Chapter 3
Problematics for Conceptualization of Multiplication

Masami Isoda and Raimundo Olfos

This chapter addresses the problematics for the conceptualization of multiplication in school mathematics and fundamental difficulties, which include semantics for defining multiplication meaningfully, syntax in relation to languages, and difficulties that originate from historical transitions. The chapter discusses the contradictions or inconsistencies in the various meanings of multiplication in school mathematics situations. Many of these problems of multiplication are originated from European languages. This discussion of these problematics provides some answers to the questions posed in Chap. 2 and provides bases for the necessity to consider the Japanese approach described in Chaps. 4, 5, 6, and 7 of this book. The terminology of multiplication discussed here is related to mathematical usages of multiplication in relation to situations and models. Educational terminology used for multiplication to explain the curriculum and task sequences for designing lessons are discussed in Chap. 4 of this book.

3.1 Definitions of Multiplication and Their Meanings in Situations in School Mathematics

Mathematics curricula look well designed and consistent for learned adults; however, they usually have a number of inconsistencies for learners. Given this essential nature of mathematics curricula, the learning sequence used for mathematics, such as the curriculum and task sequence, can be explained by reorganization of

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mathematics, such as mathematization (Freudenthal, 1973; see Chap. 1 of this book and Isoda, 2018). Here, several inconsistencies in the definitions and meanings of multiplication are confirmed.

Multiplication as an operation can be explained in several ways, depending on the context (see Freudenthal (1983)). Here, some definitions and meanings which can be seen in curriculum documents, textbooks, and research articles will be illustrated in relation to problematics. These definitions and meanings will provide some answers to the questions posed in Chap. 2 and the necessary didactic questions for considering the Japanese challenges to established teaching sequences for developing the concept of multiplication in later chapters.

3.1.1 The Concept of Multiplication in Pure Mathematics in Relation to School Mathematics

In the formal context of pure mathematics, multiplication is defined by axioms such as the field theory of numbers.[1] Multiplication is defined as a binary operation and is distinguished from addition. In relation to abstract algebra, upper secondary school mathematics usually focuses on these two operations: division should be represented by multiplication of the dividend and the reciprocal (multiplicative inverse) of the divisor, and subtraction should be represented by addition of the minuend and the opposite (additive inverse) of the subtrahend. Multiplication and addition allow the rule of commutativity as a field axiom, such as $2 \times 3 = 3 \times 2$ and $2 + 3 = 3 + 2$. On the other hand, subtraction and division change their values if the order of numbers changes: $2 \div 3 \neq 3 \div 2$, and $3 - 2 \neq 2 - 3$. It provides one of the necessity in school mathematics to reorganize the four arithmetic operations at the elementary school level into the two major operations at the university level. In relation to Set theory, multiplication can be seen as Cartesian products. The value of multiplication can be seen as a cardinal number of the set of ordered pairs.

In elementary school, students learn all four arithmetic operations on their basis of life under their languages.[2] Depending on the learning trajectories under their own curriculum, students encounter contradictions (inconsistencies), which produce several gaps between arithmetic and the two operations in field theory.[3]

[1] The axioms for numbers are not only limited to the field theory. There are theories for the number system based on the algebraic extensions from the axiom of Peano. Further extension to real numbers is done by the Dedekind cut and hyperreal numbers (Tall, 2013). Complex numbers do not maintain the axiom of order. The R-module in relation to vector space can be another perspective for the number system. Vergnaud (1983) also discussed the "multiplicative structure" in relation to modern mathematics. This chapter is written from the Japanese and Chilean authors' perspective of the bases for the Japanese approach, which was established up to 1960s and is illustrated in Part I of this book.

[2] The matter of language will be discussed in Sect. 3.2.

[3] A simple example of miscalculation is $2 \div \frac{3}{5} \times 5 = \frac{2}{3}$, instead of $\frac{50}{3}$.

In formal algebra, natural numbers are introduced with Peano's axiom and the number systems are extended through progressive introduction of the four operations, magnitude[4] relations (the equivalence relation (=)), and order relations (greater or less than (> or <)) (see Michell and Ernst (1996)).[5] In elementary school, the equivalence of numbers can also be confirmed in every operation: $1 + 4 = 2 + 3 = 3 + 2 = 4 + 1$, $2 - 1 = 3 - 2 = 4 - 3 = \ldots$, $2 \times 3 = 3 \times 2$, $2 \div 1 = 4 \div 2 = 6 \div 3 = \ldots$ On natural number, the commutativity of multiplication also illustrates the equivalence of products. Within the natural numbers, the equivalence of values in addition and multiplication are finite but that in subtraction and division are infinite.

In school mathematics, the concept of multiplication is developed through reorganization of the process for multiplication (see Chap. 1, Fig. 1.1).[6] In elementary school, multiplication is usually introduced as repeated addition. Within a few years, children have to distinguish both addition and multiplication as independent operations. The elementary school curriculum usually treats the relationships between multiplication and division, and between addition and subtraction, as inverse operations, such as division of fractions is multiplication of reciprocal numbers. Teachers need to help students reorganize the four operations into two operations when the numbers are extended to positive and negative numbers. The rules of commutativity, associativity, and distributivity are usually introduced at the earlier stage of elementary school in preparation for future reorganizations.

For introducing multiplication of whole numbers, it can be defined as repeated addition, which is useful for getting the products of the multiplication table. In developing the multiplication table, the pattern "the product increases by the multiplier" for each row is used and, mathematically, it will be explained by the distributive law. For students, the row of 1—such as 1×1, 1×2, and 1×3—is not easy to explain by repetition because the row of 1 is the same as counting and not adding. Thus, we use the permanence of form (see Table 1.1 in Chap. 1). There is no counting - objects for the row of 0, thus the row of 0 is normally never discussed. Extension of the multiplication table from 9 by 9 to 10 by 10, or more, is easier for students if we use the pattern (permanence of form) supported by the distributive law. If the multiplication table is established at once, it will provide an alternative way to get the value of multiplication as the product.[7]

As mentioned in Chap. 1, Fig. 1.1, extending the numbers to multidigit multiplication is done by the column method which is a mixture of the unit (multiplier) in the

[4] Here, the magnitude is used for the size of the number such as larger or less in mathematics without indicating a concrete unit quantity on concrete situation such as just "3," not "3 marbles," which is called a denominate number (a number with "marbles" as the denomination for the unit of quantity).

[5] English translation of Otto Holder's German text (1901), Journal of Mathematical Psychology 40, 235–252 (1996).

[6] Tall (2013, 2019) sketched the process of reorganization on his terminology of three words of mathematics.

[7] In Japan, in the process of extension, the permanence of form has been enhanced in relation to mathematical thinking (see Chap. 1, Table 1.1) since 1956. It is used in the same way as the historical meaning of the extension of numbers, such as that described by George Peacock (for example, see Eves, 1997, p. 111).

multiplication table and addition in the base ten place value system. Extension to decimals and fractions causes overgeneralization of the definition of multiplication as repeated addition because addition of natural numbers always increases; however, it does not work with decimals and fractions. This is the problematic (which means "inconsistency" in elementary school mathematics, see Chap. 1) because multiplication of decimals (and fractions) does not always increase as repeated addition does on whole numbers. To overcome this, we need to follow the idea of the base ten system to find an alternative decimal unit such as 0.1 and $\frac{1}{10}$ to see it as a unit fraction, and the vertical form multiplication algorithm (the column method) using the multiplication table. In this process, for instance, if $9 \times 8 = 72$, then $90 \times 8 = 720$; associativity and commutativity can be used, such as $90 \times 8 = 9 \times 10 \times 8 = 9 \times 8 \times 10 = (9 \times 8) \times 10 = 720$. The distributive law is also necessary to introduce the multiplication algorithm (the column method), which will be explained in Chaps. 4 and 5.

3.1.2 Multiplicative Situations, Expression, and Translations

Formally, multiplication is a binary operation to get the product, just as addition is to get the sum. It is an expression in the world of mathematics without any concrete situation.[8] On the other hand, in applying multiplication in life, several meanings depending on the situation should be learned, particularly with regard to translations (interpretation) between the situation and the multiplication expression throughout the school curriculum. These meanings are usually expressed with everyday language to represent multiplication in situations and relations (mapping/arrow/correspondence) as a translation between situations and multiplication (expressions). Everyday language is necessary to represent reasoning in elementary school; it also brings limitations, such as the row of 1 in the multiplication table, which has already been mentioned. Here, we would like to consider several meanings of multiplication in relation to situations.

3.1.2.1 Origin of Written Situations

Multiplicative situations can be found in the ancient Babylonian language, Sumerian (Muroi, 2017), represented as *A a-rá B túm A*. Here, *túm* means "carry" and implies repeated addition. It means "A, B times" ($B \times A$), however, there were no expressions to represent it as a binary operation. Kazuo Muroi translated the following inheritance text for explaining the Sumerian sense of the base 60 system:

> There were 1,1,1,1 on base 60 system (= 219661 in base 10 system) rams and 13,13 on base 60 system (= 793) shepherd boys. How many rams did each boy receive? Each boy received 4,37 on base 60 system (= 277). There were 1,1,1,1 (= 219661) rams and 13 shepherd boys.

[8] Historically, the column method appeared much earlier than the expression.

> How many rams did each boy receive? Each boy received 4,41,37 (= 277 × 61 = 16897) There were 1,1,1 (= 3661) rams and 7 shepherd boys. How many rams did each boy receive? Each boy received 8,43 (= 523).

UET 5121 (from around the eighteen century BCE) was used in Muroi's Japanese translation; see also Figulla and Martin (1953) and Friberg (2007).

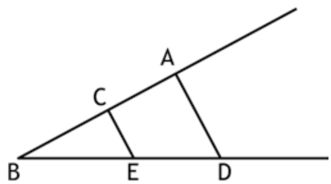
For finding the answers, the Sumerians used various tables on tablets; however, they did not write down the process of calculation. According to Muroi, the division of $a \div b$ is calculated as $a \times 1/b$ by using the reciprocal number table. For us, the quotation is a multiplicative situation; however, it is not the same as our multiplication as a binary operation. In division of the integers $a \div b$, a is not always divisible by b; it is a finite decimal or a recurring decimal. In the case of $1 \div 7$, this produces a recurring decimal. In the base 60 system, the numbers 2, 3, and 5 as factors of 60 are called *a-rá-gub-ba*, which means an ordinal factor. Seven in the base 60 system is the first number for which the reciprocal becomes a recurring decimal.[9] This implies that the number sense for multiplication in the base 60 system is not the same as that in the base ten system. For example, in the binary system, multiplication becomes addition. In this book, we focus on multiplication in the base ten system.

3.1.2.2 In Situations of Geometry with Proportionality

In Euclid's Elements, the idea of multiplication is discussed as "multiple/multiplicity" in the ancient Greek language in relation to ratio and proportion (Chemla, Chorlay, & Rabouin, 2016; Saito, 2008). It is *not* the same as the current meaning of multiplication in school, which is represented by expressions with "×" as the symbol of operation. During the era of Euclid, there was no algebraic expression. For example, a current expression such as $x^2 + a$ would have no meaning for Euclid because it would imply the addition of (a segment) to (a square). In the context of the Euclidian Elements, the product can be measured with a plane (two-dimensional) unit by associating the unit as measurable with multiplicity. For Euclid, measurable means the existence of the greatest common divisor.

To create algebraic representation as a universal language (mathematics), Descartes redefined the four operations as constructions with segments although he used "∝" instead of the current "=". Figure 3.1 was used for redefining multiplication in his book of geometry, published in 1637.

Fig. 3.1 Descartes (1637)



[9] Muroi mentioned that this is an origin of a myth which distinguishes 7 from other decimals.

In Fig. 3.1, BE:BC = BD:BA, then $BE \times BA = BC \times BD$. If BA is a unit, $BE = BC \times BD$. This is the definition of multiplication according to Descartes. This diagram was also used by Euclid. However, in the context of Euclid, $BE \times 1 = BC \times BD$ is acceptable because "an area = another area" but $BE = BC \times BD$ is not, because "a segment = an area" is inappropriate. Descartes established expressions beyond the limitations of dimension.

Descartes reorganized geometry as a part of his universal mathematics with algebraic expressions. His motivation was to shift mathematics from distinguished subjects such as geometry, arithmetic, astronomy, and music to algebra (universal mathematics). In his geometry, he needed to explain the appropriateness of using algebraic notation. In this context, the current meaning of multiplication, which is represented by expressions, becomes possible to use beyond Euclid.

We can extend Descartes's procedure of geometric construction to multiplication of negative numbers "$(-) \times (-) = (+)$" although the negative sign was not independently discussed during his time, unlike today.

3.1.2.3 In Situations with Quantities and Definition by Measurement

In the context of quantities, multiplication is the operation used to get the total quantity when the unit quantity and the number of units are known. This is the definition (explanation) in the Japanese curriculum documents, but it was not written in the textbook directory (Isoda, 2010). Here, we call it the definition of multiplication by measurement.[10] This definition degenerates to a group of groups or a set of groups, which was mentioned in Chap. 2, if it is limited to the natural numbers. It is consistent with Descartes's definition when we adapt it to geometric construction. If we apply this definition to measurement with geometric construction, it is to measure the length when the length of the unit and the number of units are known. Here, the length of the unit and the number of units can be real numbers if we extend the segments to lines (according to Euclid, the line can be extendable). On the other hand, a set of groups is usually imagined as whole numbers by students. Definition by measurement can be extended from natural numbers to real numbers. It does not contradict repeated addition such as a set of groups and can be applied to real numbers.

The Japanese textbooks from the third to the sixth grades use proportional number lines[11] (see Chap. 4 and Fig. 3.1) based on this definition (Isoda, Murata, & Yap, 2015, Grade 2, p. 9; Isoda & Murata, 2011, Grade 2, p. 9). Even in the second-grade textbooks, an approach to that meaning is provided by sentences such as "number of

[10] "Get the total quantity when the unit quantity and the number of units are known" is not actually a measuring activity; however, it is well connected with the proportional number line, which will be explained fully in Chap. 4. Definition by measurement is named by Shizumi Shimizu (Curriculum Specialist in the MEXT, personal communication). The definition was known in the 1960s at least (see Ito, 1968). Recently, Izak and Beckmann (2019) provided the same ideas for a world researchers.

[11] The proportional number line for elementary school mathematics was systematized by Ito (1972). By using the textbooks (Hitotsumatsu et al., 2005), Murata (2008) illustrated the tape diagram as the model for Zone of Proximal Development.

pieces of 3 cm tape and their lengths" (Isoda et al., 2015, Grade 2, p. 13; Isoda & Murata, 2011, Grade 2, p. 14). This definition is consistent with repeated addition when we limit the quantities to whole numbers or integers. If the measure and the value of the unit are natural numbers, the product can be seen as "repeated addition of the quantity corresponding to the unit" but when they are not, the definition applies to multiplication of decimals, fractions, and any measurement.[12] Both of these meanings have been written in the guidebook for the Japanese curriculum since the 1960s and can also be seen in Freudenthal (1983). For the extension of multiplication, this definition by measurement can also be applied to fractions and decimals with proportionality by using proportional number lines in Japan, serving as a mediational means (model/representation) for definition by measurement before formal definition of the proportion. Theoretically, the proportional number line is consistent with the Descartes[13] similarity in Fig. 3.1. Proportionality can be seen as the natural extension of multiplication in relation to definition by measurement.

Definition by measurement is not popular in the world. For example, in the Chilean curriculum (MINEDUC, 2013a, p. 152), repeated addition has been chosen as the definition. It looks like there is no inconsistency in interpreting the given example "In each of 6 boxes are 4 brushes, how many total brushes are there?" in the context of repeated addition rather than definition by measurement. However, the Chilean definition of repeated addition cannot be extended directly to decimals and fractions (see Chap. 5).

3.1.2.4 Contradictions between Repeated Addition and Situations with Quantities

In real-life situations, numbers usually appear with measurement units (quantities); these are called denominate numbers, such as "2 cups."[14] In this example, "2" is the number and "cups" is the denomination, with "a cup" as the unit of measurement to be counted. The "2" in "2 cups" can be seen as a mapping from the world of numbers in mathematics to the world of measurement in real life, setting the translation rule by seeing a cup as a counting unit. In this correspondence, the relationship of magnitude (greater than, less than, or equivalence to) is kept.

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[12] This works for real numbers. Multiplication of real numbers should be redefined for extension of real numbers to complex numbers.

[13] If the intersecting lines in Fig. 3.1 become parallel lines, they are proportional number lines. First Japanese translation of Descartes's Geometry was 1949 by Kouno.

[14] In mathematics (not in real life), quantity as magnitude is defined with the axiom of the magnitude relationship (the equivalence relationship and order relationship) without any physical unit quantity. In this section, quantity means the physical quantity and the quantities produced from physical quantities referring to a measurement quantity in real life where numbers are usually denominated with a measurement unit. In English, a denominate number such as "3 apples" refers to the measurement–quantity unit "apple," whereas in some other languages—such as Thai, Japanese, and so on—the measurement–quantity unit does not correspond to the denomination well. For example, "3 cups," "3 apples," "3 tomatoes," etc., in English are all said as *3 ko* ("3 pieces") in Japanese; *3 ko* is the denominate number. However, *ko* is not as clear as a measurement unit in English.

The Japanese definition of multiplication is introduced by situations with denominations by using measurement units such as the following: “If there are 3 apples for each dish (3 apples per dish) and 4 dishes, then the total number of apples is 12 apples.” It is not the same as just saying “3 apples and 4 dishes.” In the case of a number with a denomination, the situation can be represented by a physical expression such as “(dishes) × (apples/dish) = (apples).” Here, “per dish part” is canceled out by the quantity “dish” in the multiplication, and what remains is the measurement unit “apple.”

Multiplication is an operation in the world of numbers. However, with regard to interpretation in situations, it includes a metaphysical interpretation among physical quantities (measurement units) used in real life. As for the scaffolding used to support the interpretation and translation between a situation with physical measurement units and the world of mathematics, mathematical sentences of quantities such as “(dishes) × (apples/dish) = (apples)” are used even though they are mathematical informal–physical representations, which are not formally allowed as mathematical expressions in the world of mathematics.

The interpretation of “physical expression” in the situation (see Kobayashi, 1986) “(dishes) × (apples/dish) = (apples)” is inconsistent with the repeated addition of “(apples/dish)” in mathematics, which can be discussed as follows:

4 (dishes) × 3 (apples/dish) = 12 (apples)

≠ 3 (apples/dish) + 3 (apples/dish) + 3 (apples/dish) + 3 (apples/dish)[15]

≠ 12 (apples/dish), or ≠ (12 apples)/(4 dishes) =3 (apples/dish)

However, in mathematics textbooks, it will be as follows.

$$\underbrace{3\ (\text{apples}) + 3\ (\text{apples}) + 3\ (\text{apples}) + 3\ (\text{apples})}_{4\ (\text{dishes})} = 12\ (\text{apples})$$

This inconsistency is related to embedding the ways of explanation in the quantities (several measurements) in the situation into the world of number operations without quantity. In general, the quantity for a denomination such as apples can be added because the quantity implies the measurement unit for counting, which is an apple. However, the measurement unit (quantity) produced by the rate of different units such as “apples/dish” cannot be added. To avoid such inconsistencies, when repeatedly adding (((3 + 3) +3) +3), we should see only the part of apples by disregarding the part of the “every (or per) dish” in each term and counting “4 dishes” repeatedly. Thus, we can say that repeated addition is the way to find the product by regarding 3 “apples” and 4 “dishes” instead of regarding 3 “apples/dishes” in the situation even if it is hiding the idea to see “3 apples” as one set for the dish. The translation between situations and multiplication is only possible using specific ways of reinterpretation of the measurement unit in situations, just like the one discussed above (see Chap. 5).

[15] This sentence itself is inappropriate because the ratio of different units cannot be added.

3.1.2.5 Using the Situation of Multiplication Only for the Attribute of the Object

In the 1950s, the Association of Mathematics Instruction (AMI), Japan, proposed to introduce the meaning of multiplication using the attribute of the object in relation to the binary operation with their theory of quantity (Kobayashi, 1986) and asserted that multiplication is not repeated addition (see Chap. 1). For example, two wheels are an attribute of a bicycle. In this situation, the row of 2 in the multiplication table is represented by the total number of wheels when the number of bicycles is given. The row of 3 is represented by the attribute of a tricycle. In this manner, AMI proposed to choose the specific situation in relation to the attribute of a specific object which cannot be divided for each row by the attribute of the specific object for the introduction of the multiplication table. Even the row of 0, which is normally not in the multiplication table, is explained with the belly button of a frog because the frog does not have it.

In Chap. 5, we will revisit the treatment of the attribute of an object for multiplication in the case of the Chilean approach with a discussion of making sense (or sense making) (McCallum, 2018).

3.1.2.6 In the Situation of Area, As for Extension to Decimals and Fractions

As it will be discussed in Chap. 4, for the extension of multiplication to decimals, conversion between measurement units such as 1.5 L and 15 dL is useful because it changes decimals into whole numbers, which can be seen as repeated addition, and the multiplication table can be applied. Area (diagram) is also used for the extension to decimals and fractions.

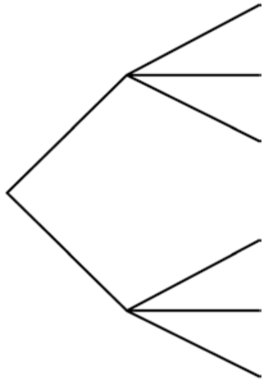
The area of a rectangle is defined by two perpendicular segments: $a \times b$, "length (longer side) × width," or "width × length." Before defining the area by multiplication, school textbooks usually introduce the dot array or block array diagrams[16] to explain multiplication (see Chap. 5). These array diagrams can be seen as a preparation to introduce the area (Mathematically, these can be seen as the idea for Cartesian Product: see 3.1.2.9). Conservation of the area of a rectangle in the dot array diagrams supports the commutative and distributive laws.

From the perspective of denominate numbers, the unit "1 cm^2" means the same area of the square as "1 (cm) × 1 (cm)." The number of unit squares in a rectangle with length 3 (cm) and width 2 (cm) is $3 \times 2 = 6$. Then, the area formula of a rectangle is "length × width." In the case of 2.5 (cm) × 1.2 (cm), it cannot be well represented by using the unit square "1 cm^2"; however, if we change the unit square to 1 mm^2 it means 25 (mm) × 12 (mm).[17] The area formula for a rectangle "length × width" supports the extension of multiplication from whole numbers to decimals

[16] The dot array diagram is also represented by parallel crosses.

[17] Japanese usually uses "dL" and "L" for the model diagram of decimals to show concepts such as $\frac{1}{10}$ L because 1 mm is too small for the model.

Fig. 3.2 Tree Diagram



and fractions through the permanence of form. In the case of 1.5 (cm) × 3 (mm) or 3 (mm) × 1.5 (cm), it has no meaning if they are expressed with different measurement units. Thus, we have to change them into 15 (mm) × 3 (mm) or 3 (mm) × 15 (mm). Changing the measurement units into the same quantity is a strategy for the extension of multiplication to decimals.

3.1.2.7 In the Situation of Tree Diagrams

In probability, multiplication can be applied in situations that can be explained by the tree diagram. In the tree diagram in Fig. 3.2, first there are two cases, then three cases that develop into six branches. If we use the term "splitting" for tree diagrams, one splits into two and then splits into three. This is written as 2 × 3. Based on the multiplication theorem of the probability for equally likely cases, it is written as $\frac{1}{2} \times \frac{1}{3}$. In tree diagrams, the operations 2 × 3 and 3 × 2 correspond to different diagrams and thus, area diagram is more preferable diagram explaining the commutativity of multiplication,

3.1.2.8 Seeing the Tree Diagram as an Operator

A multiplication on probability tree looks like an operator. In some situations, the symbol "×" shows processes such as "1 → (× 2) → 2" and then "2 → (× 3) → 6" in tree diagrams, and in situations of probability as "$1 \to (\div 2) \to \frac{1}{2}$" and then "$\frac{1}{2} \to (\div 3) \to \frac{1}{6}$" according to the multiplication theorem of probability in equally likely cases. Here, the process "→ (× 2) →" and "$1 \to (\div 2) \to \frac{1}{2}$" for indicating functions can be seen as operators. It implies that the "× 3" part of "2 × 3" or "1 × 2 × 3" and the "× $\frac{1}{3}$" part of "$\frac{1}{2} \times \frac{1}{3}$" or "$1 \times \frac{1}{2} \times \frac{1}{3}$" for showing the situations can be seen as operators.

In Indo-European languages, "× 2" should be written as "2×" and some prefer "2 ×" for showing the operator. Indeed, in $f(g(x))$, $f(x)$ is the operator for $g(x)$. However, "$1 \rightarrow (\div 2) \rightarrow \frac{1}{2}$" cannot be written as "$1 \rightarrow (2 \div) \rightarrow \frac{1}{2}$." In arithmetic, operators in "÷ 5" or "– 5" do not mean "5 ÷" or "5 –" because commutativity does not work for division and subtraction (except if it is an identity). For explaining the four arithmetic operations as operators, "× 2" is consistent usage with "÷ 2" in usage. Under the compartmentalization of knowledge, many Indo-European language users feel comfortable in using "2 ×" and "÷ 2" at the same time. However, preferring "× 2" is reasonable as long as it enhances the consistency of representations in the four arithmetic operations as operators. It is a kind of unary operator in mathematics. Indeed, even though European Language, the unary operator "^" is written in the right hand such as 3^2 (3 square). The matter of language will be discussed in the next section.

3.1.2.9 Activity of Elementary School and Cartesian Product

In Portugal's curriculum (Ministério da Educação e Ciencia, Portugal, 2013, p. 9) as mentioned in Chap. 2, the situations of multiplication for repeated addition are distinguished from those for combinatorics: "Solve one-step or two-step problems involving additive, multiplicative situations and combinatorial." The product of multiplication is also given by the counting activity in combinatorics: "Perform a given multiplication by fixing two disjoint sets and counting the number of pairs that can be formed with one element each by manipulating objects and by drawing." If we draw a diagram under this instruction, it should be a counting activity as shown in Fig. 3.3.

The combinatorial counting diagram in Fig. 3.3 can be seen as a part of tree diagram (Fig. 3.2). In the case of Portugal, it is introduced as another definition. In many countries, multiplication using a tree diagram is discussed after elementary school as combinatorics.

Watanabe (2003) explained Cartesian Product, A X B={(a,b): a∈A, b∈B}, as a meaning of multiplication. It can be seen from the perspective of probability tree because Fig. 3.3 can be seen from the perspective of ordered pares. On Cartesian Products, products by numbers of elements for A and B is a number of elements A X B. On set theory for Cartesian Products, commutativity and associativity do not work.

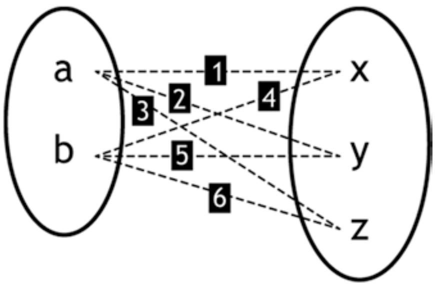


Fig. 3.3 Combinatorial counting

3.1.2.10 In Situations of Splitting as for Partitive Division

Confrey (1988) first presented splitting as a "multiplicative interpretation of partitive division" (p. 255) although repeated addition looks like a multiplicative interpretation of quotative division. Then, Confrey (1994, p. 292) defined splitting as "an action of creating simultaneously multiple versions of the original, which is often represented by a tree diagram." Confrey focused on the development of ratios and proportional reasoning, including scaling, similarity, and exponentiation. All of these involve the coordination of two or more quantities or dimensions, which may or may not consist of levels of units that are commensurable.

Harel and Confrey (1994) point out that the idea of disaggregating or splitting is a powerful tool for teaching multiplication, which favors the extension of multiplication to decimals and fractions, providing a geometric, and not only an arithmetical, view of multiplication.

According to Steffe (2003, p. 240), the splitting operation is the simultaneous composition of partitioning and iterating, where partitioning and iterating are understood as inverse operations. Steffe (2003) and Hackenberg (2007) provide definitions focused on the unit (and coordination of a unit of units). Steffe's splitting builds multiplication as repeated addition, based on counting, addition, and subtraction. The focus has been on the coordination of levels of units in students' development of fractions, assuming equal-sized groups.

According to Harel and Confrey (1994), the operation that determines the total number of elements arranged in groups of equal quantity is of multiplicative character.

Following Confrey, in Fig. 3.4, equipartitioning/splitting indicates cognitive behaviors that have the goal of producing equal-sized groups (from collections) or pieces (from continuous wholes) as "fair shares" for each of a set of individuals. Equipartitioning/splitting is not breaking, fracturing, fragmenting, or segmenting in which there is a creation of unequal parts. Equipartitioning/splitting is the foundation of division and multiplication, as well as ratios, rates, and fractions (see Chap. 4).

Confrey maintains that the technique of splitting promotes early work with units that are not a singleton, diminishing the difficulty that children have in conceptualizing ratios and proportions and other areas of multiplicative structures. For Confrey, the appropriate conceptions regarding ratios and proportions are built not on the basis of multiplication as repeated addition but, rather, as a parallel numbering system that can be developed on the basis of a splitting operation. Confrey postulates that the foundation of the parallel system is developed naturally by children, and that the nature of such a system could have a powerful effect on the comprehension

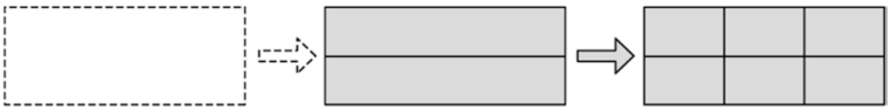


Fig. 3.4 Splitting equally: representation of 2×3 using splitting from the second rectangle to the third one. As well as the probability tree in Figs. 3.2 and 3.3, the splitting is consistent with multiplication as the operator: $1 \rightarrow (\times 2) \rightarrow 2$, $2 \rightarrow (\times 3) \rightarrow 6$. Here the unit for counting number 6 is a smallest part of the rectangle in the right

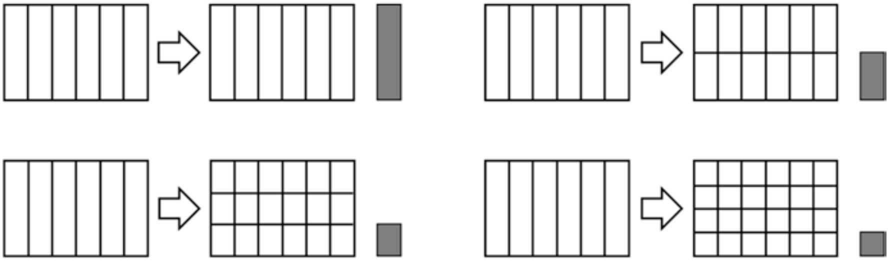


Fig. 3.5 Splitting changes the units' figures for products in the diagrams

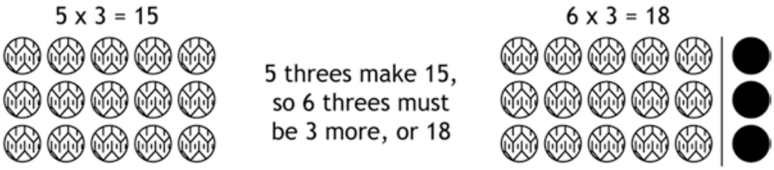


Fig. 3.6 Extending multiplication

of multiplicative concepts. Children could build a multiplicative world parallel to, complementary to, and interdependent on the additive world.

Splitting links multiplication and division because it includes the meaning of equal distribution (partitive division); however, it is inconsistent with repeated addition. Fig. 3.5 can be read as $6 \times 1 = 1 + 1 + 1 + 1 + 1 + 1$ and $6 \times 2 = 2 + 2 + 2 + 2 + 2 + 2$ and so on, but the basic units for counting the answers "6" and "12" are different. Splitting changes the unit of measurement before and after. In this context, the multiplicative world under the idea of splitting is consistent with equal division, partitive division, but independent of the additive world, as has been discussed regarding the rate of different units.Given this inconsistency with repeated addition, splitting in multiplication is inconsistent with definition by measurement according to the Japanese. Because splitting changes the units before and after multiplication, in Fig. 3.4, the whole rectangle on the left is 1 before the multiplication but is divided into 6 equal pieces after the second.

Considering this consequence, Portugal can be seen as a unique country as it introduces both meanings of multiplication (group of groups and combinatorics), as mentioned in the introduction to the discussion in Chap. 2.

3.1.2.11 Another Usage: Splitting in Relation to the Distributive Law

The terminology of "splitting" is also used in relation to the distributive law (van den Van den Heuvel-Panhuizen, 2001) but it is outside Confrey's claim in relation to partitive division. It is used in splitting, as in Fig. 3.6. Here, the knowledge of 5×3 (5 threes) helps to give meaning to 6×3: "If 5 threes make 15, how many are 6 threes?" For this expression, it is $5 \times 3 + 3$ and also can be seen as $(5 + 1) \times 3$.

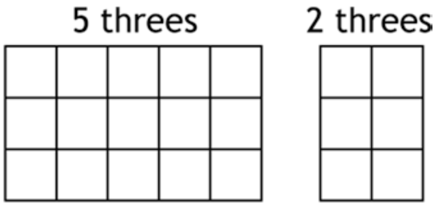


Fig. 3.7 Splitting and distribution

Table 3.1 Row of 2 and Row of 3 produce Row of 5

	1	2	3	4	5	6	7	8	9
Row of 2	2	4	6	8	10	12	14	16	18
Row of 3	3	6	9	12	15	18	21	24	27
Row of 5	5	10	6 + 9	8 + 12					

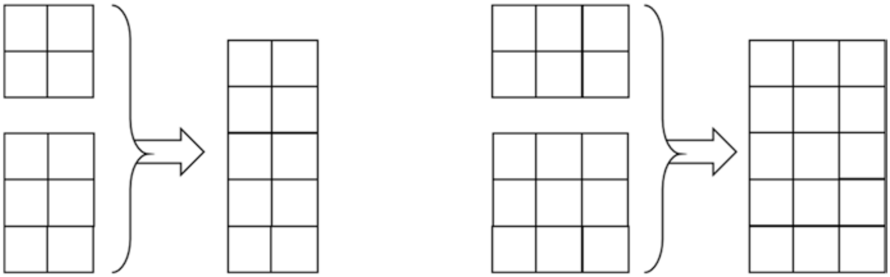


Fig. 3.8 Row of 5 from rows of 2 and 3 using the distributive law

On another usage of the word 'Splitting', it is used to explain the distributive law such as "If 5 threes are 15 and 2 threes are 6, then 7 threes must be 15 + 6, which is 21." $5 \times 3 + 2 \times 3 = (5 + 2) \times 3$. The splitting on meaning of distribution is a key idea to extend the multiplication table and multiplication for multidigit numbers (Fig. 3.7).

In the multiplication table, (row of 3) + (row of 2) = (row of 5) if we adapt the distributive law (Table 3.1).

Here, splitting is used for inverse operation of distribution but not for equal division. It keeps the unit for counting. It is consistent with the array diagram and area. Japanese textbooks such as those from Gakko Tosho (Hitotsumatsu et al., 2005; Isoda, Murata & Yap, 2015; Isoda & Murata, 2011) use this idea to enable students to extend the multiplication table and adopt it by and for themselves (see Chaps. 6 and 7). The activity for this meaning of splitting can be explained by the theorem in action for the distributive law (Vergnaud, 1990; see also Tall, 2013, pp. 183–188) (Fig. 3.8).

3.1.2.12 Limitations of Every Model for Multiplication

According to Freudenthal (1983), multiplication is used to find a number, called the product, that is to the multiplier what the multiplicand is to the unit, such as 6:3 = 2:1 (6 is to 3 as 2 is to 1). It is related to proportionality and is consistent with definition

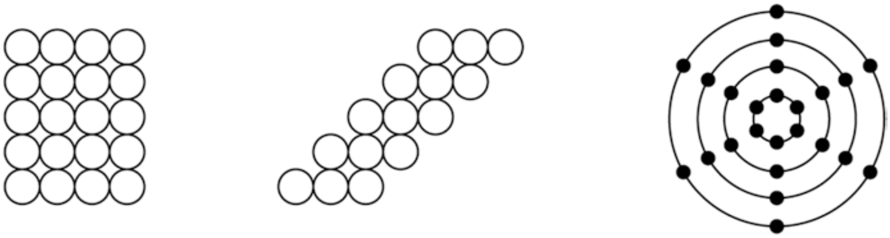


Fig. 3.9 Multiplication task variation (see Gakko Tosho textbooks—for example, Hitotsumatsu et al. (2005), Grade 2, Vol. 2, p. 12)

by measurement as used by the Japanese since the 1960s and Descartes's diagram (Fig. 3.1). In natural numbers such as 3 × 2, the multiplier 3 shows the number of repetitions of 2 for preferring multiplication in additive situations.

Even though the dot models in Fig. 3.9 can be used, there are various ways to find the units. In the models for repeated addition, all the units for counting should be seen as the same. Seeing the models from this perspective is possible when we have the idea of multiplication. At the same time, every model has its own nature as a mediational means. For example, the area diagram (model) for multiplication can be used for the extension of multiplication to decimals and fractions and positive (and 0) real quantities, and is appropriate to explain commutativity. However, it cannot be a model for multiplication of negative quantities. Descartes's constructions and proportional number lines, which are consistent with definition by measurement, can be applied for negative numbers, but commutativity cannot be seen instantly. Confrey links multiplication and partitive division; however, this is inconsistent with repeated addition because the unit for measurement changes.

From the viewpoint of magnitude, magnitude relationships (equivalent relations and order relations) can be illustrated by using models such as Descartes's construction, area diagrams, and dot diagrams[18] when their units of measurement are clearly embedded in the models. However, splitting and tree diagrams change their units. As Miwa (1983) mentioned, models function as a joint between mathematics and the real world. Gravemeijer (2008) discussed the model of a situation and the model for a form. Tall (2013) explained the conceptual difference and the development of the three worlds of mathematics by the terminologies "embodiment," "symbolism," and "formalism" and also the cognitive obstacles in one's development, which he termed "met-before." Freudenthal (1973) also explained the process of reorganization by mathematization. The Japanese use these inconsistencies as part of their curriculum content by explaining it as extension and integration for the opportunity to develop mathematical thinking (Chap. 1).

Depending on the context, the roles of the models are different. The number lines are bases for Cartesian coordinates to represent the changes in the graph of function and the figure defined by an equation. Descartes's construction of multiplication in his geometry is the origin of the Cartesian coordinates. Depending on the context, the roles of models change in the world of mathematics.

[18] Historically, Pythagorean schools used a dot diagram to represent properties of numbers.

Every model for multiplication has limitations in its nature. Every model for a specific situation is usually used for scaffolding. However, the reasoning when using the models is not the same as the formal reasoning even though they support mathematical–conceptual reasoning itself. In the case of Japan, the terminologies “concrete objects,” “semi-concrete objects,” and “abstract objects” have been used to discuss the different functions of the models and situations, and extension and integration have been the principles of the teaching sequence, corresponding to reorganization for mathematization.

3.1.2.13 Conceptual Fields for Multiplication

Vergnaud (1990) studied the conceptual field for multiplicative structures and distinguished three types of problems: isomorphism of measures, product of measures, and single measure space. This categorization provides a framework to distinguish conceptual difference in relation to multiplicative situations in teaching.

A problem of the first type, isomorphism of measure, is “A bag has 7 sweets. How many sweets are there in 6 bags?” A scalar resolution to the problem is “If there are 7 sweets per bag, in 6 bags there will be 42 sweets (7 sweets/bag × 6 bags).” A functional resolution is “If there are 6 bags, and in each bag, there are 7 sweets, then there will be 42 sweets (6 bags × 7 sweets/bag).” In the functional resolution, there is a movement from one measure (bags) to another (units of sweets). It is consistent with definition by measurement.

A problem of the second type, product of measures, is “We have 3 different shirts and 4 different skirts. How many combinations of shirts and skirts are possible?” This situation includes two fields of measurements that are composed without constituting a proportional function that associates the two fields. It is consistent with combinatorics and the probability tree.

A problem of the third type, unique measure space, is “Andres has thrice (3 times) the number of pencils that Jose has. How many pencils does Andres have if Jose has 4?” It is consistent with definition by measurement.

Vergnaud’s categorization for multiplicative situations can be also seen in our terminologies for meanings of multiplication in situations (see Figs. 4.20 and 4.21 in Chap. 4).

This section has illustrated various meanings of multiplication; however, it has not discussed the curriculum design itself. As explained in Chap. 1, these terminologies distinguish the difference of content necessary for considering the curriculum and the task sequence. For example, the framework of the multiplicative structure must distinguish combinatorics and others, and combinatorics is consistent with splitting. Such discussions are bases to establish the sequence, but it does not explain well why only Portugal’s curriculum introduces combinatorics from the beginning. The terminologies promote to distinguish conjectural difference but do not explain the curriculum sequence itself. The principle of extension and integration, or reorganization for mathematization, to develop mathematical thinking provide the sequence under the distinguished concepts (Chap. 1). The sequence will be discussed in Chap. 4 with further terminology.

3.2 Problems with Multiplication that Originate from Languages

Vygotsky (1934/1986, 1934/1987) and Wertsch (1991) enhanced the roles of mediational means to develop thinking. Under their perspective, children develop their mathematical thinking through mediational means used for communicating their language, such as speaking and writing, for making sense of what they learn.

Language enables us to verbalize numbers, such as "eleven, twelve, thirteen." However, writing does not always correspond with our way of speaking. For example, "twenty-five" is written as "25" under the base ten system and not as "205" (the way it is said). In English, as well as in other Indo-European languages, the way the four operations are spoken does not usually correspond to their algebraic expressions: "Add B to A" is A (augend) + B (addend), read as "A plus B." "Subtract B from A" is A (minuend) − B (subrahend), read as "A minus B." "Divide 12 by 4" is "12 (dividend) ÷ 4 (divisor)" but "multiplied 3 by 4" is "4 (multiplier) × 3 (multiplicand)": "–r" or "–d," which one is operato "–r"? Depending on Vygotskian claim, those inversions between grammatical structure and mathematical notation may set some limitations for learning mathematics in English, even though adults' users of English do not perceive any difficulties and inconsistencies in their usage. In Japanese, the grammatical expressions and algebraic expressions correspond well and there is no such inverted correspondence between their daily expression and mathematical expressions. In multilingual countries, the differences are more complicated. For example, the official language of Indonesia is inverted like English. However, like Japanese, the Javanese language of the central island of Java in Indonesia has no such inversion. In Javanese, 2 × 3 means "2, 3 times" as well as Japanese.

In the case of English and Spanish, the "×" symbol in the multiplication expression, which is read as "by" and *por* ("by"), respectively, does not necessary refer to the order of numbers. In English, there is no order if we say "multiply A and B."

However, if the expression is associated with the word "times" in English (or *veces* in Spanish) in real life—and, as such, the multiplier—the number of groups is placed to the left, as in Fig. 3.10. As for the language, there is a good ordinal correspondence.[19]

In Indo-European languages, when they introduce the multiplication table to be consistent with their languages, there is a syntactic contradiction between models A and B in Fig. 3.11 using "times."

The row of 2 in the multiplication table is usually shown below:

2 × 1 = 2, 2 × 2 = 4, 2 × 3 = 6, 2 × 4 = 8, 2 × 5 = 10, 2 × 6 = 12, 2 × 7 = 14, 2 × 8 = 16, 2 × 9 = 18.

[19] As we discuss later, the daily usage of language and algebraic expression do not always correspond. For example, in English (Latin), the limited words for multipliers (such as "single," "double," "triple," and "quadruple") already include the meaning of "times" but are not applicable to the multiplication of any natural numbers. In real life, "double" in tea implies 2 cups of tea, with 1 cup as the unit. As in "half of something," the "of" implies the multiplication symbol "×". "Multiply 3 by 2 to get 6" in daily usage is "3 multiplied by 2 equals 6" in an algebraic sentence. However, "multiply 5 and 2" enhances commutativity and does not consider the order of the multiplier and the multiplicand.

Fig. 3.10 "×" as "times"

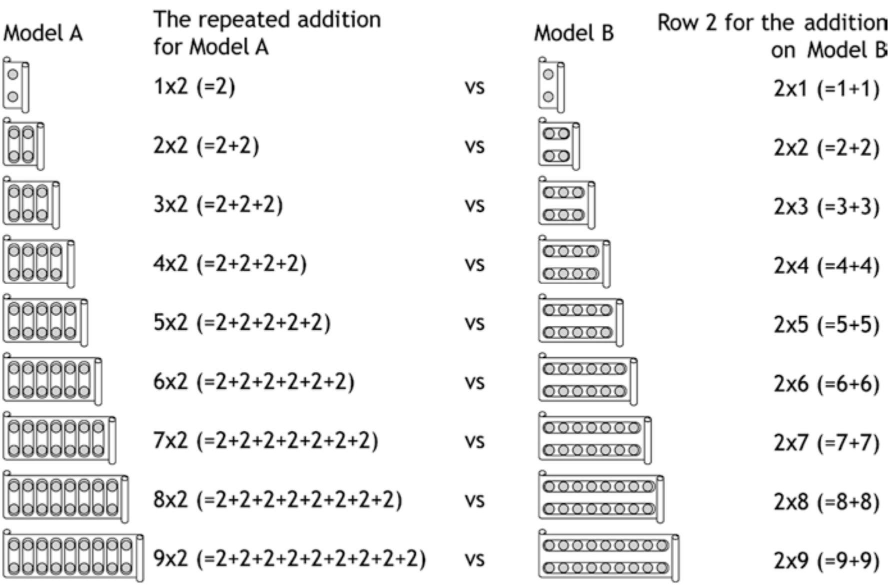


Fig. 3.11 What is repeated addition in a European language?

The difference between the consecutive products is + 2, the same as the multiplier. This property is used to proceduralize the sequence of the row of 2.[20] When we try to explain this constant difference with repeated addition, it will be understandable and reasonable for children to explain that "2 × 3 is 2 added 3 times, and 2 × 4 is 2 added 4 times, thus the difference corresponds to 2 added once." In model A, it should be written as 3 × 2 and 4 × 2. As long as we read the multiplication symbol "×" as "times," the appropriate interpretation based on the repeated addition is "2 × 3 [two times three] is 3 + 3, and 2 × 4 is 4 + 4, like model B. Every term increases by 1—that is, both 3s become 4s, and since there are two terms, the increase is 2." Adding 2 in repeated addition and the interpretation of the two terms will be contradictory for children as long as the definition of multiplication is repeated addition.

The reason for keeping the multiplier for the row number in the multiplication table—in this case, the multiplier for the row of 2—is based on multiplication in vertical form, which is called a multiplication algorithm in US English and the column method in UK English. For multiplying 43 × 2 in vertical form, the row of 2 is used for calculation from the lower line number 2 to the upper line number 43 (see Chap. 7). In multiplication in vertical form, multiplying from the lower number to the upper number is usually used not only in countries that speak Indo-European languages but also in countries that speak non-Indo-European languages, such as Japan.

In Fig. 3.11, the image of "increase by two in the row of 2" looks like model A. However, the row of 2 should be explained by model B. But model B cannot clearly explain the constant difference in the consecutive products. To avoid this contradiction that students may meet, there are two well-known traditional approaches:[21]

- The first approach enhances commutativity for applying repeated addition in model B: 2 × 1 = 1 × 2 = 2, 2 × 2 = 2 × 2 = 2 + 2, and 2 × 3 = 3 × 2 = 2 + 2 + 2.

[20] As explained briefly in Fig. 1.1 of Chap. 1, this procedure is known as an automatized algorithm. Proceduralization means to produce an algorithm with meanings. In the Japanese approach, "thinking about how to calculate" is an objective, as well as understanding and achieving proficiency. Thus, it is recommended that the procedure is produced by students on the basis of the meaning they already know (see Isoda & Olfos, 2009, pp. 127–144). The Japanese use the meaningful pattern increase by the unit for memorizing the multiplication table. In Eastern culture, historically, the table should be memorized using the Chinese–Japanese abacus. In Western culture, memorization in mathematics education is usually discouraged because the word "memorize" often implies "without understanding" and the table is used for reference. From the Eastern cultural perspective, Western images of memorization look like a stereotype discussion. In East Asia, historically, people only used Chinese characters for academic subjects. Even if the word pronunciations were the same, they could reason by applying different characters to represent appropriate meaning. People were able to distinguish the meaning from the visible characters. The current simplified Chinese (pinyin) changed the tradition. Hangeul, and French-based Vietnamese alphabets become phonograms that have no intrinsic meaning for characters. However they still keep the tradition of meaningful memorization.

[21] Several approaches to vertical form will be discussed in Chap. 7.

- The second approach prefers that in the row of 2 defined by model A, the multiplicand is the constant, here 2, to be consistent with repeated addition, such as the following: $1 \times 2 = 2$, $2 \times 2 = 2 + 2 = 4$, $3 \times 2 = 2 + 2 + 2 = 6$, $4 \times 2 = 2 + 2 + 2 + 2 = 8$, ..., $9 \times 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 18$.

Both approaches can be seen in countries that influenced from Indo-European languages and are considered to introduce the multiplication table reasonably. In the case of a number table without models and situations, the multiplication table is used to show the products of multiplication. In the multiplication table, if the multiplier is 2 and the multiplicand is 3, then the intersection, 6, is the product. In the multiplication algorithm in vertical form, mental calculation and mechanical writing of the products is necessary (see Chap. 7) and the first approach is preferred by many countries because it is well connected with the multiplication algorithm. Indeed, the order in the multiplication table, such as the multiplier and multiplicand, and the row and column, are related to multiplication and division in vertical form (the column method, algorithm, and long division) and the representation of proportion, $y = ax$ (see Chap. 4).

The reason why the second approach is not easily chosen is because it is inconsistent with the vertical form (column method), where multiplication is from the lower line to the upper line: If the row of 2 is "$1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6$, $4 \times 2 = 8$, $5 \times 2 = 10$, $6 \times 2 = 12$, $7 \times 2 = 14$, $8 \times 2 = 16$, $9 \times 2 = 18$," 43×2 in vertical form becomes upper line to lower line and making decision of applying row of 2 from $2 \times [\ \ ]$ to $[\ \ ] \times 2$. And if so, the proportion changes to $y = xa$.

Against these two approaches, splitting has been proposed as an alternative approach in place of the traditional approaches. Indeed, Portugal considers both repeated addition and combinatorics (similar to the tree diagram) in introducing multiplication in the second grade. Due to the inconsistency between models A and B, it may be reasonable that Portugal introduces a number of cases from the second grade. It may be complicated for some of students if different situations cannot be seen as one operation for them. It will be supportive if students can use the idea of splitting to find the product, such as to split a rectangle horizontally into 2 and vertically into 3 (see Fig. 3.4). In English and other European languages, only splitting and the tree diagram are not complete approaches, unlike the others, because they change the meaning of the unit and thus are not consistent with repeated addition. On the other hand, in the Japanese syntax, the notation under the Japanese grammar does not produce such inconsistences (see Fig. 3.12) (Isoda, Arcavi, & Mena, 2007, p. 281). If the Japanese notation 3×2, which is written 3 [×] 2 here, is translated into English, it means "3, two times." In Fig. 3.12, "the difference in the row of 3 is the constant 3 (the constant property of the difference)" is explained consistently with repeated addition as follows: 3 [×] 1 = 3, 3 [×] 1 + 3 = 3 + 3 = 3 [×] 2, 3 [×] 2 + 3 = (3 + 3) + 3 = 3 [×] 3, 3 [×] 3 + 3 = (3 + 3 + 3) + 3 = 3 [×] 4, and so on.

Thus, in Japanese notation, there is no contradiction between repeated addition and the property of constant difference between consecutive products.

In some Indo-European-language-speaking countries that are supported by the Japan International Cooperation Agency (JICA), the Japanese notation is preferred for overcoming contradictions. Because as the discussion on Section 3.1.2.8, and Fig. 3.10, from the perspective of division operator, multiplier will be seen as the

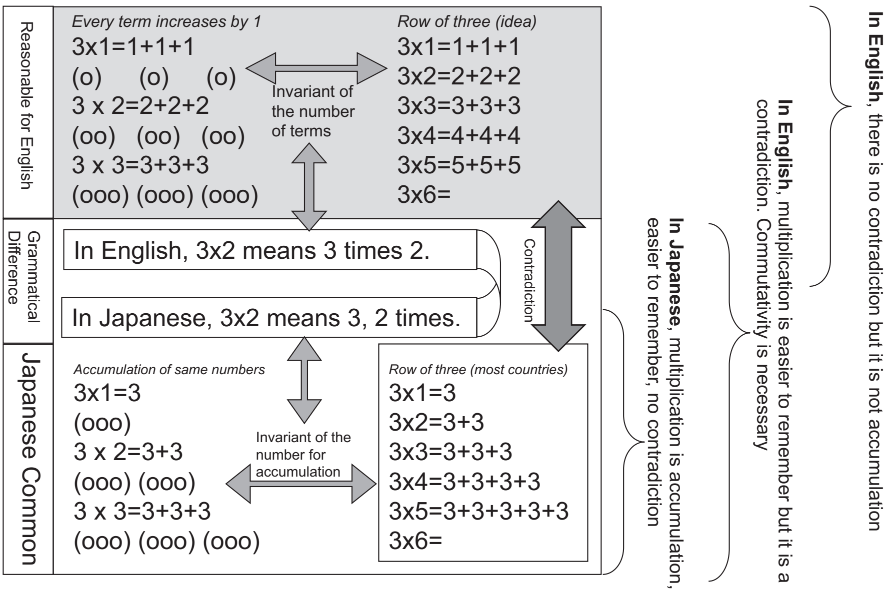


Fig. 3.12 Meanings and approaches of 3 × 2 in English and Japanese for applicable to traditional multiplication table and vertical form (column method) which multiply from the each digit in the bottom to the each digit in the top (see Chap. 7)

second number. For example, "'divide a by b' and then 'multiplied by c'" might be fine to be written as a÷b×c. It will be strange if we have to write is as c×a÷b in any time because it is read as 'c multiplied by a' and then 'divided by b'. In this approach, the terms "times" and *veces* create confusion in explaining and reading the multiplication symbol "×"; it should be read as "multiplied by," "by (por)," "of," or "and" instead of "times (*veces*)") because originally 3 [×] 2 meant "3, two times." Here, we cannot read the symbol "×" as "times." These syntactical changes are preferred by the curriculum departments in governments that have had deep discussions on historical tradition and current convenience. These countries use multi-languages on their histories and enhance the commutativity of multiplication.

The problem of inconsistency in English and Spanish originated from the difference between natural languages and mathematical notation.[22] Several difficulties might appear because the natural language should be preferred in school mathematics at the begging for referring to situations with quantities in real life. In the world of mathematics without situations, such confusion never appears.[23] Problematic appears in Indo-European languages but not in Japanese.

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[22] Fischbein, Deri, Sainati, and Sciolis (1985, p. 5), and Vergnaud (1990) also discussed the problematics of English but did not mention other languages. In this book, the roots of these contradictions are discussed in Chaps. 6 and 7.

[23] In informatics as a scientific language, mathematical notation itself can be changed. In programming language, "=" usually means substitution. In metanotation in informatics, there are Polish notations, reverse Polish notations, and others such as normal mathematical notations.

In theoretical arithmetic under normal mathematical notation, natural numbers are defined by the inductive definitions of Dedekind and Peano, and, in theory, the product is deduced inductively "$M \times (N + 1) = M \times N + M$" (Olfos, 2002). In this compete-inductive definition of multiplication, which is the same as the constant property in the table, the Japanese multiplication notation in Fig. 3.12 is consistent with the mathematical notation. The expression 3×4 refers to a group of three as the unit and 4 as the number of groups/units. Consequently, $3 \times 5 = 3 \times (4 + 1) = 3 \times 4 + 3$, as the unit is added to the initial groups. On the other hand, as previously mentioned, the English usage of "times" corresponds to $3 \times 5 = 5 + 5 + 5$. To see the sequence increase by three in English notation as for the repeated addition of 3, it must be changed, like $3 \times 5 = 5 + 5 + 5 = (4 + 1) + (4 + 1) + (4 + 1) = 3 \times 4 + 3$. It is an interpretation far from the inductive definition of multiplication.

The inconsistencies of expressions between natural language and mathematical notation in the Indo-European languages are problems not only for multiplication but also for the other three operations, as already mentioned. These inconsistencies produce difficulty for explanation of arithmetic in the said languages. As a consequence, there are projects that prefer the Japanese notation system in Central America, Thailand, and other places. To maintain consistency between language and mathematics, Japanese textbooks have established a sequence for extension that can be seen as attractive in being understandable (see Chap. 4).

If you feel uncomfortable about discussion of the Japanese notation of multiplication and not your notation, this is because of your familiarity with your mother tongue. However, we should note that our acquired usage itself can be seen as the result of our achieved curricula. There are various approaches for solving the matters in Fig. 3.12. There are further reasons why the Japanese approach is selected by some countries[24] as an alternative approach, like the idea of splitting in the US approach and combinatorics in Portugal. One reason is consistency of definitions with the extension of numbers and operations, and another reason is consistency with the multiplication table. Other reasons such as consistency of multiplication in vertical form and division and so on will be clearly illustrated in Chaps. 4, 5, 6, and 7 with explanations of the Japanese approach. The Japanese approach has rationality but it is one of the various existed approaches. The National Curriculum on Colombia introduce multiplication as 'multiplier x (multiplied by) multiplicand' at the lower grade and then,upper grades, treat 'multiplier' like an operator in relation to 'divisor' (see Section 3.1.2.8). Such an approach is normal for Latin America. On the next section, we would like to discuss the historical usages and influence to Chile.

[24] In Latin America, the countries of Honduras, Guatemala, El Salvador, Nicaragua, the Dominican Republic, and Mexico prefer Japanese notation based on Japanese textbooks.

3.3 European Languages and Their Historical Usages

Depending on historical origins such as languages and developments,[25] there have been various textbooks in different periods and regions that placed the multiplicand on the right, while others placed it on the left. In the thirteenth century, Ibn al-Bannā (from Almohades (Morocco), which included a part of Spain) explained a procedure for multiplying in columns (grids) by placing the multiplicand at the top of the column and the multiplier to the left or to the right of the column; this was later known as "Napier's bones" or "rods of Napier (1617). Ulloa (1706) indicated that the multiplier was placed below in the column algorithm. Before the predominance of modern mathematics, there were texts in Spain that presented the multiplier after the multiplicand, as the second number (Rey Pastor & Puig Adam, 1935). With the arrival of set theory, the language changed, and inconsistencies appeared in mixing arithmetic language with algebraic language. Prima-Luce (1976) stated, "We call a 'product' the cardinal of the Cartesian product. The second factor is called the multiplier. The first factor is called the multiplicand. $2 \times 3 = 3 + 3 = 6$." (Prima-Luce, 1976). There are two inconsistencies in the above description: maintaining names connected to the contexts together with formal language and exemplification with an inappropriate numerical representation.

The representation of "two hundred" can be seen as "2 times 100." Spanish grammar accepts this, saying *dos manzanas* for "two apples" although nouns usually come before adjectives in Spanish, as in *manzana roja* ("apple red" rather than "red apple"), which involves a kind of rupture. $2A$ is $A + A$ in algebraic notation. However, in the first grade, students learn arithmetic operations starting with situations like "Add something to A" or "Take something away from A." So, $A + B$ and $A - B$ initially are represented by situations that add B to A or take B away from A. A is the noun or the subject to be transformed, so A comes before B. If we adopt this approach to $A \times B$, it is possible for Spanish (Roman) to see "A" as the multiplicand, as in Japanese, because the action is done by B, the multiplier, as in the previous discussion of the operator. In reciting "2 times 3, 6," "2 times 4, 8," "2 times 5, 10" the number 2 can be seen as being multiplied by several numbers as the action. The sequence of results is 2, 2 + 2, 2 + 2 + 2, and so on. In this instance, it is like the probability tree that was discussed earlier in this chapter. In this manner, "2 times 3" implies "2, three times"; "three times" looks like part of the operator, and the first number 2 looks like the multiplicand in Japanese. In Spanish, $A \times B$ as "A times B" and "A multiplied by B" provide a polysemy, which affects the meaning of the

[25] European languages can be divided into Latin–Roman, Indian Europe (for example, German, English, and Nordic languages), and Slavic. Some languages such as Finnish and Hungarian are independent of these categories. Here, we are referring to Latin–Roman and Indian Europe, especially Spanish. Cajori (1928) explained that multiplication symbol "×" was introduced by Oughtred (1631, used Latin Edition, 1667). Oughtred used column multiplication for number and introduced "×" for his algebraic notation. He mentioned factor at introduction and discussed his column multiplication. He did not used symbol "×" for column multiplication. He discussed significance of multiplication for logistics and estimation of multiplicand and calculation of multiple on the column (p.8). It implicates that multiplicand comes upper and multiplier comes lower on column. See (Chap. 7).

expression for the multiplication table. Consequently, even students misinterpret the pattern $2 \times 6 = 2 \times 5 + 2$ as $(2 + 2 + 2 + 2 + 2) + 2$. It is not necessary to say "it should be 6 + 6" if they can see it like this in this situation. This is a possible reason why Central American countries such as Honduras, Guatemala, Nicaragua, and El Salvador prefer the Japanese notation of multiplication in JICA projects.

Freudenthal (1983) highlights the fact that the language of mathematics differs greatly from everyday language used in different countries where it has developed, and adds that the divergence between a natural language and mathematical language can in fact create learning difficulties. He also points out that "4 + 3" is a strange way to write the task "add 3 to 4," which mathematically indicates the sum of 4 and 3, and that everybody reads "four plus three" even though it does not agree with their language (English or German, and also Spanish or French). At the beginning of the twentieth century, "7 − 4" was read in German as *vier von sieben* ("4 from 7"). These antecedents are indications that in German and English, it would be natural to write the subtrahend and then the minuend, and by analogy the multiplier would precede the multiplicand.

With regard to Spanish, which originated in Castile, Vallejo (1841, p. 26) wrote, "The expression '5 − 3 = 2' means that after removing 3 units from 5, 2 are left, and is read 'five minus three equals (or is equal to) two.'"

Anglo-Saxon languages differ from Latin languages. Base twelve English measurement systems and base eight Spanish playing cards are remnants that predate the Indo-Arabic decimal system, which penetrated Europe through southern Spain. The Arab invasion of Spain during the eighth century brought the decimal system with its operative algorithms and modalities of oral expression, which surely conflicted with the existing European languages.

Research around 30 years ago by Fischbein, Deri, Sainati, and Sciolis (1985, p. 5) and Vergnaud (1990) revealed that differences between the multiplier and the multiplicand are at the root of different complexities presented by multiplication problems (which we mention in Figs. 3.11 and 3.12) and influence the decision of anticipating the operation that needs to be made.

3.3.1 The Transition in Chile

Chile inherited the Spanish language in the nineteenth century, along with textbooks that place the multiplicand first, on the left. Later, with North American influence and the universality of the International Commission on Mathematical Instruction (ICMI), Chilean mathematics programs in 1968 (MINEDUC, 1968) introduced multidigit multiplication in the fifth grade and used the term "factor" together with algebraic terminology with the idea of the multiplier on the left.

The current Chilean mathematics programs (MINEDUC, 2013b, 2013c) maintain the introduction of multiplication with the term "factor" and do not use the terms "multiplicand" and "multiplier." The current programs for the third and fourth grades identify the word "factor" as a key term, which is cited more than a dozen times in each program.

The mathematics program in the national curriculum for the second grade in Chile (MINEDUC, 2013a) presents multiplication as repeated addition, without

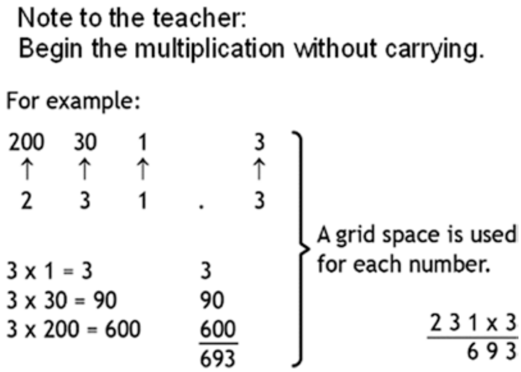

Fig. 3.13 Change the direction of multiplication from "left to right" to "right to left" for using the multiplication table

establishing associations between the factors and the meaning that each of them can take on. It does not establish connections between the number of groups and the term "times." It does not identify "times" as a pattern associated with a multiplier and a multiplicand.

The idea that the multiplier goes on the left is observed in the examples, without the term being mentioned, coinciding with the approach of textbooks from Singapore. In the second grade, the program says, "Demonstrate understanding of multiplication. using concrete and pictorial representations; expressing multiplication as the addition of equal addends. to construct the multiplication tables for 2, 5, and 10." The program for the third grade adds, "to construct the multiplication tables up to 10." The program for the fourth grade says, "Demonstrate understanding of multiplication of 3-digit numbers multiplied by 1-digit numbers" and the program for the fifth grade adds, "of 2-digit numbers multiplied by 2-digit numbers."

The same Chilean mathematics program for the fourth grade (MINEDUC, 2013c, p. 66) currently presents as an example the calculation "231 × 3," beginning the calculation on the right, although the multiplication table is introduced with the multiplier on the left, as can be observed in the bottom part of Fig. 3.13.

In Chile, textbooks and even curriculum standards have adapted influences from other countries and the tendencies in mathematical education of each period. Simultaneously, old textbooks still circulate in the country. In some textbooks and in the language of some parents and tutors, the teaching of the multiplication table—and, to a greater degree, the use of the procedure for multiplying from right to left—persist with the multiplier on the right. Despite all of these, for adults (and even for primary mathematics teachers), "3 × 2" means "3 times 2" and "3 multiplied by 2" without distinction, as the order of the factors does not change the product.

3.4 Final Remarks

This chapter has addressed the problematics in the conceptualization of multiplication in school mathematics—including definition of multiplication by measurement, various meanings of multiplication, and the problem of syntax in relation to languages and grammar—and has discussed historical transitions and adaptations to a country such as Chile.

The discussions of those problematics provide some answers to the related questions posed in Chap. 2; however, this chapter has not mentioned the curriculum and the task sequence themselves, which are necessary to consider for designing lessons. The mathematical terminology in this chapter provides a basis for the necessity to consider the Japanese approach in Chaps. 4, 5, 6, and 7. The terminology for the curriculum and the task sequence will be discussed in Chap. 4.

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