Chapter 11
Can We Explain Students' Failure in Learning Multiplication?

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11.1 Problem Presentation

Volumes have been written about teaching multiplication, and no didactic manual is without at least one chapter dedicated to this issue. Precisely for this reason, it is paradoxical that the teaching of multiplication, to which much time is dedicated, continues to be so deficient, and the results of students' learning of it is so mediocre. This issue is not trivial if one considers that it is knowledge that should be acquired in compulsory elementary education and that is aimed at giving future citizens the necessary general education to deal with common problems in everyday life.

The problems students encounter, at least in Spain, are of four kinds.

First, to give up on memorizing results, long considered an outdated and aberrant pedagogical method that has resulted in poor mastery of the multiplication table, which makes students take a long time to carry out multiplication of, for example, three digits by two digits, making the activity tedious, as well as leading to many errors in the results. This circumstance seems to exceed the limits of a given country. As such, at a conference held in Santiago, Chile, in February 2003, Guy Brousseau stated that:

> In recent times, French teachers made students (and thus their parents) responsible for learning the multiplication table, given that they considered learning it to be too repetitive and non-technical. When teachers today assume this responsibility again, they do so, using the same methods as parents (simple repetition). Emptied of content and of mathematical supports, this learning loses part of its interest and efficacy.[1]

[1] Conference presentation by G. Brousseau in Santiago, Chile, in February 2003.

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Research on brain function has shown that memorization through oral recitation has a high cost, as well as not being mathematically pertinent:

> Starting school supposes a radical change in mental arithmetic. One moves from an intuitive knowledge of numerical quantities, in which counting dominates, to arithmetic learned by memory. This great change, not coincidentally, is concurrent with the first difficulties in mathematics. Often, progressing in mathematics implies storing in memory great quantities of numerical information, a task that our brains are not prepared for. Children adapt to this as well as they can, but, as we will see, they often lose all intuitive understanding of arithmetic operations (Dehaene, 2003).

Dehaene (1997) postulates that to master elemental arithmetic, our brains use at least two formats to represent numbers: a symbolic format, based on our language faculties, which is used for manipulating symbols and numerical algorithms; and a kind of language-independent representation that is located in brain circuits associated with visual and spatial processing, which is used for approximate calculation of numerical quantities. Elemental arithmetic capacities are obtained as the result of the dynamic integration of these two kinds of representations.

Second, the multiplication algorithm universally taught and used socially—the Fibonacci algorithm—is not precisely the most adequate and it presents innumerable inconveniences: the necessity of retaining in memory the amount carried while a result from the multiplication table is being found; placement of the partial results obtained by multiplying the multiplicand by each digit of the multiplier, in a way that is difficult for students to understand and is often unjustified; errors in placement when there are intermediate zeros in the multiplicand or multiplier; lack of control, when an error is produced, in finding its origin; etc.

Third, the understanding of the meaning of multiplication is not worked on enough, which leads to not identifying situations that can be solved with a multiplicative calculation. So, we find ourselves with schoolchildren who can apply the multiplication algorithm but are unable to resolve a simple multiplication problem, and ask their teachers the classic questions "Is it with addition?" "Is it with multiplication?" etc.

Finally, it must be said that we have practically never seen schoolchildren taught, simultaneously with the operative techniques, control mechanisms that allow them to evaluate, with the teachers' sanction, if the result obtained when carrying out multiplication has an aspect of verisimilitude or, on the contrary, is clearly incorrect or even ludicrous. The reigning didactic contract indicates that the responsibility of the student ends when he or she provides a number as a result of the multiplication, without ever including, as part of the student's work, deciding whether or not it is correct, which is a competency only of the teacher.

Classical learning of multiplication is based on mechanization; this mechanization reaches both the multiplicative repertoire and the learning of the algorithm—an algorithm given to the student ready-made, without an express concern for the student discovering the usefulness and pertinence of the intervening mechanisms, which necessarily leads to lack of motivation and interest. The wide array of alternative algorithms (lattice multiplication or gelosia multiplication, Egyptian multiplication, Russian multiplication, etc.) are not contemplated to give the student the choice of the algorithm that is most understandable or best suited to the numbers in question.[2]

It is evident that these four problems, the causes of which we analyze below, are interrelated and reinforce each other, and that one cannot be competent in calculation when conceptual understanding is not guaranteed and the calculation methods utilized are not understood.

11.2 Multiplication of Natural Numbers in the Curriculum

Operations with natural numbers have made up part of the elementary education curriculum in all countries of the world since long ago, and, as such, the contents are fixed and not up for discussion, although the same does not occur with the issue of how they should be taught.

The National Council of Teachers of Mathematics (NCTM, 2003) indicates in its curricular standards—as goals from the third grade to the fifth grade, in the part regarding understanding of the meaning of operations—the following:

- Understand diverse meanings of multiplication and division.
- Understand the effects of multiplying and dividing natural numbers.
- Identify and utilize the relations among operations (division as the inverse operation of multiplication, for example) to solve problems.
- Understand and utilize properties of the operations, for example, the distributive property of multiplication with respect to addition.
- With regard to fluency and estimation of calculations, it indicates:
- Develop fluency in the basic combinations of multiplication and division and utilize them to mentally carry out calculations related to them, for example, multiplying 30 times 50.
- Develop fluency in the four basic operations with natural numbers.
- Develop and utilize strategies for estimating the results of calculations with natural numbers and judge the reasonableness of these results.
- Choose and use appropriate methods and tools (mental calculation, estimation, calculators, pencil and paper) to calculate with natural numbers, according to the context and nature of the calculation in question.

These indications would be accepted today in almost all countries, although it does not follow from this—and this is what is curious—that the methodology applied in classrooms leads in all cases to achieving these goals.

[2] This fact is so notorious that during the initial education of future elementary teachers, the teaching students are amazed when they are told that there are other ways to multiply and confess that they have always thought that there is only one way to do it: the traditional way that they learned in school. Also, they tend to be unable to justify the placement of the partial results and need to be convinced that the multiplication of the digits of the multiplier must always be done in the order units, tens, hundreds . . .

Entering a bit more into a vision of the future of what teaching calculation can lead to, the results of the Kahane Commission, created by the French Ministry of Education to reflect on mathematics teaching, have been published. One of the chapters in this nearly 300-page study by Kahane (2002) is dedicated to teaching calculation, and some of its recommendations that we consider most insightful are the following:

- Mental calculation can play an important role in linking calculation and reasoning, and exact calculation and approximate calculation in elementary school.[3] If we want to achieve this role, it should not be the result of routine and memorization but should be associated with diverse calculations strategies.
- For mental or written calculation to be effective, it must be supported by a minimum memorized repertoire.
- Working on thinking calculation[4] is essential for developing mathematical properties and concepts.
- The importance given to calculation algorithms is in decline, as exact numerical calculation done today with a pencil and paper is very limited, so it does not seem reasonable for the school to dedicate so much time to it, nor to demand a high level of competency from students in this area. Having available a reliable algorithm for simple cases seems sufficient.
- Greater interaction between calculation with a calculator and calculation with a pencil and paper, as a function of the goals of each situation, is desirable.

The reductionist image of calculation as a mechanical, automatable, and unintelligent activity must be fought against, as well as the idea that learning it is a purely repetitive process. Calculation should be thoughtful, beginning with initial education, and related to reasoning and proof.

The Spanish curriculum is regulated by Royal Decree 1513/2006, which establishes educational minimums in primary education[5] and defines mathematical competency regarding number algorithms as follows:

> Mathematical competency implies the ability to follow certain processes of thinking . . . and apply some calculation algorithms.

Later, in block 1, dedicated to numbers and operations, it gives the following methodological indications:

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[3] In the work previously cited, Dehaene explains how the human brain is gifted with continuous and approximate representation. When our brains are presented with a number in a symbolic form such as “8” they immediately make an effort to convert it into a continuous quantity, and do so automatically and unconsciously. In this way, our brains allow us to find meaning in the symbol “8” as a quantity contained between 7 and 9, closer to 10 than to 2.

[4] Thinking calculation is not the same as mental calculation; it is halfway between mental and written calculation. Intermediate steps can be written, but procedures more similar to mental calculation than written calculation tend to be used.

[5] It can be obtained online at http://www.educacion.es/educacion/que-estudiar/educacion-primaria/contenidos.html.

> Numbers should be used in different contexts, with the knowledge that understanding of the processes developed and the meaning of the results is a prior and priority context compared to skill in calculation. Of principal interest is the ability to calculate with different procedures and the decision in each case regarding which is the most adequate.

In each of the cycles, the corresponding contents are detailed:

First cycle (first and second grade):

- Utilization of multiplication in familiar situations to calculate the number of times
- Oral expression of the operations and the calculation
- Construction of the multiplication tables for 2, 5, and 10, based on the number of times, repeated sum, arrangement in grids . . .
- Development of personal strategies for mental calculation . . . for calculating doubles and halves of quantities
- Approximate calculation; estimation and rounding of the result of a calculation to the nearest ten, choosing among various solutions and evaluating reasonable answers

Second cycle (third and fourth grade):

- Utilization of multiplication as an abbreviated sum, in rectangular arrangements, and combinatorics problems in familiar situations
- Additive and multiplicative decomposition of numbers; construction and memorization of the multiplication tables
- Utilization of standard algorithms for adding, subtracting, multiplying, and dividing in problem-solving contexts
- Utilization of personal strategies for mental calculation
- Estimation of the result of an operation on two numbers, evaluating whether or not the answer is reasonable

We can conclude that the Spanish curriculum follows the fundamental recommendations of the NCTM, although we appreciate that certain issues that we consider vital to the understanding of the meaning of the operation are not given the weight they deserve (understanding diverse meanings of multiplication, understanding the effects of multiplying and dividing natural numbers, identifying and utilizing relationships among operations—division as the inverse operation of multiplication, for example—to solve problems). Also, few indications are given regarding how to construct multiplication tables or how to arrive at the calculation algorithm, nor are the advantages of teaching one algorithm or another analyzed.

As strengths of this curriculum, we recognize the references to the need for working on mental calculation and estimation, as well as the use of the calculator. While it is accepted that students create personal calculation procedures, these seem to be limited to the domain of mental calculation and not applicable to written calculation.

If we compare this to the Chilean curriculum, it can be appreciated principally that the latter is more detailed and explicit, providing more indications regarding what to do and how to do it. We consider the strong points of the Chilean curriculum to be the proportionality approach to multiplication and its simultaneous treatment with division. We also find the learning order of the multiplication tables reasonable (2, 5, and 10 first, as the first thing children learn is to count by twos, by fives, and by tens). We share practically all the indications in the teaching guide that we have been able to read,[6] which give very precise indications of how to proceed in the classroom to reach the definitive algorithm, and, as such, we believe that if teachers follow these indications rigorously, it will lead to the success of the students. We can summarize by saying that it is a good curriculum, and, as such, the causes of scholastic failure must be looked for in other areas—for example, in how teachers apply this curriculum or in the training they have for its concrete interpretation.

Our knowledge of the Japanese curriculum is limited to what is described by Isoda and Olfos (2009), and we have been amazed to see the degree of detail and meticulousness in the Japanese government's mathematics teachers' teaching guide in the development of content related to multiplication. We appreciate, as a distinctive feature of the Japanese curriculum, the importance granted to the manipulation of material, often considered "not very mathematical" in other cultures (e.g., in Spain), as well as to graphic representations (in particular, to numerical patterns) and how much time is dedicated to ensuring student comprehension of the meaning of an expression, without ignoring the acquisition of calculation procedures. We regard the disciplined participation of students in the development of the lesson as definitive for achieving the stated results, but we consider it difficult to extrapolate to Latin societies, where, unfortunately, the intrinsic motivation of mathematics itself is not usually enough to stimulate the desire to learn.

11.3 Contributions to Didactics

Recent research in the didactics of mathematics gives emphasis to considering multiplicative calculation, and arithmetic in general, as a means for comfortably and effectively resolving problems that present themselves in students' daily lives, giving more importance to the meaning of operations than to the speed reached in using calculation algorithms. Currently, a universally accepted methodological principle is that more time and attention should be dedicated to dealing with situations that give meaning to multiplication, with less time dedicated to memorization and repetition of the corresponding standard algorithm, as numerical competency cannot exist if it is not based on conceptual competency (Fig. 11.1).

In Gerard Vergnaud's words:

> Mathematical competency can be defined with relatively variable criteria:
>
> (a) Someone who knows how to deal with situations and solve problems is more competent than those who do not;
> (b) Someone who solves problems in the most efficient, most reliable, fastest, most general, or conceptually most elaborate way is more competent;
> (c) Someone who has a variety of alternative means for solving problems of a certain category and can choose the appropriate method as a function of the values of certain parameters of the situation is more competent (Vergnaud, 2001).

[6] We have been able to access only what is described in the book by Isoda and Olfos (2009), and not the original documents.

Fig. 11.1 Students discuss in pairs in the math lab

If we apply the previous case to multiplication, the result is that we should aspire for students to be able to distinguish in the case or when they encounter a situation that demands a multiplicative calculation, to know which is most appropriate (as a function of the numbers that appear), a) and to use a calculator, b) to use an algorithm that requires a pencil and paper, or c) to use thinking or mental calculation. From this, it is easily deduced that standard learning of multiplication—which dedicates many hours to learning the traditional algorithm and does not provide or teach alternative, personal calculation methods, ignores the existence of mental calculation, and dissociates problem solving and calculation—cannot educate schoolchildren with the necessary numerical competency.

11.3.1 What Does the Theory of Conceptual Fields Teach Us?

One of Vergnaud's most important contributions in his theory of conceptual fields (Vergnaud, 1990) has been to effectively show how some concepts relate to others and the necessity of considering the different contexts in which a concept appears. In the case at hand, it refers to not dissociating (as habitually happens) work with multiplication, division, and proportionality, as the situations that demand their use form part of the same conceptual field.

Gerard Vergnaud (1981), as early as his first texts, made manifest the necessity of carrying out an exhaustive study of the different types of situations in which multiplicative calculation participates, and which, as such, give meaning to the operation, leading to his well-known classification of multiplicative problems as isomorphism of measures, product of measures, and single measure space. This classification not only informs us about the level of difficulty of each of these types—which on its own helps us to explain many students' errors and difficulties and the different rates of success and failure in one type or another—but also shows us the different contexts in which the necessity of multiplying appears. It is necessary for the teacher to be familiar with these contexts in order to be able to provide students with all the variety of situations that give meaning to the concept of multiplication, as it must not be forgotten that recognition of situations that can be dealt with using multiplication is much more important than having an effective multiplication algorithm. Making students face this variety of situations will obligate them, in the best of cases, to adapt, modify, and generalize problem-solving procedures, and to abandon them and construct new ones in other cases. What we call learning is nothing other than an individual's capacity to decontextualize a concept or procedure and then recontextualize it again, and in doing so make necessary adaptations or changes.

Working and systematically observing the different problem-solving procedures for a multiplicative situation ("approximative" in Piaget's language, or "working on schema" in Vergnaud's[7]) helps students to discover operative invariants and is useful for the teacher not only to be able to determine with greater precision the levels of skill reached by the students, but also to follow a logical teaching progression adapted to the students' competencies, as:

> The cognitive function of a subject or of a group of subjects in situation is based on the repertoire of previously formed schemata available to each of the subjects considered individually.

As a consequence of this, there is unanimous agreement in didactics regarding the necessity of making students face, from the very beginning, situations of a multiplicative character, without needing to wait for students to have available algorithms or advanced procedures for numerical resolution. Thus, emerging techniques like drawing the situation and then counting will lead to the iterated sum of equal summands, which is useful for giving meaning to the calculations, so that students always know what they are calculating in order to respond to a concrete question. This is the technique that is developed in many situations observed and studied by Isoda and Olfos (pastries in a box with various layers, and knowing how many there are in each layer; pastries in various boxes, and knowing how many there are in each box; pastries that fit in a box, and knowing the size of the pastries; balls in various containers, and knowing how many fit in each container; pencils in stacked boxes of pencils; etc.).

[7] Vergnaud defines the concept of a schema as "the invariant organization of behavior for a given class of situations." Subjects' knowledge in action should be investigated in a schema—that is, the cognitive elements that allow for a subject's action to be operative. The expressions "knowledge in action" and "theorem in action" designate knowledge contained in a schema, which can also be designated by the more global expression "operative invariants."

11.3.2 Developing Didactic Progressions for Teaching Multiplicative Calculation

Students should experience in class something that is intrinsic to mathematics: the need to debate the truth or falsity of an affirmation, the search for more effective solutions for solving a problem, and practicing debate as a means to answering these questions. The first edition of the book by Isoda and Olfos (2009) for teaching multiplication expertly shows something that many do not consider, but that has enormous importance in learning mathematics: that mathematical knowledge is built collectively in the little society of the class, which is why student motivation is needed.

Guy Brousseau, considered the father of the modern didactics of mathematics, says this on the topic (Brousseau, 1995):

> As a social practice, proof is the legitimate method of convincing an interlocutor: the interlocutor should be respected, using nothing except his or her repertoire (logical, mathematical, scientific . . .) and the information he or she currently has available, and other means of pressure—rhetorical (formal ability), psychological (such as seduction, authority, or compassion) or material (threats, violence, etc.)—should be avoided.
>
> In mathematics, knowing how to prove an affirmation, justify a result, etc., is part of one's own learning of the material, but the practice of proof is constructed here very differently than how it tends to occur socially. There is a series of psychological barriers to overcome, as the person who is correct is not always the most powerful or the most socially valued, but rather the one who can prove their arguments to be valid, so our self-esteem is often compromised. The truth in mathematics is not associated with power, which conflicts with social habits. Even the teacher is obligated to demonstrate that what he or she says is true. Authority is not enough. Nor are things true based on voting, nor can we support our friends' answers based on loyalty if these answers are not correct. The instrument of this initiation is learning proofs, not only as official knowledge, but also as a way of practicing proofs (and of limiting them to their domain of relevance). It forms part of the individual and particularly of the rational individual, just as much as the most essential social relations do. Democracy cannot exist without a social organization that integrates the role of knowledge in decision making and without shared and correct management of knowledge, truth, and proof. In primary school, this fundamental civic formation is not formulated, but it first happens in mathematics (see Fig. 11.2).

Fig. 11.2 Researchers take notes from pairs' discussions

For debate to arise naturally in class, and not as an imposition by the teacher, a problem should be proposed that makes sense to the students and allows them to use personal or group strategies that can be compared and validated. The situations have to be designed so that the knowledge the students possess at that moment allows them, if not to solve the problem completely, at least to understand the solution and an outline of the solution (a base strategy). We should consider that whatever is being learned, students always possess prior knowledge, which is often partial or incorrect, and one of the teacher's tasks is precisely to begin with this prior knowledge and make compatible something that is very important in calculation: the use of these personal procedures and the acquisition of faster and more effective universal algorithms.

If the students' knowledge were sufficient to resolve the situation, we would be in a situation of application of prior knowledge, not a learning situation. As such, the students' base strategies must be shown to be insufficient or not very effective, and the students should progress to be able to successfully solve the problem proposed in the situation (modification of schema, generalization, or construction of new schema).

As we have seen in the text from Isoda and Olfos, numerical learning requires considerable periods of time, and, as such, a family of interconnected situations must be designed—that is, didactic engineering (see Chamorro, 1999, 2003, 2004).

One of the first examples of didactic engineering—developed at COREM (Centre d'Observation por la Recherche en Enseignement des Mathématiques de Bordeaux) in 1985 and, as such, under the supervision of Guy Brousseau himself—is about multiplication and is clearly based on his theory of situations. Despite the 25 years that have passed, and everything that has happened in didactics in that time, some of its guiding principles remain relevant today:

Part I

Introducing multiplication through the need for rapidly counting the number of elements in a collection structured, or susceptible to being structured, in equal parts. The multiplicative structure $a \times b$ appears as a comfortable and effective way of designating the total number of elements in this collection, a manipulable collection at first, and later a represented collection. The need for using writing is connected to a situation of communication between teams: sending a message with a written multiplicative expression allows the receiver to form the corresponding collection (3 sessions).

Designation as a product of a collection arranged in the form of a table, using the number of elements per row and per column (1 session) (see Fig. 11.3 and 11.4).

Designation of products in the form $a \times b$ (4 sessions).

Part II

Comparison of numbers (near 250) written in the form $a \times b$ (1 session).

First calculation methods for $a \times b$ based on a multiplicative repertoire (3 sessions).

For example (see Fig. 11.5), find the value of 7×15 using the following repertoire: $4 \times 6 = 24$, $3 \times 6 = 18$ . . . $4 \times 6 = 28$, $7 \times 7 = 49$ . . .

$$2 \times 7 = 14$$

15

Fig. 11.3 Collections arranged in the form of a table

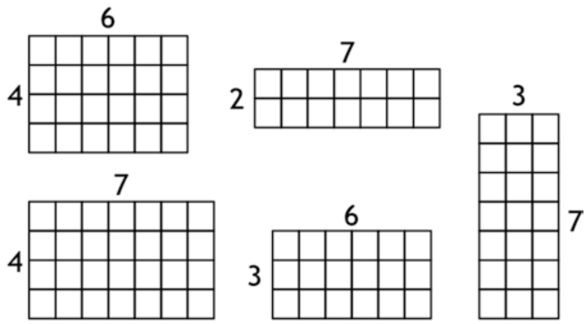


Fig. 11.4 Products as tables of rows and columns

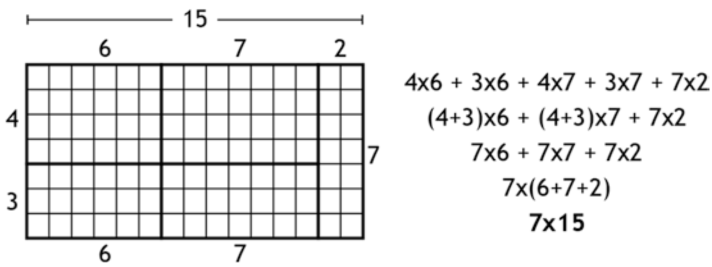


Fig. 11.5 Distributive property applied to arrays

Part III: Abandoning Graph Paper

First sessions (2 or 3). Find the total number of squares in a grid, for example 24 × 18 (see Fig. 11.6), using only blank paper and a multiplicative repertoire.

Second group of sessions (3 or 4). Progressive elaboration of a complete solution based on parts, using fundamentally the dimension 10 (see Fig. 11.7).

$$24 \times 18 = 11 \times 7 + 11 \times 8 + 11 \times 3 + 13 \times 7 + 13 \times 11$$

Last group of sessions (3 or 4). Search for the results of products provided by the teacher, without using graph paper, which can only be used for checking results.

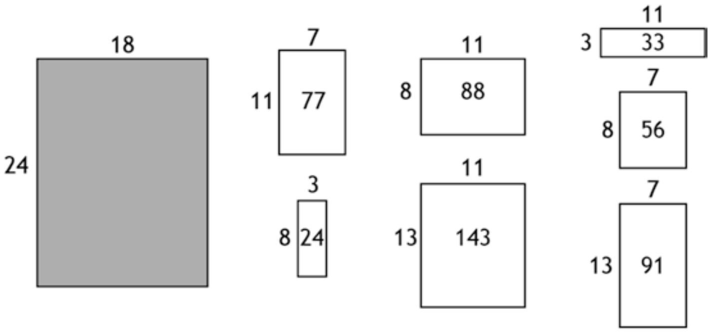
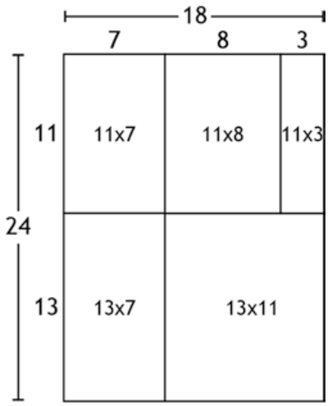


Fig. 11.6 Task visual information

Fig. 11.7 Representation of a solution



Part IV: Fine Tuning an Algorithm (Lattice Multiplication)

- Organization and observation of the product of one-digit numbers (tables) (1 session).
- Rule of zeros: calculate in 1-step products like 20 × 30, 7 × 80, general rule (4 to 7 sessions).
- Organization and arrangement of calculations, connected through additive decomposition of the factors (tens and units) and the distributive property.
- Reduction of the decomposition (3 to 6 sessions).
- Institutionalization of the algorithm, preferably lattice multiplication (1 session).

In parallel, mental calculation and solving multiplication problems are worked on.[8]

Part V

- Counting a collection (4 sessions): squares in a grid (43 × 32, 46 × 32, 56 × 37, 234 × 526 . . .) posted on the board, using blank paper or graph paper.

[8] The problem statements are of the following type: “A train has 9 cars, each with 18 seats and 4 wheels. How many children can sit in the train?” or “A construction worker wants to put tiles in a bathroom. The tiles come in boxes of 10. The worker places 5 rows of 6 tiles each. How many tiles has the worker placed?”

We could say that the ERMEL group (Equipe de Recherche des Mathématiques de l'Enseignement Elementaire)[9] continues the paths introduced by the Bordeaux IREM (Institut de Recherche pour l'Enseignement des Mathématiques), which, at the same time, was heavily influenced by research carried out by Guy Brousseau.

In the guidelines in ERMEL's latest edition, some principles to be followed in teaching multiplication can be observed:

- Reinforce what has been learned about decimal numeration.
- Introduce multiplication through iteration situations in which collections formed of subcollections of the same number of elements participate, or situations whose resolution requires a repetition of actions that imply adding or subtracting repeatedly the same quantity.
- Build the meaning of multiplication through the set of problems that belong to the multiplicative conceptual field.
- Give preference to multiplicative problems of the direct proportionality type in which the student can use known procedures that should evolve and adapt to new situations.
- Abandon graph paper, despite its advantages (easy geometric observation of the commutative property, easy management of the decomposition of products, multiplicative writing of $a \times b$ as a designation of a number and not as a calculation, etc.) due to the long and difficult process that must be followed to reach the Fibonacci algorithm if all the steps are followed.
- Do not separate multiplicative problems from the associated division problems.
- Construction of multiplication tables based on a series of multiples: discovery of the rule of zeros (using commutativity, iterated summation, or multiplicative decomposition of the numbers).
- Construction of the multiplication technique by the students.
- Insist on processes that allow for solving products through mental calculation (successive doubling, using multiples of 10, decomposing a number, etc.).
- Encourage the use of processes that can solve products through mental arithmetic (successive duplications, the use of multiples of 10, decomposition of numbers, etc.).
- Construction of the operatory multiplication technique through summing of multiples of the multiplicate of the type ×10, ×20, etc.

The above treatment is achieved over 2 years (in the third and fourth years) by presenting several different situations that must be resolved using multiplication procedures, as well as games aimed at the acquisition and memorization of sets or the use and discovery of mental arithmetic techniques (dominoes, battle games, bingo, etc.).

[9] Since 1977, this group has been publishing various manuals aimed at preschool teachers, elementary teachers, and teacher trainers—manuals that have collected practically all the research results in didactics of mathematics at the elementary level, and that, as such, constitute an obligatory reference (see https://forums-enseignants-du-primaire.com/topic/78945-ermel/). Through the various successive editions, one can appreciate the evolution that has occurred in the teaching of different mathematical concepts.

In our opinion, although the models and underlying multiplication structures are mathematically clear, there are unresolved questions in all known didactic approaches to multiplication, implying the need for in-depth study, analyzing the proposals given by the teacher in this regard. For example, if the teacher begins with situations that demand repeated addition, how can we justify that $a \times b$ is equal to $b \times a$ when one of the factors is measurement with dimensions? To find the process of calculating 4 bags of flour that cost €2 a bag, the correct answer is to do 2 + 2 + 2 + 2, since calculating 4 + 4 would be absurd and make no sense, even though 4 + 4 = 2 + 2 + 2+ 2. Nevertheless, something that can easily be seen, even without wanting to see it, is that the number of objects arranged in 2 rows of 4 is the same as when they are arranged in 4 rows of 2.

Despite this, this difficulty is mainly seen in solving problems in which it is necessary to maintain the meaning of the operations being carried out, keeping the connection with what is represented by the problem data. Thus, in the calculation of a multiplication ($2 \times 4 = 4 \times 2$),[10] the pupil must search for the best way to solve the problem, meaning that the commutative property is greatly helpful.

Perhaps the only possible solution is always to propose the answer to a problem using the form that makes the most sense, clearly separating it from the calculation stage of actual multiplication, though it is then necessary to recontextualize the result obtained in order to ensure it makes sense.

11.4 Informal Arithmetic Methods

For many years, several researchers have questioned the importance of the common practice of teaching arithmetic algorithms, relative to the lack of consideration of informal arithmetic procedures used by pupils in daily life, often in parallel with the usual algorithms from school. The result is that in the eyes of the pupils, the school has a different way of doing things from daily life, and they are unable to realize that they are dealing with procedures that aim to find solutions to the same problem.

Resnick and Ford (1990) use data obtained by Lankford to conclude the following:

1. The thought patterns/arithmetic strategies pupils develop when studying basic mathematics are highly individual, and they often do not follow orthodox models from textbooks or the classroom.
2. Differences can be seen in . . . the arithmetic strategies of pupils that are successful and those that are not.

[10] The operation written as "2 × 4" is not read the same in all cultures. In Japan it would mean "2 + 2 + 2 + 2" whereas in Spain it is read as "2 times 4" and would therefore be written as "4 + 4." We understand that although one of the two forms is more advantageous to the construction of tables, cultural tradition is far stronger, and it would be wrong for a school to go against social/mental norms.

Fig. 11.8 Grouping in five groups of five. (Reproduced from Tsubota, 2007)

3. Indications can be found for teaching that support arithmetic ability based on pattern observation . . . by pupils who do incorrect calculations.

It can also be said that the use of informal arithmetic procedures is mostly among the pupils, and not all those who use them make mistakes.

For example, for counting the quantity of spots (see Fig. 11.8) some children will see the five spots of a die in five locations on the tile, while others will move the spots from the four corners into a new location, turning the tile into a 5 × 5 square.

Many of the informal multiplication methods used by pupils are based on a common pattern: counting (2 by 2, 3 by 3, etc.). It is therefore important to include this type of exercise in mental arithmetic work. We tend to think that this method, which can appear simple and primitive, is only used by first-year pupils, but the reality is that it remains in use by pupils in later years. Lankford found that of 176 seventh-year pupils, 63 (36%) used counting when doing multiplication. It is precisely the use of the counting algorithm that causes pupils to have difficulty in dealing with and retaining multiplication results in memory work, while also making them take longer in finding the results.

Therefore, the idea of pupils memorizing multiplication tables lies in the aim of transitioning from the counting algorithm to recalling numerical facts from long-term memory; i.e., numerical facts can be recalled from long-term memory almost immediately, thus freeing up resources in the working memory for immediate results and consequently decreasing the number of errors. This does not mean that the pupil will no longer make mistakes, as it is known that remembering a numerical result is more complex than recalling it from long-term memory, since numerical facts are strongly connected, even when they apply to different operations, and it is easy to activate an incorrect result such as "2 + 7 = 14" or "2 × 7 = 9."

It is also known that the mistakes made by pupils are more systematic than random. They respond to a certain logic, and this often originates from a lack of understanding of the procedures implied in algorithms, which are therefore applied incorrectly (see Fig. 11.9). It is precisely this logic that makes many errors persistent, since the same incorrect procedure is repeatedly applied. It is therefore important for teachers to take time to observe these errors and identify which procedures they come from. If this is not done, they will be unable to help their pupils overcome the errors.

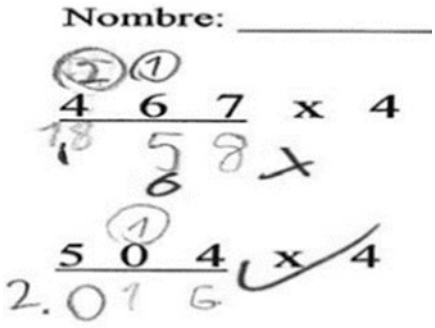


Fig. 11.9 Student' work and mistake

If the aim of teaching arithmetic is the development of understanding, research must be done into the informal procedures that pupils use in daily life to ensure that the teacher can help the child make a connection between their formal mathematics learning in school and their everyday practices. Baroody (1988) recommends that any formal expression of the type $3 \times 4 = 4 + 4 + 4 + 4$ is always linked to real experiences that have meaning for the pupil. The pupil can then establish connections with her/his own informal knowledge, and the formal symbolism of the mathematics avoids becoming something hollow.

This becomes all the more significant when considering some results of research into how the human brain functions. For example, when asking why the results of memorizing multiplication tables are so mediocre (a lot of time is spent on memorization and repetition of the tables, with very poor results, as pupils get confused and forget many of the multiplications despite the number of hours spent on it), Dehaene provides some very interesting clues, such as the very structure of the multiplication tables themselves. Furthermore, to make the difficulty experienced by children when learning this for the first time more understandable to adults, he replaces the list of numbers 0, 1, 2, 3, . . . with a list of names and replaces the multiplication with a workplace, giving a table such as the following:

- Carl David works in Richards-Brown Street ($3 \times 4 = 12$)
- Carl William works in Brown-Richards Street ($3 \times 7 = 21$)
- William Pierce works in Carl-Pierce Street ($7 \times 5 = 35$)

It is obvious that memorizing the results above is a very difficult task. This is because our memory is not structured like that of a computer. It is associative[11] and it weaves several different connections between very different pieces of information; this is at the same its strength and weakness.

[11] Since human memory is associative, it weaves innumerable connections between very different pieces of information that in turn are activated regardless of whether they proceed or not, which happens from a very early age. When learning multiplication tables, it is vitally important that numerical facts are not mixed with facts relating to other operations, giving the result of a sum or a difference instead of a product. However, the human memory has difficulty saving the results of different operations separately. As a result, it is easier to notice that "$2 \times 4 = 9$" is wrong than that "$2 \times 3 = 5$" is wrong.

As Dehaene says, it is interesting to recall the behavior of a lion when we see a tiger, but it is disastrous to activate knowledge of 7 + 6 or 7 × 5 when we want to know 7 × 6. Interference and inappropriate association are the basis of the failure to memorize the multiplication tables.

The errors are not random, and incorrect answers are always numbers that are on the multiplication tables somewhere, often in the same line or column as the result the pupil is looking for. Considering that our brains use continuous and approximative representation, it is reasonable that when searching our long-term memory for the answer to 7 × 8, the results of 7 × 9 and 6 × 8 are also activated. The brain also has difficulty saving additions and multiplications separately. This explains why we are quicker to see that "2 + 4 = 8" is wrong and slower to spot whether "2 × 4 = 6" is wrong. Similarly, it is easier to see that "2 × 4 = 7" is wrong than to see that "2 × 4 = 6" is wrong. It is also known that the difficulty in recalling a numerical fact from long-term memory depends particularly on the number of associations that cause interference—so-called interfering associations (Bideau and Lehalle, 2002)—which vary with the development of the individual and are activated when looking for an answer to a problem.

About 80% of errors arising when learning the multiplication tables are of the type described above, and the so-called distance effect can be seen (van Hout, Meljac, and Fischer, 2005). For example, the error "7 × 8 = 42" has a result from the adjoining table (7 × 6 = 42). Only 13% of errors are not related to inverted numbers, such as "8 × 7 = 54"; since 54 is 6 × 9, it is not in the 8 table or the 7 table. When multiplication and some other operation appear together in the same class or in the same problem, the errors that appear are consistent with swapping the operations (e.g., "8 × 7 = 15" or "4 + 2 = 8"), and this type of error can account for up to 30% of errors. Only 7% of errors are those of the type where the answer is not related to the numbers or the operations, for example: "5 × 9 = 26."

Since it is known that the brain is associative, if the table has been built and learned by the pupils by establishing connections between the results—as shown in the guidelines of PROMETAM [Proyecto Mejoramiento en la Enseñanza Técnica en el Área de Matemática] (Secretaría de Educación, Honduras, 2007) and described in classes by Professor Tsubota (2007) or in the texts of ERMEL (1993, 1995)—activating 7 × 5 can be helpful if we know that the next answer is found by adding another 7. This means that if we want to be more effective with less effort, we should adapt the way tables are taught to what is known about how the brain stores and recalls information in the long-term memory, favoring semantic learning of multiplication tables.[12] However, pupils in primary education do not in general spontaneously seek out this type of method, meaning that it is necessary to encourage discovery of the properties of multiplication.

[12] The term "semantic" assumes comprehension and serves to differentiate between rote learning (which is the most common type and is not based on comprehension or connections) and associations between products. Semantic learning of multiplication tables uses the associative characteristic of memory, which is useful to find, for example, 6 × 9, using procedures such as (6 × 10) − 6, or (6 × 6) + (6 × 3).

Of the multiplication tables, special mention must be made of the 1 and 0 tables, since they can be learned by the general rules: anything multiplied by 0 is 0, and anything multiplied by 1 is itself. It has been shown that access to numerical facts does not work in this case, such that results presented as $n \times 1 = n$ and $m \times 0 = 0$ are recalled from memory through selective rules that can be lost or confused, meaning that errors affect all the answers in the table and not just certain numerical facts, as is the case with the other numbers.

Some researchers, such as McCloskey and Macaruso (1994), posit that the cognitive system related to numerical treatment is structured into modules and comprises:

- A comprehension system
- A production system
- An arithmetic system

The first two can in turn be divided into two subsystems, one related to Arabic numbers and the other to verbal names. The third has three components: knowledge of the operation symbols, the arithmetic procedures, and the numerical facts saved in long-term memory. According to this model (see Fig. 11.10), each operation has a network of different representations, which can easily explain the disassociation between operations in the minds of many schoolchildren.

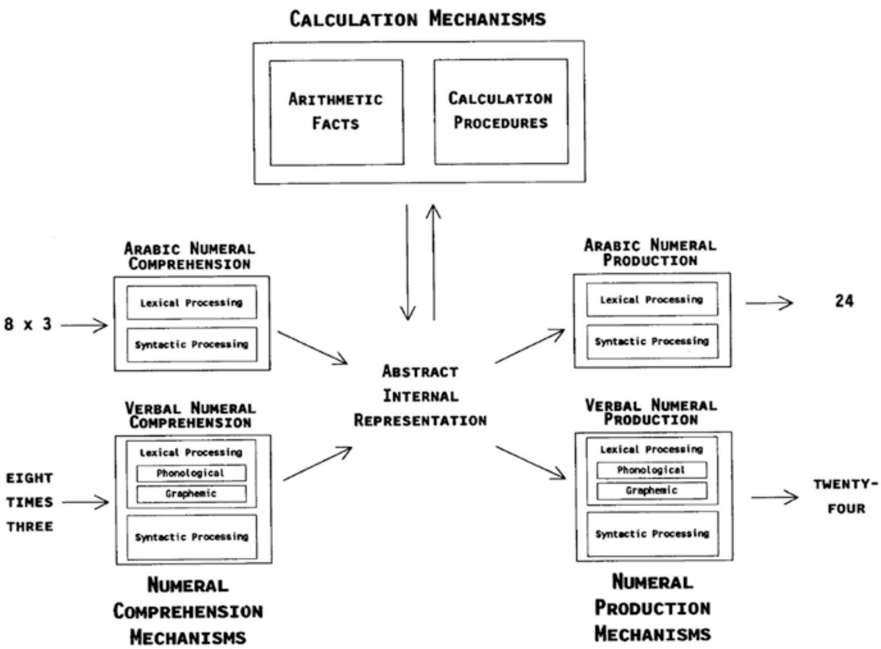


Fig. 11.10 The McClosky model. (From McCloskey, 1992, p. 113)

For Dehaene, each number is represented by an analogous code in the form of a number line, an audiovisual code, and a visual–Arabic code, and each of these codes is used for different tasks. Specifically, multiplications and some simple additions, learned routinely by memory by some pupils, are coded verbally, while the results of subtractions and divisions are learned and solved by the application of rules that involve semantic manipulation (e.g., $68 - 17 = 68 - 20 + 3 = 48 + 3 = 48 + 2 + 1 = 51$), and therefore the analogous representation of the quantities. This fact is neurologically linked with tasks carried out by each of the two hemispheres of the brain; thus, arithmetical operations are only possible for the left hemisphere, while both hemispheres can recognize whether two numbers are identical and perform counting, though the latter is done more easily by the left hemisphere.

For the treatment of calculation difficulties, educators/teachers should insist on the presentation of situations (to provide activities for students) in the varieties of codes, use verbal, written and Arabic numerals interchangeable.

The Japanese method of teaching the multiplication tables, as is done in schools in many countries, also involves memorization through repetition—i.e., using verbal memory to store phrases such as "three times four is twelve" easily in the memory. It should be noted that the verbal memory stores this phrase on the same level as the phrase "two thousand bees appeared on the honeycomb"—i.e., a sentence without any numerical meaning.

There are many studies, dating from 1967, that confirm Asian superiority in mathematics and, in particular, that of Japanese[13] students. Some factors that explain this superiority are the following:

- Schoolwork is of a large quantity and high quality, with pupils dedicating considerably more extracurricular time to schoolwork than South American students. In particular, as stated in the text cited above, they spend a lot of time not only on systematic work to learn arithmetic but also on solving situations that require the application of that arithmetic.
- The attitude of parents, being more demanding and ambitious with the progress of their children.
- The culture of competition within schools (the text by Isoda and Olfos (2009) describes this aspect very well), putting additional pressure on students to obtain good results in school.
- Motivation based on the idea that work and effort are important virtues that are absolutely necessary for success in later life. These aspects are clearly seen in the classroom studies described in the aforementioned text.

In addition to these factors, there is also the numerical language. The uniqueness of the Japanese[14] language allows for much shorter sentences than those possible in Spanish, since they omit the word "times," thus facilitating memorization.

[13] In the text by Fischer (2002, pp. 215–237), several studies can be found on the implementation of these aspects, particularly in the USA.

[14] See Isoda and Olfos (2009), p. 50.

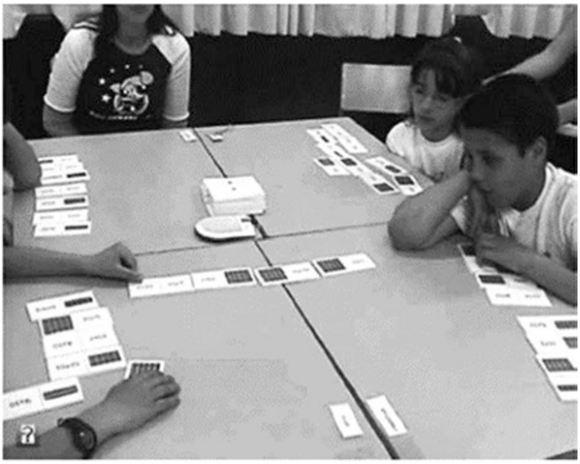


Fig. 11.11 Dominoes for connecting arrays with products

With regard to oral numeration, Asian systems of number words are fully regular, while oral numeration in Spanish is very irregular and uses the different powers of ten as its basis, with each one given a specific word: *diez* ("ten"), *cien* ("one hundred"), *mil* ("one thousand"), *diez mil* ("ten thousand"), *cien mil* ("one hundred thousand"), *un millón* ("one million"), etc. Furthermore, a different word is needed to designate each of the numbers from zero to fifteen. The words *once* ("eleven"), *doce* ("twelve"), *trece* ("thirteen"), *catorce* ("fourteen"), and *quince* ("fifteen") are plainly irregular, as are *veinte* ("twenty"), *treinta* ("thirty"), *cuarenta* ("forty"), *cincuenta* ("fifty"), *sesenta* ("sixty"), *setenta* ("seventy"), *ochenta* ("eighty"), and *noventa* ("ninety"). Asian systems, on the other hand, are fully regular and the composition of a number is clearly apparent in its name; for example, the word for "eleven" transliterates as "ten one," the word for "twenty-five" as "two tens and five," etc. All of these make the names of the numbers easier to learn for Asian pupils in general and for the Japanese in particular. This oral construction of numbers also makes it easier to avoid many errors that commonly arise in arithmetic, by combining the cardinal meaning with the name of the number.

However, doing arithmetic in Spanish requires a pre-established connection between the written number and the number words used in oral numeration (since the multiplication tables are learned orally)—i.e., understanding the quantitative meaning of the written form, which is evidently more complex (see Fig. 11.11).

11.5 Do We Have to Teach Algorithms?

It is evident that learning arithmetic algorithms is more costly in terms of classroom hours and the effort and failure of pupils, leading us to ask the question as to whether this effort is worthwhile in mathematics teaching.

Before answering this question, we would like to examine one of the most significant causes of pupils' failure in arithmetic, which goes unnoticed by many teachers: the lack of understanding of decimal numbers.

The positional principle that governs decimal numbers[15] is based on a considerable mathematical apparatus. It should not be forgotten that although all numbers involve an expression of a polynomial of 10 to different powers, their normal abbreviated written form, which removes the powers and leaves only the coefficients, works with a norm for reading and writing the numbers based on the value of the position—i.e., it is the position that allows us to interpret the value of the number.

Kamii conducted an experiment to determine children's level of comprehension of the place value (Kamii, 1985). Basically, the test consisted of asking children to associate the number 16 with a number of corresponding tokens and then indicate how many tokens each of the numerals 1 and 6 represented in a drawing. The results were surprising: only 51% of the fourth-year pupils, 60% of the sixth-year pupils, and 78% of the eighth-year pupils drew ten tokens to represent the 1 in 16.

It is clear that, as such, a large percentage of pupils have difficulties understanding place value, even in older age groups. The number of pupils who will fail in arithmetic, particularly in applying classical arithmetic algorithms, will also be very large, since they are almost all based on the properties of decimal numbers. The solution that many pupils find to this problem is rote learning without understanding the steps of the algorithm; thus, they lose control over what they are doing. For them, the path, the act of placing numbers in classical multiplication, going to one place if there is a zero in the multiplier, etc., are purely mechanical acts, lacking explanation; it is done like that merely because it is, and, as Baroody states:

> Although children recall basic information learnt by memory, this does not guarantee intelligent use of that information. Deep down, many of them learn arithmetic but do not learn mathematics. These problems are made worse when the exercises and repetitions lack any interest and meaning. All too often, mass teaching becomes an obstacle to meaningful learning, thought, and problem solving (Baroodi, 1988, p. 55).

We work to help students learn automated procedures mechanically as arithmetical algorithms (learning arithmetic), but they do not know how they are built and what they are for (learning mathematics). The pupils accumulate easily assessable knowledge, but they cannot use it in a meaningful way because it is not part of their interests or the solution to any problem. Adding to the difficulties that children have, for the reasons detailed above, when relating the name of a number to its cardinal meaning, the panorama facing teachers when teaching algorithms is not promising.

We should also ask ourselves about the usefulness of algorithms in daily life and their frequency of use. Many of us have never done a multiplication with a three-digit number after leaving school, and when it has been necessary, we have used a calculator or an estimate, depending on whether an exact answer was needed. This cannot be denied, but it is not sufficient to conclude that schools should adopt measures to promote other type of arithmetic, both mental and use of a calculator, instead of spending time on learning algorithms.

[15] See Isoda and Olfos (2009), p. 50.

In our opinion, using a written arithmetic algorithm to multiply numbers with three digits is a waste of time and takes a great deal of effort for most pupils. However, these same pupils can acquire knowledge of number theory solely through observation, which can be done simply by letting them use a calculator freely. This can be enhanced by discoveries guided by questions proposed by the teacher: magic squares, numbers whose squares are palindromes, numbers whose products do not change when the numbers are written backward (e.g., 36 × 84 = 63 × 48), random numbers, the pole of a number, etc.

For multiplication of two-digit numbers by two-digit numbers, we recall the use of the distributive property and the automation of simple results, mainly multiples of 10, later adding the results without the need for putting them in place, as is done with the Fibonacci method.

To calculate 36 × 28, we can do the following:

$$36\times 28=(30+6)\times(20+8)=30\times 20+30\times 8+6\times 20+6\times 8$$
$$30\times 20=3\times 2\times 10\times 10=600$$
$$30\times 8=3\times 8\times 10=24\times 10=240$$
$$6\times 20=6\times 2\times 10=12\times 10=120$$
$$6\times 8=48$$
$$600+240+120+48=960+48=1008$$

or use mental arithmetic strategies, depending on the level achieved by the pupils—for example, using doubles which are often automated easily.

$$36\times 28=(36\times 30)-(36\times 2)=(36\times 20)+(36\times 10)-(36\times 2)=$$
$$720+360-72=1080-721008$$
$$36\times 28=(40\times 28)-(4\times 28)=(28\times 2\times 2\times 10)-(28\times 2\times 2)=$$
$$(56\times 2\times 10)-(56\times 2)=1120-112-1008$$

Other nonconventional algorithms, such as Egyptian multiplication, are based on the process of doubling. In the case of the multiplication above, we have the following:

1	36	1	28
2	72	2	56
4	144	4	112
8	288	8	224
16	576	16	448
___	___	32	896
28	1008	___	___
		36	1008

The Russian peasant algorithm.

* Write each number at the head of a column.
* Double the number in the first column, and halve the number in the second column.
* If the number in the second column is odd, divide it by two and drop the remainder.
* If the number in the second column is even, cross out that entire row.
* Keep doubling, halving, and crossing out until the number in the second column is 1.
* Add up the remaining numbers in the first column.
* The total is the product of your original numbers.

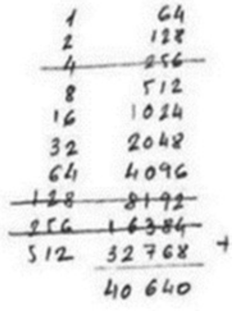


Fig. 11.12 Russian peasant multiplication algorithm

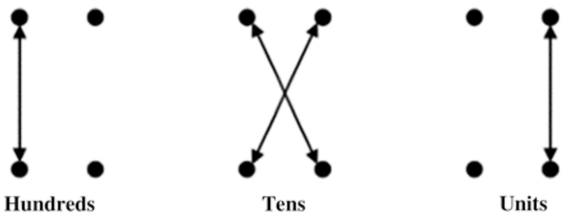


Fig. 11.13 Diagrammatic explanation of Russian multiplication

In the first case, we double 36, obtaining 4, 8, and 16 times 36 (in bold text), which can then be summed to find 28 times 36. On the right we can see that the result is the same if we double 28, obtaining 4 and 32 times 28, which are summed to give 36 times 28.

The diagram below is also useful, Fig. 11.13, as it can be followed mentally to find the product of two two-digit numbers (c.f. Fig. 11.12). For numbers with more than two digits, we believe that mental arithmetic is not appropriate; a calculator is.

If we apply the diagram above to 28 × 36, we have:

- Units: 8 × 6 = 48; we write the "8" and carry the 4 to be added to the tens figure.
- Tens: 2 × 6 = 12, 3 × 8 = 24, 12 + 24 = 36, 36 + 4 = 40; we write the "0" and carry the 4 to be added to the hundreds.
- Hundreds: 2 × 3 = 6, 6 + 4 = 10; we write the "10".

The result is 1008. The process can be done mentally, noting only the final result, but we can aid the process with a pencil and paper, writing down the intermediary steps, as described above.

In conclusion, we are left only to underline one of the ideas already described above: that arithmetic is not an end in itself but a means of solving problems quickly and effectively. Therefore, learning numerical facts or algorithms to the detriment of understanding and the meaning of the operation should be avoided at all costs. Attaining speed with arithmetic should not be an objective in school, and using fingers or other objects should not be seen as embarrassing or something to be discouraged in pupils. We can learn from the results of neuropsychology research, making us more understanding and tolerant of pupils' mistakes, allowing us to adapt our teaching methods to how the brain actually works, as this is the root of many failures in learning arithmetic.

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