Chapter 2
Multiplication of Whole Numbers in the Curriculum: Singapore, Japan, Portugal, the USA, Mexico, Brazil, and Chile

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This chapter shows how the teaching of multiplication is structured in national curriculum standards (programs) around the world. (The documents are distributed by national governments via the web. Those documents are written in different formats and depths. For understanding the descriptions of the standards, we also refer to national authorized textbooks for confirmation of meanings). The countries chosen for comparison in this case are two countries in Asia, one in Europe, two in North America, and two in South America: Singapore, Japan, Portugal, the USA (where the Common Core State Standards (2010) are not national but are agreed on by most of the states), Mexico, Brazil, and Chile, from the viewpoint of their influences on Ibero-American countries. (The National Council of Teachers of Mathematics (NCTM) standards (published in 2000) and the Japanese and Singapore textbooks have been influential in Latin America. Additionally, Portugal was selected to be compared with Brazil). To distinguish between each country's standard and the general standards described here, the national curriculum standards are just called the "program." The comparison shows the differences in the programs for multiplication in these countries in relation to the sequence of the description and the way of explanation. The role of this chapter in Part I of this book is to provide the introductory questions that will be discussed in Chaps. 3, 4, 5, 6, and 7 to explain the features of the Japanese approach. (As is discussed in Chap. 1, the Japanese approach

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includes the Japanese curriculum, textbooks, and methods of teaching which can be used for designing classes, as has been explored in Chile (see (Estrella, Mena, Olfos, Lesson Study in Chile: a very promising but still uncertain path. In Quaresma, Winslow, Clivaz, da Ponte, Ni Shuilleabhain, Takahashi (eds), Mathematics lesson study around the world: Theoretical and methodological issues. Cham: Springer, pp. 105–22, 2018). The comparison focuses on multiplication of whole numbers. In multiplication, all of these countries seem to have similar goals—namely, for their students to grasp the meaning of multiplication and develop fluency in calculation. However, are they the same? By using the newest editions of each country's curriculum standards, comparisons are done on the basis of the manner of writing, with assigned grades for the range of numbers, meanings, expression, tables, and multi-digit multiplication. The relationship with other specific content such as division, the use of calculators, the treatment of multiples, and mixed arithmetic operations are beyond the scope of this comparison. Those are mentioned only if there is a need to show diversity.

2.1 Comparison of Curricular Standards' Descriptions for Introducing Multiplication in Different Countries

In the various programs (National Curriculum Standards and related documents), the meaning of multiplication is usually given using situations for multiplication. The way to find the answer (product) of multiplication is known as repeated addition. However, the sequence of descriptions and ways of explanation are very different in terms of their format and terminology. Thus, here, we would like to briefly illustrate the differences in format, terminology, and ways of explanation in their introduction of multiplication.

In Singapore (Ministry of Education, Singapore, 2012), the term "multiplication" appears with "division" as one joint category—"multiplication and division"—from the first grade until the third grade. In the first grade (Ministry of Education, Singapore, 2012, p. 35), multiplication and division are explained: concepts of multiplication and division, use of "×," multiplying within 40, dividing within 20, and solving one-step word problems involving multiplication and division with pictorial representations. The students should be given opportunities to experience the following:

- Making equal groups using concrete objects and counting the total number of objects in the groups by repeated addition using phrases such as "2 groups of 5" and "2 fives"
- Sharing a given number of concrete objects/picture cutouts and explaining how the sharing is done and whether the objects can be shared equally
- Dividing a set of concrete objects into equal groups and discussing the grouping and sharing concept of division

In Japan (Ministry of Education, Culture, Sports, Science and Technology; MEXT, 2017), the program is written in two categories: (A) knowledge and skills;

and (B) competencies for thinking, making decisions, and representing. Mathematical activity is autonomous and objective oriented, with the aims of self-directedness, interactivity, and deep learning. In this framework, multiplication is introduced in the second grade as described below.

Students should be nurtured to be able to acquire the following through mathematics activities for multiplication (MEXT, 2017, p. 51):

- To acquire the following knowledge and skills:
  - Understand the meaning of multiplication and know situations where multiplication is used
  - Represent situations where multiplication is used with algebraic expressions and interpret these expressions
  - Understand simple properties that hold for multiplication
  - Learn the multiplication table up to 9 × 9 and multiply 1-digit numbers accurately
  - Know ways of multiplication of a 2-digit number by a 1-digit number in simple cases
- To acquire the following competency for thinking, making decisions, and representing:
  - Focusing on mathematical relations, thinking about the meaning and ways of calculation (operation), finding the properties of multiplication, and, by using these properties, utilizing calculation and confirming the result of calculation
  - Focusing on mathematical relations and utilizing multiplication in daily life

In Portugal (Ministério da Educação e Ciencia, 2013), the teaching of multiplication begins in the second grade. In the first grade, addition and subtraction are written in different categories but in the second grade, they are written in the same category. Multiplication and division appear in different categories in the second grade. This implies that depending on the grade level, the content is seen from different perspectives. The term "multiplication" is explained for the second grade as follows (however, this clearly implies that the content is short) (Ministério da Educação e Ciencia, Portugal, 2013, p. 7):

- Additive and combinatorial meaning
- The symbol "×" with the terms "factor" and "product"
- The product of 1 and 0
- Multiplication tables for 2, 3, 4, 5, 6, and 10
- The terms "double," "triple," "quadruple," and "quintuple"
- One- or two-step problems that involve multiplicative situations in the additive and combinatorics sense

In the USA, instead of the national standards, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers (NGAC and CCSSO), 2010) have a hierarchy—grade, domain, cluster, and standards—and formal multiplication is

specified in the third grade. The domain "Operations and Algebraic Thinking" has seven standards, categorized into the following three clusters for multiplication:

- Representing and solving problems involving multiplication and division
- Understanding the properties of multiplication and the relationship between multiplication and division
- Multiplying and dividing within 100

In the other domains, multiplication is also mentioned. Additionally, the memorization of multiplication tables is done in the third grade.

In Mexico (Secretaría de Educación Pública, 2017a, 2017b), the curriculum framework is written under the following hierarchy: domain, learning expectation, didactical orientation, and assessment. Multiplication is introduced in the second grade.

The learning expectation for multiplication is as follows (Secretaría de Educación Pública, México, 2017a):

- Solve multiplication problems with natural numbers less than 10

In didactic orientation, the following processes of teaching are described:

- Problems regarding repeated quantities
- Making multiplication explicit
- Problems of counting in rectangular arrays
- Mental calculation and application of the product of digits in the cases of 5 and 2

In Brazil (Ministério da Educação, 2017), a National Common Curricular Base is written under the following hierarchy: thematic units, object of knowledge, and abilities according to the competencies of mathematics.

For the thematic unit of numbers in mathematics in the second grade (Ministério da Educação, 2017, pp. 280–281), the object of knowledge is as follows:

- Problems that imply addition of equal groups (multiplication)
- Problems that imply meanings of double, half, triple, and third

The abilities are:

- To solve and elaborate problems of multiplication (by 2, 3, 4, and 5) with the idea of adding equal parcels by means of strategies and forms of personal registry with or without the support of images and/or manipulative materials
- To solve and elaborate problems involving double, half, triple, and third, with the support of images or manipulative materials, using personal strategies

In the Chilean Curricular Framework (MINEDUC, 2012, p. 104) for the second grade, the skills that students should be able to acquire to show that they understand multiplication are as follows:

- Using concrete and pictorial representations
- Expressing multiplication as the addition of equal summands

- Using the distributive property as a strategy for building the multiplication tables of 2, 5, and 10
- Solving problems that involve the multiplication tables of 2, 5, and 10

Based on the comparison of the programs above, there are differences in the format and terminology used to introduce multiplication.

- Some countries do not mention the meaning of multiplication. Combinatorics meaning is uniquely introduced in Portugal. In Japan, the word "meaning" is just said without referring to any idea. What is the meaning of multiplication?
- Some countries introduce complete multiplication tables in the same grade while others introduce 2 and 5 in the upper grades or in several grades. How they are different.
- Some countries introduce factors and others do not. Why?
- There is a variety in the ranges of numbers when introducing multiplication. In Brazil, multiple is mentioned as half and third. It already implies multiplication of fractions when it is introduced. In the case of other countries, how do they prepare to extend multiplication to larger whole numbers, decimals, and fractions?

Those differences provide some perspectives on what we shall compare.

With regard to differences in the format and terminology, as well as the grade level when multiplication is introduced, there are differences in the depth of writing. For example, some countries discuss the method of teaching and assessment of their programs, and others do not. For comparison of the depth of writing, their guidebooks and authorized textbooks should be referred to.

2.2 Comparison of the Assigned Grade Levels for Multiplication

How are the assigned grades and teaching sequence of multiplication different? The tables in this chapter—in relation to the range of numbers, meanings, tables, and multidigit multiplication—highlight the differences in specific aspects in dealing with multiplication according to the programs of the countries.

2.2.1 Range of Digits

When is it possible to discuss multiplication and multidigit multiplication? We can use the number of digits as an indicator (Table 2.1). Addition and subtraction are deeply dependent on the base ten place value system for making ten. On the other hand, the counting unit in multiplication is produced depending on the multiplicand and is not only limited to ten as a unit. The numbers up to 100 are necessary for

Table 2.1 School grades in which different countries introduce extensions of the range of digits

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Up to 100 or so	1	1	1	1	1	1	1
Up to 1000	2	2	2	2	2	–	2
Up to 10,000	4	3	4	–	3	2	–
Up to 100,000	–	4	5	–	4	3	–
Up to 1 million	–	5	–	3	5	–	4
Up to thousand, or millions	5	–	–	4	–	3	–

learning the multiplication table, and further extension of numbers produces the need for multidigit multiplication. For teaching multiplication of a 3-digit number by a 2-digit number, it is necessary to deal with numbers up to 100,000.

Based only on the viewpoint of the range of numbers, if the multiplication table is up to 10 by 10, it is not impossible to introduce the multiplication table from the first grade in all countries. When do they actually introduce multiplication?

2.2.2 The Meaning of Multiplication

For discussing the meaning of multiplication, every country introduces counting by 2s or by 5s in the first or second grade. Most countries, except Singapore and the USA, introduce multiplication from the second grade. The USA begins to introduce repeated addition in the second grade but the definition of multiplication is provided in the third grade. Singapore introduces multiplication within 40 in the first grade.

Multiplication as a group of groups and length based on unit length in a tape diagram are not mentioned in Mexico. An array diagram with a rectangular shape is not mentioned in the Portugal and Singapore programs. The area of a rectangle is introduced in the upper grades in every country. Combinatorics is discussed only in Portugal, and proportionality is mentioned only in Brazil and Japan (Table 2.2).

From Table 2.2, the following questions emerge: Is an array diagram an alternative for a group of groups? Is repeated addition the same as a group of groups? How do the different meanings contribute to understanding of multiplication? Is repeated addition the only way to obtain the result when multiplication is introduced? Why is combinatorics used when multiplication is still being introduced? Is the use of equal amounts in a group the way to explain the meaning of multiplication?

Table 2.2 School grades in which different countries introduce various meanings of multiplication

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Counting by 2s	1	1	1	2	1	1[a]	2
Situation for adding equal quantities	2	2	2	2	1	2	2
Repeated addition	2	2	2	2	1	2	2
Group as a unit or group of groups (without repeated addition)	2[a]	–	3	2	1	2	3
Length based on unit length (tape diagram)	2	–	3	2	1	2	3
Array diagram (or rectangular shape)	3	2	3	–	–	2	2
Area (rectangle)	4	4	5	3	3	4	3
Proportionality	–	–	4	–	–	3[b]	–
Combinatorics	–	–	–	2	–	–	–

[a]Includes interpretation of examples provided in the guidebook
[b]Means a proportional number line or tape diagram

Table 2.3 School grades in which different countries introduce mathematical expressions of multiplication

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Use of "×" symbol	2[a]	2	3[b]	2	1	2	3
Multiplication expression explained in Table 2.2	2?	2	3[b]	2	1?	2	3
Multiplier Multiplicand	–	–	4 5	–	–	2	–
Product	4	2	5	2	3	2[c]	3
Factor	4	3	–	2	4	–	3

[a]In the second grade, Chile uses the dot, for instance a Croix, as a multiplication symbol
[b]This is not explicit in the National Common Curricular Base
[c]In Japan, the term "product", as well as sum, difference and quotient, is the content of teaching for students at the fourth grade. Until fifth grade, student call it as a value of multiplication. On the other hands, the term "product" itself is appeared from the second grade in the guidebook

2.2.3 The Definition of Multiplication

The "×" symbol is introduced in different grades in different countries (see Table 2.3). In particular, the USA introduces repeated addition in the second grade, and introduces multiplication and symbol "×" in the third grade. Brazil, on the other

hand, introduces them in later grades. It is only in Japan that the multiplier and multiplicand are introduced at the beginning. In contrast, factors are not introduced in Brazil and Japan.

As a binary operation, multiplication is defined as $N \times N$ and provides the image/value for every pair of natural numbers. The numerical expression of multiplication can be introduced with various meanings as provided in Table 2.2. The meaning is not only limited to repeated addition. Some countries introduce different terms such as factors, multiplier, and product providing exemplary numerical expressions with the symbol "×" for introducing multiplication. Why do some counties use it? How do the terms, symbols, and explanations contribute to the definition of multiplication?

2.2.4 Multiplication Tables

Multiplication tables are completed up to the second or third grade (see Table 2.4). Singapore spends 3 years on them, while Portugal, the USA, and Japan spend only 1 year.

Why do some countries divide the multiplication table into several grades and others do not? What criteria are used to select the order for discussion of the table?

2.2.5 Use of Algorithm or Column Method for Multiplication

The teaching of techniques to calculate the product of multidigit numbers varies among countries (see Table 2.5). Japan starts to teach T0 × U in the second grade (the simplest case of TU × U); the others start in the third grade. Some countries require 3 years, such as Chile, and others only 1 year, such as Portugal.

Comparing the countries, questions emerge, like the following: Why is the product of a 3-digit number by a 2-digit number considered? Why is KHTU × TU not considered?

Table 2.4 School grades in which different countries introduce multiplication tables

	School grade in which concept is introduced						
Country	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
The rows of 2 and 5	2	2	2	2	1 (2)	2	3
The rows of 3 and 4	3	2 and 3	2	2	2	2	3
The rows of 6 and 8	3	2 and 3	–	2 and 3	3	2	3
The rows of 7 and 9	4	3	2	2	4	2	3
Multiplying by 10	2	3	3	3	2	2	3

Table 2.5 School grades in which multiplication of multi-digits of numbers and vertical forms are introduced

	School grade in which concept is introduced						
Country	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
TU × U	3	3	3	3	3	2	3
HTU × U	4	3[a]	4	3	3	3	–
KHTU × U	–	4[a]	4	3	4	4	4
TU × TU	5	3[a]	4	3	4	3	4
HTU × TU	–	4	4[b]	3	4	3	5
Algorithm in vertical form (column methods)	4	4	4	3	3	3	5

U units, *T* tens, *H* hundreds, *K* thousands

[a]Programs in Mexico demand that the product has at most three digits in the third grade and five digits in the fourth grade, so 20 × 30 is acceptable in the third grade but 30 × 40 is not

[b]In Brazil, the teaching of algorithms in the fourth grade is up to 5-digit numbers

2.2.6 Comparing the Results with Previous Research

The results of this comparison are not the same as the results of the comparisons made in the 2000s in our previous Spanish-language edition (Isoda and Olfos, 2009). Ten years ago, there was more diversity under the previous curricula. The teaching of multiplication at the various grade levels was very much different. For example, in the case of Chile, multiplication tables are presently taught in the second and third grades, but they were taught in the third and fourth grades 10 years ago, so now it is much easier to share teaching approaches beyond individual nations.

2.3 Questions for Later Chapters

Those comparisons in the tables show how the teaching of multiplication is assigned at each grade level. The sequences and content of teaching are different. From the comparisons, we raise several questions, which will be discussed in Chaps. 3, 4, 5, 6, and 7. Here the questions are summarized as follows:

- In relation to the meaning of multiplication: What is the meaning of multiplication? Repeated addition? A group of groups? What is combinatorics? How many meanings do we have for multiplication? (See Chaps. 3 and 5.)
- In relation to the expression of multiplication: What are the multiplier and multiplicand? Why do some countries teach them and others do not? Why do some countries introduce factors instead of the multiplier and multiplicand? (See Chaps. 3 and 7.)

- In relation to the multiplication table: Why do some countries teach it at different grade levels? What is the difference? (See Chap. 6.)
- Some countries introduce the complete multiplication table in the same grade level while others introduce 2 and 5 at the upper grade level or at several grade levels. How are they different? (See Chap. 6.)
- Why do some countries teach multiplication algorithms (the column method or vertical form) in later grades? Why do they not teach it at the same time as the teaching of multiplication of several digits? (See Chap. 7.)
- How is the teaching of multiplication of whole numbers related to other numbers such as fractions and decimals, and how is it related to other operations such as division? (See Chap. 4.)

These questions can be explained in every country's context. In Chaps. 3, 4, 5, 6, and 7, these questions will be answered in comparison with the Japanese approach by explaining the reasons why the Japanese prefer such a teaching sequence and ways of teaching. This provides alternative perspectives for the other countries, as has been explored in Chile (Estrella, Mena, & Olfos, 2018). Because of the difference of the format of the national curriculum documents, we did not analyze the philosophical, mathematical, educational reason of every curriculum sequence. Every curriculum for teaching multiplication exists under its school system, educational culture, historical background, and reform issues. This comparison is done for posing these basic questions which make clear the perspectives to explain Japanese Approach for multiplication.

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