

Chapter 2

Subject-Matter Didactics



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Abstract In the development of didactics of mathematics as a professional field in Germany, subject-related approaches play an important role. Their goal was to develop approaches to represent mathematical concepts and knowledge in a way that corresponded to the cognitive abilities of the students without disturbing the mathematical substance. In the 1980s, views upon the nature of learning as well as objects and methods of research in mathematics education changed and the perspective was widened and opened towards new directions. This shift of view issued new challenges to subject-related considerations that are enhanced by the recent discussions about professional mathematical knowledge for teaching.

Keywords Subject-oriented didactics · Basic ideas · Mental representations · Didactics of arithmetic · Didactics of calculus

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2.1 Introduction

In the development of mathematics education as a professional field in Germany, subject-related approaches played an important role from the very beginning (see also Hefendehl-Hebeker 2016). We consider subject-matter didactics as an approach to contents of teaching, which comprises detailed insight into the subject matter such as:

- Essential concepts, procedures and relationships including appropriate formulations, illustrations and arrangements for teaching
- Essential structures and domain-specific ways of thinking
- The inner network of paths by which the components are connected and possible learning paths throughout the domain.

This general approach is open to multifaceted refinements and extensions, and its perspectives and functions can vary under the influence of changing circumstances.

In Sect. 2.2 we outline an overview of origin, main issues, theory and methods of subject-matter didactics (L. Hefendehl-Hebeker). We will also introduce central aspects of the concept of basic ideas (*Grundvorstellungen*), which is an important theoretical concept of German subject-matter didactics concerning the role of mental representations and procedures (R. vom Hofe). Then we will consider two main fields of application in more detail: In Sect. 2.3 the concept of basic ideas in primary school, which focuses on numbers, fractions, basic operations and their representations (S. Wartha and A. Schulz) is explored, and in Sect. 2.4 the question about clarity and rigor in calculus, which was a main theme of German traditions of subject-matter didactics and which is still an actual problem area (A. Büchter and H. Humenberger).

2.2 Origins and Main Issues of Subject-Matter Didactics

2.2.1 *The Origins*

In the course of the Humboldtian educational reform in the beginning of the 19th century, mathematics changed from a marginal to a major teaching subject. Mathematics was considered as a constitutive part of ‘Bildung’, the development of a person’s personality, namely as a key component for developing the capacity of autonomous thinking. Thus, mathematics was no longer confined to imparting practically useful arithmetic skills but was to be taught as a training of cognitive abilities. This general attitude initially resulted in a preference for pure mathematics (see Chap. 1 in this issue and Schubring 2012).

During the second half of the 19th century, both the economy and the industry experienced an enormous upswing, exposing the necessity for an adapted modern education whilst taking into account actual developments of science and technology. At gymnasium (grammar school), a climate of reforming education developed.

Specialized journals on the teaching of mathematics and science were founded and books on methods of teaching mathematics started to be published. Some professors of mathematics began to offer special courses on school mathematics for future teachers at the gymnasium.

Felix Klein (1849–1925), professor at the University of Erlangen, the Technical College in Munich, the University of Leipzig and finally at the University of Göttingen, had always been engaged in educational aspects of mathematics. He undertook the challenge to instigate a profound reform of mathematics teaching, forging a broad alliance of teachers, scientists and engineers to support his ambitious plans.

The slogan for his Meran reform program was the famous notion of *functional reasoning*, which should pervade all parts of the mathematics curriculum ‘like an enzyme’ (Klein 1904). As Krüger (1999) pointed out, this notion should be considered as an overarching curricular guideline and orientation, which refers to a renewal of content and teaching methods of the mathematics curriculum at once.

With respect to content, the aim of the mathematics curriculum in the Meran reform program was to gradually create in the pupil’s mind a consciousness for the variability of quantities—in arithmetic as well as geometric contexts and thus to raise a habit of thinking, to prepare an access to analysis and the differential and integral calculus and to bridge the gap between secondary and higher education. With respect to didactic guidelines and concrete pedagogies, the aim was to reject systematic-deductive arrangements and to turn to heuristic and genetic approaches as well as to change from rather static logical conclusions to more flexible ways of thinking. Thus the reform movement determined the course for a profound modernization. However, the teaching practice often failed to fulfill its purpose due to an obvious tendency to deal with rigid subjects of instruction instead of developing habits of flexible thinking (Krüger 1999, p. 304).

The Meran reform movement also entailed extensive publication activities in subject specific journals and monographs, for instance reports of experiences with the new subjects, proposals for the structure of the syllabus and for the design of teaching with regard to content and methods, and finally complete textbooks and task collections (Krüger 1999, p. 149). Klein himself created an example of continued relevance by his lessons on *Elementary Mathematics from a Higher Standpoint*, which later appeared in a series of books in several editions (Klein 1968). He intended to give an overview over the academic discipline of mathematics as a whole including the historical development and so to convey a general education (*Allgemeinbildung*) to the teachers.

W. Lietzmann (1880–1959), who was directly involved in the Merano reform, created a classic with his ‘teaching methods for mathematics teaching’ (Lietzmann 1923). Working in the tradition of Felix Klein, his purpose was to provide practicing teachers with a detailed insight into the subject matter, to propose appropriate formulations and illustrations such as possible learning paths, and to indicate obvious difficulties.

In the beginning of the 20th century, new educational methods (like the so-called *Reformpädagogik*) gained increasing influence especially on primary teaching. They aspired to replace the old ‘learn and drill school’ by an education towards self-acting

and consequently self-reliance. On this basis, J. Kühnel (1869–1928) developed his didactics of arithmetic (Kühnel 1916), which claimed that education and teaching should be oriented towards the natural development of mind (‘start from the child’). In the 1920s and 1930s, psychologist J. Wittmann (1885–1960) established his ‘holistic didactics of reckoning’, which was considered as an application of Gestalt psychology (Wittmann 1939). His concept of arithmetic started with doodle patterns, which children had to arrange, rearrange, compose, decompose and compare. Thus they were trained to discover relationships, which served as an illustrative base for the subsequent systematic consideration of numbers. Many of his figurative patterns are in use today. These approaches marked a rejection of formalist teaching methods, which mainly consisted of training procedures and memorizing rules. Mathematics was seen as offering an opportunity for developing cognitive abilities and forming the mind even at an early age, with a new challenge to create appropriate subject-based teaching material.

Thus, in the 19th and the beginning of the 20th centuries the development of didactics of mathematics as a professional field originated in different traditions which partially started from opposite ends of the field. The secondary education tradition was clearly orientated towards the subject as a scientific domain with a more or less pronounced awareness of psychological needs. The new developments in primary education were mainly oriented towards the contemporary psychology, and hardly any connections to mathematics as a field of science and research were established (Müller and Wittmann 1984, p. 147). Some decades later, the development should lead to syntheses of the approaches, which also resulted in a more differentiated view of the subject matter.

2.2.2 New Developments After 1945

The development of new structures within the educational system, the introduction of new curricular contents, and a refined consciousness of facets of subjective representations of knowledge created the need for appropriate literature on subject-oriented didactics as well as discussions about the nature and place of such contributions. F. Drenckhahn (1894–1977) wrote a clarifying contribution, wherein he defined didactics of mathematics to be “the presentation of the subject matter with respect to teaching” (Drenckhahn 1952/1953, p. 205). As he pointed out, the difference between mathematics as a scientific discipline and didactics of mathematics primarily exists in the aspects guiding the presentation, which are linked to different aims:

- Mathematics as a scientific discipline strives for a tight systematization and logical compression. The presentations reproduce the final stage of mathematical insight (according to the latest scientific findings), where the tracks of the thought process are covered up.
- In didactics of mathematics, not only formal logic but also the inner logic of the subject plays an important part. First of all, didactics of mathematics has

the task to reveal (uncover) the images, notions, ideas, concepts, judgements and conclusions, but also the impulses and working methods, which originally result from the subject matter with logical necessity. This results in a new subject-related architecture of different layers of mathematics with distinct subject-related logics. For example, in this sense the rules for the multiplication of fractions or negative numbers follow another subject-related logic than the empirically based rules for calculation with natural numbers.

With these clarifications, Drenckhahn took up the secondary school tradition and influenced the main orientation up to the 1970s. Mathematics remained the primary academic discipline for referral by didactics of mathematics (Burscheid 2003).

W. Oehl (1904–1991) accentuated the importance of the essential structures of the subject and claimed that subject-related considerations and domain-specific ways of thinking should play a central role in mathematics education from the very beginning (Oehl 1962, 1965).

On this basis ‘didactically oriented content analysis’ was developed as a tool for research in didactics of mathematics, resulting from the ambition for solid foundations and conducted with the aim to present the contents in a way that is compatible with the standards of the field, and at the same time appropriate to the learners and the requirements of teaching (Griesel 1974).

In a first period, the emphasis of the ‘didactically oriented content analyses’ was on the lower secondary school level, especially in the domain of arithmetic and algebra, complemented by an analysis of the concept of function (Vollrath 1974). Kirsch presented far-reaching analyses of the foundations of proportional reasoning as well as of linear and exponential growth (Kirsch 1969, 1976a).

After an initial concentration on the primary and lower secondary levels, the didactically oriented content analyses were extended to the domains of upper secondary school teaching. Here the contents already had a solid scientific foundation and the problems were mostly opposite to those with respect to the lower stages. The question was how mathematical theories and concepts could be simplified and made accessible without falsifying the essential mathematical content. W. Blum and A. Kirsch suggested more intuitive approaches (at least for basic courses) with the original naive ideas of function and limit and sequential levels of exactitude, which could be achieved according to the capacity of the learners (Blum and Kirsch 1979; Kirsch 1976b).

A general goal was to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, while simultaneously simplifying mathematical material without distorting it from its original form, with the aim of making it accessible for learners (Kirsch 1977). The simplifications introduced into mathematical subjects should be ‘intellectually honest’ and ‘upwardly compatible’ (Kirsch 1987). That is, concepts and explanations should be taught to students with sufficient mathematical rigour in a manner that connects with and expands their knowledge of the subject. Such goals also caused a search for guiding orientations in a local and global sense and produced paramount constructs of subject matter didactics, among these the concept

of *Grundvorstellungen* and the tension between *clarity and rigour*, which will be discussed in more detail below.

Special challenges had resulted from the New Math movement in the mid-1960s, which pursued the idea that mathematics education should be science-oriented from the very beginning, and which led to a comprehensive conceptual system of school mathematics adopting the structure-oriented view of the Bourbaki group together with a high level of formalization. A new national curriculum framework (KMK 1968) chose sets, structures, mappings, functions and logical concepts as content of teaching. Numerous contributions in journals and textbooks unfolded these concepts with respect to teaching (for a partial overview, see for example Vollrath 2007). However, teachers and educators were widely unprepared for the reform and it finally failed. The advocates of the reform concentrated overly on the contents of the curriculum, neglected accompanying experimental approaches (Schubring 2014) and underestimated the epistemological obstacles of the new arrangements (Damerow 1977).

Within a changed view of the nature of learning, the focus of teaching had to be shifted from the conveyance of knowledge to the organization and inspiration of learning processes. Thus, wider programs of ‘mathematics education *emerging from the subject*’ (Wittmann 2012) were developed (for more details, see Hefendehl-Hebeker 2016; Hußmann and Prediger 2016, as well as Nührenböcker et al. in Chap. 3 of this issue).

2.2.3 *The Concept of Grundvorstellungen*

An important theoretical concept of German subject-matter didactics is the concept of *Grundvorstellungen*, abbreviated GVs (Oehl 1962; vom Hofe 1998). The German word ‘Grundvorstellungen’ consists of two sub words: The prefix ‘Grund’ means ‘ground’ or ‘basis’, the second word ‘Vorstellung’ roughly means ‘idea’ or in the context of GVs ‘conception’ or ‘notion’. So *Grundvorstellung* can be roughly translated as ‘basic notion’ or ‘basic idea’. Below we use the abbreviation GV, this is meant to more adequately articulate the specificity of this concept.

The idea of the GV concept was developed during the heyday of the ‘New Math’, where ‘set theory’ was forced into elementary schools, and particularly rigorous *academic* curricula were commonly present in lesson plans, textbooks and in the classrooms of secondary school education. It was during this time that early subject-matter didactics was developed (cf. Griesel 1968, for fractions; Kirsch 1969, for everyday arithmetic; or Blum and Kirsch 1979, for calculus). The goal of this new approach was not to simply continue importing more precise academic mathematics into schools. Rather, the goal was to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, while at the same time simplifying mathematical material without distorting it from its original form (see above).

Subject matter didactics, in other words, resisted the tendency of formalizing concepts and procedures exhibited by the New Math, and instead placed more value on constructing viable and robust mental representations—i.e. *Grundvorstellungen*—with which to capture mathematical concepts and procedures. On the one hand, GVs should be able to accurately fit to the cognitive qualifications of students, and on the other hand, capture the substance of the mathematical content. The concept of *Grundvorstellungen* describes the relationships between mathematical content and the individual concept formation, referring especially to three main characteristics:

- The *constitution of meaning* of a mathematical concept by linking it back to familiar knowledge or experiences, or to (mentally) represented actions.
- The *generation of a corresponding mental representation* of that concept; that is, an ‘internalization’, which (following Piaget) enables operative action at the level of thought.
- The *ability to apply* a concept to real-life situations by recognizing a corresponding structure in subject-related contexts or by modelling a subject-related problem with the aid of mathematical structures.

2.2.4 Examples of Grundvorstellungen

First, we consider an example of GVs and take the concept of subtraction. Figure 2.1 shows GVs and areas of application, symbolized as a tree. The GVs are pictured as roots, which give each student the capability of dealing with the applications pictured in the greenery of the tree. The different GV aspects correspond to different areas of application:

The aspect ‘taking away’ has the structure state-change-state (S-C-S). It corresponds to situations like this: *Tim has €12.30. He spends €4.30. How many € does he have left?*

The next situation is structurally different: *Lily has €12.30, Luce €4.30. How many more € does Lily have compared to Luce?* In this situation nothing happens, nothing is taken away, everybody keeps what he has. It’s a static situation, the structure is state-state-state (s-s-s) and the corresponding GV aspect is comparing states. The third application area refers to comparing of changes. And the fourth example refers to the GV aspect of complementing: *Jacob has €12.30, Mary has €4.30. How many more € does Mary need to have the same as Jacob?* This aspect is difficult for several students, who for example try to solve tasks like *How much is 42 minus 39* by counting backward. This usually results in miscounts and errors because they can’t activate the GV of complementing.

Over time the learning individual develops a growing system of networked GVs and GV aspects, which builds an important basis of mathematical understanding. Of course the development of GVs doesn’t stop with the basic operations. Further examples will be specified in the following sections.

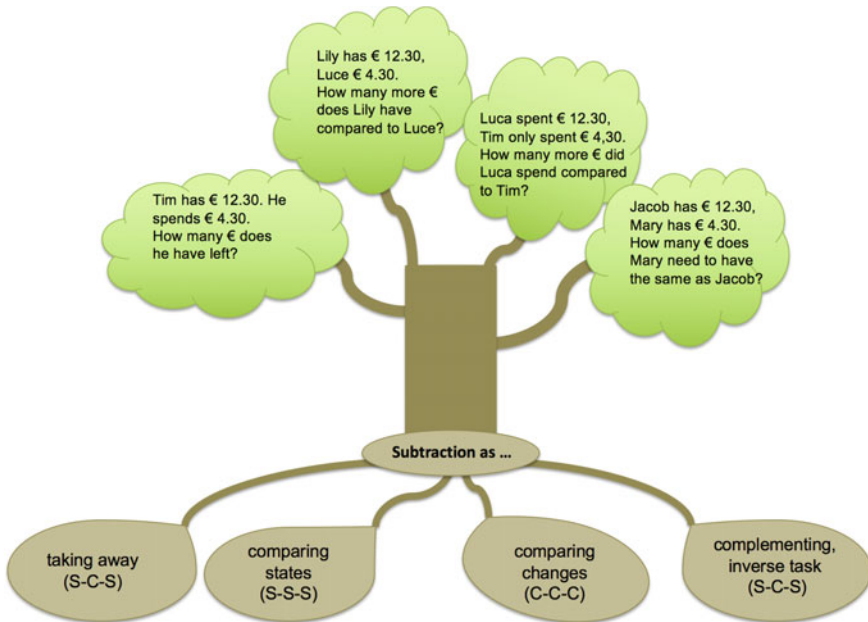


Fig. 2.1 GVs of subtraction

2.2.5 Central Aspects of the Concept of Grundvorstellungen

Normative, descriptive, and constructive aspects

An important quality of the GV concept is the combination of normative and descriptive working methods. In this context we can distinguish between two aspects:

- *Normative working methods* are used to deduce GVs as *normative notions*. They work as *educational guidelines*, following a particular educational goal and describing *adequate interpretations* of the use of mathematical concepts. An example is the normative description of GVs of subtraction and their attribution to appropriate areas of application above (see Fig. 2.1).
- *Descriptive working methods* are used to get insight into the *mental representations, individual images and explanatory models* that students have in fact. These individual representations usually deviate more or less from the GVs that are intended as normative guidelines.

Comparing and detecting potential conflicts between normative and descriptive aspects, which is between normative GVs intended by the teacher and observable actual images and explanatory models of the student, can provide constructive insight in learning problems of the student and give hints for removing misconceptions.

Primary and secondary GVs

Another important quality of the GV concept is the distinction between primary and secondary GVs (vom Hofe and Blum 2016):

- *Primary GVs* are based upon concrete actions with real objects. The corresponding concepts can be represented by real objects and actions for instance by joining or dividing real sets of things. Primary GVs for this reason possess a *concrete character*.
- *Secondary GVs* are based on mathematical operations with symbolic objects. Constituent of the corresponding mathematical structures is dealing with mathematical objects, such as number lines, terms, and function graphs. Secondary GVs for this reason are said to have a *symbolic character*.

The distinction between primary and secondary GVs largely corresponds to the distinction given by Fischbein between ‘primary’ and ‘secondary intuitions’ (cp. Fischbein 1987).

Grundvorstellungen and modelling

Primary and secondary GVs play a key role in the process of modelling, that is, in the translation between mathematics and the ‘rest of the world’. The important steps of modelling can be considered as: (1) constructing a situation model, (2) simplifying it to a real problem, (3) mathematizing, (4) working mathematically, (5) interpreting, (6) validating, and (7) exposing (cp. Blum and Leiss 2007). Central mental activities work here at step 3, when a real world problem situation is given and a corresponding mathematization has to be found. Or at step 5, when a mathematical result has to be interpreted in relation to the problem situation.

For this process, one needs GVs to decide which mathematical content or method can fit a particular problem situation, or, vice versa, which problem situations can be modelled with specific mathematical content.

Furthermore, modelling processes often require translation between different levels of representation within mathematics, for example between algebra and geometry (step 4 of the modelling process). Again, GVs are required here to assign, for example, geometric representations of concepts such as slope or monotonicity to their corresponding algebraic representations. These processes of translation are typically accomplished with secondary GVs.

All in all, GVs can be considered as means of translation between mathematics and reality or between different representation levels of mathematics—more generally—as objects of transition between different mathematical representation systems. These relationships demonstrate the important role of GVs for the development of mathematical competencies. This development would ideally be accompanied by the formation of both primary GVs and, with progressive learning also secondary GVs, into a growing and networked system. In particular, the ability to apply mathematical skills is based, according to this view, on the quality of development and the degree of the cross-linking of GVs, as well as on the ability to activate and coordinate GVs.

Relations between GVs and other concepts of mental representations

The formation of the GV concept as well as its interrelated basic assumptions can be considered in the context of other concepts dealing with mental mathematical representations. We shortly focus on three important approaches: the research on intuition in mathematical thinking, the work on concept image and concept definition, and the theory of conceptual change.

According to Fischbein et al. (1990), even at higher levels the process of mathematical problem solving is always combined with intuitive images and assumptions that can affect the way of problem solving more or less unconsciously (see Fischbein 1987, 1989). Fischbein's research focuses on the contrast between the intuitive level and the formal level, and the finding that many errors in students' mathematical thinking and acting are based on intuitive tendencies which interfere with correct reasoning. Fischbein's insight, where serious problems of understanding and communication found in mathematics lessons are based on conflicts concerning the intuitive level, corresponds clearly to the findings of the early advocates of subject-matter didactics. His work has significantly influenced the generation of the GV concept (see vom Hofe 1995). However, a major difference between Fischbein's approach and the GV concept concepts is that, in the work of Fischbein, intuitions are used exclusively as a descriptive notion while the GV concept uses *Grundvorstellungen* primarily as a normative notion.

Another important idea in this area is the theory of concept image and concept definition by Tall and Vinner (1981). The term concept image describes "the total cognitive structure that is associated with the concept, including mental pictures and associated properties and processes. It is built up over the years through experience of all kinds, changing as the individual meets new stimuli and matures" (Tall and Vinner 1981, p. 2). It can be more or less in accordance with, or in contrast to, the concept definition. These ideas developed similarly to the GV approach during the area of New Math and provide the insight that teaching mathematics on a formal level does not lead to appropriate understanding of the students. The contrast of concept image and concept definition may seem parallel to the distinction between the descriptive and the normative aspect of the GV concept, but this would be a misunderstanding: both the descriptive and the normative aspect of the GV concept refer to the field of concept image and describe actual detectable individual images (descriptive aspect) and didactically intended GVs (normative aspect).

Finally, we take a look at the theory of conceptual change. A key assumption of this theory is that individuals generate robust and early concepts by interpreting their daily lives, which then could become inadequate in the course of time when facing new information. Therefore, processes of conceptual change are needed in order to extend the existing mental structures and adapt them to the new requirements. This concept has roots in Piaget's theory of assimilation and accommodation and was developed especially in science education (Posner et al. 1982). Furthermore, this concept has also been applied to the learning of mathematics, particularly concerning the advancement from natural to rational numbers (Vosniadou and Verschaffel 2004; Kleine et al. 2005; Prediger 2008). Considering that the generation of *Grund-*

vorstellungen in the long run is supposed to be a dynamic process with changes, reinterpretations and substantial modifications (vom Hofe 1998), affinities between the idea of conceptual change and the generation of GVs are obvious.

In summary, we can state that certain aspects of the GV concept can also be interpreted from the perspective of the above-mentioned theories. However, a substantial difference exists concerning the main foci of these concepts: While research concerning the above mentioned concepts has mainly or exclusively a descriptive emphasis, the approach of the GV concept combines normative and descriptive methods with a constructive aim. In this sense, analyses of students' work based on the GV concept typically do not remain at the descriptive level but lead to indications of a constructive 'repairing' of the analyzed problems.

2.3 Numbers, Fractions, Operations and Representations—Grundvorstellungen in Primary School

In the following sub-chapter, we would like to show that through the comparison of normative and descriptive interpretations of *Grundvorstellungen* it is possible to gain constructive insights for research and teaching (Sect. 2.2). To this end, we will use examples of *Grundvorstellungen* of natural numbers, fractions, subtraction and multiplication.

2.3.1 *Grundvorstellungen and Levels of Representation*

Grundvorstellungen are models that build a bridge between various levels of representation. They allow translations from one level to another (see Fig. 2.2; see also vom Hofe 1995; Wartha and Schulz 2012; Sect. 2.2).

Translations can be performed both *between* and *within* different levels of representation. Typical levels of representation are images, actions (e.g. manipulatives), word problems (of realistic situations) and spoken or written mathematical symbols. 'Understanding' means that mathematical ideas are not linked to one isolated level, but they can be used and translated in various levels of representation. The idea that the 'mathematical understanding' can be diagnosed and developed through the translation *between* and *within* levels of representation has been described by many authors with similar models (e. g. Kuhnke 2013; Duval 1999; Janvier 1987; Söbbeke and Steinbring 2007). Furthermore, these levels of representation can be more concrete or more abstract (see Fig. 2.2).

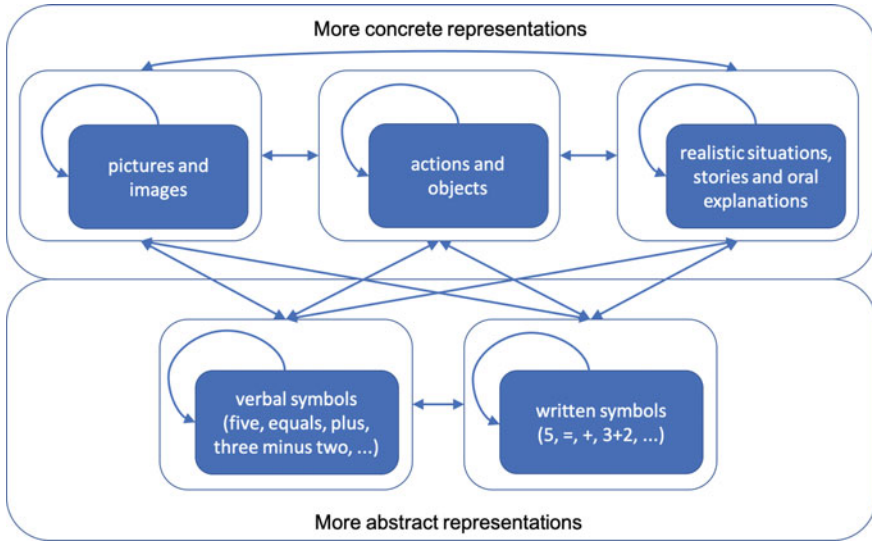


Fig. 2.2 GV and levels of representation (Schulz 2018, p. 6)

2.3.2 GV of Natural Numbers

Natural numbers can be interpreted in many different ways. On the normative level we can distinguish, for example, between two aspects of the understanding of natural numbers: one is the cardinal aspect, which is based on the comparison of quantity of numbers (described through equivalence classes), the other one is the ordinal aspect of numbers, which refers to the position of a given number in an order of numbers (Klein 1968; Padberg and Benz 2011; Padberg et al. 2001). In the following excerpt of an interview (Table 2.1) we can identify these two aspects from a descriptive perspective:

Matteo's case is an illustrative example for the empirical finding that children can have two different *Grundvorstellungen* of numbers:

- a sequenced scheme, which means an ordinal understanding of number or a mental number line (he has problems with determining the number 'just before', if this number is a multiple of ten)
- a quantitative scheme, which means a cardinal understanding of numbers (he is able to name the correct multiple of ten when asked: *take one away* from 61)

Ever since Resnick's publications on the development of number understanding, we have assumed that robust number sense means integration of these two aspects (e. g. Resnick 1989). Early research shows that number sense develops by integrating the cardinal and ordinal interpretation of numbers. Fuson (1988) has already pointed out that the development of concepts of ordinal and cardinal numbers is closely linked, and that more cardinal prompts (more than/fewer than) are interpreted differently

Table 2.1 Matteo transcript (a 9-year-old pupil at the end of the second year of primary school)

I	Count backwards from 78	I	Imagine I'm showing the number 61 now
M	Backwards?	I	What does it look like?
I	Backwards	I	Can you show me this up in the air? [...]
M	78, 77, 76, 75, 74, 73, 72, 71, 60	M	10, 20, 30, 40, 50, 60, 1 (shows with fingers)
I	Go on	I	If I take this one away, what number will I get?
M	59, 58, 57, 56, 55, 54, 53, 52, 51, 40 , 39	M	Take this one away. 60.
I	OK. Thank you Please count backwards once again from 92	I	And another one away?
M	91, 80, 79, 78, 77, 76	M	((60) ... 59
I	OK. Thank you	I	OK

than more ordinal prompts (just before, just after). In Matteo's case, both approaches could be perceived: there is a difference in his behavior depending on whether Matteo is offered an 'ordinal prompt' or a 'cardinal prompt'. Sayers et al. (2016) have recently presented a somewhat new approach to number sense. By intensive investigations they have elaborated an 'eight component framework' in order to be able to characterize a so called 'foundational number sense' (FoNS). This approach also stresses the correlation between cardinal and ordinal number aspect (see also Gerlach 2007).

Possible implications for teaching and the education of teachers are that instruction can only be appropriate to students' learning processes if the individual underlying concepts and misconceptions are well known and can be taken into account (Schulz 2014). This requires profound knowledge about how students *typically* learn a specific content, which conceptions they bring with them, and also knowledge about possible, empirically documented misconceptions and mistakes that commonly arise during this learning processes. For the present content this means that teachers should be informed of the *existence* of different aspects of numbers and of the *difference* between them, namely for both: a normative and a descriptive level (Clarke et al. 2011; Lindmeier 2011; Schipper 2009; Schulz 2014; Wartha and Schulz 2012). Only in this way are they able to appropriately support the pupils' processes of learning, especially by fostering a simultaneous development of both aspects of number.

In the next section, we will show that the relations between aspects of numbers and between representations can not only support pupils' learning processes, but that these translations between representations can be interpreted as the core of mathematical learning.

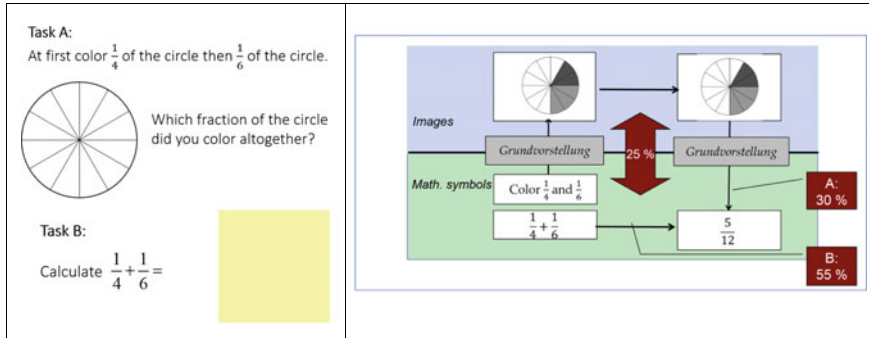


Fig. 2.3 Fraction Tasks A and B, description with *Grundvorstellungen*, percentage of correct answers

2.3.3 *Grundvorstellungen of Fractions*

There are many different normative descriptions of how fractions can be interpreted. One thing that many interpretations have in common is that a fraction describes a part-whole-relation as the comparison of a quantity to a dividable unit. In this sense, fractions can be interpreted as quotients or measures and represented as areas or countable parts of an object. This interpretation can be seen as continuation of the cardinal aspect of natural numbers. On the other hand, fractions can be interpreted as measure numbers in real life situations. Hope and Owens (1987) state: “Measure numbers are commonly represented as points on a number line partitioned in smaller segments.” This interpretation can be seen as a continuation of the ordinal aspect of natural numbers.

From a descriptive perspective, two content-related fraction tasks investigate the part-whole-*Grundvorstellung* of learners (Fig. 2.3). These two items (Fig. 2.3, left) were developed by Hasemann (1981). In his research, he wanted to assess the role of the level of representation on solving processes (Wartha 2009).

A normative analysis shows that the task A does not require calculation skills with fractions. The fractions $\frac{1}{4}$ and $\frac{1}{6}$ are given as written symbols and have to be translated into a given image. Two and three parts of the circle should be colored and put together to five parts. In a last step, 5 of 12 parts have to be translated into the symbolic expression $\frac{5}{12}$. *Grundvorstellungen* have to be activated at least twice. In contrast, there is no translation required to solve task B. The sum can be calculated within the symbolic representation by using fraction addition rules. The problems were given to $N = 1010$ 6th grade pupils in Bavaria of all school forms who finished the course on learning fractions.

Less than a third of the pupils are able to activate a *Grundvorstellung* whereas more than 50% are capable of solving an arithmetic expression containing the same numbers. Although the tasks followed each other immediately in the tests, only a quarter of all pupils produced the same solution.

Apparently three out of four pupils do not realize that the mathematics in both problems is the same—just in different levels of representation. They operate with numbers of which they do not have any *Grundvorstellung*. Numerous other investigations (Hasemann 1997; Mack 1995; Prediger 2008, 2011) also demonstrate that solving processes are influenced significantly by the level of representation and whether translations are required or not.

The comparison of normative analysis and descriptive empirical results shows that translations between representations themselves are the core of mathematical thinking. The request of these translations can be regarded as a didactic principle for both diagnoses and furtherance. On the one hand, the (un-) successful translation can give insight in how well a given content is understood. On the other hand, the constant request to switch between symbols, non-symbolic representations and real or mental models can lead to robust *Grundvorstellungen*.

The term *Grundvorstellung* can thus be regarded also as an operationalization of mathematical ideas for planning curricula, preparing lessons and analyzing children's answers, solving processes and supporting improvements (Padberg and Wartha 2017). The fundamental idea relies on the translation between symbols and *appropriate* models. The adequacy of models is discussed in the next example, *Grundvorstellungen* of subtraction.

2.3.4 *Grundvorstellungen of Operations: Subtraction*

Hefendehl-Hebeker and vom Hofe (Sect. 2.2) have already provided some examples for different applications and the corresponding GVs of subtraction. These GVs are based on the following two fundamental different models of subtraction (Selter et al. 2012; Usiskin 2008; Wessel 2015)

The mathematical definition of subtraction is linked to addition: $a - b$ is a number c , for which a equals $b + c$. For the *meaning* of subtraction, we can distinguish two fundamental different models or *Grundvorstellungen* (Wessel 2015)—determining a remaining quantity and determining a difference. Further aspects in order to categorize interpretations of subtraction can be dynamic versus static aspects, given and unknown data. Word problems in additive and subtractive situations are often classified in the categories change, combine, compare and equalizing (Schipper 2009; see also Sect. 2.2).

Determining a remaining quantity by ‘taking away’ (dynamic) or ‘comparison’ (static):

In the example 7-4, the first number 7 can be translated as a dot on the number line. ‘-4’ means an arrow back to a *position*, where the result can be determined (see Fig. 2.4 top left). Word problems such as:

- “I have 7 pets, 4 are taken away, how many are left?”
- “Tim has €12.30. He spends €4.30. How many € does he have left?”

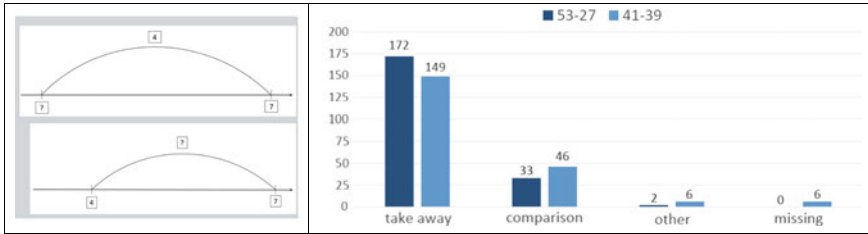


Fig. 2.4 Take away and comparison model normative (left, see Wessel 2015, p. 29) and descriptive (right)

- “Chris has 7 bottles of water, 4 more than I have. How many do I have?”

are associated with this model. The first two are dynamic situations and the third one is a static situation.

Determining a difference by ‘complementing’ (dynamic) or ‘comparison’ (static):

In contrast to determining the remaining quantity, both numbers of the term, e.g. 7-4 can be located on the number line and the result is seen in the *difference* between them (see Fig. 2.4 bottom left). Examples for word problems are:

- “Mat has 7 cars, Tom has 4. How many more cars does Mat have?”
- “Mat has 7 cars, Tom has 4. How many more cars does Tom need in order to also have 7?”
- “Lily has €12.30, Luce has €4.30. How many more € does Lily have in comparison to Luce?”

We want to stress that all of these categories come down to these two *Grundvorstellungen* of subtraction. In other words, every situation that requires a minus sign to be entered into a calculator to solve it contains a *Grundvorstellung* of subtraction.

In empirical studies, many colleagues showed that solution processes strongly depend on which *Grundvorstellung* that has to be activated.

- Stern (1998) pointed out that preschool children deal significantly better with dynamic take away situations than with static comparison situations in word problems.
- Schipper et al. (2011) demonstrated that about 80% of 2000 second-year pupils can solve take away situations correctly, but only 60% can do so in the case of static comparison word problems—with comparable numbers.

Studies conducted in calculation processes of pupils showed a clear preference of using ‘take away’ strategies, even if determining the difference would offer more effective solutions:

- Selter (2000) showed that the term 701-698 is very difficult to third graders and only a few of them could use a comparison or complementing GV to interpret subtraction as the difference between 701 and 698.

- Wartha and Benz (2015) stressed that only 15% of fifth graders used a comparison or complementing GV to calculate the term 601-598—even if more pupils knew that the difference between the numbers was 3, they could not use this information for the calculation.
- In a further study conducted by Wartha, student teachers were asked to create appropriate word problems by translating the terms 53-27 and 41-39. The vast majority of the (approximately) 200 students created word problems by activating the take away GV. Only a quarter of the students used a word problem based on the comparison or complementing model containing the term 41-39 (see Fig. 2.4, right).

The limited interpretation of subtraction as ‘take away’ is problematic, as the comparison or complementing model:

- provides efficient calculation strategies for terms like 41-39
- is relevant for the determination of differences in real life situations
- is an important mathematical interpretation of subtraction which is used to construct integers for example.

As a consequence, teachers, textbook authors, and curriculum developers should:

- know *both* models of subtraction
- emphasize the comparison model
- choose and use appropriate models and representations to construct both *Grundvorstellungen*.

The previous sections emphasized the interpretation of *Grundvorstellungen* as the ability to translate between different representations and models concerning a given content. In the next section the important role of choosing *appropriate* representations will be discussed.

2.3.5 Representations: The Case of Multiplication

As mentioned before, the normative and descriptive analysis of *Grundvorstellungen* can be seen as a useful precondition for planning assessment, promotion and support—especially for analyzing and choosing an appropriate representation for a given content. Using the example of multiplication, it can be clarified that a subject matter didactic analysis can help to choose an appropriate model or type of representation for teaching and learning.

As multiplication of natural numbers is frequently interpreted as repeated addition, school books often show illustrations or models using this kind of idea. For example, the problem 3×6 is illustrated by placing six apples on three plates each time (see Fig. 2.5a). This representation is appropriate for a first contact with the idea of multiplication.

A closer subject matter analysis reveals the shortcomings of this representation:

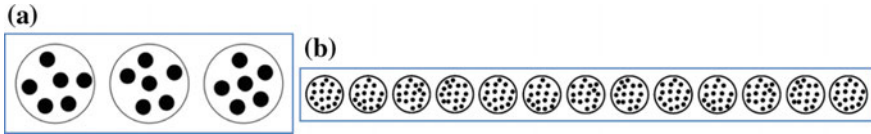


Fig. 2.5 a and b: Illustration for multiplication interpreted as repeated addition (3×6 and 13×16)

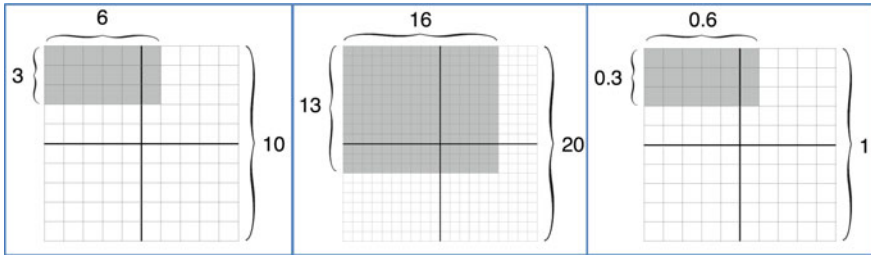


Fig. 2.6 Illustrating multiplication using rectangle representations

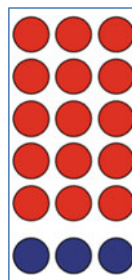
- (1) This type of representation is hardly suitable for an appropriate extension for larger numbers—reconstructing the model *mentally* would hardly be possible (see Fig. 2.5b).
- (2) Furthermore, it is impossible to illustrate the multiplication of decimal fractions (like 0.3×0.6) using this type of representation.
- (3) Also, the illustration of fundamental arithmetic laws (e.g. commutative law) is hardly possible using this type of representation.
- (4) For these reasons, this type of representation does not serve for the development of sustainable and long-term calculating strategies (for example using the distributive law).

To overcome these shortcomings, it is worthwhile to provide and use a better type of representation. The interpretation of multiplication using the scheme of a rectangle is a sustainable and extendable model, employing either dots or squares. The factors are represented by the number of columns and rows, the product being equal to the total number of visible objects or to the generated total area. This representation can be used from primary level (multiplication of natural numbers) up to secondary school (e.g. to illustrate the multiplication of decimal fractions) (see Fig. 2.6), and could be developed from the first representation e. g. by restructuring the given dots.

From a subject matter content analysis standpoint, the rectangle model seems to be very appropriate for teaching and learning multiplication. But how do learners interpret the given representation? Is the ‘power of the model’ sufficient enough to be self-explanatory? The interview reported in Table 2.2 is an illustrative example of the fact that even a suitable and sustainable representation is not necessarily applied without difficulties.

Table 2.2 Stephanie's interpretation of a rectangle model

S	I see: There are red dots and blue dots. 5, 10, 15. And the blue ones: Plus 3.
I	Would you also see a multiplication task there?
S	I could as well, yes.
I	Which one would that be?
S	Namely 15 times 3.
I	And why 15 times 3?
S	Because I've looked at those rows. The red rows amount to 15. And the blue rows.
I	OK.



Numerous empirical results show that pupils use didactical beneficial material rather unfavorably or even incorrectly; and faulty interpretations and mainly faulty 'inner images' of materials and representations have also been repeatedly proved empirically (Lorenz 1998; Rottmann and Schipper 2002; Söbbeke 2005). Obviously, a sustainable representation is not self-explanatory. This gap between the normative potential of a given type of representation and the empirically proved interpretation of these representations can lead to a constructive point of view. A possible conclusion might be that representations, their purpose and structure have to be interpreted *together with the children*. This means amongst other things discussing on *mathematical relevant* structure and relationships, and discussing different views on the same representation to widen the mathematical perspective of teachers and learners. Concerning this constructive point of view, there still is a gap in empirical research and in design (e. g. in mathematics education as well as in developmental and cognitive psychology). Which representations, models, prompts, discussions are fruitful, sustainable and suitable for the development of mathematical thinking and thus for the development of robust *Grundvorstellungen*?

2.3.6 Summary

To sum up, we would like to deduce some consequences for education, teaching and research.

- (1) The goal of mathematics education should be to establish *Grundvorstellungen*. They should remain sustainable for as long as possible and describe the meaning of numbers and operations comprehensively. At best, *Grundvorstellungen* of mathematical contents should be operative tools to be used from kindergarten to university.
- (2) We underlined the fact that a given content could be described by more than one *Grundvorstellung*. For this reason, we (teachers, educators, researchers)

should know which *Grundvorstellungen* (plural) describe a given content—both normative as well as descriptive.

- (3) This is necessary for planning assessment, promotion and support. We should have knowledge on representations, sustainable models, and on how mathematical contents are translated between representation levels.
- (4) Beyond that, we have to keep in mind that the results of normative and descriptive consideration are hardly ever congruent and not every learner is able to use the given *Grundvorstellungen*.
- (5) Based on these differences between subject matter didactic analyses and the empirical findings, constructive suggestions can be derived not only for assessment and teaching, but also for further investigations. Mathematics education as ‘design research’ provides the essential contributions to this task—not only in competition or as a complement to subject matter didactics or to empirical research on instruction, but rather as a necessary conclusion (e. g. Wittmann 1995).

2.4 Clarity and Rigor in Teaching Calculus

2.4.1 *Intended Understanding and Students’ Concept Images—The Example of Tangents*

‘Clarity and rigor in teaching calculus’ is a classical topic in subject-matter didactics. In this first part of our paper, we refer to an extended understanding of subject-matter didactics, which focuses explicitly on the learner’s perspective. The starting point is the difference between the intended understanding of mathematical concepts and the actual students’ concept images. We will show our considerations by referring to an exemplary concept in calculus: the concept of tangent. The goal is to create constructive proposals for the development of curriculum, textbook and lessons. Therefore, we refer to a model of different levels of curriculum, which is suitable for our considerations.

A Model of Curriculum Levels

The four-level curriculum model has been used in large scale assessments to analyze students’ achievement; however it is also suitable for research on students’ concept images:

- On the first level of this curriculum model, the *intended curriculum* includes the targets and objectives defined by the officials at state level (e.g., core syllabus, teaching plan demands, and central examinations).
- The *potential curriculum* on the second level includes textbooks and other learning materials, which must be often authorized by the ministry of education. This level can be regarded as a superset of learning opportunities.

- On the third level, the *implemented curriculum* is about the classroom practice ('Which topics are introduced and how?'). This level can only be examined by classroom observation.
- Lastly, the *achieved curriculum* on the fourth level concerns the content learned by the students. For our research, we will be focusing on the individual student's concept images.

In the following sections, we begin by outlining the syllabus specifications of the tangent concept ('intended curriculum'). Afterwards, we introduce individual students' concept images of the tangent concept ('achieved curriculum'). Then, we attempt to explain the differences between the intended understanding and the students' concept images by means of excerpts from text books ('potential curriculum') and we propose consequences for the development of learning units in textbooks and in differential calculus ('implemented curriculum'). Finally, we reflect on our analysis regarding the topic of 'clarity and strictness'.

Syllabus Specifications About the Tangent Concept

The concept of tangent plays an important role in three course units in lower and upper secondary education in most of the German federal states:

- Tangents to circles are introduced in the *Euclidean geometry* around *grade 7*.
- Tangents to parabolas are introduced and calculated in *analytic geometry* in *grade 10*.
- During *differential calculus* in *upper secondary*, the concept of tangent is made an object of discussion again by using curves, which are considered as graphs of functions in order to achieve a higher level of generalization of the same concept.

Thus, students are required to develop an analytic understanding of tangents at upper secondary level. However, a glance at the actual students' concept images about the tangent reveals that the learning opportunities in Euclidean and analytic geometry are more formative, as the next section shows.

Students' Concept Images About the Tangent

We present two selected examples from the results of a study about students' concept images of tangent. All students who have been interviewed had attended an introductory course and partly an advanced course in differential calculus. The student task consisted of answering the question: 'What is a tangent?', and no examples or hints about use of representations were provided. The following excerpts represent exemplary solutions, which were quite common among the data collected.

In the first example (Fig. 2.7), one can observe that the student activates the definition of tangent to a circle in Euclidean geometry. The striking characteristics in this solution are the global view of the situation as well as the property of having only one point in common, and the perception of touching. The reference to calculus is vague and turned upside down from a content perspective: the derivative is required for an appropriate determination of the tangent line at a curve and not the other way round.

The second typical example (Fig. 2.8) can be positioned more strongly in the world of functions. Even though the starting point is a function, the student refers to

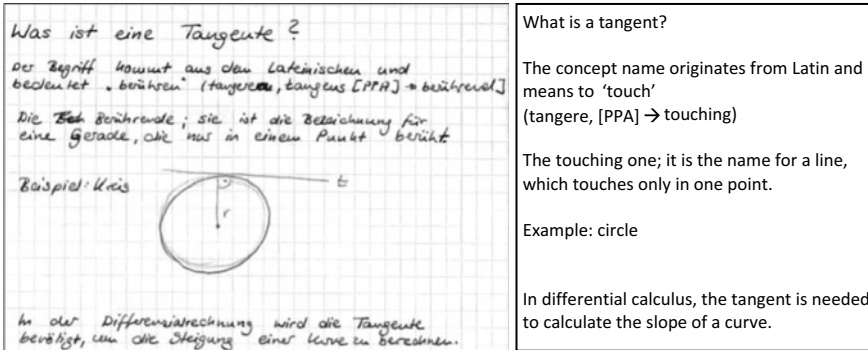


Fig. 2.7 Example for the strong activation of aspects from Euclidean geometry

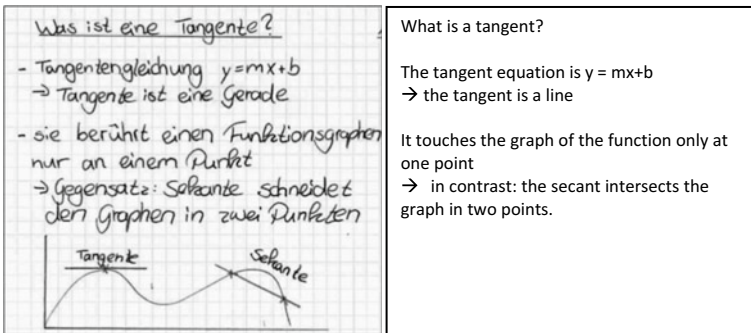


Fig. 2.8 Example for an answer from the world of functions

the characteristic of having only one point in common and the touching phenomenon, hence the perspective is still global. The comparison of the tangent to the secant is generally not sustainable.

Both examples represent several properties of a large number of student solutions obtained:

- A global perception of the situation
- Only one common point between the curve and the tangent line
- ‘Touching’ to describe the common behavior.

Furthermore, the students did not often establish a relationship between the tangent and derivative. If they had described this relationship in their solutions, then it was mostly false.

Considering the differences between the intended and the achieved understanding of the tangent concept, the next step is to investigate how these differences emerge and where they can be avoided. An appropriate way to analyze these questions is to take a look at textbooks.

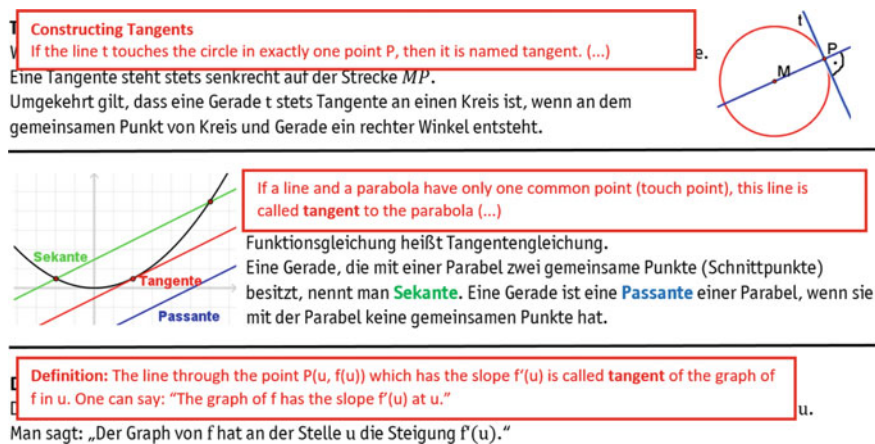


Fig. 2.9 Excerpts from textbooks

The Concept of Tangent in Textbooks

The following excerpts are from a typical popular schoolbook series from Grades 7, 10 and 11 respectively (Fig. 2.9).

- In the first excerpt, the concept of tangent is introduced as a tangent line to a circle in the Euclidean geometry. Here, only the properties of a global perception of the situation are mentioned: exactly one point in common and the touching property.
- In the second one, tangents are introduced in analytic geometry. This consideration is still global and the main properties mentioned are the common point between the tangent and the curve and the touching phenomenon. The differences among tangent, secant and passant are explained by referring to the number of common points.
- The third excerpt shows that the concept of tangent is introduced by a formal definition in differential calculus, without referring to previous concept images. The further analytic development of the available tangent concept is not made visible.

In the following sections we would like to make some constructive proposals to support the continuous concept development of tangent in terms of generalization. For instance, in Euclidian geometry one can consider other geometrical objects rather than only circles in order to investigate the tangent concept. For example, consider a triangle and a line through one vertex (Fig. 2.10, left). Students can discuss whether such a line can also be called a ‘tangent’. In this case, an adequate argumentation would be that the line and the triangle have exactly one point in common. But if that line is also called a tangent, then different tangents can exist (Fig. 2.10, right)—finally infinitely many. Hence, the uniqueness of the tangent is no longer valid.

These considerations lead to the question: Why are circles appropriate to construct just one tangent to a specific point? This way, one can activate the core aspects of the tangent concept: the local linearity of the figure and the tangent as the line of best

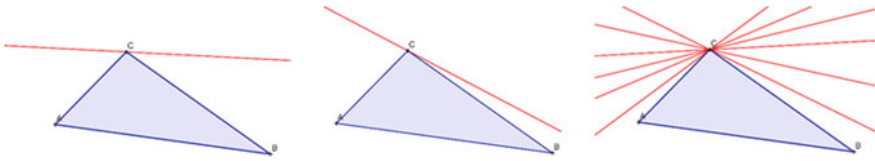


Fig. 2.10 Should these lines also be named ‘tangent’?

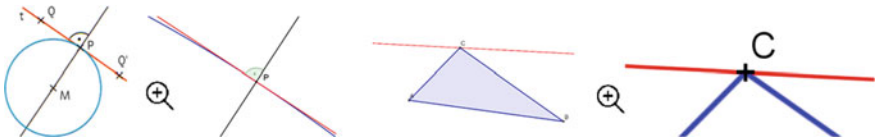


Fig. 2.11 ‘Figure microscope’ and local linearity

approximation. The idea is that of zooming in the figure (Fig. 2.11, left), which was proposed as ‘function microscope’ by Kirsch in order to examine the local linearity of differentiable functions (cf. Kirsch 1979; Blum and Kirsch 1979). Consider a triangle to highlight the characteristics of the concept of tangent (Fig. 2.11, right). Upon zooming in on the intersection point between the line and the triangle vertex, one can realize that nothing changes. There will still be the possibility of constructing several different lines through the vertex, which has one point in common with the triangle.

The phenomena of local linearity of the curve and the tangent as the line of best approximation are therefore already present in circle geometry. They are suitable as core aspects of the tangent concept on which we can build upon in differential calculus. In the following section, we discuss the question of how previous knowledge or experiences from geometry can be used to develop an analytic concept of tangent as an adequate generalization.

Dealing with the Tangent Concept in Differential Calculus

Differential calculus builds up on previous experiences and knowledge from circle geometry and analytic geometry. The concept development should be considered as an extension and generalization. At the beginning of the calculus course, the students’ actual concept images concerning tangents should be discussed in order to serve as foundations on which the concept development process can be further built upon. For the generalization of the tangent concept, crucial questions for the further concept development should be discussed:

- Which aspects of the tangent concept remain unchanged?
- Which aspects of the tangent concept do change?
- Which new aspects of the tangent concept do emerge?

The concept development should be supported by adequate examples that show the particularity of the analytic concept of tangent (Fig. 2.12).

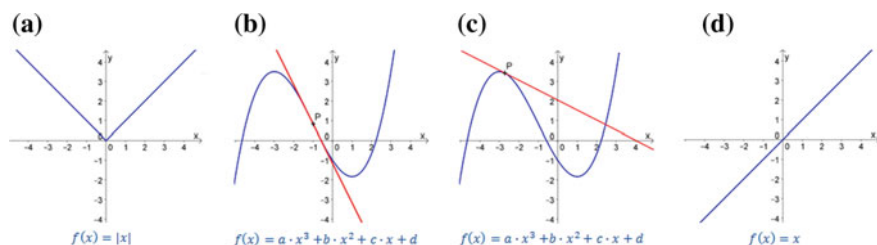


Fig. 2.12 Useful examples for the development of the differential concept of tangent

- The absolute value function (Fig. 2.12a) is a well-known example of a function that is not differentiable at one point. At 0 the situation is similar to the situation of a vertex of a triangle.
- Polynomial functions of order 3 show simple examples of functions with inflection tangents (Fig. 2.12b). In this case it becomes clear that the property of ‘touching’ cannot be transferred to the analytic definition of a tangent. Graphically it is more ‘intersecting’ than ‘touching’. Even though set theoretically, the ‘touching’ is also an ‘intersection’, one can consider a geometrical difference between ‘touching’ and ‘intersecting’. It is important to emphasize that a tangent can ‘touch’ or ‘intersect’ the graph.
- By using the same graph (Fig. 2.12c), one can describe the local character of the analytic tangent concept. The tangent could have more common points with the graph.
- A linear function and its tangents are identical (Fig. 2.12d). The consideration of such special cases is also part of a sustainable concept development.

‘Clarity and Rigor’

Finally, we reflect on the tangent concept by referring again to the balance between clarity and rigor:

- Blum and Kirsch stated that clear and pre-formal proofs can also be rigorous: “[A] preformal proof [is] a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises” (Blum and Kirsch 1991; cf. also Wittmann 1989). According to Blum (2000), clarity and vague or superficial are not the same. He emphasizes that strict must not necessarily mean formal.
- Vollrath emphasizes the interplay between ‘clarity’ and ‘strictness’. If both are understood appropriately, then both are dependent on each other: “Strict ideas can only be understood if one has the corresponding clear ideas. Appropriate clear ideas can only develop from strict considerations” (Vollrath 1993).

What does this mean for the development of the tangent concept? The strict aspects—local linearity of the curve and the tangent as the line of best approximation, which are sustainable for calculus—should already be highlighted in the Euclidean geometry. This can happen visually by means of clarity, as we have shown above. Based on these core aspects, special cases and phenomena can be considered, for

example, the phenomena of touching and of exactly one point in common in the cases of circles and parabolas.¹ Additionally, the relationship between tangent and derivative should be explicitly focused on in differential calculus. Therefore, it is important to reflect on the way of attaining mathematical knowledge:

- The assumption about the possible existence of the tangent can be used to develop the idea of the tangent as the limit of the family of secants, and of understanding the algebraic transition from the difference quotient to the differential quotient.
- The tangent's uniqueness can be defined by using the derivative.

2.4.2 *Extreme Value Problems and the Monotonicity Theorem for Intervals*

The topic of intuitive accessibility of abstract issues and rigor in calculus is renowned in the German subject-matter didactics during the last decades (e.g. Blum and Törner 1983; Kirsch 1996; Blum and Kirsch 1991; Danckwerts and Vogel 2005; Greefrath et al. 2016, etc.). In particular, we refer to the paper *Der Hauptsatz – anschaulich?* (Kirsch 1996), in which the author wants to show how to achieve a clear, intuitively accessible understanding of the fundamental theorem of calculus using appropriate *Grundvorstellungen* (see Sect. 2.2 of this chapter) of the derivative. Anyone teaching calculus at any level has to deal with the problem: What should the balance of clarity and rigor look like? And the answer to this question is surely not the same for courses at university level and at school level.² This problem is not reduced to calculus, it is a problem in principle, as H. Winter stated already (1983, p. 66; translated by the author): “There is a sort of tension between the intuitive accessibility of phenomena and considering them systematically as parts of theories, i.e. between intuition and proof, but this relationship is much more complex than...”.

In the following two sections, we would like to focus on two selected aspects of calculus courses at high school or secondary school. The first one deals with the question: Extreme value problems—how and when should these be introduced in the curriculum of differential calculus? The other one deals with the role of the ‘criterion for monotonicity in intervals’ (why is it important at school? Is there a good way of reasoning for it at school?).

Extreme Value Problems at School

In German or Austrian mathematics textbooks one can find the following way of dealing with extreme value problems (EVP). At first, one does not only have to understand what a derivative is and how to calculate it for polynomial functions, but also how to know second derivatives, positive and negative curvature, and criterions

¹In the case of parabolas, vertical straight lines (with the equation $x = a$, $a \in \mathbb{R}$) should be examined to see that exactly one point in common is not sufficient for a tangent.

²With ‘school’ we mean the level at which students get to know calculus the first time (high school or secondary school; note this is different in other countries).

that are used in curve sketching etc. EVP are then dealt with as an application of the learned techniques while dealing with curve sketching problems. Therefore, in most cases the second derivative is used when solving EVP.

These steps are quite elaborate and time-consuming for students to learn and achieve before considering EVP, thus we propose a more efficient way of reaching the same goal. Let us consider some typical problems:

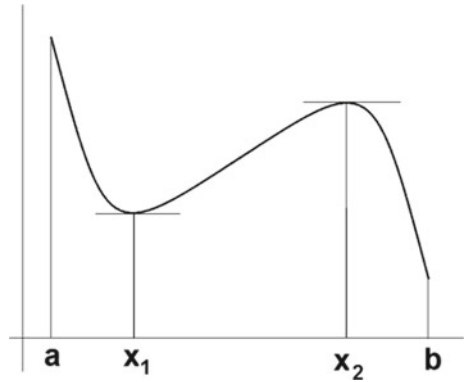
- Solving EVP means to calculate *global* maxima or minima. In curve sketching, one must consider all the *local* decisions (which of the ‘critical points’ are local maxima or minima?). We must distinguish between these aspects more precisely (also at school level).
- EVP can only be introduced at a late stage in the curriculum of differential calculus (one requires knowledge concerning curve sketching); however extreme values offer new and motivating perspectives—not only for students (Guinness World Records)—that should be used at a previous stage. Increasing motivation in an early phase of differential calculus is definitely desirable, since it is beneficial for teaching concepts in calculus.
- Nowadays, the ‘classical way’ of curve sketching (i.e. the graph at the end is the aim) is not a good motivation for differential calculus as computers do a better and quicker job in plotting graphs of functions.
- There is missing rigor in using the second derivative. Students often implicitly/explicitly conclude: “I have found a (local) maximum, therefore this must be the solution of the EVP (meant: global maximum)—I have even checked the situation with the second derivative.” But the second derivative can provide only local decisions. At this point, many false conclusions are made and in particular the values on the boundary are ignored. Also when avoiding the second derivative and focusing on *does f' change its sign at a point?* one has the same problem—this leads to only local decisions not global ones.
- There is also missing clarity hidden in this way: involving the second derivative is not very clear to many students, since students have difficulties in understanding its concept.

The ‘usual method’ at school for solving EVP is: (1) Knowing what the zeros of the first derivative (‘critical points’) are; (2) one can then decide which of these critical points lead to local maxima or local minima with the help of the second derivative; and (3) the ‘conclusion’ is made, the minimum/maximum calculated is the solution of the EVP. However, such an argumentation is wrong in the general case, for instance, as it can be observed when dealing with polynomials of degree 3 (quite popular in school mathematics, see Fig. 2.13), because the global maximum or minimum of such polynomials is at the boundary of the interval.

We propose another way for solving EVP (differentiable function f , closed interval $[a, b]$): (1) What are the zeros of f' (‘critical points’; finitely many³: x_1, \dots, x_n ?). This first item is the equivalent to the first one described in the ‘usual method’ above; (2)

³This is the case with usual problems at school, therefore it is not a major restriction.

Fig. 2.13 EVP solutions at the boundaries of the interval $[a, b]$



calculate the function values $f(a), f(x_1), \dots, f(x_n)$; and (3) the smallest of these values is the solution of the minimum problem, the biggest of the maximum problem.

Advantages of this method:

- The second derivative is not needed
- This method offers better clarity and understanding of EVP among students
- Furthermore, the method is more strict and precise (the consideration of the boundary is deeply integrated in the method; one can hardly skip it)
- This method can be used at an early stage in the curriculum of differential calculus. As soon as students can differentiate polynomials, they can work on several problems (also involving functions like \sqrt{f} , $\frac{1}{f}$, $\exp(f)$, $\log(f)$, ... where f is a polynomial).

EVP and Curve sketching can be seen as different items (local decisions are not relevant in EVP), therefore the corresponding techniques should be different. In case of open intervals, the corresponding limits $\lim_{x \rightarrow a(b, \infty)} f(x)$ must be calculated instead of $f(a)$ and $f(b)$. But in most cases this method is still easier than involving the second derivative.

Only in the specific situation ‘there is only one zero x_0 of f' ’ the conclusion

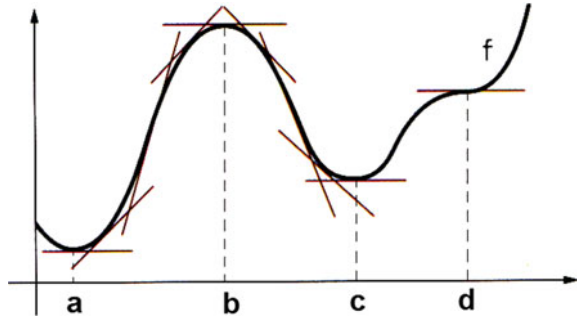
$$\text{‘local minimum at } x_0 \Rightarrow \text{minimum at } x_0\text{’} \quad (2.1)$$

is correct. One would have to think about this ‘theorem (1)’ more deeply, but this is usually not the case when students apply (2.1), therefore this conclusion is rather critical.

We think the role of EVP should be changed (in the differential calculus curricula at school) with regards to three aspects:

- More emphasis on EVP (as a major important aspect of differential calculus, with high potential for motivation).
- Earlier (in the curriculum of differential calculus) emphasis, without the second derivative.

Fig. 2.14 Graphical reasoning for the theorems



- EVP are not only an application of curve sketching techniques; there should be a further distinction between.
 - local* and *global* maxima/minima
 - open* and *closed* intervals.

The Role of the ‘Monotonicity Theorem for Intervals’

Monotonicity theorem for intervals I :

$$f'(x) > 0 \forall x \in I \Rightarrow f \text{ is strictly increasing in } I \quad (2.2)$$

Why is this theorem important at school?

The core of curve sketching at school can be described using four steps:

1. Find the local minima and maxima
2. Find the points of inflection
3. Find the intervals of monotonicity
4. Find the intervals of positive and negative curvature.

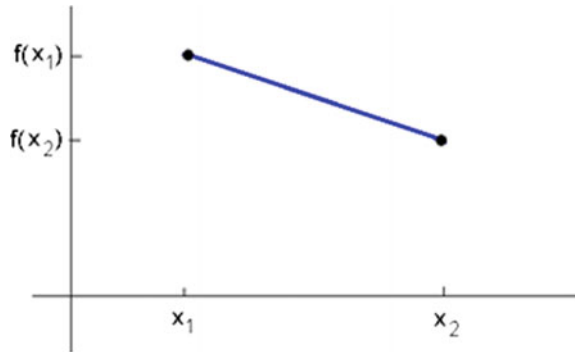
With the information of points 1–4 above, one can sketch the graph of a function. What are the crucial (underlying) theorems?

In points 1 and 2, the crucial theorem is: If f (differentiable) has an inner local maximum/minimum at x , then $f'(x) = 0$. In points 3 and 4, the crucial theorem is the monotonicity theorem (above). These two theorems are very often used at school level (although this may happen only implicitly rather than explicitly), particularly in curve sketching problems and EVP. Let us consider possible ways of introducing the content of these theorems at school level. First, one could argue only graphically (see Fig. 2.14), i.e. one can perceive the theorem’s statements in the figure. Such reasoning is probably vivid and clear to students but of course lacks precision.

What is the usual way of arguing at university level (for the monotonicity theorem for intervals)? The proof relies mostly on:

- (1) Intermediate value theorem for continuous functions
- (2) Extreme value theorem for continuous functions
- (3) Rolle’s theorem
- (4) Mean value theorem.

Fig. 2.15 Non positive slope



This way of argumentation is highly precise and abstract, hence it is obvious that it cannot be applied at school level (too theoretical; the concept of ‘continuity’ is not that important at school—often it comes even after dealing with derivatives, making things more precise after having dealt with them on a somehow intuitive level). The question is: Is there something ‘in between’? And the answer is: Yes, for the theorem that f' vanishes at positions of an inner local maximum/minimum the methods are well known. Regarding the ‘monotonicity theorem’, there is a good possibility to show the idea of a proof (see Danckwerts and Vogel 2005, 59ff). In this case, the principle of nested intervals is needed and it emphasizes the fact that the whole content of calculus requires the *completeness* of the reals. This method of finding arguments would be a reasonable compromise between clarity and rigor: The argumentation is not only graphical (also on a somehow theoretical but elementary level); if it is undertaken this way, the monotonicity theorem for intervals is not just a corollary of other theorems (unproven at school), but it serves as a core theorem for argumentation.

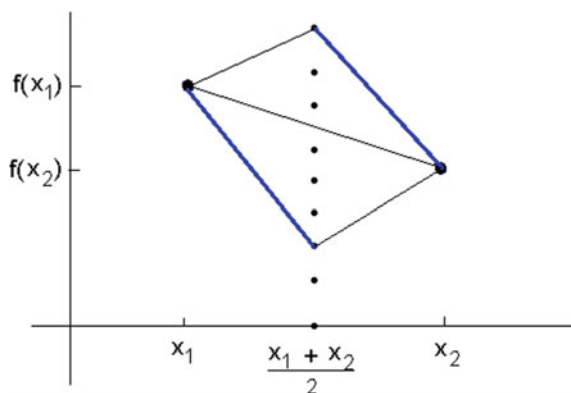
Idea of a proof of (2):

We argue indirectly and assume that f is not strictly increasing in I . Then points $(x_1|f(x_1))$ and $(x_2|f(x_2))$ ($x_{1,2} \in I; x_1 < x_2$) exist so that the slope of the connective straight line is not positive (Fig. 2.15).

Then regardless of $f\left(\frac{x_1+x_2}{2}\right)$, at least one of the corresponding slopes in the left-hand half and in the right-hand half of the interval $[x_1, x_2]$ is also not positive (see Fig. 2.16; because if both were positive also the ‘total slope’ in Fig. 2.3 would also be positive).

Under the consideration of the above aspects, the idea of the proof can be outlined. By using continuous bisection of the intervals, one gets a sequence of connective straight lines with a non-positive slope. The principle of nested intervals yields a point $x_0 \in I$ (completeness of \mathbb{R} !). At this point, the derivative $f'(x_0)$ cannot be positive because a sequence of *not positive* values cannot come ‘arbitrarily near’ to a *positive* one. But this leads to a contradiction concerning the precondition of

Fig. 2.16 Bisection of the interval



theorem (2): $f'(x_0) > 0$ for all $x \in I$. A more detailed proof with basically the same idea can be found in Greefrath et al. (2016), p. 196.

2.5 Conclusion

In these two chapters, we provided two selected examples about *how calculus courses at school* can be different from *calculus courses at university*. Of course, the level of rigor at schools cannot be the same as at universities. In this case, we do not only mean the frequency or depth of formal proofs, but emphasize primarily the way of approaching important items of calculus (theorems, applications, etc.) that are needed at school—for instance the theorems (1) and (2) and solving EVP.

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