total?”
(ii) To complete the following story, drawing what they are told: “5 friends go to a store and each one buys 2 figurines . . .”

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[16] This Chilean approach using attributes is the same idea as Toyama’s approach (see Kobayashi, 1986, Sect. 5.2, Chaps. 1 and 3). It tries to express the meaning of multiplication by using a specific model for every row.

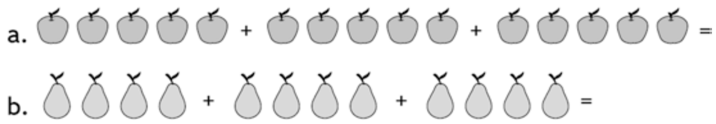


**Fig. 5.24** Repeated addition or counting by each: MINEDUC, 2017, pp. 151–154

Four legs are an attribute of a cat. This can be generalized to a situation like one person for two figurines, using the example of the cat as the metaphor for the attribute. It is also enhanced to see the situation as the base for the binary operation.

The third example follows the reverse scheme postulated by the definition.

In activity 3, the students are asked to express the quantities in Fig. 5.24 as a repeated sum and then as multiplication in the form of "times" and then give the answer.

In a of Fig. 2.54, for example, the students are expected to calculate the quantity by using the expression "3 times 5 [for '3 · 5' because a dot ('·') is used for '×' in Chile] is 5 + 5 + 5 = 15."

Here, multiplication can be seen as repeated addition. However, the answer can also be obtained by counting by each in the diagram and not necessarily by adding. In relation to activities 1 and 2, the task sequence implies that multiplication is introduced by the metaphor of the attribute of the object and reorganized as repeated addition.

From the discussion in Chap. 3, we can explain the reason why the Chilean program enhances the sequence from activity 1 to activity 3. If we represent the denomination of quantity clearly, "3 × 5 = 15" means "3 (dishes) × 5 (apples/dish) = 15 (apples)." However, students cannot directly understand the meaning of "apples/dish" as ratio. Thus, the Chilean program introduces the part of "apples/dish" using the metaphor of the attribute such as 5 apples for each dish. The attribute of the dish is 5 apples. For using the attribute as a metaphor for the binary operation, multiplication has to be introduced, such as the ears of humans and the legs of cats. It can be seen as an effort to make sense of multiplication as a binary operation and as repeated addition.

However, as long as we use the attribute of animals, we encounter the difficulty of asking students to overgeneralize the attribute of the original model because we do not discuss a person with six fingers or with two heads. If it is an attribute, students cannot change the unit of measurement. In addition, even though we introduce the unit of measurement by attributes, it cannot be connected well to repeated addition because it should be understandable if we write it as follows:

$$\underbrace{5\ (\text{apples}) + 5\ (\text{apples}) + 5\ (\text{apples})}_{3\ (\text{dishes})} = 15\ (\text{apples})$$

As explained in Chap. 3, it is not the same as “3 (dishes) × 5 (apples/dish) = 15 (apples).”

Through the comparison of the Chilean and Japanese approaches, we can recognize well why the Japanese approach enhances the setting of various units of measurement by students and asks them to count the number of units for setting the multiplication expression under the definition of multiplication by measurement (see Chap. 3). The most necessary activity for the introduction of multiplication is to see the situation by various units of measurement and find or set the number of units. As well as the usage of times (*bai* in Japanese) directed for proportionality represented by the proportional number lines (see Chap. 4), is a key idea of mathematical thinking to see the situation with mathematical ideas—in this case, ideas of a set (a group) and multiplication (see the idea of set and unit in Chap. 1, Table 1.1). For setting the unit of measurement, we can move the object (as in Figs. 5.3, 5.14, and 5.16, and in Fig. 4.3 in Chap. 4). In the case of Chile, the attribute of a given object is used to let students see the number as a unit.

## 5.4 Final Remarks

On the comperison of Chilean and Japanese Approaches, the Chilean approach analyzed to make sense of multiplication in the situation from the teachers’ side, and the Japanese approach analyzed to develop the sense-making activity of students who are able to set the measurement unit and to try to make clear the number of units by and for themselves, as well as making sense.[17] This chapter has illustrated this feature with two lesson study exemplars, a survey of student development from the first grade to the third grade before and after introduction of multiplication in Japan, and a comparison with Chile. Even we conclude Chilean Approach using attribute is an approach for making sense rather than sense making like Japanese approach, we should note the differences were originated from the behind school system and teaching culture. For example, Ministry of Education Chile distribute the different companies’ textbooks to the different grades as for the national textbooks. For example, first grade textbooks are published from the company A and second grades’ textbooks are published from company B. On this setting of Chile, it is difficult to teach based on what students already learned and preparing future learning. Indeed, if the textbooks are different depending on the grades, students’ sense making beyond grades is difficult because the ways for make sense are not the

[17] McCallum (2018) explained the sense-making stance as “the process perspective: mathematics as pattern seeking, mathematics as problem solving, big ideas have in common what I call the sense-making stance” (p. 2). He also mentioned, “Where the sense-making stance sees a process of people making sense of mathematics (or not), the making-sense stance sees mathematics making sense to people (or not). These are not mutually exclusive stances; rather they are dual stances jointly observing the same thing. The making-sense stance is related to the content perspective described by Schoenfeld, without the unappetizing ‘carving content into bite-sized pieces.’ It views content as something to be actively structured in such a way that it makes sense” (pp. 2–3). Both perspectives are necessary for curriculum development.

same amongst several textbook companies. If Chile try to shift to the sense making stance from making sense stance, it have to change the textbook free distribution system itself.[18] On this setting, Chilean make sense approach for multiplication can be seen as a best consideration on the current Chilean setting. In the countries such as England and USA, textbooks are not referenced as the minimum essentials but functioning as the one of the sources for the worksheets which teachers prefer every day. Such countries might be much more difficult to establish consistent sense making teaching sequence beyond the grades like Japan as we discussed at Chap. 4.

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[18] On 2019, the ministry of education chile decided to adapt Gakkotosho textbooks (Isoda and Murata, 2011) for them and Chilean using the Chilean adapted edition (Isoda et al., 2020; Isoda and Estrella, 2020).

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