Chapter 4
Introduction of Multiplication and Its Extension: How Does Japanese Introduce and Extend?

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In Chap. 1, the Japanese theories related to lesson study which oriented the development of students who learn mathematics by themselves through the development of mathematical thinking were summarized by the aims and objectives under the national curriculum standards, the terminology to distinguish content, the task sequence to develop students, and the teaching approach. In Chap. 2, the questions to make clear the Japanese Approaches were posed through the comparison to other countries. In Chap. 3, the difficulties to learn multiplication from using national languages towards mathematical form was described. In this chapter, Japanese curriculum sequence will be over-viewed from the perspective to make clear the extension and integration process shown on Fig. 1.1 of Chap. 1 by using the terminology and task sequence related to multiplication. It also describes related content such as the unit, division, decimals, fractions, and proportionality, and how each content is embedded for the preparation of future learning for sense making. The necessity to distinguish multiplier and multiplicand will be explained to sequence of these contents. The significance for the definition of multiplication by measurement in Chap. 3 will be also confirmed in relation to proportional number line.

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4.1 The Introduction of Multiplication Using the Japanese Approach

As discussed at Chap. 2, the process of teaching multiplications are usually fixed depending on the national curriculum standards in every county. For example, Hulbert, E. T. et al (2017) brought their research for teaching multiplication and division to the classrooms in USA under the Common Core State Standards Mathematics (2010) and show their teaching process clearly. Here we would like to illustrate how Japanese sequence of teaching multiplication are consistent with the related content under the principle of Extention and Integration (see Chap. 1).

Under Japanese grammar, the definition of multiplication in the second grade consists of associating the mathematical sentence A [×] B,[1] "A, B times" with the answer C (A [×] B = C), which corresponds to the total number of elements, with A, the number of elements in each group, and B, the number of the same groups under the definition of multiplication by measurement in Chap. 3. The mathematical expression "3 [×] 2" codifies the operative procedure "three, two times", in English or Spanish. The Japanese syntax of multiplication gives independent meanings for A (*kakerareru-su*; "multiplicand" in Japanese) and B (*kakeru-su*; "multiplier" in Japanese), which, as they are associated with situations in context, make it necessary to consider that A [×] B and B [×] A refer to different situations, even though they give the same numerical result, C.[2]

On the other hand, in the case of English and Spanish notations, to avoid inconsistencies (which were discussed in Chap. 3), commutativity is enhanced from the beginning. Then, students do not care about the independent meanings of multiplier and multiplicand, thus $A \times B$ and $B \times A$ will be seen as the same from the introduction of multiplication. Some countries such as Brazil just call them factors (see Chap. 2). However, if we do or do not distinguish A (multiplier) and B (multiplicand) in multiplication, how will this influence other teaching content?

Here, we explain why the Japanese curriculum has a consistent teaching sequence, and then we explain the hidden inconsistencies seen in other countries. For explanation, we go back to the definition mentioned in the Japanese mathematics curriculum guide (Isoda, 2005, 2010; Isoda and Chino, 2006), which points out that multiplication is used to find the total based on "how many units there are when a unit is given." This was explained as definition by measurement in Chap. 3. For the second grade, the guide proposes the use of groups as a unit. Here a unit means an arbitrary measure wherein any number can be a unit in Descartes's definition (see Chap. 3). In the 1989 teaching guide, translated into English (Isoda, 2005), it is interpreted that:

- The study of multiplication begins as an efficient means to express a unit repeated several times. The unit can be the cardinality of a set or a group. So, if a group of

[1] In Japanese grammar, in the official placement of multiplication, the unit is on the left. In this chapter, we write Japanese multiplication using "[×]" instead of "×" to highlight this. In A [×] B, A is the multiplicand and B is the multiplier.

[2] In the ancient Mesopotamian language, Sumerian, the order of words is the same as that in Japanese (Muroi, 2017); it is represented as *A a-rá B túm*. Here, *túm* means "carry" (see Chap. 3).

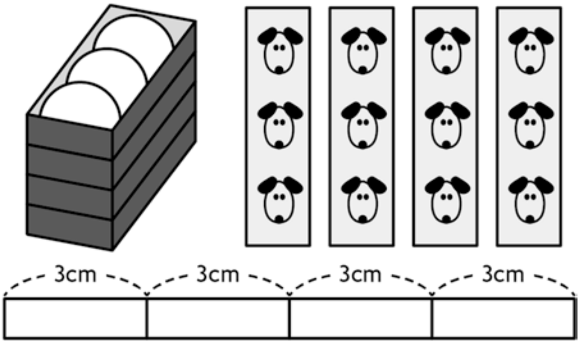


Fig. 4.1 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, p. 18

3 elements is repeated 4 times, there are 3 + 3 + 3 + 3 elements, which is abbreviated as 3 [×] 4.

- The definition of multiplication arises from the assignation of the name of every quantity as an object of measurement; that is, the definition is set as the way of measurement in tape diagrams, which can be adopted into proportionality in the tape diagrams later.
- The meaning of multiplication is addressed gradually in the extension from restricted situations for repeated addition and times[3] up to decimals and fractions. The introduction of every row of the multiplication table begins with situations for quantities and extends multiplication up to 10 times. Units that are larger than 10 are discussed in the next grade. Continuous units are discussed, particularly with the centimeter as a unit of measurement[4] (they already know measurement by 1 cm), for extending it to decimals and fractions (quantities of length produce continuous numbers) in later grades (Fig. 4.1).

The introduction of multiplication in the second grade in Japan (See Situation A and Meaning A in Fig. 1.1 of Chap. 1) is based on the operation to get the total quantity when the unit quantity and the number of units are known. It means, for natural numbers, a number in every group (unit) and counting the number of the same groups (units). It is the definition of multiplication by measurement and the whole number at this stage—that is, a set of groups or a group of groups (see Chaps. 2 and 3). Two different quantities using denominate numbers are necessary; for example, each plate has 3 apples and there are 4 plates (see Chap. 3). This is significant in two ways. One is to explain the situation briefly, which enables students to distinguish ordinary addition. Another is repeated addition in situations which is the starting point in the proceduralization from repeated addition to using the multipli-

[3] The Japanese usage of "times" (*bai*) is not only limited to the number of repetitions. The number of repetitions is usually represented by *kai* instead of *bai*; *bai* in Japanese is used up to multiplication of decimals and fractions, and for proportionality in the context of enlargement and reduction of the given number. The idea of *bai* is the key idea in development of proportionality. Its usage is rather close to "of."

[4] In Japanese textbook, the symbol "×" is read as *kakeru* or *Kake*. It is close to *por* in Spanish and "by" in English. The tape diagram is introduced later after the redefinitions of "×" as *bai* (times). Bai is defined using the tape diagram. It is used for extension of numbers to decimals and fractions.

cation table. As discussed in Chap. 3, repeated addition is not multiplication as a binary operation, even though it is the only way to find the answer at the start. Multiplication as a binary operation begins with the multiplication table.

At the introduction of the multiplication table on Procedure A of Fig. 1.1, repeated addition is necessary. On the extension of the multiplication table in every row, the pattern of products increases by the unit of the row. This becomes the principle to construct the multiplication table (see the discussion on permanence of form in Chap. 3 and Table 1.1). On the teaching of the multiplication table, the pattern of products is introduced and then all rows are combined to form the multiplication table. Knowing the properties of the multiplication table promotes the change in the meaning of multiplication from conceptual to procedural without concrete situations (see Chap. 1 and 3, especially Fig. 1.1). The properties of the table itself provide the procedural meaning of multiplication in the world of mathematics, which exists as patterns for sets of products without quantities. Chapter 7 of this book shows teaching of multidigit multiplication in the third grade in Japan. The proceduralized table with patterns also results in the conceptual meaning of multiplication (this emerges as a procedure in the second grade; see Chap. 1, Fig. 1.1), which is used for developing the procedure in vertical form (the column method). The multiplicative procedure of multidigit numbers can be created based on the conceptual and procedural knowledge of the multiplication table and the base ten system up to the second grade.

The dualities through conceptualization and proceduralization in the Japanese teaching sequence for the conceptual development of multiplication are not limited to multiplication but also apply to all teaching sequences for mathematics in Japan (Isoda, 1996; Isoda and Olfos, 2009, pp. 127–144). Those gradual conceptual development processes are well illustrated in the Gakkotosyo textbooks (Hitotsumatsu, 2005; Isoda and Murata, 2011; Isoda, Murata, and Yap, 2015).

4.1.1 The Way to Initiate the Situation for Multiplication Before Repeated Addition in the Japanese Approach[5]

The first task consists of challenging the students with multiplicative situations so they are able to distinguish them from additive situations (see Figs. 4.2 and 4.3). Figure 4.3 is discussed at the introduction of the symbol "×" (multiplied by) with the expression "multiplied by" (*kakeru* in Japanese) or just "by" (*kake* in Japanese). For future extensions to fractions and decimals, the Japanese textbooks prefer the multiplication symbol to be read as "multiplied by" or just "by" instead of times.

The introduction of multiplication is enhanced to form groups (sets) of an equal quantity (set) and to determine the total number based on the number of groups. It is a simple activity for an adult. However, to distinguish it as a binary operation from ordinal additive situations, these tasks are necessary for second-grade students. Thus, the second-grade textbooks include various situations for multiplica-

[5] This section explains the outline. In Chaps. 5 and 6, it will be explained more concretely.

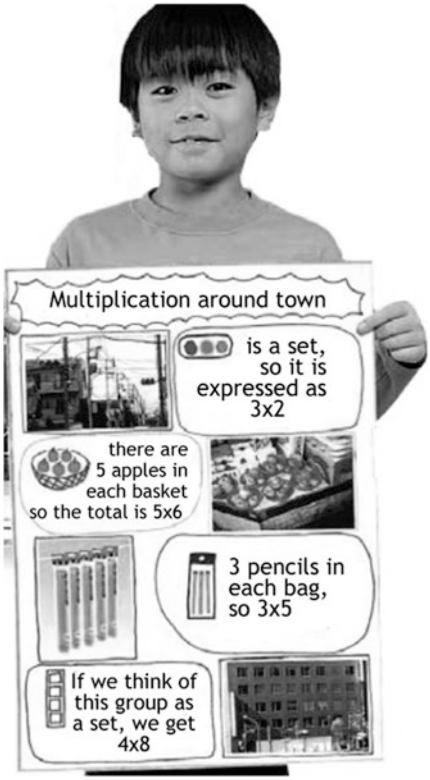


Fig. 4.2 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, p. 47

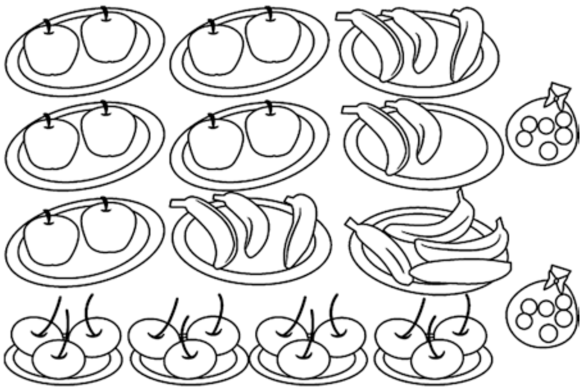


Fig. 4.3 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, pp. 2–3, "Look at the banana plates. Is it a multiplicative situation?"

tion based on the number of groups, where the group represents the unit. For example, in Fig. 4.2, if each stoplight has 3 lights, 2 stoplights have 6 lights. In Fig. 4.3, if we move one banana to another plate, we produce a situation where there are 3 bananas for every plate and 4 plates. Based on those tasks, teachers enable students to see the situation as a multiplicative situation by seeing the repeated

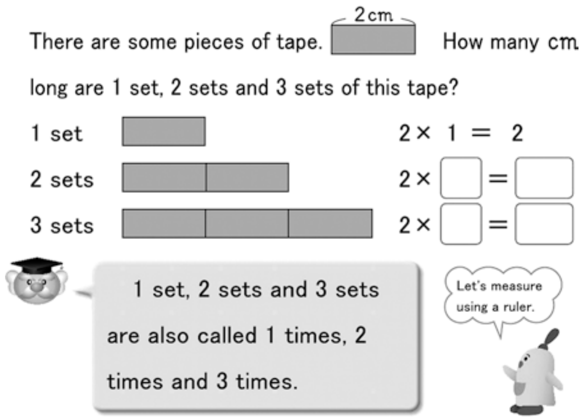


Fig. 4.4 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, p. 10., pp. 2–3

quantity as the unit, and the expression of multiplication is introduced. When the students are able to distinguish the repetition of these situations with other additive situations, we can say that they are able to see the world by multiplication, as in Fig. 4.2.

In the beginning, the students will find the product by repeated addition or counting; however, in the addition form, it is necessary to make clear the number of repetitions. For this necessity, a multiplication expression is introduced. For knowing this reasonable and effective way of representation, the Japanese textbooks show various situations to count the number of times (*kai* in Japanese), for the students to be aware of its reasonableness and the simplicity of its form.

As shown in Fig. 4.4, a unit length of tape such as 3 cm is introduced. The tape model (diagram) is necessary for later extension to continuous numbers in relation to proportionality. After the definition of multiplication with a situation as a binary operation based on definition by measurement (see Chap. 3), the tape diagram in Fig. 4.4 is introduced and the term *bai* looks the same as "times" (in English) and *veces* (in Spanish) at this stage. The Japanese usage of *bai* ("times") is not the same as the English and Spanish usage of the symbol "×"; in Japanese, *bai* is the terminology used to explain proportionality. Later, the term *bai* can be used for extension from whole numbers to decimals and fractions by using proportional number lines (see Sect. 4.3), and then it becomes the base to define proportions in relation to ratio.

For finding the answer or product for a binary operation, the term *bai* is also connected to repeated addition (*kai* in Japanese). As mentioned in Chap. 3, the repeated addition meaning of "times" in English and *veces* in Spanish is inconsistent with the multiplication table in relation to the order of expression.

After this, the multiplication table is introduced with the rows of 2 and 5, which have already been learned as ways of counting. Those rows are convenient for students because they already knew the answers from their experience of counting by 2s and 5s. Recognizing that the products can be increased by the units becomes the basis to extend multiplication in each row up to 9. After the exploration of the properties of the multiplication table, a project to find multiplicative situations (as in Fig. 4.2) is done for students to explain the significance of multiplication.

4.1.1.1 Repeated Addition and Challenges to Difficulty

When the students begin the study of multiplicative situation, it is normal for them to see the situations as a kind of addition because they have learned only addition and subtraction, and they are recommended to use what they have already learned in the Japanese approach. To distinguish multiplication from ordinary addition, they need to reorganize their knowledge from counting by one to counting by the unit number (such as 2, 3, . . . , 9) as sets in the manner shown in Fig. 4.3 (see also Chap. 5, Sect. 5.2). Up to the introduction of multiplication in the Japanese approach, the Japanese textbooks provide opportunities for students to learn that any magnitude can be a unit for counting. If the teachers have only discussed counting by one in the first grade, this becomes an obstacle for learning multiplication in the second grade. Thus, the first-grade Japanese textbooks enhance the activity of measurements to set the tentative unit for measuring as well as counting by 2s and 5s. The introduction of multiplication from verbalizing of the grouping such as in Fig. 4.3 looks like repeated addition to adults. However, this verbalizing activity enhances the ability of the students to explain the situation by a number in every group (unit) and to count the number of the same groups (units), by using different denominate numbers such as “3 apples for each plate and 4 plates.”

During the introduction of multiplication, if the teachers do not include the denominations of quantity (see Chap. 3), such as apple and plate, and just say “3 for each and 4,” the students lose the point of learning at the beginning even though it is routine for those students who have learned it well. If we compare it with “3 apples for each dish and 4 dishes,” they may understand what the object of counting is. The number of dishes should be clearly mentioned for showing the unit of counting. First, the Japanese textbooks ask the students to explain the situation of grouping to develop the notion of the unit (Fig. 4.3) and, later, shift to repeated addition. In this context, “How many apples are there? And how many dishes are there?” and “Which have the same number of fruits in the dishes?” are not the same because the first questions can be a question of counting and the second one is a question for explaining a multiplicative situation. To clarify such differences, the Japanese textbooks consider tasks that distinguish the situations by sets of groups.

Repeated addition in situations is necessary for developing the multiplication table. However, as was mentioned in Chap. 3, it is not so much reasonable to represent repetition of 1 and 0 in situations such as 1 + 1 + 1 and 0 + 0 + 0 + 0 by multiplication. If the counting unit is 1, it is not addition but just looks like counting. Japanese textbooks introduce the row of 1 by the permanence of form: the situation “2 apples for every dish and 3 dishes” is 2 [×] 3; if it is “1 apple for every dish and 3 dishes” how shall we express it? This is the question for the permanence of form. In the case of 0, the Japanese textbooks in the second grade discuss 10 times (*bai*) instead of multiplying 0.

4.1.1.2 Use of the Multiplicand and Multiplier for Students to Think of Division Situations by and for Themselves

When Japanese students begin multiplication with a situation of the type “2 multiplied by 3” in English, they learn that this is *3 ga 2 ko* (“3, 2 times”)—writing it as “3 [×] 2”—and they learn to read it as *3 kakeru 2 ha 6*. This Japanese notation may be misread by English readers, however; as mentioned in Chap. 3, the notations of multiplication in English and Spanish include contradictions.[6]

The multiplicand as the first element in multiplication and the multiplier as the second element in Japanese are introduced in the multiplication table and used in the extension of multiplication to fractions, decimals, and negative numbers (see the discussion of Descartes in Chap. 3).

In the Japanese curriculum sequence, the “multiplicand [×] multiplier” in multiplication is directly connected to division. Students who are able to define division by themselves are expected to use the idea of multiplication in situations involving division. In this context, teachers have to enable students to distinguish the first and second numbers by identifying the multiplicand and multiplier. The two different situations in division, which are called partitive and quotative division, can be distinguished by this identification.

Students are able to think by and for themselves, and are able to reorganize what they already know by using their daily language as well as their learned mathematical language. In this curriculum sequence, Japanese primary school teachers in the lower grades ask students to codify the situation for the expression “2 [×] 4” and distinguish to represent it as “4 [×] 2.” Japanese teachers try to develop students to develop mathematical sense to make sense by and for themselves based on what they have learned and to elaborate the definitions in their classes (see Chap. 5). They are asked to analyze the situations and formulate them by using their everyday language (see Fig. 4.5).

4.1.1.3 Commutativity and Order in Expression

In Japan, it is expected that all students will memorize the multiplication table in the second grade. For developing the table, the property “increase by 3 if the row is 3” is used.[7] For memorizing the multiplication table, the teachers shorten an expression such as *3 kakeru 2 ha 6* to *3 kake 2 ha 6* and to *3 2 ga 6*[8] by characteristic abbrevia-

[6] In Japan, only primary school teachers recognize the difference between 3 × 2 and 2 × 3, explaining the meaning of multiplication in each situation. In secondary school, teachers never distinguish these two because they do not feel any necessity to do so in their teaching. Primary teachers have to consider it on their curriculum sequence.

[7] Here, we call this a “procedure with meaning” (Isoda and Olfos, 2009). Students memorize the table using properties (patterns), meaningfully.

[8] The term *ga* is only used in the event that the product is less than 10. If it is more than 10, even *ga* is omitted, such as 3 [×] 4 = 12 (“3 4, 12”).

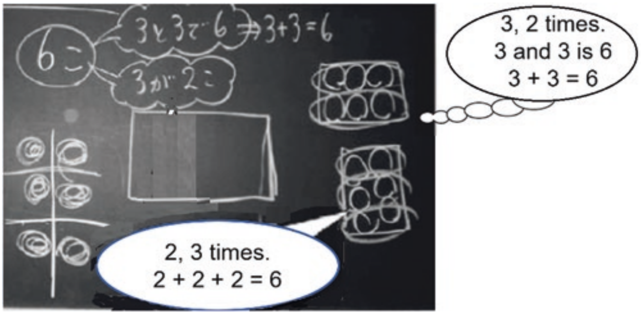


Fig. 4.5 How we read and express from Mr. Satoshi Natsusaka's lesson

tions in the Japanese language which in English mean "three one, three" (3 [×] 1 = 3) and "three two, six" (3 [×] 2 = 6)."

Once the students have completed and memorized the multiplication table up to 9 × 9,[9] they lose the need to use repeated addition to get the product within the range of the multiplication table. They get the product of the binary operation automatically, without referring to situations and repeated addition. In the task to find the properties of the multiplication table, without considering situations in multiplication, the students can find many of them. The numbers in the table have a symmetrical property on the diagonal. For example, 2 × 3 and 3 × 2 are the same value since the answer does not change even though the order of the multiplicand and multiplier is changed. This discussion is on multiplication expression. On the other hand, in a concrete problem-solving situation, teachers and students continue to distinguish which one is the multiplicand and which is the multiplier in situations such as partitive division and quotative division in the third grade (Isoda, 2010). It is useful up to ratios and rates for considering which one is the base unit quantity in the situation.

4.1.1.4 Differences in the Multiplier and Multiplicand in an Array and a Block Diagram

In Japanese classes, the teachers usually ask how to read the array or block diagram like those in Figs. 4.5 and 4.6 (see also Chap. 3). This is an opportunity to identify and distinguish the multiplicand and multiplier.

Figure 4.7 is a representation of 4 plates (the unit or group). Each plate (unit or group) has 2 sweets—that is, "four times two" in English—and this is codified using the mathematical expression "4 × 2." In Japanese, it is represented as "2 [×] 4" ("2, 4 times"). In Chap. 3, the Japanese notation is consistent with the property

[9] The Japanese numeral system follows the base ten numeral system. The base ten numeral system can be well recognized from twenty in the case of English and from hundred in the case of Spanish.

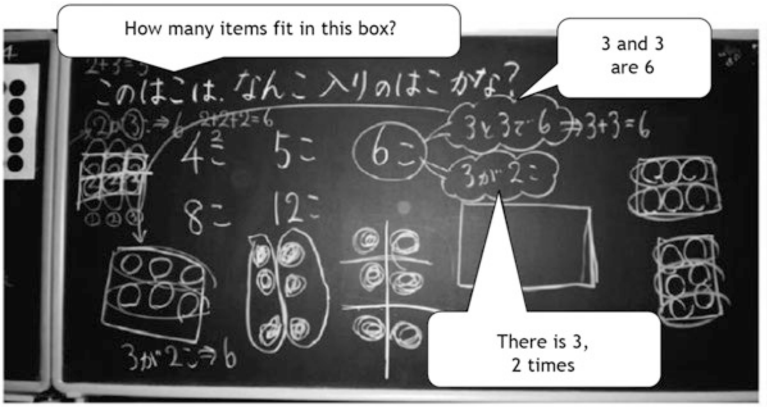


Fig. 4.6 How we read and express from Mr. Satoshi Natsusaka's lesson (see Chap. 5)

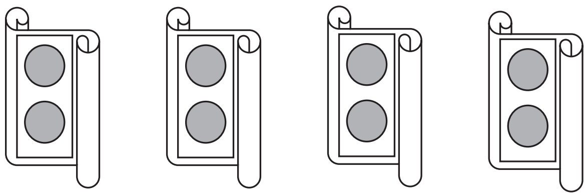


Fig. 4.7 4 × 2 in English and 2 × 4 in Japanese

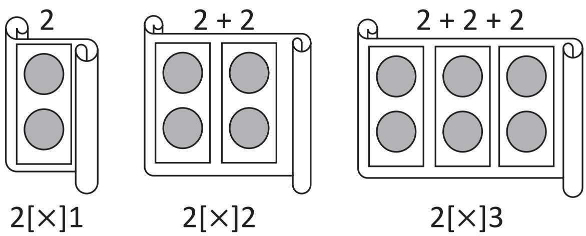


Fig. 4.8 Increase by 2 on row 2

of row 2 (Fig. 4.8). In this comparison, how to see the array is the point of the discussion, such as vertically and horizontally in Fig. 4.9.

Figure 4.9 is a model to illustrate commutativity—why it produces the same products and added the information for the order of multiplier and multiplicand in relation to Fig. 3.12.

4.1.1.5 Revisiting Which Notation Is Better and Why

"3 apples on each plate, and 2 plates" is 3 [×] 2 (3 apples, 2 times) in Japanese; the multiplier is on the right. In English, it is 2 × 3 (2 times 3 apples); the multiplier is on the left.

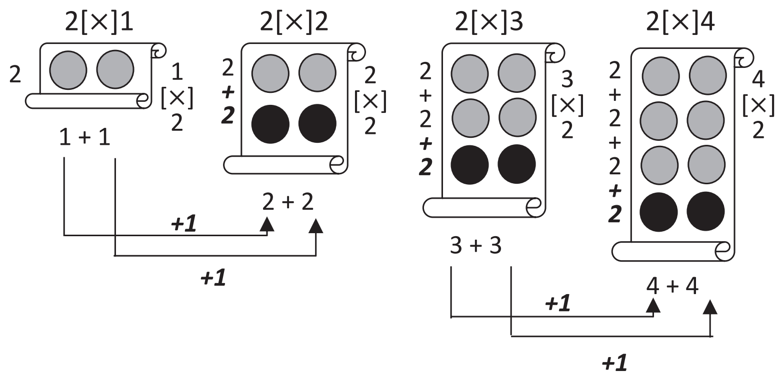


Fig. 4.9 How to read the array diagram horizontally and vertically

There is a grammatical–syntactical difference. As long as preferring the teaching language is fixed first, there is no choice. Thus, the discussion on which is better is unsolvable because it is a cultural matter; however, the question is necessary to design the curriculum sequence. Some usages of daily language such as "times," "dividing into equally," and "equally likely" are also learned in mathematics class at first. In this context, some Latin American countries already prefer the Japanese notation of multiplication for themselves.[10]

The Japanese form has the following significance (see Chap. 3):

- It facilitates the construction of the multiplication table: if the multiplier increases by 1, the product increases by the quantity of the multiplicand.
- The multiplication table is consistent with the multiplication algorithm.
- The meaning of multiplication consistently applies to the two meanings of division in situations.
- The meaning of multiplication and the multiplication table is consistent with the algebraic expression (constant) × (variable).

Additionally:

- It agrees with the traditional Spanish arithmetic book by Rey Pastor and Puig Adam (1935) on the use of the terms "multiplicand" as the first factor and "multiplier" as the second factor.
- It first presents the multiplicand, the unit, the size, or the quantity of the elements in each group, which the students consider in order to be able to decide if multiplication is appropriate in this situation.

The English form has the following significance:

- It agrees grammatically with the use of the term "times" in English and the terms used in most of European languages, such as *veces* in Spanish.

[10] Many Central and South American countries use Spanish as their national language; however, they use multilanguage in relation to their mother tongues.

As will be explained in the next section, the Japanese notation produces consistency in the curriculum sequence. In terms of this consistency, the Japanese teaching sequence in textbooks is considered more deeply as compared with that used in Chile.

4.2 Preparation for Multiplication in the Japanese Curriculum and Textbooks

One of the features of the Japanese course of study (the national curriculum standards) and authorized textbooks is that the sequence is well prepared for future learning for sense making (see also Chap. 5), which is explained by the extension and integration principle.[11] In terms of this principle, the following sections describe the teaching sequence beginning in the first grade, the extension of multiplication to new numerical domains through proportionality, and the extension to other content such as division, rates, and fractions, up to proportions, in the Japanese context.

4.2.1 Preparation for Introduction of Multiplication in the First Grade

The teaching sequence of Japanese textbooks is well prepared for future learning, which means that each part of the teaching content includes preparation of the necessary underlying ideas for use in the future, such as the idea of the "number of units," according to the principle of learning based on what the students have already learned. In the following sections, the four preparations for introducing multiplication in the first grade are explained.

4.2.1.1 Composition and Decomposition of Cardinal Numbers for Binary Operations

First, the textbooks from the publisher Composition and decomposition of numbers in Gakkotosyo (Isoda and Murata, 2011) intensify the development of the idea of a number as the cardinal of a set up to 10. They teach composition and decomposition

[11] As explained in Chap. 1, the extension and integration principle was used in the course of study in 1968 (Ministry of Education, 1968). The meaning is almost the same as the reorganization of experience which was defined by Freudenthal (1973) with his terminology of "mathematization" under his reinvention principle, although the term "mathematization" has been used officially in Japan since 1943 (Sugimura, Simada, Tanaka, and Wada, 1943). Preparation for future learning, conversely, is done using learned knowledge and skills from the perspective of students. However, the students do not know which of them should be used. Students have to know how to extend or to use the known. It is a source of problematics which should be solved in the lesson (See Fig. 1.1 in Chap. 1). It is also a source from which the students produce misconceptions by their own overgeneralization of their learned knowledge and skills. It is the task for a dialectical style of communication between appropriate and inappropriate use of what they have learned in the classroom (Isoda, 1996).

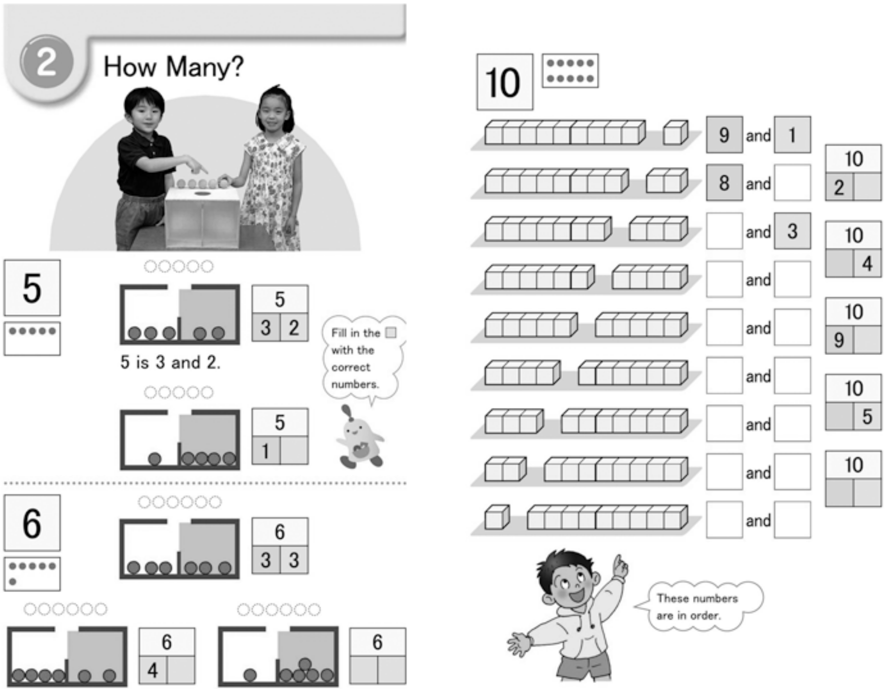


Fig. 4.10 Gakkotosyo (Hitotsumatsu, 2005), Grade 1, p. 26 and p. 29

of numbers before dealing with addition and subtraction (p. 26 in the first-grade textbook; see Fig. 4.10). This is done with the purpose of teaching the association of a number (cardinal) with a set and preparing for mental calculation for addition and multiplication.

If composition and decomposition of numbers are not taught before addition and subtraction, the students can only obtain the result of addition and multiplication by counting. If these are taught before addition and subtraction up to 10, the students can also obtain the answers as sets of objects and not necessarily by counting. Addition up to 10 is based on the composition of numbers, while subtraction up to 10 is based on the decomposition of numbers.

When the students encounter addition of more than 10 up to 20, they will be able to use manipulatives, using the idea of making 10, such as $8 + 3 = 8 + (2 + 1) = (8 + 2) + 1 = 10 + 1 = 11$. Here, $3 = 2 + 1$ is a decomposition of the number 3, and $8 + 2 = 10$ is a composition of the number 10. On this learning trajectory, addition is extended/reorganized from the composition of numbers to the combination of decomposition and composition of numbers for making 10 in relation to carrying.

Mathematically, addition and subtraction are both binary operations. When students study addition on this trajectory, they can see addition as a binary operation and the sum as the value obtained. Otherwise, they can only use counting in instances such as “3 + 2 is three, four, five.” It becomes 3 + 1 + 1 when we represent the process by using the plus sign. Getting the sum by counting is not a binary operation even though it is a strategy to get the sum. Here, counting is used as a method to justify the answer.

To learn addition as a binary operation, it is necessary that the two numbers refer to two sets. If we do not prefer this trajectory, counting will still remain as the method used to find the answer. The students may keep on counting as long as they can count. If students learn composite and de-composite of numbers.

4.2.1.2 Counting by Twos or by Fives as the Base for the New Unit to Count

Second, in the extension of numbers beyond 10, students are taught to count by 2s or by 5s as “ways of counting”. Here, the students become proficient in the number sequence for counting by 2s or by 5s. It becomes the basis for learning the multiplication table and, for this reason, Japanese textbooks address the multiplication table starting with the rows of 2 and 5 in the second grade. Base 10 system itself is the base for column multiplication by using distribution in the later grade (See Chap. 7 and Meaning of B in Fig. 1.1, Chap. 1).

4.2.1.3 Polynomial Notation

Third, multiplication is a binary operation and the answer is given by repeated addition at the introduction of multiplication (Fig. 4.11). To get the answer in multiplication, the students have to know polynomial notation first before they can interpret the meaning of polynomial notation.

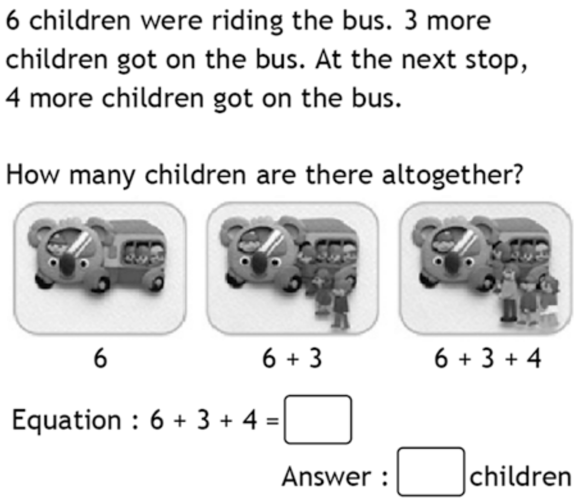


Fig. 4.11 Preparation of repeated addition, Direct comparison, In direct comparison, and Arbitrary Unit, Gakkotosyo (Hitotsumatsu, 2005), Grade 1, p. 97

4.2.1.4 Production of Tentative/Arbitrary Units

Fourth, in the introduction of "measurement" in the first grade (Fig. 4.12), the students learn "how to compare" and not the measurement quantity itself such as "cm." For "how to compare," students study direct comparison, indirect comparison, and arbitrary units. For indirect comparison, through the comparison of *A*, *B* and *C*, students make an order and visualize transitivity, clearly: if *B* is smaller than *A* and *B* is smaller than *C*, then *C* is smaller than *A*. For direct and indirect comparisons, the differences are usually discussed. These are necessary to produce arbitrary units. The differences can produce a unit for measuring (a Euclidean algorithm). In Japan, students learn how to produce arbitrary units in this way. The standard units for measurement quantities such as "cm" are introduced in later grades. Those activities are the bases to understand that any object can be seen as a unit (See Table 1.1 in Chap. 1: the idea of unit). And this processes are prepared for students who are able to learn how to produce the necessary unit. It can be seen as learning trajectory by Szilagyi, Clements, & Sarama (2013).

Those preceding four preparations are the bases for the introduction of multiplication in the first grade. In addition, there are other preparations. For example, multiplication is the base for proportionality. The number line is a key preparation for representing times and extending it to proportionality. In the first grade, it is implicitly introduced as a line of numbers by using repetitions of the unit tape (Fig. 4.13).

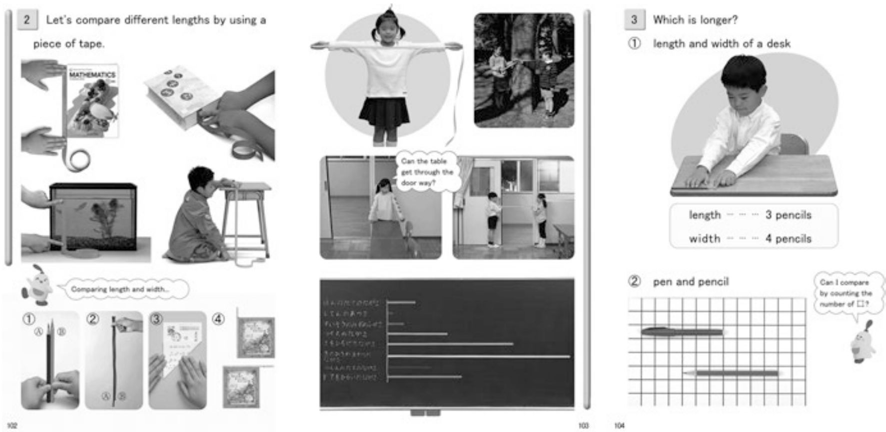


Fig. 4.12 Gakkotosyo (Hitotsumatsu, 2005), Grade 1, pp. 103–104

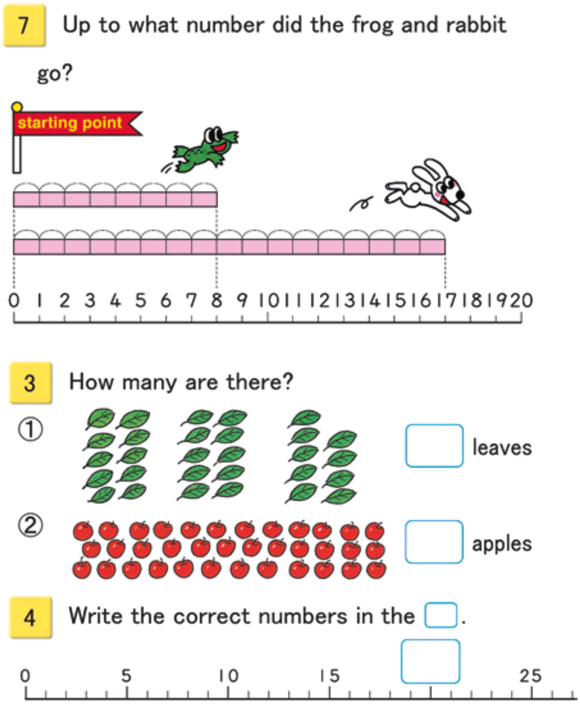


Fig. 4.13 Preparation of Number line, Gakkotosyo (Hitotsumatsu, 2005), Grade 1, p. 67, p. 70

4.3 Proportionality for Extension of Multiplication

Extension of numbers is part of the curriculum sequence in any country. In Japan, the key idea for multiplication in extension of numbers is proportionality (properties of proportion) using a tape diagram and a table with the rule of three before teaching the formal definition of proportions. Since the 1960s, proportionality has been embedded in the Japanese textbooks by the following sequence for extension of numbers.

In the introduction of multiplication in the second grade, times (*bai*) is introduced with a tape diagram (see Fig. 4.4) and repeated addition of the unit length tape, which corresponds to the constant difference in the multiplication table. The tape diagram is used for the extension of numbers to decimals and fractions as proportional number lines (see Meaning C of Fig. 1.1 in Chap. 1).

4.3.1 Introduction of Proportional Number Lines and Their Adaptation for Extension

In the third grade, the Gakkotosyo textbooks (Isoda and Murata, 2011; Isoda, Murata, and Yap, 2015) extend this tape diagram to two-dimensional lines, which the Japanese call “proportional number lines” (see Figs. 4.14 and 4.15). The proportional number line is a model that represent the definition of multiplication by measure-

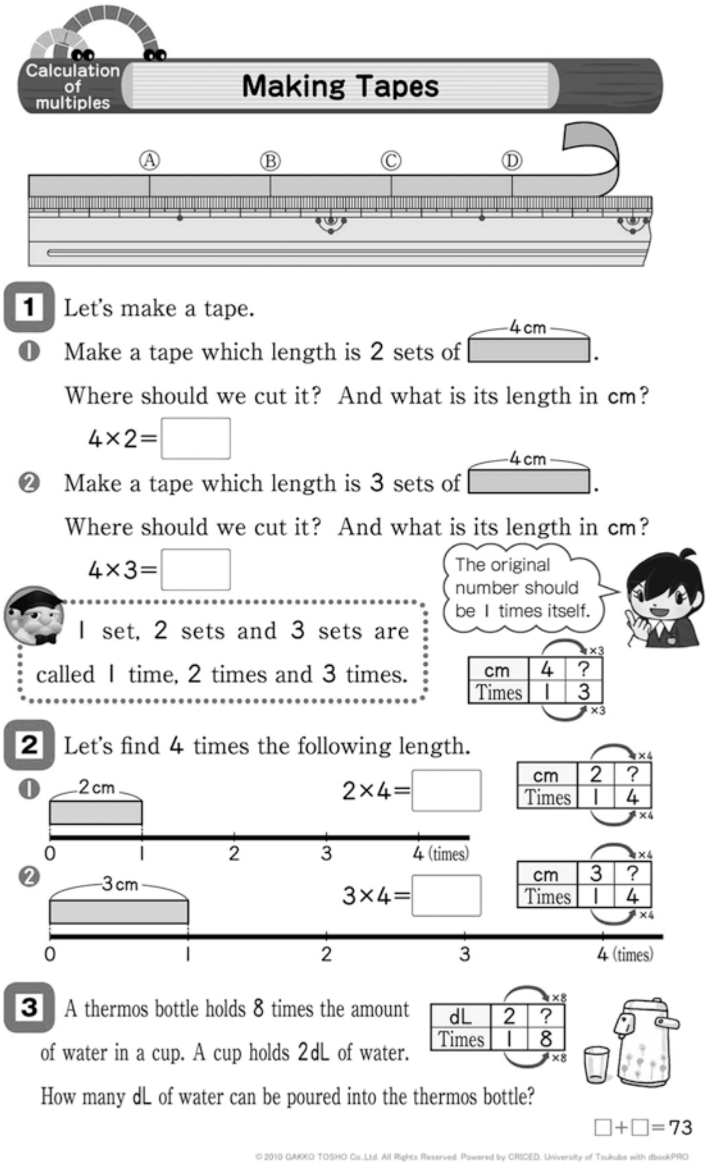


Fig. 4.14 Proportional number line is introduced by the number line with tape diagram in the case of Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 73

ment in all Japanese primary schools' mathematics textbooks (see Chap. 3). In particular, the Gakkotosyo textbooks enhance the rule of three by using arrows to show the pattern on the table (see the four-column tables in Figs. 4.14 and 4.15). Proportionality is also embedded in these tables.

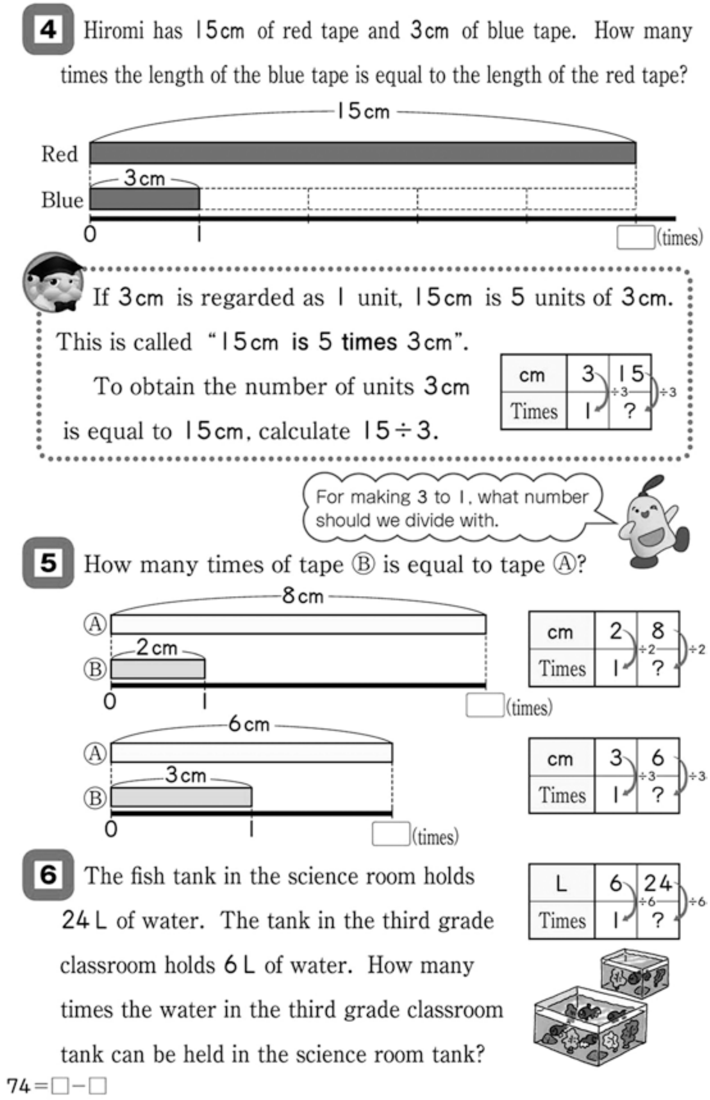


Fig. 4.15 Proportional number line is introduced by the number line with tape diagram in the case of Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 74

Those two types of representations—proportional number lines and the table for the rule of three—are not necessary to find the answer in situations involving multiplication at Grade 3. However, they are necessary to prepare for the extension of multiplication from whole numbers to decimals and fractions in upper grades (See Extension of B to C, Fig. 1.1, Chap. 1). Thus, the model representations in Figs. 4.14 and 4.15 are the preparations made in the third grade for future learning in later grades as for sense making.

4.3.2 Extension of Multiplication by Using Proportional Number Lines

The following are the first four pages of the fifth-grade textbook on the extension to decimals (Isoda and Murata, 2011):

In Figs. 4.16, 4.17, and 4.18, there are tape diagrams to show proportionality and, at the same time, the rule of three in the table in Figs. 4.16 and 4.18. There are also further strategies that can be seen for extensions using the properties of multiplication sentences at 10 times and 1/10 in Fig. 4.18 and area diagrams in Fig. 4.19. When students discuss the mutual relationship of their ideas in Fig. 4.18, it is an opportunity for them to develop the idea of proportionality.[12]

In this manner, Japanese textbooks prepare for future learning by consistently developing and using the same representations. In these preparations, students are able to challenge further learning such as proportion (see Sects. 4.3.4 and 4.4) by and for themselves.[13]

4.3.3 Partitive and Quotative Divisions Using Multiplication

In case of divisibility (with no remainder), division is represented by (dividend) ÷ (divisor) = (quotient). Division is the inverse operation of multiplication, which is (dividend) = (divisor) × (quotient) or (quotient) × (divisor). If multiplication is repeated addition of the same number, then division can be seen as repeated subtraction of the same number. However, we should note that commutativity does not hold in division.

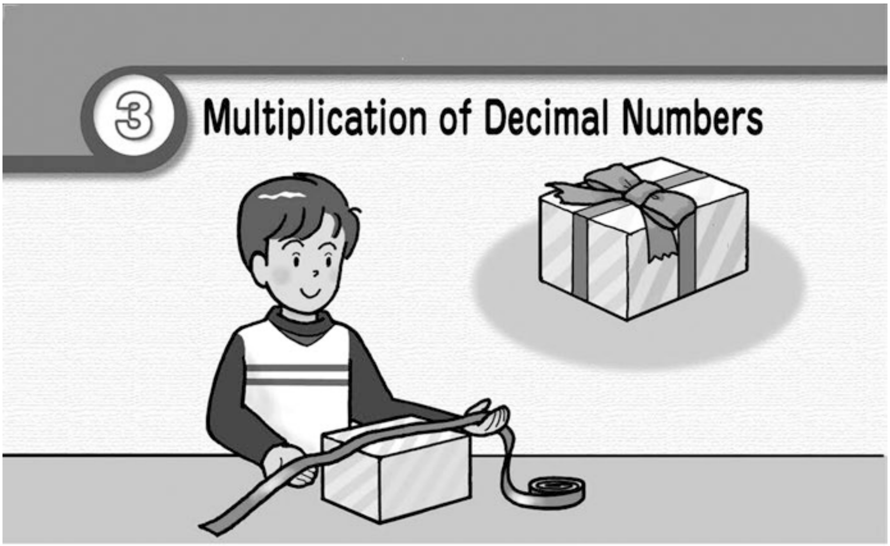
Two meaning of division are shown in Fig. 4.20.

The situations of the two activities on division in Fig. 4.20 are different. The situation of the partitive division activity establishes the number of equal partitions. The situation of the quotative division activity distributes the same amount recursively until there is no more left to distribute. However, if we compare only the left part of the diagram showing partitive division with that showing quotative division in Fig. 4.20, it looks the same as repeated subtraction. This correspondence provides a reason for students to explain these different situations to be integrated as one operation.

---

[12] The term “proportion” is learned in the fifth grade in the 2011 edition and in the sixth grade in the 2005 edition.

[13] As explained in Chap. 1, in Japan, the problem-solving approach is enhanced based on sequential preparations of applying already-learned knowledge to unknown tasks for extension and integration. Learned knowledge is not limited to the procedure but also includes ways of meaningful representations. Such preparations are beyond the strategy of teaching in Pólya’s articles. On this basis, the Japanese problem-solving approaches are very far from just the solving of nonroutine problems under the Pólya framework. In this context, Japanese teachers try to develop students’ mathematical thinking every day.


③ Multiplication of Decimal Numbers

1 Calculating (Whole Numbers) × (Decimal Numbers)

▶▶ Keita is thinking about wrapping the box with a ribbon around it. He needs 2.4m of ribbon.

1 The price of the ribbon is 80 yen per meter. Let's find out how much it would cost for 2.4m.

❶ Draw a number line with a taped diagram.

❷ Write an expression.

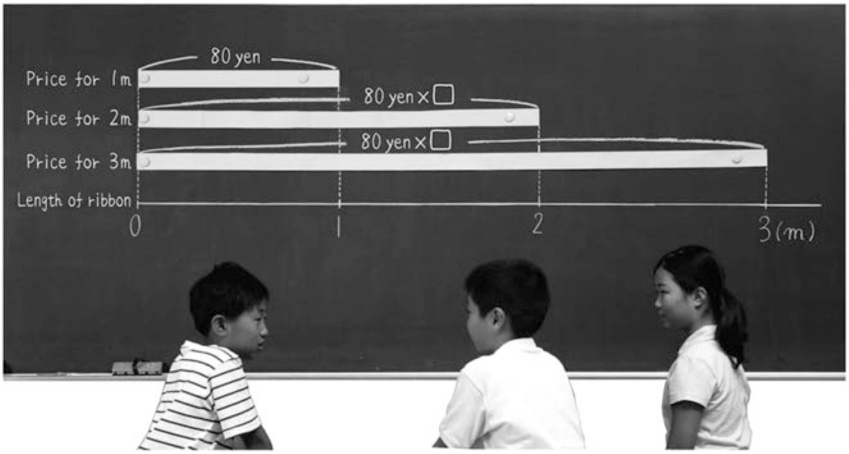
Price (Cost)	80	?
Length of ribbon (m)	1	2.4

Expression : □

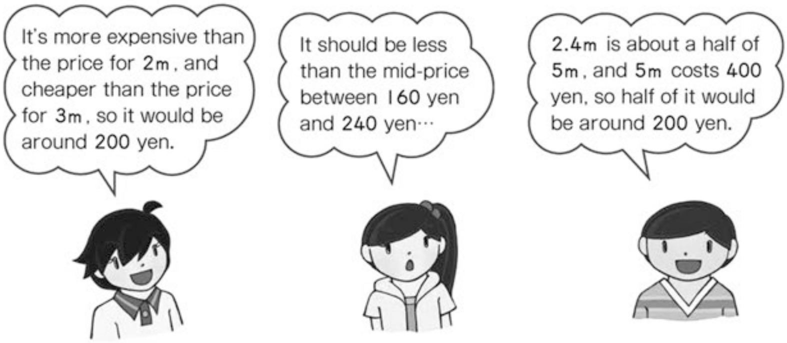
30=□×□

Fig. 4.16 Extension to decimals, Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 30

Multiplication in those situations is (number of each unit) [×] (amount of unit) = (product; total number) in Japanese notation, and (amount of unit) × (number of each unit) = (product; total number) in English notation. The partitive division situation corresponds to finding the amount of each unit (per dish or child), where the quotient means the amount of each unit. On the other hand, the quotative division situ-



❸ Approximately, how much would the cost be?

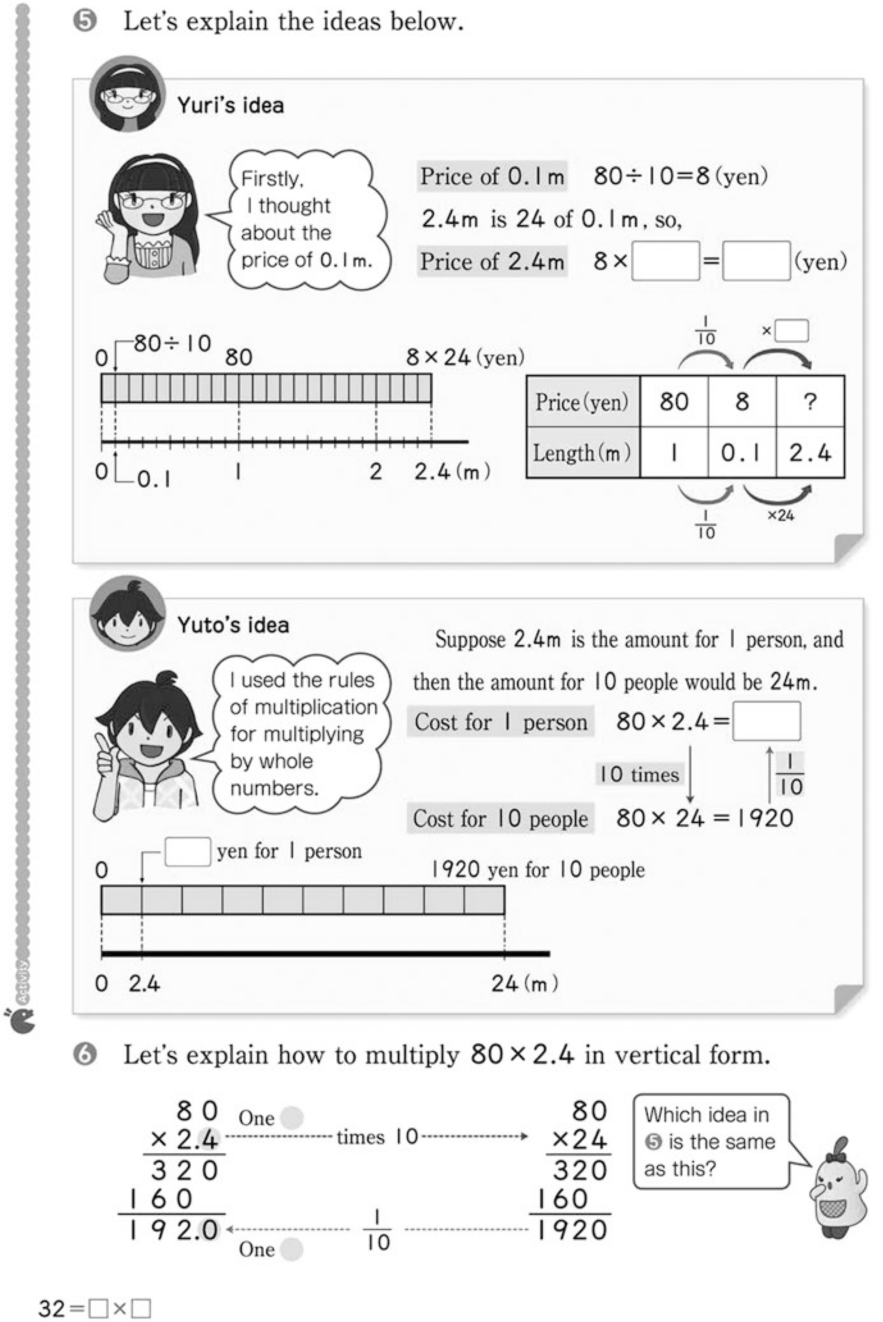


As shown with the length of ribbon, when the multiplier is a decimal numbers instead of a whole number, the expression is the same as for multiplication of whole numbers.

❹ Let's think about how to calculate.

□÷□=31

Fig. 4.17 Right page of Fig. 4.16 (continuous), Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 31



❺ Let's explain the ideas below.

Yuri's idea

Firstly, I thought about the price of 0.1 m.

Price of 0.1 m $80 \div 10 = 8$ (yen)

2.4 m is 24 of 0.1 m, so,

Price of 2.4 m $8 \times \square = \square$ (yen)

0 $80 \div 10$ 80 8×24 (yen)

0 0.1 1 2 2.4 (m)

	$\frac{1}{10}$	$\times\square$	
Price (yen)	80	8	?
Length (m)	1	0.1	2.4
	$\frac{1}{10}$	$\times 24$	

Yuto's idea

I used the rules of multiplication for multiplying by whole numbers.

Suppose 2.4 m is the amount for 1 person, and then the amount for 10 people would be 24 m.

Cost for 1 person $80 \times 2.4 = \square$

10 times $\frac{1}{10}$

Cost for 10 people $80 \times 24 = 1920$

0 $\square$ yen for 1 person 1920 yen for 10 people

0 2.4 24 (m)

❻ Let's explain how to multiply 80×2.4 in vertical form.

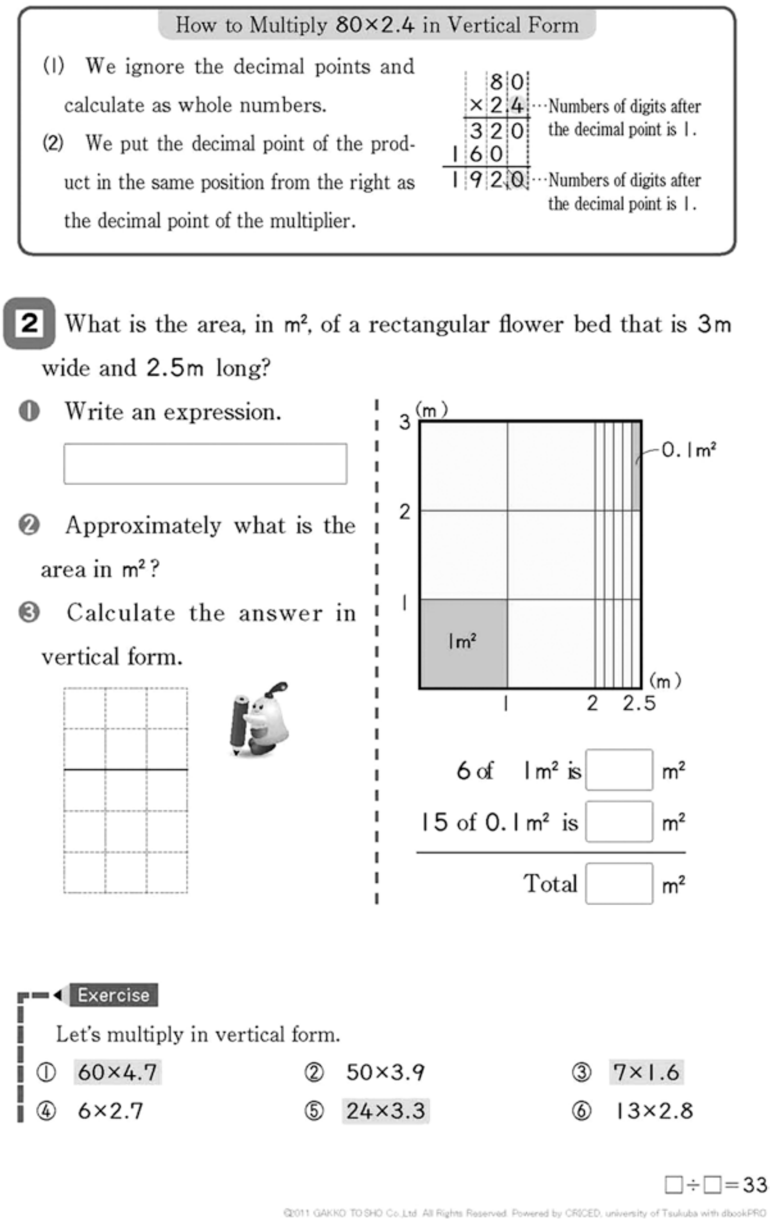
$$\begin{array}{r} 80 \\ \times\ 2.4 \\ \hline 320 \\ 160 \\ \hline 192.0 \end{array} \quad \xrightarrow{\text{times } 10} \quad \begin{array}{r} 80 \\ \times 24 \\ \hline 320 \\ 160 \\ \hline 1920 \end{array}$$

One ◯ times 10 · $\frac{1}{10}$ One ◯

Which idea in ❺ is the same as this?

$32 = \square \times \square$

Fig. 4.18 Continuous from Figs. 4.16 and 4.17, Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 32


How to Multiply 80×2.4 in Vertical Form

(1) We ignore the decimal points and calculate as whole numbers.

(2) We put the decimal point of the product in the same position from the right as the decimal point of the multiplier.

```
   8 0
 × 2.4 …Numbers of digits after the decimal point is 1.
 3 2 0
1 6 0
1 9 2.0 …Numbers of digits after the decimal point is 1.
```

2 What is the area, in m², of a rectangular flower bed that is 3m wide and 2.5m long?

❶ Write an expression.

❷ Approximately what is the area in m²?

❸ Calculate the answer in vertical form.

6 of 1m² is ☐ m²

15 of 0.1m² is ☐ m²

Total ☐ m²

Exercise

Let's multiply in vertical form.

① 60×4.7 ② 50×3.9 ③ 7×1.6

④ 6×2.7 ⑤ 24×3.3 ⑥ 13×2.8

☐÷☐=33

Fig. 4.19 Right page of Fig. 4.18, Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 33

ation corresponds to finding the number of units, where the quotient means the number of units. Thus, the partitive division situation is represented by (total number) ÷ (amount of unit) = (number of each unit), which is the representation of the inverse operation (total number) = (*number of each unit*) [×] (amount of unit). The quotative division situation is represented by (total number) ÷ (number of each unit) = (amount

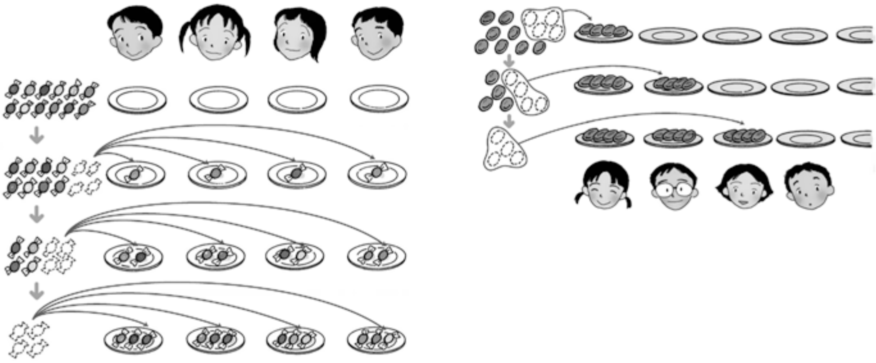


Fig. 4.20 Partitive division (*left*) and Quotative division (*right*), Gakkotosyo (Hitotsumatsu, 2005), Grade 3, Vol. 2, p. 4 and p. 8

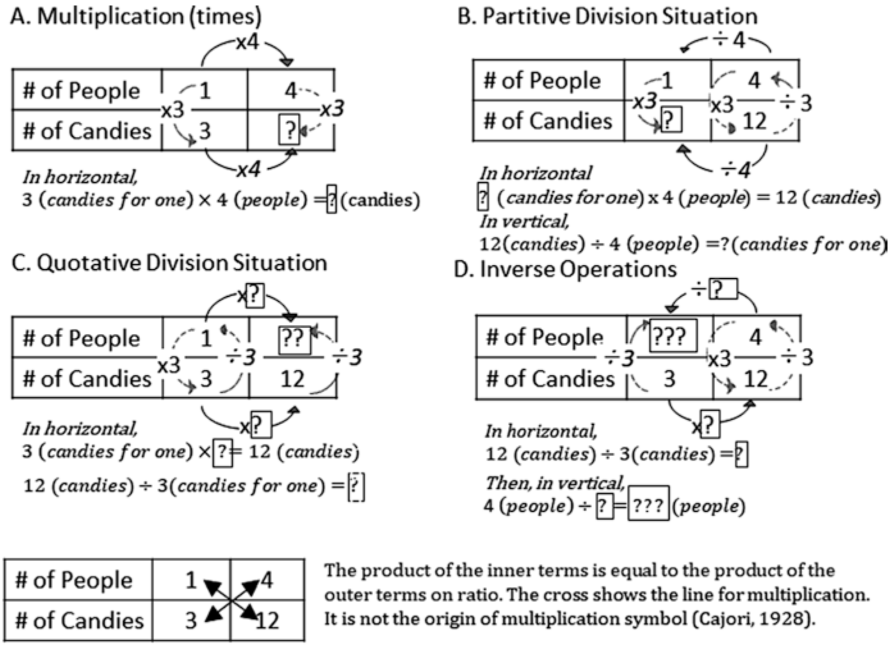
of unit), which is the representation of the inverse operation (total number) = (number of each unit) [×] (*amount of unit*). Those are two different meanings of quotient, depending on the situation. Left vertical arrows in partitive division on the left of Fig. 4.20 and left vertical arrows in quotative division on the right of Fig. 4.20 both, can be seen as repeated subtraction of the same numbers. Repeated subtraction is a key to seeing both situations as the division operation for integration.

For teachers teaching mathematics in Indo-European languages, this discussion is not so clear because they do not use the terms "multiplier" and "multiplicand" (see Chaps. 2 and 3) and may not feel the necessity to do so because it is customary for them to use commutativity in their minds and expressions for finding the answer in multiplication. From the viewpoint of the Japanese approach, teachers who do not feel any necessity to do so can be seen as teachers who are less likely to teach mathematics by using what their students have already learned. If teachers are able to see that the two meanings are not exactly the same, they may understand the difficulty that students have in seeing the different situations as one operation. If they can make the distinction, they can really understand what content they should teach. The Japanese distinguish it clearly based on consistency of multiplication.[14]

4.3.4 Relationships Among the Rule of Three, Multiplication, and Division

The well-memorized and proceduralized multiplication-table is adapted to the tables that embed proportionality by multiplications and division rules. For filling in the blanks in the table, division is treated as the inverse operation of multiplication, as shown in different contexts in Fig. 4.21.

[14]The Japanese use two different characters for division. Partitive division is *waru* ("splitting") which implies dividing equally. Quotative division is *jyo* ("subtraction") which implies repeated subtraction. Due to this difference, both partitive division and quotative division are necessary terminologies to specify what they teach even though they do not teach these words to students. In Japan, division using the abacus was introduced in the sixteenth century.



Adaptation of proportionality to different situations for the ratio of the same quantity

First Usage of Ratio (find ratio (times); table C treatment):

How many times (M) equals (N) when base amount (M) and comparing amount (N) are known; "Taro has 30 balls and Hiro has 45 balls. How many times of number of balls does Hiro have when we compared his number of balls with Taro's?"

ratio	1	?
balls	30	45

Second Usage of Ratio (find the comparing amount; table A treatment):

When the based amount (M) and ratio (p) are known, to find the amount (N) to be compared with (M); "Mie has 20 dolls and the number of Chie's dalls is 1.5 times of the number of Mie's dolls. How many dolls does Chie have?"

ratio	1	1.5
dolls	20	?

Third Usage of Ratio (find the base amount: table B treatment):

When the compered amount (N) and ratio (p) are known, to find the base amount (M); "Taro's height is 156cm which is 1.2 times of the height of Mie. How tall is Mie?

ratio	1	1.2
height	?	156

Table arrangement changes depending on how we draw a table and ways to read. In Japanese textbooks, normally, ratio (times) comes the bottom row such as Figures 4.14 and 4.15.

Fig. 4.21 Various preparation for ratio, rate and proportion by using the table in relation to rule of three, multiplication and division

Historically, these calculations using proportionality are done under the name of the "rule of three." The "?" blanks in tables A, and "??" in B and C in Fig. 4.21 can be represented by multiplication as follows: (A) $3 \times 4 = 12$, (B) $(?) \times 4 = 12$, (C) $3 \times (?) = 12$.[15] If students learn division as inverse operation of multiplication, tables can be seen in various ways by using the idea of proportionality like Fig. 4.21. For finding these answers using multiplication, it is not necessary to refer to the original situations as long as the numbers in every table are placed in the appropriate columns under the proportionality. In Japan, the three usages of the ratio on the situations likely Fig. 4.21 are summarized as kinds of formulas and such discussions were existed before World War II. If teacher teach those different usages just different formulas, it produce difficulty for students. To recognize the proportionality, multiplication and division in the table treatments like Fig. 4.12 make it meaningful (see Fig. 1.1). Thus, Gakkotosyo textbook introduce it from Grade 3 in relation to multiplication and division and prepare the extension of multiplication and division into decimals and fractions, and ratio and proportion in Grades 5 and 6.

4.3.5 From Division to Ratios and Rates Using the Multiplicative Format

Division by a different quantity (partitive division) results in the rate, which is the unknown quantity and is represented by a quantity per another quantity, such as speed (km/h = distance (km)/duration (hour)) or population density (population/km^2).

Division by the same quantity, which corresponds to quotative division, results in the ratio of the same quantity which produces the ratio value to show the coefficient.[16] However, if we carefully read the task for quotative division, we may recognize that it is not exactly division of the same quantity (denomination). It can be represented by (number of students) = (candies) ÷ (candies per student). Students usually see both the numbers 12 and 4 as the same candies and thus do not easily recognize them as different quantities in the sentence. Knowing the unit as a quantity per another quantity in those situations is a key to deriving expressions. In Japanese, terminology "per" is used for ratio on Grade 5, thus at Grade 2, Japanese use terminology "for each amount" instead of "per" to be expendable to ratio.

[15] In Chap. 3, we referred Vergnaud (1983) to explain the situations for multiplication. Currently, the rule of three is explained by algebraic expressions. However, if we ask students to distinguish the expressions as different formula, it produce difficulties. Historically, the rule of three existed as methods to find the answers with the three numbers alignment before the emergence of algebraic expressions. With regard to ratios, the "× 4" arrows in the tables in Fig. 4.21 are explained as *bai* ("times"). In Japanese terminology, *bai* is a key word to explain proportionality. Division is alternated reciprocal number of multiplication. The division treatments on tables in Fig. 4.21 are alternated to reciprocal-number times at Grade 6.

[16] This is the Japanese usage of "times" (*bai*) like multiple.

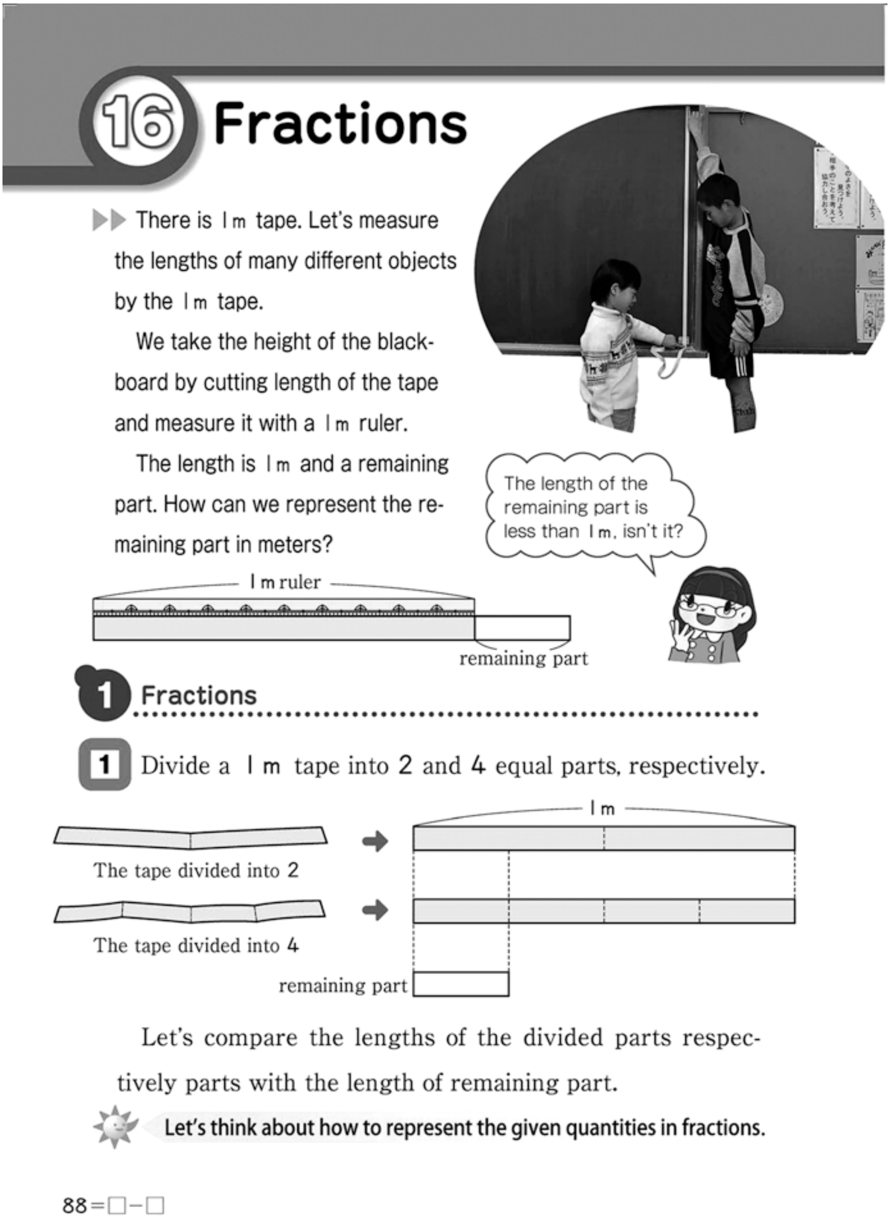
4.4 Various Meanings of Fractions Embedding the Meanings of Division Situations

Fraction in situations can be distinguished in various perspectives and contexts (see Isoda, 2013).

A fraction as a part–whole relationship usually corresponds to the partitive division situation; the Japanese call this a "dividing fraction." A fraction can also be seen as quotative division. The situation of a fraction in quotative division is called an "operational (taking away, measuring) fraction." A fraction with a denomination to indicate the unit quantity by the denomination is called a "fraction with a quantity." A fraction used for showing the number and used for the value of division as a quotient is called a "quotient fraction." A fraction as a ratio is called a "fraction as the value of a ratio". Japanese distinguish these five meanings of fractions.

Before providing explanations, first we describe the existing English-language terminologies used to distinguish the meanings of fractions. In English, the following meanings of fractions are distinguished: according to Reys, Lindquist, Lambdin, and Smith (2012), from the USA, the meanings of fractions in situations are distinguished as part–whole, the quotient, and the ratio. Part–whole corresponds to the Japanese "dividing fraction" and the "operational (measuring) fraction" means the measuring by unit such as the measuring by using reminder like 4.22. However, it is not so much clear to distinguish these two ideas. According to the USA Common Core State Standards for Mathematics (CCSSM) (2010) "Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$. (CCSS.MATH.CONTENT.3.NF.A.2.A)." It is a dividing fraction. And CCSSM continues "Represent a fraction a/b on a number line diagram by marking off a length $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line. (CCSS.MATH.CONTENT.3.NF.A.2.B)." This can be seen as an "operational fraction" if the scaling is done by $1/b$; however, it is unclear if it is operational fraction. It implicates that CCSSM does not use terms like Japanese to establish conceptual consistency between the two division meanings and meanings of fractions even though it embedded the ideas. Indeed, Watanabe (2006) in USA makes clear the activity of the operational fraction and explains the uniqueness of the Japanese way to introduce fractions. According to Haylock (2010), also from the USA, a fraction is (a) a part of a whole or unit, (b) a part of a set, (c) a modeling division problem, (d) a ratio and it is unclear if this is an operational fraction. According to Van de Walle, Karp, Bay-Williams, and Wray (2015), from the UK, there is no such manner to distinguish different types of fractions in different situations. According to Kupferman (2017), from Israel, there are various analyses. Petit, M. et al (2016) also explained various models of fraction under CCSSM well and we can read the ideas in it with Japanese terminology of fraction but their wording is following the CCSSM. These articles implicate that the Japanese approach clearly adopts two meanings of divisional situations into meanings of fraction in situations as "dividing fractions" and "operational fractions."

The following is the introduction to the fraction in Gakkotosyo textbooks (Isoda and Murata, 2011) using a situation for an operational fraction (Figs. 4.22 and 4.23).



16 Fractions

There is 1 m tape. Let's measure the lengths of many different objects by the 1 m tape.

We take the height of the black-board by cutting length of the tape and measure it with a 1 m ruler.

The length is 1 m and a remaining part. How can we represent the remaining part in meters?

1 Fractions

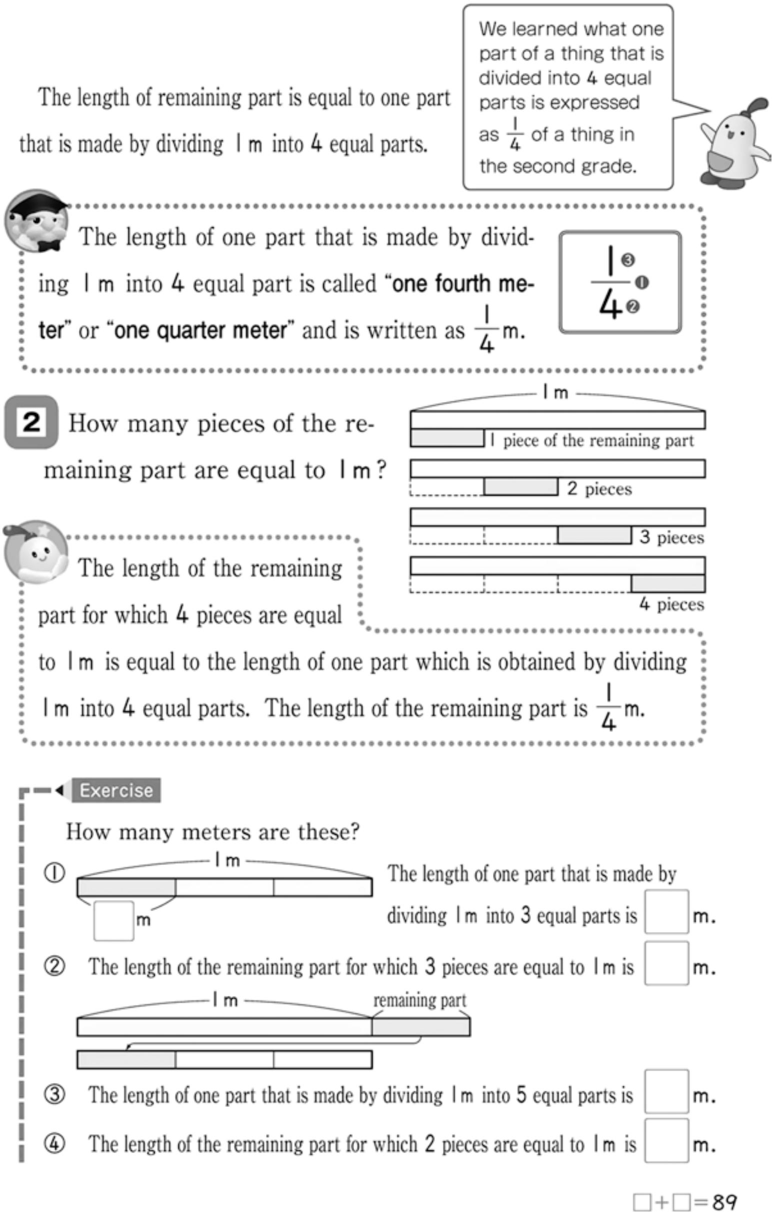
1 Divide a 1 m tape into 2 and 4 equal parts, respectively.

Let's compare the lengths of the divided parts respectively parts with the length of remaining part.

Let's think about how to represent the given quantities in fractions.

88=□−□

Fig. 4.22 Measuring by the remaining part. Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 88



We learned what one part of a thing that is divided into 4 equal parts is expressed as $\frac{1}{4}$ of a thing in the second grade.

The length of remaining part is equal to one part that is made by dividing 1 m into 4 equal parts.

The length of one part that is made by dividing 1 m into 4 equal part is called "**one fourth meter**" or "**one quarter meter**" and is written as $\frac{1}{4}$ m.

$\frac{1}{4}$ ❸ ❶ ❷

2 How many pieces of the remaining part are equal to 1 m?

1 m

1 piece of the remaining part

2 pieces

3 pieces

4 pieces

The length of the remaining part for which 4 pieces are equal to 1 m is equal to the length of one part which is obtained by dividing 1 m into 4 equal parts. The length of the remaining part is $\frac{1}{4}$ m.

Exercise

How many meters are these?

① 1 m, □ m. The length of one part that is made by dividing 1 m into 3 equal parts is □ m.

② The length of the remaining part for which 3 pieces are equal to 1 m is □ m.

1 m, remaining part

③ The length of one part that is made by dividing 1 m into 5 equal parts is □ m.

④ The length of the remaining part for which 2 pieces are equal to 1 m is □ m.

□+□=89

Fig. 4.23 Continued from Fig. 4.22, Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 89

In Figs. 4.22 and 4.23, we can see three different meanings of a fraction. One is an operational fraction, which is measured by the remaining part. This remaining part as the unit for measurement is called a “unit fraction.”[17] The other one is fraction with quantity, here 1m tape.[18]

In this context, the size of the fraction cannot be compared without fixing the unit such as “m.” The size of a half a pizza in a dividing fraction cannot be compared without fixing the size of the original (whole) pizza before dividing it. According to this meaning, dividing and operational fractions cannot be explained well as comparative sizes of numbers without fixing what the whole is. Especially, in dividing fractions, fixing the whole is easily forgotten than operational fraction. If forgot, the sizes of the fractions cannot be compared. To see a fraction as a number to compare sizes, Japanese textbooks show that meaning of the fraction in a context with a quantity such as 3/4 L and 1/2 L. It is a “fraction with quantity.” The fraction with quantity enables it to be put on the number line, clearly, because it fixes the unit for the magnitude.

The answer to division is called a quotient, which is a number without a denomination (quantity). The equivalence of fractions can be seen on the number line. The fraction which is the answer to division is called a “quotient fraction” in the Japanese approach. By introducing the quotient fraction and equivalence of fractions, a fraction becomes a number because it begins to function as part of the operation of other numbers—that the division of numbers should have their answers on the number line.[19]

A fraction as the “value of a ratio” or the rate is not always a part–whole relationship. Even the ratio of the width to the length of a rectangle with the same quantity is not a part–whole relationship physically because the width never belongs to the length of the rectangle directly.[20] Contextually, the value of the ratio can be seen as a quotient fraction if it is a fraction. It is usually used for (a/b) times (quantity). The Japanese consider this fraction as a ratio such as “half of a bottle.” Here, “half” is the value of the ratio and “of” implies multiplication. In Japanese, *bai* (“times”) is usually used in this context (“of”). The rate is represented by the division of different quantities and results in a new quantity such as “km/h.” It is related to partitive division but not to part–whole relationships because of the differences in quantities. In the Japanese teaching sequence, ratios and rates can be seen as the extended adaptation of multiplication and partitive and quotative divisions in relation to proportion-

[17] The unit is the unit for measurement which is based on the definition of multiplication by measurement. The unit can be a decimal or a fraction.

[18] In the case of Gakkotosyo (2005), dividing fraction appeared the same pages. In the case of 2005 edition, fraction is introduced from this page at the same time because students are not yet learned how to fold the 1m tape into 4. In the case of 2011 edition, paper folding for fraction is learned at 2nd grade. Those differences originated from the difference of national curriculum standards.

[19] Numbers can be seen as numbers when this becomes a number system which discusses existence, magnitude (greater or less, equality, comparison), and the four operations. The quotient fraction is the answer of division. Division is defined by the inverse operation of multiplication. Before the quotient fraction, Japanese textbooks already addressed addition and subtraction of fractions.

[20] In the Euclidean algorithm for finding the greatest common divisor of two segments of a rectangle, we have to move the width to length by using a compass. As mentioned in the introduction of measurement, it functions to find the common unit for the measurement and also addition with different denominators.

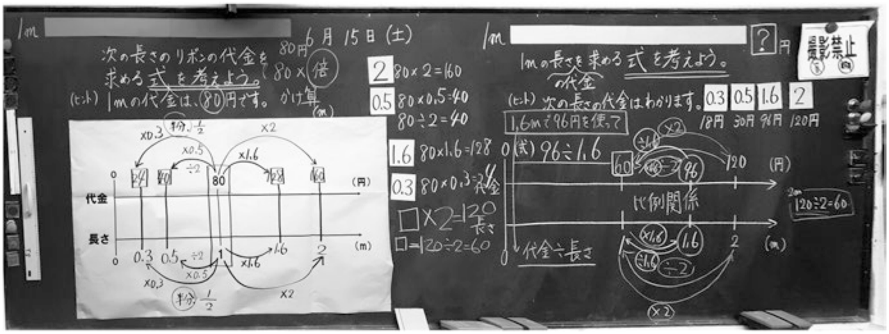


Fig. 4.24 Open class by Takao Seiyama (June 15, 2019 at the Elementary School at the University of Tsukuba). Learned task is multiple (see Fig. 4.21, A) (*left*): "Let's find the price of ribbon when the price of 1 m is given." Unknown task for today (see Fig. 4.21, B; *right*): "Let's find the price of 1 m of ribbon when the price for 2 m is given." Before this class, students have already learned about proportions using proportional number lines. The major objective of today's class is as follows: Using the proportional number line, recognize that division is the inverse operation of multiplication, such as 0.5 times is one half times or division by 2 (as review; *left*), and adopting learned to the new division task which can be seen as multiplication through finding the unit price at first and then finding the value by multiplication (*right*). By using the proportional number line, multiplication and division are integrated on the tasks for ratio like tables in Fig. 4.21

ality (See Fig. 4.21). In the Japanese curriculum, the different meanings of fractions are well distinguished and sequenced in the curriculum up to ratios and proportions, and up to Integration of multiplication and division to multiplication by representing division as multiplication of reciprocal number.[21]

In the Japanese curriculum sequence, the definition of multiplication by measurement is consistent with the multiplication table, division, fractions, and ratio and proportion. This consistency is supported by the model representations, proportional number lines (see Fig. 4.24) and the table in relation to the rule of three (see Fig. 4.21). This is the reason why Japanese distinguishes the multiplier and multiplicand at the introduction of multiplication. Actually, if we do not distinguish both, we can not distinguish partitive and quotative division, then, can not distinguish dividing and operational (measurement) fraction. After this consistency and extended adaptation of multiplication, the formal definition of proportions is introduced. To introduce Proportion formally, Gakko Tosyo Textbooks evolve the proportional number line from tape diagrams on Fig. 4.4 to a tape and a number line on Fig. 4.14, and apply it for the extension of numbers on Fig. 4.16, and replace it to the parallel number lines on Fig. 4.24 as for the preparatory representations of the proportion. It illustrates the sequence to develop sense making for using multiplicative reasoning and proportionality on the principle of extension and integration. This sequence and representations make possible to apply the definition of multiplication by measurement to different teaching content (see Chap. 3). With this

[21] On this integration, "÷3" becomes "× (1/3)" which changes the view of multiplier from the first number to likely second number as an operator in the case of India-European language (see Chap. 3).

meaning, Japanese can introduce multiplication as preparation for division, fractions, and proportions. In the Japanese approach, the teaching content and sequence are usually preparation for future learning. It is not only just making sense for reasonable explaining at every moment but also to develop sense making for extending and integrating by and for themselves.

4.5 Further Challenges to Distinguish Additive and Multiplicative Structures

There are a number of misconceptions between additive and multiplicative structures in relation to ratios, rates, and proportions, such as misusing addition in multiplicative or divisional situations. An origin of this type of misconception is originated from the properties of the multiplication table (see Fig. 4.25).

When students learn multiplication in the second grade, they find and use this additive property. On the other hand, the multiplicative property is not easy for students because they have just learned the table and still use the additive property for explaining the table, Explaining such as two times of multiplier produces two times (double) products, and three times of multiplier produces three times (triple) product, and so on are not easy because the symbol "×" itself can be read "times." On the multiplicative property there are "×" meaning of times and "left arrows and right arrows between expressions" meaning of times appeared at once on the multipliers in the table and the number of times (see Fig. 4.25; Isoda, 2015). If students extend multiplications to fractions in the upper grades, they can realize the difference of these two properties, such as half of two in Fig. 4.26. Thus, in the second grade, they

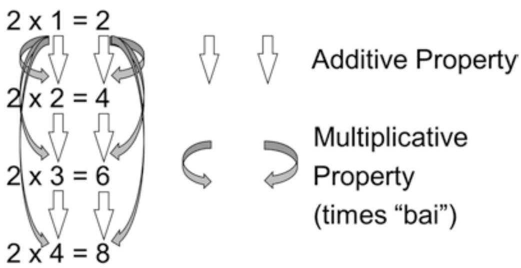


Fig. 4.25 Additive property and multiplicative property of the multiplication table

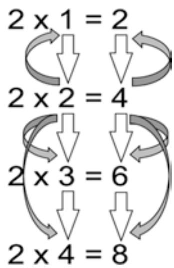


Fig. 4.26 After the extension of multiplication to proportionality of numbers such as decimals and fractions, students are able to discuss ×0.5 and ×(1/2)

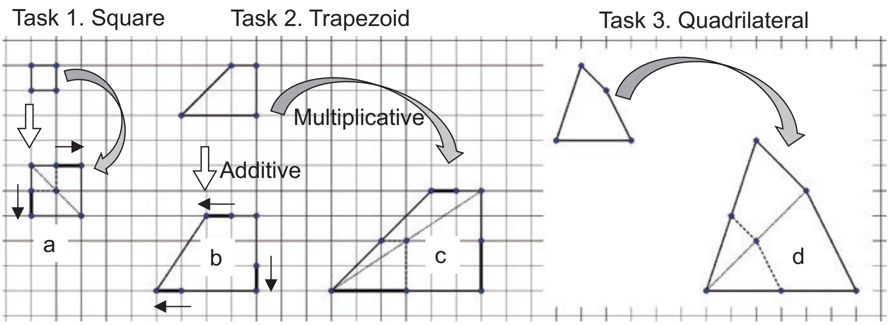


Fig. 4.27 The sides of the square, trapezoid, and quadrilateral are enlarged two times on 1-cm-squared paper (by Suzuki in Isoda (1996), revised by Isoda)

cannot easily distinguish additive and multiplicative structures in the table like the one shown in Fig. 4.25, but not like the one shown in Fig. 4.26.

In Japan, a proportion is defined in the fifth grade by using "2 times (*bai*), 3 times (*bai*), . . . , of *x* corresponds to 2 times (*bai*), 3 times (*bai*), . . . , of *y*, respectively" and the constant of the quotient is the property of the proportion. Even though they have learned the use of "times" (*bai*) in proportions in the fifth grade, students are still confused, depending on the daily context; for example, the usage of *bai-bai* means "three times" or "quadruple."[22] Even though proportionality is learned in the fifth grade, students have to extend its usage and meet the problematic (see Chap. 1 and Tall, 2013) to distinguish additive and multiplicative properties on every occasion. In the sixth grade, students extend it to the enlargement of figures. In the task sequence shown in Fig. 4.27, both additive and multiplicative strategies appear.

Figure 4.27 shows a task sequence to develop students who learn mathematics by and for themselves by using what they have learned through extended tasks.

In task 1, all students easily draw "a." The students use three drawing strategies: just adding 1 cm (↓), doubling of sides, and using the diagonal. They cannot distinguish these three on the drawing in task 1. In task 2, there are two drawings ("b" and "c") and they meet the problematic. The students who draw "b" use the strategy of "adding 1 cm." The students who draw "c" use the strategy of "doubling of sides" or "using the diagonal." In task 3, which is posed on nonsquared paper, students draw "d" by "using the diagonal" and compare it with the other two strategies ("b" and "c"). Then, the teacher asks the students to summarize what they have learned through this task sequence and continues the lesson on how to draw by using diagonals with the idea of proportions.

[22] In Japanese, just *bai* without a number implies "double." Thus, *bai-bai* means "quadruple." In English grammar, there are several types of numerals: cardinal or set numbers such as one, two, and three; ordinal numbers such as first, second, and third; multiplicative numbers such as once, twice, and thrice; multipliers such as single, double, and triple; and fractional numbers such as half and quarter. In Japanese, ordinal numbers, multipliers, and multiplicative numbers are expressed with denominations to the number such as first (1 *banme*), second (2 *banme*), and third (3 *banme*); single (1 *bai*), double (2 *bai*), and triple (3 *bai*); and once (1 *kai*), twice (2 *kai*), and thrice (3 *kai*).

This task sequence for the problem-solving approach (See Chap. 1) was produced under the curriculum and task sequence theory of Isoda (1992, 1996) based on the theory of conceptual and procedural knowledge by Hiebert (1986) (see Chap. 1, Fig. 1.1). In Task 1, the students already know the word “enlargement” in the daily context with images. At the beginning, it functions as the meaning. Thus, “a” is appropriate based on this meaning and it recognizes additive and multiplicative procedures. In Task 2, based on this meaning, “b” is inappropriate because it is an overgeneralization of additive procedure, and “c” (and “d”) is appropriate because it is an adaptation of a multiplicative (proportional) procedure. While comparing the procedures, students are able to recognize the difference in additive and multiplicative procedures. The task sequence functions to reconceptualize it with an appropriate procedure. From Tasks 2 to 3, the procedure “using diagonals” is the only one that works. It is used to reconceptualize the meaning of enlargement without an additive strategy, and students are able to define enlargement based on proportion with the point (homothetic center) of enlargement by using diagonal. After this, the students continue to learn the case of other figures such as enlargement of polygons. Consequently, using diagonals becomes the procedure for enlargement of figures.

4.5.1 Redefinition of Proportionality at Junior High School

In Japan, proportionality is extended to negative numbers and redefined as a function at the junior high school level. The difference in additive and multiplicative structures is discussed using positive and negative numbers from four arithmetic operations to two operations by using the reciprocal and inverse (see Chap. 3). The equation of the function $y = ax$ can be seen algebraically as a generalized equation of the multiplication table such as in Figs. 4.25 and 4.26. The variable x is consistent with the multiplication table (see the discussion of Figs. 3.11 and 3.12 in Chap. 3). For $y = ax + b$ and $y = ax$, only $y = ax$ keeps the multiplicative property, and both keep the additive property.

The enlargement of figures in the sixth grade is the base for the definition of similar figures in relation to the center of similarity at the junior high school level. The line and point similarity of figures also becomes the source of problematics on this redefinition.

4.6 Final Remarks

The Japanese approach to multiplication at the elementary levels provides a consistent sequence for preparing future learning in the curriculum in the context of extension and integration, up to proportions. The introduction of multiplication in the second grade is a preparation for division in the third grade and a preparation for proportions in the fifth and sixth grades. The reason why the Japanese try to maintain a consistent sequence to develop and extend ideas is based on the aims of education, which tries to develop students who think and learn by and for themselves.

It is the process to develop the sense for making sense. The curriculum sequence enables students to think about what they already know and how they can challenge themselves to extend ideas by and for themselves. In the Japanese curriculum and textbooks, the problem-solving approach is enhanced based on a well-designed task sequence for applying already-learned knowledge to unknown tasks as well as preparation for future learning. Learned knowledge is not limited to the procedure but includes ways of thinking, the methods of representations, and values and attitudes regarding mathematics.

Solving non-routine problems under the Pólya framework is not the same in usage as the Japanese problem-solving approach in Chap. 1 because the task sequence, which is designed by the teacher, is basically defined under the curriculum sequence for enabling students to think by and for themselves in future learning. The second-grade tasks are unknown problems for the students; however, they become a routine problem after they have learned them or in later grades. The objectives of the tasks which produce unknown and problematic situations can be explained by using these terminology in the curriculum sequence. Under the shared curriculum, Japanese teachers has been engaging in lesson study by using these terminology to explain and distinguish the objective of every class and produceing the textbooks for enabling them to practice. For developing students' mathematical thinking and ideas for future learning through problem solving, the terminology is necessary to explain task sequence beyond just using a method of teaching simply changing every task to be open ended for just solve an independent problem. Similar terminology is existed in teacher education in the world but it is not always functioning among classroom teachers. Beside, Japanese teachers have to use it on their lesson study activities systematically.[23]

The Japanese approach, which is based on the consistency of the curriculum sequence under explained terminology described in this chapter, is preferred for projects in Central America, Southeast Asia, Central Asia, Africa, and so on. In relation to this, the Japanese definition and notations of multiplication are also preferred because of the following consistencies:

- Consistency among situations, repeated addition, and the multiplication table
- Consistency with other content such as measurement, division, fractions, ratios, and proportions in relation to distinguishing the multiplier and multiplicand
- Expandability to decimals and fractions by using consistent representations such as dot–area diagrams, proportional number lines, and tables for the rule of three
- Consistency in proportionality

The specified consistencies support the process of extension and integration at future learning. Multiplication provides strong bases up to proportions in this process. This specified feature is one of the reasons why the Japanese approach has been preferred by other countries. Such an approach, including the terminology,

[23] For example, Izak & Beckmann (2019) systematically explained the role of proportional number line however, in Japan, it was introduced for lesson study, already systematically embed the into textbooks 50 years ago (see Chap. 1) and now, it progressively changes its representations beyond the grades.

which is explained in this chapter, existed in the textbooks until the curriculum of 1968 and has been used in several Japanese official development assistance (ODA) programs around the world, from Singapore in the late 1970s to Africa and Central and South America from the 1990s onward.

There is an additional discussion on consistency in relation to the vertical form of multiplication in Chap. 7.

References

Cajori, F. (1928). A History of Mathematical Notations, Vol. 1. La Salle, Ill, USA: Open Court Pub. Co.

Common Core State Standards Initiative. Common core state standards for mathematics [CCSSM]. (2010). Common Core State Standards Initiative. Retrieved from http://www.corestandards.org/Math/Content/3/NF/

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Riedel.

Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Dordrecht: Kluwer.

Haylock, D. (2010). *Mathematics explained for primary teachers* (4th ed.). Los Angeles: Sage.

Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale: Lawrence Erlbaum.

Hitotsumatsu, S. (2005). *Study with your friends: Mathematics for elementary school*, 11 volumes. (English translation of Japanese textbook.) Tokyo: Gakkotosyo.

Hulbert, E. T., Petit, Marjorie, M., Ebby, C. B., Cunningham, E. P., Laird, R. E. (2017). *A focus on multiplication and division: Bringing resarch to the classroom*. NY, USA: Routledge.

Isoda, M. (1992). *Designing problem solving approach with cognitive conflict and appreciation*. Iwamizawa: Hokkaido University of Education. (in Japanese).

Isoda, M. (Ed.). (1996). *Developing lesson on mathematics problem solving approach: Conflict and appreciation on the dialectic among conceptual and procedural knowledge*. Tokyo: Meijitosho. (in Japanese).

Isoda, M. (2005). *Elementary school teaching guide for the Japanese course of study: Arithmetic (grade 1–6)*. (English translation of the 1989 edition published by the Ministry of Education, Japan.). Tsukuba: CRICED, University of Tsukuba.

Isoda, M. (2013). *Fraction for teachers: Knowing what before how to teach*. Retrieved from http://math-in-fo.criced.tsukuba.ac.jp/museum/dbook_site/Fraction%20for%20Teachers%20dbook-Pro%202013-pub/files/EText.html

Isoda, M. (2010). *Elementary school teaching guide for the Japanese course of study: Mathematics (grade 1–6)*. (English translation of Japanese curriculum standards, 2008; and guide, 2009.) Tsukuba: CRICED, University of Tsukuba. Retrieved from http://www.criced.tsukuba.ac.jp/math/apec/ICME12/Lesson_Study_set/Elementary_School_Teaching_Guide-Mathematics-EN.pdf

Isoda, M. (2012). Introductory chapter: Problem solving approach to develop mathematical thinking. In M. Isoda & S. Katagiri (Eds.), *Mathematical thinking: How to develop it in the classroom* (pp. 1–28). Hackensack: World Scientific.

Isoda, M. (2015). *Mathematization for mathematics education: An extension of the theory of Hans Freudenthal applying the representation theory of Masami Isoda with demonstration of levels of function up to calculus*. Tokyo: Kyoritsu (in Japanese).

Isoda, M. (2016). Introducción. El enfoque de resolución de problemas para desa-rrollar el pensamiento matemático. In M. Isoda, & S. Katagiri (A. Jel-drez, Trans.), *Pensamiento matemático: Cómo desarrollarlo en la sala de cla-ses*. Santiago: CIAE, Universidad de Chile.

Isoda, M., & Chino, K. (2006). *Elementary school and lower secondary school teaching guide for the Japanese course of study: Arithmetic and mathematics (grade 1–9)*. (English transla-

tion of the 1999 edition published by the Ministry of Education, Japan.). Tsukuba: CRICED, University of Tsukuba.
Isoda, M., & Murata, A. (2011). *Study with your friends: Mathematics for elementary school*, 12 volumes. (English translation of Japanese textbook.) Tokyo: Gakkotosyo.
Isoda, M., Murata, A., & Yap, A. (2015). *Study with your friends: Mathematics for elementary school*, 10 volumes. (English translation of Japanese textbook.) Tokyo: Gakkotosyo.
Isoda, M., & Olfos, R. (2009). *El enfoque de resolución de problemas: en la en-señanza de la matemática a partir del estudio de clases*. Valparaíso: Ediciones Universitarias de Valparaíso.
Izak, A., & Beckmann, S. (2019). Developing a coherent approach to multiplication and measurement. *Educational Studies in Mathematics, 101*, 83–103.
Kupferman, R. (2017). *Elementary school mathematics for parents and teachers* (Vol. 2). Singapore: World Scientific.
Ministry of Education. (1968). *Course of study for elementary school mathematics*. Tokyo: Ministry of Education.
Muroi, K. (2017). *Sumerians' mathematics*. Tokyo: Kyoritsu.
Petit, M. M., Laird, R. E., Marsden, E. L., Eddy, C. B. (2016). *A focus on fractions: Bringing research to the classroom* (2nd ed.). NY, USA: Routledge.
Rasmussen, K., & Isoda, M. (2019). The intangible task—a revelatory case of teaching mathematical thinking in Japanese elementary schools. *Research in Mathematics Education, 21*(1), 43–59.
Rey Pastor, J., & Puig Adam, P. (1935). *Bachillerato. Matemáticas tercer curso. Tomo I aritmética*. Madrid: Unión Poligráfica.
Reys, R., Lindquist, M., Lambdin, D., & Smith, N. (2012). *Helping children learn mathematics* (10th ed.). Hoboken: Wiley.
Sugimura, K., Simada, S., Tanaka, R., & Wada, Y. (1943). *The guidebook of secondary school mathematics, cluster I and II*. Tokyo: Secondary School Textbook Publisher.
Szilagyi, J., Clements, D., & Sarama, J. (2013). Young children's understandings of length measurement: Evaluating a learning trajectory. *Journal for Research in Mathematics Education, 44*(3), 581–620.
Tall, D. (2013). How humans learn to think mathematically: Exploring the three worlds of mathematics. New York, USA: Cambridge University Press.
Van de Walle, J., Karp, S. K., Bay-Williams, J. M., & Wray, J. (2015). *Elementary and middle school mathematics* (9th ed.). Essex: Pearson.
Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York: Academic.
Watanabe, T. (2006). Teaching and learning of fraction: A Japanese perspective. *Teaching Children Mathematics, 12*(7), 368–374.