# Chapter 1
# Algebra

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It gives me great pleasure to present this brief commentary on some of T. P. Speed's papers on algebra. It may come as a surprise to many of Speed's colleagues to know that his 1968 PhD thesis was entitled *Some Topics in the Theory of Distributive Lattices*. Moreover, of his first 15 papers only one was in probability theory with the remainder in algebra. Nevertheless, this fruitful excursion into algebra has its roots in the foundations of probability theory. In the introduction to his PhD thesis, Speed writes:

> In July 1965, the author began to look at the lattices associated with intuitionistic logic which are called variously – relatively pseudo-complemented, brouwerian or implicative lattices. This was under the direction of Professor P. D. Finch and aimed towards defining probability measures over these lattices. It was hoped that a probability theory could be developed for the intuitionistic viewpoint similar to the Kolmogorov one for classical logic.

Speed never returned to the search for an intuitionistic probability theory for, as he says later in the introduction to his thesis, he became "*sold on distributive lattices*". In the summer of 1968–1969, between my third and honours years, I spent three months on a Monash University Graduate Assistantship during which I read Speed's PhD thesis. By the end of that summer I was also *sold on distributive lattices* and have been ever since [2].

Between 1969 and 1974, Speed published 17 papers on a range of algebraic topics: distributive lattices, including their topological representation (9), Baer rings (3), Stone lattices (2), semigroups (2), and $\ell$-groups (1). In the commentary below, I will discuss five of these papers. Only one of these papers, the first discussed, comes from Speed's thesis.

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S. Dudoit (ed.), *Selected Works of Terry Speed*, Selected Works in Probability and Statistics,
DOI 10.1007/978-1-4614-1347-9_1, © Springer Science+Business Media, LLC 2012 

## Distributive lattices in general

Most of Speed's work on distributive lattices revolves around the role of particular sorts of prime ideals, with an emphasis on minimal prime ideals. In this section, we will look at two of the seven papers that fall into this category, namely, *On rings of sets* [10] and *On rings of sets. II. Zero-sets* [16].

In the first of these papers, Speed provides a unified approach to a number of representations of distributive lattices as rings of sets, that is, as lattices of subsets of some set in which the operations are set-theoretic union and intersection. Each of these characterisations was originally given in terms of the existence of enough elements of a special form, and their proofs looked quite different. Given cardinals $\mathfrak{m}$ and $\mathfrak{n}$, a lattice $L$ is called $(\mathfrak{m},\mathfrak{n})$-complete if it is closed under the operations of least upper bound and greatest lower bound of sets of at most $\mathfrak{m}$ and $\mathfrak{n}$ elements, respectively. An $(\mathfrak{m},\mathfrak{n})$-complete lattice of sets is an $(\mathfrak{m},\mathfrak{n})$-*ring of sets* if $\mathfrak{m}$-ary least upper bounds and $\mathfrak{n}$-ary greatest lower bounds are given by set union and intersection, respectively. For example, the open sets of a topological space form an $(\mathfrak{m},2)$-ring of sets for every cardinal $\mathfrak{m}$. Speed introduces $\mathfrak{n}$-prime $\mathfrak{m}$-ideals and employs them to give natural necessary and sufficient conditions for an $(\mathfrak{m},\mathfrak{n})$-complete lattice to be isomorphic to an $(\mathfrak{m},\mathfrak{n})$-ring of sets. As Speed remarks in the introduction to the paper, *It is interesting to note that the elementary methods used in representing distributive lattices carry over completely and yield all these results, although this is hardly obvious when one considers special elements of the lattice*.

In *On rings of sets. II. Zero-sets* [16], Speed turns his attention to an important example of $(2,\omega)$-rings of sets, the *lattice* $\mathbf{Z}(X)$ *of zero-sets* of continuous real-valued functions on a topological space $X$. The paper, which is deeper and somewhat more technical than the first, includes lattice-theoretic characterisations of $\mathbf{Z}(X)$ in two important cases, when $X$ is compact (Theorem 4.1) and when $X$ is an arbitrary topological space (Theorem 5.9). In both cases, the characterisations involve minimal prime ideals. Along the way he proves a result (Theorem 3.1) that very nicely generalises Urysohn's Lemma for normal topological spaces and the fact that, in a completely regular space, disjoint zero-sets can be separated by a continuous function.

## Distributive lattices—Priestley duality

About the same time that Speed was writing his PhD thesis at Monash University, H. A. Priestley was writing her DPhil at the University of Oxford. Speed was amongst the first to realise the importance of the new duality for bounded distributive lattices that Priestley established in her thesis (see Priestley [8, 9] and Davey and Priestley [2]).

In *On the order of prime ideals* [13], Speed addresses the question, raised by Chen and Grätzer [1], of characterising *representable* ordered sets, that is, ordered sets that arise as the ordered set of prime ideals of a bounded distributive lattice. By using Birkhoff's duality between finite distributive lattices and finite ordered sets, he shows that an ordered set is representable if and only if it is the inverse limit of an inverse system of finite ordered sets. Speed observes that, when combined with deep results of Hochster [5], this tells us that an ordered set is isomorphic to the ordered set of prime ideals of a commutative ring with unit if and only if it is isomorphic to an inverse limit of finite ordered sets. This cross fertilisation in Speed's work between commutative rings with unit and bounded distributive lattices will arise again in Section 1.

Soon after writing Speed [13], Speed became aware of Priestley's results. He quickly realised that, since an inverse limit of finite sets is endowed with a natural compact topology, his characterisation of representable ordered sets could be lifted to a characterisation of compact totally order-disconnected spaces, the ordered topological spaces that arise in Priestley duality (and are now referred to simply as *Priestley spaces*). In *Profinite posets* [12], he proved that an ordered topological space is a Priestley space if and only if it is isomorphic, both order theoretically and topologically, to an inverse limit of finite discretely topologised ordered sets.

## Baer rings

Speed's PhD thesis was strongly influenced by the seminal paper *Minimal prime ideals in commutative semigroups* [6]. He took ideas from Kist's paper and reinterpreted them in the context of distributive lattices. Speed saw that there was some informal connection between the commutative Baer rings introduced and studied in Kist [6] and Stone lattices, a class of distributive lattices introduced by Grätzer and Schmidt [4]. A commutative ring $R$ is a *Baer ring* if, for every element $a \in R$, the annihilator $\text{ann}(a) := \{ x \in R \mid xa = 0 \}$ is a principal ideal generated by a (necessarily unique) idempotent $a^*$. A bounded distributive lattice $L$ is a *Stone lattice* if, for every element $a \in L$, the annihilator $\text{ann}(a) := \{ x \in L \mid x \wedge a = 0 \}$ is a principal ideal generated by an element $a^*$, and in addition the equation $a^* \vee a^{**} = 1$ is satisfied. While quite different looking, the requirements that $a^*$ be an idempotent, in the ring case, and the identity $a^* \vee a^{**} = 1$, in the lattice case, guarantee that the elements $a^*$ form a Boolean algebra and correspond precisely to the direct product factorisations of the ring or lattice.

While the proofs will typically be quite different, it is often true that a result about Baer rings will translate to a corresponding result about Stone lattices and vice versa. For example:

(i) Grätzer [3] proved that Stone lattices form an equational class; Speed and Evans [17] proved that Baer rings also form an equational class. (In both cases, $^*$ is added as an additional unary operation.)
(ii) Grätzer and Schmidt [4] proved that, in a Stone lattice, each prime ideal contains a unique minimal prime ideal; Kist [6] proved that precisely the same condition holds in a Baer ring.

In separate papers on Stone lattices [11] and Baer rings [14], Speed proves that there are broad classes of distributive lattices and rings, respectively, within which Stone lattices and Baer rings are characterised by the property that each prime ideal contains a unique minimal prime ideal.

In his third and final paper on Baer rings [15], Speed considers the question of embedding a commutative semiprime ring $R$ into a Baer ring $B$. Two such embeddings had already been given: the first by Kist [6] and the second by Mewborn [7]. In both cases, the Baer ring $B$ was constructed as a ring of global sections of a sheaf over a Boolean space. Speed shows that, in fact, there is a hierarchy of Baer extensions of $R$, the smallest being Kist's and the largest Mewborn's. Moreover, he is able to replace the sheaf-theoretic construction with a purely algebraic one similar in nature to one that had been used previously in the theory of lattice-ordered groups. The underlying lattice of a lattice-ordered group is distributive, so again we see Speed's fruitful use of the interplay between rings and distributive lattices.

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# ON RINGS OF SETS

T. P. SPEED

(Received 16 March 1967)

## 1. Introduction

In the past a number of papers have appeared which give representations of abstract lattices as rings of sets of various kinds. We refer particularly to authors who have given necessary and sufficient conditions for an abstract lattice to be lattice isomorphic to a complete ring of sets, to the lattice of all closed sets of a topological space, or to the lattice of all open sets of a topological space. Most papers on these subjects give the conditions in terms of special elements of the lattice. We thus have completely join-irreducible elements – G. N. Raney [7]; join prime, completely join prime, and supercompact elements – V. K. Balachandran [1], [2]; $\mathscr{N}$-sub-irreducible elements – J. R. Büchi [5]; and lattice bisectors – P. D. Finch [6]. Also meet-irreducible and completely meet-irreducible dual ideals play a part in some representations of G. Birkhoff & O. Frink [4].

What we do in this paper is define a new kind of prime ideal – called an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal – and show that all the above concepts correspond to a particular kind of $\mathfrak{n}$-prime $\mathfrak{m}$-ideal. Here and throughout we mean $\mathfrak{m}$ and $\mathfrak{n}$ to be (possibly infinite) cardinals, always greater than 1. Also the symbol $\infty$ will be used to denote an arbitrarily large cardinal number. A class of lattices called $(\mathfrak{m}, \mathfrak{n})$-rings of sets is then defined and some theorems proved which cover all the representation theorems mentioned above. It is interesting to note that the elementary methods used in representing distributive lattices carry over completely and yield all these results, although this is hardly obvious when one considers special elements of the lattice.

I wish to express my gratitude to Professor P. D. Finch, whose paper [6] was the inspiration for this work.

## 2. Notations and Definitions

We assume a familiarity with the elementary notions of lattice theory as outlined in G. Birkhoff [3].

DEFINITION 2.1. A lattice $\mathscr{L} = \langle L; \vee, \wedge\rangle$ is said to be $(\mathfrak{m}, \mathfrak{n})$-*complete* if the join of not more then $\mathfrak{m}$ elements of $L$ belongs to $L$, and the meet of not more than $\mathfrak{n}$ elements of $L$ belongs to $L$.

Thus an $(\mathfrak{m}, \mathfrak{n})$-complete lattice may be considered as an algebra with the $\mathfrak{m}$-ary operation of join and the $\mathfrak{n}$-ary operation of meet.

DEFINITION 2.2. An $(\mathfrak{m}, \mathfrak{n})$-complete lattice of sets $\mathscr{L} = \langle L; \vee, \wedge \rangle$ is called an $(\mathfrak{m}, \mathfrak{n})$-*ring of sets* if the $\mathfrak{m}$-ary operation of join corresponds to set union, and the $\mathfrak{n}$-ary operation of meet corresponds to set intersection.

EXAMPLE. The lattice of all open sets of a topological space is an $(\infty, 2)$-ring of sets.

DEFINITION 2.3. An ideal $P$ of the $(\mathfrak{m}, \mathfrak{n})$-complete lattice $\mathscr{L} = \langle L; \vee, \wedge \rangle$ is called an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal if

(i) For $\{x_\gamma : \gamma \in \Gamma\} \subseteqq L$ with $|\Gamma| \leqq \mathfrak{m}$ we have:

$$x_\gamma \in P \;\forall \gamma \in \Gamma \;\Leftrightarrow\; \bigvee_{\gamma \in \Gamma} x_\gamma \in P$$

(ii) For $\{y_\delta : \delta \in \Delta\} \subseteqq L$ with $|\Delta| \leqq \mathfrak{n}$ we have:

$$y_\delta \notin P \;\forall \delta \in \Delta \;\Leftrightarrow\; \bigwedge_{\delta \in \Delta} y \notin P.$$

REMARKS. 1. An ordinary prime ideal is a 2-prime 2-ideal in the above notation.

2. The definition is obviously not the most general possible but it will suffice for the purpose of this paper.

3. If $P$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal then $L \setminus P$ is an $\mathfrak{m}$-prime $\mathfrak{n}$-dual ideal with the obvious (dual) definition of the latter.

DEFINITION 2.4. An homomorphism $\psi$ between two $(\mathfrak{m}, \mathfrak{n})$-complete lattices is called an $(\mathfrak{m}, \mathfrak{n})$-*homomorphism* if $\psi$ preserves joins of $\mathfrak{m}$ elements and meets of $\mathfrak{n}$ elements.

Note that a lattice of sets is not assumed to have set union and intersection as lattice operations unless stated, although the partial ordering is set inclusion.

## 3. $(\mathfrak{m}, \mathfrak{n})$-rings of sets

In this section we clarify the notion of $(\mathfrak{m}, \mathfrak{n})$-ring of sets.

PROPOSITION 3.1. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-complete lattice of subsets of a set $S$. Then $\mathscr{L}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets if and only if for any $s \in S$*

(i) *$s \notin \bigvee \{l \in M : s \notin l\}$ for any $M \subseteqq L$ with $|M| \leqq \mathfrak{m}$*

(ii) *$s \in \bigwedge \{l \in N : s \in l\}$ for any $N \subseteqq L$ with $|N| \leqq \mathfrak{n}$.*

PROOF. If $\mathscr{L}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets, then the $\mathfrak{m}$-ary join and the $\mathfrak{n}$-ary meet operations correspond to set union and intersection respectively. It is thus clear that (i) and (ii) hold in this case.

For the converse we assume (i) and (ii). Observe that we must always have (for $M \subseteqq L$ with $|M| \leqq \mathfrak{m}$)

$$\bigvee \{l : l \in M\} \supseteqq \bigcup \{l : l \in M\}.$$

Now if $s \notin \bigcup\{l : l \in M\}$ then $s \notin l \ \forall l \in M$ and thus by (i) we see that $s \notin \bigvee \{l : l \in M\}$. The reverse inclusion is hence proved and we obtain $\bigvee \{l : l \in M\} = \bigcup\{l : l \in M\}$. Similarly $\bigwedge\{l : l \in N\} \subseteqq \bigcap \{l : l \in N\}$ always holds for $N \subseteqq L$ with $|N| \leqq \mathfrak{n}$, and (ii) implies the reverse inclusion giving

$$\bigwedge \{l : l \in N\} = \bigcap \{l : l \in N\}.$$

The proposition is thus proved.

Our next result is a direct generalisation of G. Birkhoff's theorem for distributive lattices ($=$ (2, 2)-rings of sets), [3] p. 140.

PROPOSITION 3.2. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-complete lattice. Then $\mathscr{L}$ is isomorphic to an $(\mathfrak{m}, \mathfrak{n})$-ring of sets if and only if $\mathscr{L}$ has a faithful representation as a subdirect union of a family $\{\mathscr{L}_\alpha : \alpha \in A\}$ of replicas of* **2** *in which each projection $\pi_\alpha : \mathscr{L} \to \mathscr{L}_\alpha$ is an $(\mathfrak{m}, \mathfrak{n})$-homomorphism.*

PROOF. Assume first that $\mathscr{L}$ has a sub-direct union representation with the stated properties. This is equivalent to the existence of an isomorphism $\psi$ of $\mathscr{L}$ onto a lattice $\langle \mathscr{A}, \cup, \cap \rangle$ of subsets of the index set $A$; explicitly

$$\psi : l \to l\psi = \{\alpha \in A : l\pi_\alpha = 1\}, \quad \mathscr{A} = \{l\psi : l \in L\}.$$

It is clear that $\langle \mathscr{A}; \cup, \cap \rangle$ is a $(\mathfrak{m}, \mathfrak{n})$-complete lattice. We show it is a $(\mathfrak{m}, \mathfrak{n})$-ring of sets. Take $\mathscr{M} \subseteqq \mathscr{A}$ with $|\mathscr{M}| \leqq \mathfrak{m}$, and an arbitrary $\alpha \in A$.

Now $\quad \bigvee \{K \in \mathscr{M} : \alpha \notin K\}$

$= \bigvee \{l\psi \in \mathscr{M} : \alpha \notin l\psi]$ since every $K \in \mathscr{M}$ is of the form $l\psi$, $l \in L$

$= [\bigvee \{l \in M : \alpha \notin l\psi\}]\psi$ where $M = \mathscr{M}\psi^{-1} \subseteqq L$

$= [\bigvee \{l \in M : l\pi_\alpha = 0\}]\psi$ since $\alpha \notin l\psi \equiv l\pi_\alpha = 0$.

Further, $[\bigvee \{l \in M : l\pi_\alpha = 0\}]\pi_\alpha = 0$ since $|M| \leqq \mathfrak{m}$ and the $\pi_\alpha$ are $(\mathfrak{m}, \mathfrak{n})$-homomorphisms, so that $\alpha \notin \bigvee\{K \in \mathscr{M} : \alpha \in K\}$ for $\mathscr{M} \subseteqq \mathscr{A}$ with $|\mathscr{M}| \leqq \mathfrak{m}$.

Similarly $\quad \bigwedge \{K \in \mathscr{N} : \alpha \in K\}$

$= \bigwedge \{l\psi \in \mathscr{N} : \alpha \in l\psi\}$

$= [\bigwedge \{l \in N : \alpha \in l\psi\}]\psi$

$= [\bigwedge \{l \in N : l\pi_\alpha = 1\}]\psi$ for $\mathscr{N} \subseteqq \mathscr{A}$ and $\alpha \in A$.

This gives $\quad [\bigwedge \{l \in N : l\pi_\alpha = 1\}]\pi_\alpha = 1$ if $|\mathscr{N}| = |N| \leqq \mathfrak{n}$

since the $\pi_\alpha$ are $(\mathfrak{m}, \mathfrak{n})$-homomorphisms, so that $\alpha \in \bigwedge\{K \in \mathscr{N} : \alpha \in K\}$, and we have shown that (i) and (ii) of Proposition 3.1 are satisfied. Hence $\mathscr{L}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets.

For the converse assume $\mathscr{L}$ is isomorphic to an $(\mathfrak{m}, \mathfrak{n})$-ring of sets $\mathscr{L}'$. Then $\mathscr{L}'$ has a representation as a subdirect union of replicas of **2** and the working above readily reverses to establish the fact that the $\pi_\alpha$ are $(\mathfrak{m}, \mathfrak{n})$-homomorphisms.

## 4. $\mathfrak{n}$-prime $\mathfrak{m}$-ideals

We now discuss the notion of $\mathfrak{n}$-prime $\mathfrak{m}$-ideal. The first result is straightforward but the corollary is used to establish the equivalence between our ideals and the various concepts mentioned in the introduction. These concepts are not defined here – we refer to the papers concerned – for this reason the corollary is presented without proof.

PROPOSITION 4.1. *Let* $\mathscr{L} = \langle L; \vee, \wedge \rangle$ *be an* $(\mathfrak{m}, \mathfrak{n})$*-complete lattice. Then there is a one-one correspondence between*

(i) $\mathfrak{n}$*-prime* $\mathfrak{m}$*-ideals,*

(ii) $\mathfrak{m}$*-prime* $\mathfrak{n}$*-dual ideals,*

(iii) $(\mathfrak{m}, \mathfrak{n})$*-homomorphisms onto* **2**.

PROOF. It has already been remarked that (i) and (ii) are in one-one correspondence. Let $\psi : \mathscr{L} \to \mathbf{2}$ be an $(\mathfrak{m}, \mathfrak{n})$-homomorphism onto **2**. Then it is easy to see that $\{1\}\psi^{-1}$ is an $\mathfrak{m}$-prime $\mathfrak{n}$-dual ideal and $\{0\}\psi^{-1}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal. Conversely if $P$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal, we may define a map $\pi : \mathscr{L} \to \mathbf{2}$ by setting $l\pi = 0$ or $1$ according as $l \in P$ or $l \notin P$. $\pi$ may be checked to be an $(\mathfrak{m}, \mathfrak{n})$-homomorphism and our proposition is proved.

COROLLARY (Special Cases). Under the conditions of the proposition, with the appropriate values of $\mathfrak{m}$ and $\mathfrak{n}$, there is a one-one correspondence between the objects in the following groups.

A. $(\mathfrak{m} = 2, \mathfrak{n} = \infty)$

(i) prime principal dual ideals

(ii) join prime elements (V. K. Balachandran [2]); lattice bisectors (P. D. Finch [6]); $\mathscr{N}$-sub-irreducible elements for a certain $\mathscr{N}$ (J. R. Büchi [5]).

(iii) $(2, \infty)$-homomorphism onto **2**; lower complete homomorphisms onto **2** (P. D. Finch [6]).

B. $(\mathfrak{m} = \infty, \mathfrak{n} = 2)$

(i) $\infty$-prime dual ideals; completely prime dual ideals (G. Birkhoff & O. Frink [4]).

(ii) prime principal ideals

(iii) $(\infty, 2)$-homomorphisms onto **2**.

C. ($\mathfrak{m} = \infty, \mathfrak{n} = \infty$)

(i) completely prime principal dual-ideals

(ii) completely join prime elements (V. K. Balachandran [2]); supercompact elements (V. K. Balachandran [1]); completely join irreducible elements (G. N. Raney [7]).

(iii) $(\infty, \infty)$-homomorphisms onto $\mathbf{2}$; complete homomorphisms onto $\mathbf{2}$ (G. N. Raney [7]).

LEMMA 4.2. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ and $\mathscr{L}' = \langle L', \vee, \wedge \rangle$ be two $(\mathfrak{m}, \mathfrak{n})$-complete lattices. Suppose there is an $(\mathfrak{m}, \mathfrak{n})$-homomorphism*

$$\pi : \mathscr{L} \to \mathscr{L}'.$$

*Then if $P'$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}'$, $P = P'\pi^{-1}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}$.*

PROOF. $P$ is well known to be an ideal of $\mathscr{L}$. We first show that $P$ is an $\mathfrak{m}$-ideal. Let $\{l_\gamma : \gamma \in \Gamma\} \subseteqq P$ be such that $|\Gamma| \leqq \mathfrak{m}$. Then

$$(\bigvee_{\gamma \in \Gamma} l_\gamma)\pi = \bigvee_{\gamma \in \Gamma} l_\gamma \pi$$

and since $l_\gamma \pi \in P'$, $\forall \gamma \in \Gamma$, $\bigvee_{\gamma \in \Gamma} l_\gamma \pi \in P$, and we deduce that $\bigvee_{\gamma \in \Gamma} l_\gamma \in P = P'\pi^{-1}$.

Finally we show that $P$ is $\mathfrak{n}$-prime. Suppose $\{l_\delta : \delta \in \Delta\} \subseteqq L$ is such that $|\Delta| \leqq \mathfrak{n}$ and $l_\delta \notin P \ \ \forall \delta \in \Delta$.

Then $(\bigwedge_{\delta \in \Delta} l_\delta)\pi = \bigwedge_{\delta \in \Delta} l_\delta \pi$ and since $l_\delta \notin P \ \ \forall \delta \in \Delta$ we have $l_\delta \pi \notin P'$ $\forall \delta \in \Delta$. Thus, since $P'$ is $\mathfrak{n}$-prime, $\bigwedge_{\delta \in \Delta} l_\delta \pi \notin P'$ and so $\bigwedge_{\delta \in \Delta} l_\delta \notin P = P'\pi^{-1}$. The result is proved.

LEMMA 4.3. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-ring of sets, subsets of a set $\mathscr{X}$. Then for any $x \in \mathscr{X}$, $P_x = \{l \in L : x \notin l\}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}$.*

PROOF. $P_x$ is clearly an ideal of $\mathscr{L}$. We show it is an $\mathfrak{m}$-ideal.

Let $\{l_\gamma : \gamma \in \Gamma\} \subseteqq P_x$ be such that $|\Gamma| \leqq \mathfrak{m}$. Since $x \notin l_\gamma$ for $\gamma \in \Gamma$, Proposition 3.1 (i) tells us that $x \notin \bigvee_{\gamma \in \Gamma} l_\gamma$ or $\bigvee_{\gamma \in \Gamma} l_\gamma \in P_x$.

Similarly let $\{l_\delta : \delta \in \Delta\} \subseteqq L$ be such that $|\Delta| \leqq \mathfrak{n}$ and $l_\delta \notin P_x \forall \delta \in \Delta$. Then Proposition 3.1 (ii) tells us that $x \in \bigwedge_{\delta \in \Delta} l_\delta$ or $\bigwedge_{\delta \in \Delta} l_\delta \notin P_x$.

We have thus proved $P_x$ is $\mathfrak{n}$-prime and so it is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal.

## 5. Representation of lattices by $(\mathfrak{m}, \mathfrak{n})$-rings of sets

In this section we give a fundamental representation theorem and then show all such representations are of this form.

PROPOSITION 5.1. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-complete lattice and $\mathscr{P} = \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n})$ the set of all $\mathfrak{n}$-prime $\mathfrak{m}$-ideals of $\mathscr{L}$. We assume $\mathscr{P} \neq \square$. Let $\mathscr{X}$ denote a non-empty subset of $\mathscr{P}$ and define a lattice $\mathscr{R}_{\mathscr{X}} = \langle R; \vee, \wedge \rangle$ by*

*$\mathscr{R}_{\mathfrak{X}} = \mathscr{L}\rho$ where $\rho = \rho_{\mathfrak{X}}$ is defined by $\rho : \mathscr{L} \to \mathscr{R}_{\mathfrak{X}}$, $l\rho = \{P \in \mathscr{P} : l \notin P\}$. Then $\mathscr{R}_{\mathfrak{X}}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets and $\rho$ is an $(\mathfrak{m}, \mathfrak{n})$-homomorphism.*

Proof. We show that $\rho$ is an $(\mathfrak{m}, \mathfrak{n})$-homomorphism and it will then follow that $\mathscr{R}_{\mathfrak{X}}$ is an $(\mathfrak{m}, \mathfrak{n})$-ring of sets. Take $\{l_\gamma : \gamma \in \Gamma\} \subseteqq L$ with $|\Gamma| \leqq \mathfrak{m}$.

Since

$$l_\gamma \leqq \bigvee_{\gamma \in \Gamma} l_\gamma$$

we deduce that

$$l_\gamma \rho \supseteqq (\bigvee_{\gamma \in \Gamma} l_\gamma)\rho$$

and hence

$$\bigcup_{\gamma \in \Gamma} l_\gamma \rho \supseteqq (\bigvee_{\gamma \in \Gamma} l_\gamma)\rho.$$

Now if $P \in \bigcup_{\gamma \in \Gamma} l_\gamma \rho$, then $l_\gamma \notin P$ for some $\gamma \in \Gamma$. Thus $\bigvee_{\gamma \in \Gamma} l_\gamma \notin P$ and hence $P \in (\bigvee_{\gamma \in \Gamma} l_\gamma)\rho$. We have proved $\rho$ preserves joins of $\mathfrak{m}$ elements.

Next take $\{l_\delta : \delta \in \Delta\} \subseteqq L$ with $|\Delta| \leqq \mathfrak{n}$:

$$\bigwedge_{\delta \in \Delta} l_\delta \leqq l_\delta \quad \forall \delta \in \Delta$$

and so $(\bigwedge_{\delta \in \Delta} l_\delta)\rho \supseteqq l_\delta \rho$, giving

$$(\bigwedge_{\delta \in \Delta} l_\delta)\rho \supseteqq \bigcap_{\delta \in \Delta} l_\delta \rho.$$

For the reverse inclusion take $P \in (\bigwedge_{\delta \in \Delta} l_\delta)\rho$. Then $\bigwedge_{\delta \in \Delta} l_\delta \notin P$ and so, since $P$ is $\mathfrak{n}$-prime, we must have $l_\delta \notin P$ $\forall \delta \in \Delta$;

Thus $P \in \bigcap_{\delta \in \Delta} l_\delta \rho$ and we have

$$(\bigwedge_{\delta \in \Delta} l_\delta)\rho = \bigcap_{\delta \in \Delta} l_\delta \rho.$$

$\rho$ is now proved to be an $(\mathfrak{m}, \mathfrak{n})$-homomorphism and the statements in the proposition all follow.

Our next result is basic.

Proposition 5.2. *Let $\mathscr{L} = \langle L; \vee, \wedge \rangle$ be a $(\mathfrak{m}, \mathfrak{n})$-complete lattice, and $\phi$ a $(\mathfrak{m}, \mathfrak{n})$-homomorphism of $\mathscr{L}$ onto a $(\mathfrak{m}, \mathfrak{n})$-ring of sets $\mathscr{K} = \langle K; \cup, \cap \rangle$, subsets of some set $\mathscr{Y}$. Then there is a nonempty subset $\mathfrak{X}$ of $\mathscr{P} = \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n})$ and an isomorphism $\theta : \mathscr{K} \to \mathscr{R}_{\mathfrak{X}}$ such that $\phi \circ \theta = \rho_{\mathfrak{X}}$.*

Proof. Let us first look at $\mathscr{K}$. Since $\mathscr{K}$ is a $(\mathfrak{m}, \mathfrak{n})$-ring of subsets of $\mathscr{Y}$, $P_y = \{k \in K : y \notin k\}$ is an $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{K}$ by Lemma 4.3. Also, since $\phi$ is a $(\mathfrak{m}, \mathfrak{n})$-homomorphism of $\mathscr{L}$ onto $\mathscr{K}$, $P_y\phi^{-1}$ is a $\mathfrak{n}$-prime $\mathfrak{m}$-ideal of $\mathscr{L}$ by Lemma 4.2.

Define $\mathfrak{X} \subseteqq \mathscr{P}$ by $\mathfrak{X} = \{P_y\phi^{-1} : y \in \mathscr{Y}\}$. In the statement of the proposition $\mathscr{R}_{\mathfrak{X}}$ and $\rho = \rho_{\mathfrak{X}}$ are defined as in Proposition 5.1. It remains to check that $\theta$ defined by $\phi \circ \theta = \rho_{\mathfrak{X}}$ is an isomorphism of $\mathscr{K}$ onto $\mathscr{R}_{\mathfrak{X}}$.

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(i) $\theta$ is well defined. For suppose $l_1\phi = l_2\phi$ for $l_1, l_2 \in L$.
Then

$$\{y \in \mathscr{Y} : l_1\phi \notin P_y\} = \{y \in \mathscr{Y} : l_2\phi \notin P_y\}$$

and so
$$\{y \in \mathscr{Y} : l_1 \notin P_y\phi^{-1}\} = \{y \in \mathscr{Y} : l_2 \notin P_y\phi^{-1}\}.$$

Thus
$$\{P \in \mathscr{X} : l_1 \notin P\} = \{P \in \mathscr{X} : l_2 \notin P\}$$

and so
$$l_1\rho = l_2\rho.$$

(ii) $\theta$ is an injection. For suppose $l_1\phi\theta = l_2\phi\theta$. Then $l_2\rho = l_2\rho$ by definition of $\theta$, and the lines above reverse completely to prove $l_1\phi = l_2\phi$.

(iii) $\theta$ is clearly a surjection, for $\rho_{\mathscr{X}}$ is a surjection and so is $\phi$.

(iv) We finally check that $\theta$ is an homomorphism. Take $k_1, k_2 \in K$ such that $k_i = l_i\phi$. Then

$$k_1 \vee k_2 = l_1\phi \vee l_2\phi = (l_1 \vee l_2)\phi$$

whence

$$(k_1 \vee k_2)\theta = (l_1 \vee l_2)\phi \circ \theta = (l_1 \vee l_2)\rho = l_1\rho \vee l_2\rho$$
$$= (l_1)\phi\theta \vee (l_2)\phi\theta = k_1\theta \vee k_2\theta.$$

Similarly $(k_1 \wedge k_2)\theta = k_1\theta \wedge k_2\theta$ and $\theta$ is established to be an isomorphism. The proposition is thus proved.

We close with a theorem which determines when faithful representations exist. For the theorem, let $\mathscr{P}^d(\mathscr{L}; \mathfrak{m}, \mathfrak{n})$ denote the set of all $\mathfrak{m}$-prime $\mathfrak{m}$-dual ideals of $\mathscr{L}$.

THEOREM 5.3. *Let $\mathscr{L} = \langle L; \vee, \wedge\rangle$ be an $(\mathfrak{m}, \mathfrak{n})$-complete lattice. Then the following are equivalent:*

(i) *$\mathscr{L}$ is isomorphic with an $(\mathfrak{m}, \mathfrak{n})$-ring of sets.*

(ii) $(l] = \bigcap \{P \in \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in P\}$ *for all* $l \in L$.

(iii) $[l) = \bigcap \{D \in \mathscr{P}^d(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in D\}$ *for all* $l \in L$.

PROOF. Assume $\mathscr{L}$ is isomorphic with an $(\mathfrak{m}, \mathfrak{n})$-ring of sets. Then by Proposition 5.2 there must be a set $\mathscr{X} \subseteqq \mathscr{P}$ such that $\rho_{\mathscr{X}}$ is one-one. Thus we see that the map $l \to \{P \in \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in P\}$ is also one-one and hence

$$(l] = \bigcap \{P \in \mathscr{P}(\mathscr{L}; \mathfrak{m}, \mathfrak{n}) : l \in P\}.$$

So (i) $\Rightarrow$ (ii).

It is clear that (ii) and (iii) are equivalent. Let us assume (ii). Then the map $\rho$ is seen to be one-one and so $\mathscr{L}$ has a faithful representation as an $(\mathfrak{m}, \mathfrak{n})$-ring of subsets of $\mathscr{P}$. The proof of the theorem is now complete.

REMARK. We do not deduce all possible corollaries. It suffices to illustrate the method by taking $\mathfrak{m} = \mathfrak{n} = \infty$ and deducing the result.

COROLLARY. (G. N. Raney [7], V. K. Balachandran [1]). *A complete lattice $\mathscr{L}$ is isomorphic with a complete ring of sets if and only if $\mathscr{L}$ possesses a join basis of completely join irreducibles.*

PROOF. Take $\mathfrak{m} = \mathfrak{n} = \infty$ in Theorem 5.3 parts (i) and (ii). An $\infty$-prime $\infty$-dual ideal is equivalent to a completely prime principal dual ideal and its generator is thus a completely join irreducible element. Since the intersection of a family of principal dual ideals is the principal dual ideal generated by the join of the generators of the family, we see that (iii) tells us that for any $l \in L$

$[l) = \bigcap_{\nu \in V} [j_\nu) = [\bigvee_\nu j_\nu)$ where the $j_\nu$ are completely join irreducible. This is equivalent to $l = \bigvee j_\nu$ and our Corollary is proved.

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BULL. AUSTRAL. MATH. SOC. MOS 54F05, 06A35

VOL. 6 (1972), 177-183.

# Profinite posets

## T.P. Speed

The class of ordered topological spaces which are projective limits of finite partially ordered sets (equipped with the restriction of the product of the discrete topologies) is shown to coincide with the class of compact totally order-disconnected ordered topological spaces. Hence this is another category of spaces equivalent to the category of distributive lattices with zero and unit.

## 1. Introduction

In her papers [7], [8], Miss Priestley has discussed in detail the equivalence of the category of compact totally order-disconnected ordered topological spaces (with continuous monotone maps) and the category of distributive lattices with zero and unit (with zero and unit preserving lattice homomorphisms). More recently it has been shown [10] that the partially ordered set (= poset) of all prime ideals of such a lattice must be of the form $\varprojlim_{\alpha\in I} X_\alpha$ where each $X_\alpha$ $(\alpha \in I)$ is a finite poset. A synthesis of these two results immediately suggests itself, and we prove the following:

THEOREM. *Let* $X$ *be an ordered topological space. Then* $X$ *is compact and totally order-disconnected iff* $X \cong \varprojlim_{\alpha\in I} X_\alpha$ , *where* $\{X_\alpha, f_{\alpha\beta}\}$ *is an inverse system of finite posets each equipped with the discrete topology.*

We prove this theorem in §§3, 4. An ordered topological space which

Received 8 October 1971. The author would like to thank Miss Hilary Priestley for the preprint of her paper [8].

is of the form $\varprojlim_{\alpha \in I} X_\alpha$ for an inverse system $\{X_\alpha, f_{\alpha\beta}\}$ of finite discretely topologised posets will be called a *profinite poset* by analogy with the group case. Thus the theorem above is an analogue of the well known characterization of profinite groups; see also [*6*] for other related results.

## 2. Preliminaries

The notation and terminology of [*7*], [*8*] will be adopted without further comment. Let us write $A \nleq B$ for subsets $A, B$ of a poset $(X; \leq)$ iff for all $a \in A$, $b \in B$ we have $a \nleq b$.

LEMMA 1. *Let* $(X, \tau, \leq)$ *be a compact totally order-disconnected space. Then for disjoint closed sets* $A, B$ *we have* $A \nleq B$ *iff there is an order-disconnection* $(U|L)$ *such that* $A \subseteq U$, $B \subseteq L$.

Proof. Assume $A \nleq B$. Then since $X$ is totally order-disconnected, for any $x \in A$, $y \in B$ there is an order-disconnection $(U_{x,y}|L_{x,y})$ such that $x \in U_{x,y}$, $y \in L_{x,y}$. Fix $x$. Then the family $\{L_{x,y} : y \in B\}$ constitutes an open cover of $B$, and so there exists a finite sub-cover $\left\{L_{x,y_j} : j = 1, 2, \ldots, n\right\}$. Put $U_x = \bigcap_{j=1}^{n} U_{x,y_j}$ and $L_x = \bigcup_{j=1}^{n} L_{x,y_j}$ and we observe that $(U_x|L_x)$ is an order-disconnection with $x \in U_x$, $B \subseteq L_x$. Now the family $\{U_x : x \in A\}$ is an open cover of $A$ and so has a finite subcover $\left\{U_{x_i} : i = 1, 2, \ldots, m\right\}$. Put $U = \bigcup_{i=1}^{m} U_{x_i}$ and $L = \bigcap_{i=1}^{n} L_{x_i}$ and we have an order-disconnection $(U|L)$ such that $U \supseteq A$, $L \supseteq B$ as required.

REMARK. This lemma shows that, as one would expect, compact subsets behave in much the same way as points in compact ordered spaces. For further evidence of this see Theorem 4, p. 46 of [*4*]. When the order is trivial, Lemma 1 reduces to a well known result for boolean algebras.

Let $(X; \leq)$ be a poset and $\rho \subseteq X \times X$ an equivalence relation on

$X$ . Then one way of defining a quasi-order on $X/\rho$ is to write $x/\rho \leq' y/\rho$ iff there exists $x_1 \equiv x\ (\rho)$ , $y_1 \equiv y\ (\rho)$ such that $x_1 \leq y_1$ . Unfortunately this relation $\leq'$ is not always a partial order on $X/\rho$ ; when it is we say that $\rho$ is *order compatible*. Thus the equivalence $\rho$ on $X$ is order compatible iff for any $x_1, y_1$ in $X$ , if $x_1 \equiv x_2\ (\rho)$ and $y_1 \equiv y_2\ (\rho)$ and $x_1 \leq y_1$ , $x_2 \geq y_2$ then $x_1 \equiv x_2 \equiv y_1 \equiv y_2\ (\rho)$ . Equivalently, $\rho$ is order compatible iff for any $x, y \in X$ such that $x \not\equiv y\ (\rho)$ , we have either
$\{x_1 : x_1 \equiv x\ (\rho)\} \nleq \{y_1 : y_1 \equiv y\ (\rho)\}$ or
$\{x_1 : x_1 \equiv x\ (\rho)\} \ngeq \{y_1 : y_1 \equiv y\ (\rho)\}$ .

## 3. First proof of the theorem

Suppose $X = \varprojlim_{\alpha \in I} X_\alpha$ where $\{X_\alpha, f_{\alpha\beta}\}$ is an inverse system of finite posets each equipped with the discrete topology, and $I$ is a directed set. Then $X$ is certainly a compact space ([1], Chapter I, §9.6, Proposition 8). For any $\alpha \in I$ and $x'_\alpha \in X_\alpha$ write $U_{x'_\alpha} = \{x \in X : x_\alpha \geq x'_\alpha\}$ , $L_{x'_\alpha} = \{x \in X : x_\alpha \leq x'_\alpha\}$ and $T_{x'_\alpha} = \{x \in X : x_\alpha = x'_\alpha\}$ , where $x = \langle x_\alpha \rangle_{\alpha \in I}$ denotes a typical element of $X$ . Then $T_{x'_\alpha}$ is clopen, and (since each $X_\alpha$ is discrete) so are $U_{x'_\alpha}$, $L_{x'_\alpha}$ . Further $U_{x'_\alpha}$ is increasing and $L_{x'_\alpha}$ is decreasing. We now prove that $X$ is totally order-disconnected. Suppose $x \nleq y$ in $X$ ; then for some $\alpha \in I$ we must have $x_\alpha \nleq y_\alpha$ . Thus $\left(U_{x_\alpha} \middle| L_{y_\alpha}\right)$ is an order-disconnection and $x \in U_{x_\alpha}$ , $y \in L_{y_\alpha}$ , and so the result is proved.

For the converse we suppose that $X$ is compact and totally order-disconnected. Let $R$ denote the family of all clopen order compatible equivalences $\rho$ on $X$ , that is, all order compatible equivalences of the form $\rho = \bigcup_{i=1}^{m} V_i \times V_i$ for some finite partition $\{V_i\}$ of $X$ into open sets. Then $X_\rho = X/\rho$ is a finite poset, and, when equipped with the discrete topology, is a continuous monotone image of $X$ under the canonical projection $pr_\rho : X \to X/\rho$ .

Now Lemma 1 implies that the equivalence $\rho$ is order compatible iff $V_i \neq V_j$ implies that there exists an order disconnection $(U|L)$ such that $V_i \subseteq U$ , $V_j \subseteq L$ or $V_j \subseteq U$ , $V_i \subseteq L$ . We now prove that the family of all clopen order compatible equivalences is directed, and that $\bigcap\{\rho : \rho \in R\} = \Delta$ , the diagonal of $X \times X$ . The last remark is easy, for if $x \neq y$ then either $x \nleq y$ or $y \nleq x$ . Suppose $x \nleq y$ ; then there is an order-disconnection $(U|L)$ such that $x \in U$ , $y \in L$ . But it is easily checked that $\{U, L, U^c \cap L^c\}$ is a partition which induces an order compatible equivalence $\rho$ , and hence $x \nleq y \ (\rho)$ .

Suppose $\rho$ and $\rho'$ are two clopen order compatible equivalences induced by the partitions $\{V_i : i = 1, 2, \ldots, m\}$ and $\{V'_j : j = 1, 2, \ldots, n\}$ respectively. Then the partition

$$\{V_i \cap V'_j : i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \ V_i \cap V'_j \neq \emptyset\}$$

induces an order compatible equivalence $\rho \vee \rho'$ . For if $V_i \cap V'_j \neq V_{i_1} \cap V'_{j_1}$ , then either $V_i \neq V_{i_1}$ or $V'_j \neq V'_{j_1}$ , say the former. Then either $V_i \nleq V_{i_1}$ or $V_{i_1} \nleq V_i$ , again suppose the former. By Lemma 1 there is an order-disconnection $(U|L)$ such that $V_i \subseteq U$ and $V_{i_1} \subseteq L$ . But now $V_i \cap V'_j \subseteq U$ and $V_{i_1} \cap V'_{j_1} \subseteq L$ which proves that $V_i \cap V'_j \nleq V_{i_1} \cap V'_{j_1}$ and so $\rho \vee \rho'$ is order compatible.

We now collect the foregoing results: the system $\{X_\rho : \rho \in R\}$ where for $\rho \subseteq \rho'$ the canonical map $f_{\rho\rho'} : X_{\rho'} \to X_\rho$ is continuous and monotone, and $R$ is directed, becomes an inverse system $\{X_\rho, f_{\rho\rho'}\}$ . The map $\phi : X \to \varprojlim_{\rho \in R} X_\rho$ given by $\phi(x) = \left\langle pr_\rho(x) \right\rangle_{\rho \in R}$ is continuous, bijective, and an order isomorphism, and so $X$ and $\varprojlim_{\rho \in R} X_\rho$ are homeomorphic as required.

## 4. Second proof of the theorem

We quickly sketch an alternative, shorter, proof of the theorem. It does however, have the disadvantage of using results from [2], [5], [10] of a non-topological nature, but is the way the theorem was originally deduced.

Suppose $X = \varprojlim_{\alpha\in I} X_\alpha$ is a projective limit of finite, discretely topologised posets. Then $X_\alpha = \text{Patch} A_\alpha$ for a unique distributive lattice $A_\alpha$. Thus $X = \varprojlim X_\alpha \cong \varprojlim \text{Patch} A_\alpha \cong \text{Patch}(\varinjlim A_\alpha) = \text{Patch}\, A$ where $A = \varinjlim A_\alpha$ is the direct limit of the direct system $\{A_\alpha, f^*_{\alpha\beta}\}$, and where $f^*_{\alpha\beta} : A_\alpha \to A_\beta$ is the dual map to $f_{\alpha\beta} : X_\beta \to X_\alpha$ for $\alpha \le \beta$. By the main result of [7] and some remarks of [2], $X = \text{Patch}\, A$ is compact and totally order-disconnected.

Conversely, suppose $X$ is compact and totally order-disconnected. By the main result of [7], $X = \text{Patch}\, A$ for a unique distributive lattice $A$. Write $A = \varinjlim A_\alpha$ as a direct limit of its finitely generated (finite) sublattices $A_\alpha$. Then

$$\text{Patch}\, A \cong \text{Patch}(\varinjlim A_\alpha) \cong \varprojlim \text{Patch}\, A_\alpha = \varprojlim X_\alpha$$

where $\{X_\alpha\}$ is a family of finite posets equipped with discrete topologies. The details of this proof can be reconstructed from [2], [5].

In a notice which appeared after this note was written, Joyal [3] states a theorem closely related to our main result. His proof is probably more like the one sketched above.

## 5. Final remarks

The theorem of this note and other results show that the following categories are equivalent:

(i) distributive lattices with zero and unit (with zero and unit preserving homomorphisms);

(ii) spectral spaces (with spectral maps);

(iii) compact totally order-disconnected spaces (with continuous monotone maps);

(iv) profinite posets (with continuous monotone maps).

The study of the relations between (i) and (ii) was begun by Stone in [11]; some further details are in [9] and the forthcoming part II, while much useful information is in [2]. The relation (i) ↔ (iii) is the object of [7], [8], and the connections between (i), (ii) and (iii) are being studied at the moment.

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# ON THE ORDER OF PRIME IDEALS

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T. P. SPEED

A poset $X$ is isomorphic to the poset of all prime ideals of a (distributive) lattice with zero and unit if, and only if, $X$ is the projective limit of an inverse system of finite posets.

## 1. Introduction

The problem of characterising posets of the form $X_A$ where $X_A$ denotes the set of all prime ideals of the (distributive) lattice $A$ with zero and unit, ordered by inclusion, was raised by C. C. Chen and G. Grätzer [2]. In a similar context M. Hochster [3] discussed the same problem for commutative rings with identity, and gave a solution in terms of a certain family of order preserving maps. We note below that these problems have a common solution.

Let us call a poset $X$ *profinite* if $X \cong \overset{\lim}{\longleftarrow} X_\alpha$ for an inverse system $\{(X_\alpha), (\phi_{\alpha\beta})\}$ of *finite* posets defined over some directed set I. In terms of this notion we will prove the following:

THEOREM. *A poset $X$ is isomorphic to the poset of all prime ideals of a (distributive) lattice with zero and unit if, and only if, $X$ is profinite.*

COROLLARY 1. *A poset $X$ is isomorphic to the poset of all prime ideals of a commutative ring with identity if, and only if, $X$ is profinite.*

COROLLARY 2. *A poset $X$ is isomorphic to the poset of all prime ideals of a (distributive) lattice [resp. lattice with zero, lattice with unit] if, and only if, $X$ with largest and smallest [resp. with largest, with smallest] element adjoined, is profinite.*

## 2. Preliminary lemmas

For any (finite) distributive lattice $A$ the set $X_A$ of all prime ideals of $A$ ordered by inclusion is a (finite) poset; further if $f: A \to A'$ is a zero and unit preserving lattice homomorphism between distributive lattices $A$ and $A'$ with zero and unit, there is an induced order preserving map $f^*: X_{A'} \to X_A$.

Also, if $X$ is a finite poset, there is a finite distributive lattice $A_X = A$, unique up to isomorphism, such that $X \cong X_A$; again if $\phi: X \to X'$ is an order preserving map between finite posets $X$ and $X'$, there is an induced zero and unit preserving lattice homo-

*Presented by G. Grätzer. Received November 24, 1971. Accepted for publication in final form April 4, 1972.*

morphism $^*\phi: A_{X'} \to A_X$. All these results are well known [1], and we summarise them in:

LEMMA 1. *The assignment* $A \mapsto X_A, f \mapsto f^*$, *defines a contravariant functor between the category* $\mathscr{D}_F$ *of all finite distributive lattices (with zero and unit preserving lattice homomorphisms) and the category* $\mathscr{P}_F$ *of all finite posets (with order preserving maps).*

*This functor is a category equivalence.*

Let $I$ be a directed set and suppose $\{(A_\alpha), (f_{\alpha\beta})\}$ is a direct system in $\mathscr{D}_F$ over $I$. Denote by $X_\alpha$ the poset of all prime ideals of $A_\alpha$ and by $X$ the poset of all prime ideals of $A = \underrightarrow{\lim} A_\alpha$. Then we can see that each $x \in X$ defines a thread $(x_\alpha)$, $x_\alpha \in X_\alpha$, $\alpha \in I$, such that if $\alpha \leqslant \beta$, $x_\beta f_{\alpha\beta}^{-1} = x_\alpha$: we simply put $x_\alpha = x f_\alpha^{-1}$ where $f_\alpha : A_\alpha \to A$ is the canonical map into the direct limit. Conversely each such thread $(x_\alpha)$ can be readily seen to define an element $x \in X$: we put $x = \bigcup_\alpha x_\alpha f_\alpha$. This correspondence can be shown to be bijective and order preserving in both directions, and we then have

LEMMA 2. $X_A \cong \underleftarrow{\lim} X_\alpha$.

## 3. Proofs of the main results

We first prove the theorem. Let $X$ be a profinite poset i.e. $X \cong \underleftarrow{\lim} X_\alpha$ where $\{(X_\alpha), (\phi_{\alpha\beta})\}$ is an inverse system in $\mathscr{P}_F$ relative to a directed set $I$. By Lemma 1 we then have a direct system $\{(A_\alpha), (f_{\alpha\beta})\}$ in $\mathscr{D}_F$ with $X_{A_\alpha} = X_\alpha, f_{\alpha\beta} = {}^*\phi_{\alpha\beta}$. Put $A = \underrightarrow{\lim} A_\alpha$. By Lemma 2 $X_A \cong X$ and we have proved that $X$ arises as $X_A$ for a suitable distributive lattice $A$ zero and unit.

Conversely, let $A$ be a distributive lattice with zero and unit. Then we may write $A = \underrightarrow{\lim} A_\alpha$ where $\{(A_\alpha), (f_{\alpha\beta})\}$ is the direct system in $\mathscr{D}_F$ of all finite sublattices of $A$ containing the zero and unit of $A$, with connecting maps $f_{\alpha\beta}: A_\alpha \to A_\beta$ when $A_\alpha \subseteq A_\beta$ being the canonical injections. By Lemma 1 we then have an inverse system $\{(X_\alpha), (\phi_{\alpha\beta})\}$ in $\mathscr{P}_F$ with $X_\alpha = X_{A_\alpha}$, $\phi_{\alpha\beta} = f_{\alpha\beta}^*$. Put $X = \underleftarrow{\lim} X_\alpha$. By Lemma 2 $X \cong X_A$ and we have proved that $X_A$ is profinite.

This completes the proof of the theorem.

Corollary 1 can be proved using Proposition 12 of [3]; we omit the details.

If a distributive lattice fails to have a zero [resp. unit, zero and unit] we can simply add one, thereby adding a smallest [resp. largest, smallest and largest] prime ideal. By the theorem the poset of all prime ideals obtained must be profinite, and there is a natural converse. Thus we have Corollary 2.

## 4. Final remarks

Since a first draft of the above results was written, equivalent results were announced (without proof) by A. Joyal [4].

In conclusion I would like to thank Professor G. Grätzer for his remarks concerning the first draft, and also Brian Davey for his interest.

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# ON RINGS OF SETS II. ZERO-SETS

**Dedicated to the memory of Hanna Neumann**

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(Received 2 May 1972)

Communicated by G. B. Preston

## Introduction

In an earlier paper [11] we discussed the problem of when an $(m, n)$-complete lattice $\mathbf{L}$ is isomorphic to an $(m, n)$-ring of sets. The condition obtained was simply that there should exist sufficiently many prime ideals of a certain kind, and illustrations were given from topology and elsewhere. However, in these illustrations the prime ideals in question were all principal, and it is desirable to find and study examples where this simplification does not occur. Such an example is the lattice $\mathbf{Z}(X)$ of all zero-sets of a topological space $X$; we refer to Gillman and Jerison [5] for the simple proof that $\mathbf{Z}(X)$ is a $(2, \sigma)$-ring of subsets of $X$, where we denote aleph-zero by $\sigma$.

Lattices of the form $\mathbf{Z}(X)$ have occurred recently in lattice theory in a number of places, see, for example, Mandelker [10] and Cornish [4]. These writers have used such lattices to provide examples which illuminate a number of results concerning annihilators and Stone lattices. We also note that, following Alexandroff, a construction of the Stone-Čech compactification can be given using ultrafilters on $\mathbf{Z}(X)$; the more recent Hewitt realcompactification can be done similarly, and these topics are discussed in [5]. A relation between these two streams of development will be given below.

In yet another context, Gordon [6], extending some aspects of the work of Lorch [9], introduced the notion of a zero-set space $(X, \mathscr{Z})$. This is a structure abstracted from the system consisting of a set $X$ and the family $\mathscr{Z}$ of zero-sets of the functions in a uniformly closed ring of real-valued functions defined on $X$. Gordon's axioms naturally embody some of the lattice-theoretic properties of $\mathbf{Z}(X)$ for a topological space $X$, but as we shall see below, they are more general.

We can now explain the contents of this paper. After listing our notation and terminology, we give some lattice-theoretic results which are necessary for subsequent analysis, but not without interest separately. We then give some constructions, similar to Urysohn's, of certain functions separating disjoint sets. They are more delicate than the usual since the family of sets used is closed under (finite unions and) countable intersections only, and hence the notion of closure is not available. Also these results enable us to give alternative proofs of some results of Gordon [6], thus avoiding the use of proximity spaces and the consequent application of Čech's difficult version of Urysohn's lemma, valid for uniformizable proximity spaces. In §§4,5 we turn to the main task which is find properties of $\mathbf{Z}(X)$ in addition to those which follow from its being a $(2,\sigma)$-ring of sets. Our results include algberaic characterisations of $\mathbf{Z}(X)$ for $X$ a compact, respectively arbitrary, topological space.

To conclude this introduction we gratefully thank Drs. J. W. Baker and C. J. Knight for listening to, and helpfully commenting upon, early versions of the material presented below. Also the referee is to be thanked for pointing out an incorrect result stated in the first version, and for remarks leading to some shortening of proofs.

## 1. Notation and terminology

(1.1) *Lattice theory.* Most of the concepts from lattice theory we need are defined somewhere in Birkhoff [1], while the more special ones relating to rings of sets and special prime ideals are discussed in [11]. All our lattices will be assumed to possess a zero (least element) 0 and unit (greatest element) 1, and all sublattices will be assumed to contain the same zero and unit. The join and meet operations are denoted $\vee$ and $\wedge$ respectively, and thus a lattice can be considered as an abstract algebra $\mathbf{L} = (L; \vee, \wedge, 0, 1)$ with carrier $L$; we use the partial order on $L$ without comment. Typical elements of $L$ will be denoted $a, b, c, d, \cdots$; typical prime or minimal prime ideals will be denoted $w, x, y, \cdots$. We will abbreviate the term $(2,\sigma)$-prime (see [11]) to $\sigma$-prime, in accordance with usual practice. A lattice is said to have *enough* ideals of a specified type if distinct elements of the lattice can be separated by ideals of that type. The lattice $\mathbf{L}$ is said to be a *$(2,\sigma)$-regular sublattice* of the lattice $\mathbf{L}'$ if $\mathbf{L}$ is a sublattice of $\mathbf{L}'$ such that countable meets of elements in $\mathbf{L}'$ which exist in $\mathbf{L}'$ or $\mathbf{L}$ exist in both and coincide.

(1.2) *Topology.* Our general reference in this sphere is Bourbaki [2], while the reference for the less common concepts used below, such as zeroset, $z$-filter, realcompactification etc. is Gillman and Jerison [5]. We will reserve $W$, $X$, $Y$ for topological spaces; generic elements will be denoted by the corresponding lower case letter; typical subsets will be written $A, B, C, \cdots$; typical open sets $G, \cdots$; typical closed sets $F, \cdots$.

(1.3) *General.* For subsets $A$, $B$ of a set $X$ we write $A \cup B$, $A \cap B$ for set union and intersection respectively, and $\complement A$ for the complement of $A$ in $X$. The empty set is denoted $\phi$. If $f : X \to Y$ is a map, we write $fA$ for the direct image of

$A \subseteq X$ and $f^{-1}B$ for the inverse image of $B \subseteq Y$; parentheses will only be included where necessary. The unit interval $\{t \in \mathbb{R}: 0 \leqq t \leqq 1\}$ is denoted $[0, 1]$.

## 2. Some lattice-theoretic results

Our first definition is based on the work of Cornish [4]; see also Kerstan [7] §6, Definition 2 for a closely related definition.

DEFINITION 2.1. A lattice $\mathbf{L}$ is *normal* if for any pair $a$, $b \in L$ with $a \wedge b = 0$, there exists $c$, $d \in L$ such that $a \wedge c = b \wedge d = 0$ and $c \vee d = 1$.

It is not hard to see that a Hausdorff space $X$ is normal if, and only if, the lattice $\mathbf{F}(X)$ of all closed subsets of $X$ is a normal lattice. Further it has been known for some time that the lattice $\mathbf{Z}(X)$ of all zero-sets of a topological space $X$ is a normal lattice.

A number of equivalent formulations of 2.1 in the case $\mathbf{L}$ a distributive lattice are given in [4], and although we need none of these, we note the following: a distributive lattice $\mathbf{L}$ is normal if, and only if, every prime ideal contains a unique minimal prime ideal. This last result is known for $\mathbf{Z}(X)$ in the form: a prime $z$-filter is contained in a unique $z$-ultrafilter, ([5] 2.13). We also refer to [4] for many consequences of normality. For later use we note that any Boolean lattice is normal.

Another topologically inspired concept we need is that of a $G_\delta$-element of a lattice $\mathbf{L}$, and again we note that a similar idea occurs in [7].

DEFINITION 2.2. An element $a \in L$ is a $G_\delta$ in the lattice $\mathbf{L}$ if there exists a sequence $\{a_n : n \geqq 1\}$ of (not necessarily distinct) elements of $L$ with the following properties:

($\alpha$) $a \wedge a_n = 0$ for all $n$;
($\beta$) if for $b \in L$ we have $b \wedge a_n = 0$ for all $n$, then $b \leqq a$.

Our final definition in this section is the following abstraction of the analogous topological property.

DEFINITION 2.3. A lattice $\mathbf{L}$ is *perfectly normal* if ($\alpha$) $\mathbf{L}$ is normal; and ($\beta$) every $a \in L$ is a $G_\delta$.

Clearly a Hausdorff space $X$ is perfectly normal if, and only if, the lattice $\mathbf{F}(X)$ is perfectly normal. Also it is easy to prove ([6] 2.3) that for any topological space $X$, the lattice $\mathbf{Z}(X)$ is perfectly normal.

We turn to some algebraic consequences of the definitions.

LEMMA 2.4. *A lattice* $\mathbf{L}$ *in which every* $a \in L$ *is a* $G_\delta$ *is disjunctive.*

PROOF: Take $a \nleqq b$ in $L$. By 2.2 ($\beta$) there must exist an $n$ such that $a \wedge b_n \neq 0$ while by 2.2 ($\alpha$) $b \wedge b_n = 0$. This proves the result.

A deeper result which we use frequently below requires the characteristic property of a minimal prime ideal, Kist [8] viz: a prime ideal $x$ of a distributive lattice **L** is minimal if, and only if, for any $a \in x$ there exists $b \notin x$ such that $a \wedge b = 0$.

LEMMA 2.5. *Let $y$ be a $\sigma$-prime ideal in a $(2, \sigma)$-complete perfectly normal distributive lattice* **L**. *Then $y$ is a minimal prime ideal.*

PROOF: Let $a \in y$; we must find $b \notin y$ such that $a \wedge b = 0$. Since $a$ is a $G_\delta$ there exists a sequence $\{a_n : n \geqq 1\}$ with properties 2.2 $(\alpha)$, $(\beta)$. Thus $a \wedge a_n = 0$, and so normality of $L$ implies the existence of two sequences $\{c_n\}$, $\{d_n\}$ with: $a \wedge c_n = 0 = a_n \wedge d_n$ and $c_n \vee d_n = 1$ for all $n$. If, for some $n$, $c_n \notin y$, then we are through. Suppose now that $c_n \in y$ for all $n$; then $d_n \notin y$ for all $n$, and by the $\sigma$-prime property of $y$, $d = \bigwedge_n d_n \notin y$. But for all $n$, $a_n \wedge d \leqq a_n \wedge d_n = 0$ and so by 2.2 $(\beta)$ $d \leqq a$ which contradicts $a \in y$, $d \notin y$.

Hence $a \wedge c_n = 0$ for some $c_n \notin y$ and $y$ is minimal.

## 3. Constructions similar to Urysohn's

In this section we will be working with a $(2, \sigma)$-ring of subsets of a set $X$ satisfying various conditions, and a careful analysis will enable us to extend the construction of a continuous function separating disjoint closed sets to this situation. We conclude by giving an alternative, direct, proof of a result of Gordon.

THEOREM 3.1. *Let* **H** *be a $(2, \sigma)$-ring of subsets of a set $X$. Then the following are equivalent:*

1) **H** *is a normal lattice.*

2) *For any $A, B \in H$ with $A \cap B = \phi$ there exists a function $f : X \to [0, 1]$ such that*

$(\alpha)$ *$f^{-1}F \in H$ for every closed subset $F$ of $[0, 1]$;*

$(\beta)$ *$A \subseteq f^{-1}\{0\}$, $B \subseteq f^{-1}\{1\}$.*

PROOF: 1) implies 2). We will explain the proof backwards thus motivating the construction. Let $A, B \in H$ with $A \cap B = \phi$ be given. Our aim is to define a system

$$(*) \qquad \mathscr{U} = \{U(t), F(t) : 0 \leqq t \leqq 1\} \text{ where}$$

(i) $\complement U(t) \in H$, $F(t) \in H$, $0 \leqq t \leqq 1$;

(ii) $A \subseteq U(0)$, $B \subseteq \complement U(1)$;

(iii) If $0 \leqq t < t' \leqq 1$ then $U(t) \subseteq F(t) \subseteq U(t')$.

Then we will see that the well-known procedure of defining a map $f : X \to [0, 1]$ by writing, for $x \in X$:

$$(**) \qquad f(x) = \inf\{t : x \in U(t)\}$$

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gives a function satisfying:

(iv) $f^{-1}[0,t] = F(t)$, $f^{-1}[t,1] = \complement U(t)$.

Having done this we may take an arbitrary closed subset $[0,1]\setminus\bigcup_n(\alpha_n,\beta_n)$ of $[0,1]$ and find that

$$\begin{aligned} f^{-1}\,[0,1]\setminus\bigcup_n\,(\alpha_n,\beta_n) &= f^{-1}\bigcap_n\{[0,\alpha_n]\cup[\beta_n,1]\} \\ &= \bigcap_n\{f^{-1}[0,\alpha_n]\cup f^{-1}[\beta_n,1]\} \\ &\in H \text{ as required.}\end{aligned}$$

Thus our function $f$ so constructed satisfies ($\alpha$) and ($\beta$) of (3.1)2) above.

An so we turn to defining the system $\mathscr{U}$. To do this we first define a subsystem $\mathscr{U}_\Delta$, where $\Delta$ is the set of binary rationals in $[0,1]$:

(*)′ $$\mathscr{U}_\Delta = \{U(\delta),\ F(\delta)\colon \delta\in\Delta\} \text{ where}$$

(i)′ $\complement U(\delta)\in H$, $F(\delta)\in H$, $\delta\in\Delta$;
(ii)′ $A\subseteq U(0)$, $B\subseteq \complement U(1)$;
(iii)′ If $0\leqq\delta<\delta'\leqq 1$ then $U(\delta)\subseteq F(\delta)\subseteq U(\delta')$.

Let us suppose for the moment that $\mathscr{U}_\Delta$ is defined and satisfies (i)′, (ii)′ and (iii)′. Then if we write, for $0\leqq t\leqq 1$:

(†) $$U(t) = \bigcup_{\delta>t} U(\delta),\ F(t) = \bigcap_{\delta>t} F(\delta),$$

we clearly obtain a system $\mathscr{U}$ satisfying (i) and (ii). We check (iii). Take $t$, $t'$ with $0\leqq t<t'\leqq 1$; there exists $\delta$, $\delta'$, $\delta''\in\Delta$ with $t<\delta<\delta'<\delta''<t'$, and so by (iii)′ and (†)

$$U(t)\subseteq U(\delta)\subseteq F(\delta)\subseteq U(\delta')\subseteq F(\delta')\subseteq U(\delta'')\subseteq U(t').$$

Clearly $F(\delta)\subseteq F(t)\subseteq F(\delta')$ follows from (†) and so with the above we obtain

$$U(t)\subseteq F(\delta)\subseteq F(t)\subseteq F(\delta')\subseteq U(t') \text{ which implies (iii).}$$

Thus our problem reduces to constructing $\mathscr{U}_\Delta$ satisfying (i)′, (ii)′, (iii)′. This is done inductively, using the representation $\Delta=\{k2^{-m}\colon k=0,1,\cdots,2^m;\ m\geqq 0\}$; we define for $m\geqq 0$:

(*)″ $$\mathscr{U}_m = \{U(k2^{-m}),\ F(k2^{-m})\colon 0\leqq k\leqq 2^m\} \text{ where}$$

(i)″ $\complement U(k2^{-m})\in H$, $F(k2^{-m})\in H$, $0\leqq k\leqq 2^m$;
(ii)″ $A\subseteq U(0)$, $B\subseteq \complement U(1)$;
(iii)″ If $0\leqq k<l\leqq 2^m$ then $U(k2^{-m})\subseteq F(k2^{-m})\subseteq U(l2^{-m})$.

Then we put $\mathscr{U}_\Delta = \bigcup_{m\geqq 0}\mathscr{U}_m$.

Case $m = 0$. Put $\mathscr{U}_0 = \{U(0), F(0), U(1), F(1)\}$ where

$$U(1) = \complement B, \ F(1) = X, \text{ and}$$

$U(0)$, $F(0)$ are chosen so that $\complement U(0) \in H$, $F(0) \in H$, and

$$A \subseteq U(0) \subseteq F(0) \subseteq \complement B.$$

This can be done: for $A \cap B = \phi$ implies, by the assumed normality of $H$, the existence of $V$, $F \in H$ with:

$$A \cap V = \phi, \ B \cap F = \phi \text{ and } V \cup F = X.$$

But these relations imply $A \subseteq \complement V$, $F \subseteq \complement B$ and $\complement V \subseteq F$ so that putting $U(0) = \complement V$ with $F(0) = F$ satisfies our requirements.

Now suppose that for some $m \geqq 1$, $\mathscr{U}_{m-1}$ is defined and satisfies (i)″, (ii)″ and (iii)″, and let us consider $\mathscr{U}_m$. For even $k$ we define $U(k2^{-m})$, $F(k2^{-m})$ in the obvious way. For odd $k \geqq 1$ we note that (iii)″ implies:

$$U((k-1)2^{-m}) \subseteq F((k-1)2^{-m}) \subseteq U((k+1)2^{-m}).$$

The last inclusion can be written $F((k-1)2^{-m}) \cap \complement U((k+1)2^{-m}) = \phi$ and so we may proceed as for the case $m = 0$ with $F$ replacing $A$, $\complement U$ replacing $B$, and find elements $V$, $F$ of $H$ with

$$F((k-1)2^{-m}) \subseteq \complement V \subseteq F \subseteq U((k+1)2^{-m}).$$

Thus we may put $U(k2^{-m}) = \complement V$ and $F(k2^{-m}) = F$ and satisfy (iii)″ thus completing the inductive step.

And so we have constructed $\mathscr{U}_\Delta$ and thus $\mathscr{U}$, and it only remains to prove (iv) is valid in order to complete the proof of 1) implies 2). Recall the definition (**) of $f$.

If $f(x) \leqq t$ for some $t \in [0, 1]$ then for any $\delta \in \Delta$ with $\delta > t$ we have $x \in U(\delta) \subseteq F(\delta)$ whence $x \in \bigcap_{\delta > t} F(\delta) = F(t)$.

On the other hand, if $x \in F(t)$, then for any $\delta \in \Delta$ with $\delta > t$ we may find $\delta' \in \Delta$ with $\delta > \delta' > t$ and so $x \in F(\delta') \subseteq U(\delta)$. Thus $f(x) < \delta$ for each $\delta > t$ whence $f(x) \leqq t$, and we have proved that $f^{-1}[0, t] = F(t)$. The other part of (iv) is proved similarly.

2) implies 1) Let us assume that $A \cap B = \phi$ for $A$, $B \in H$. By 2) there exists $f: X \to [0, 1]$ such that $A \subseteq f^{-1}\{0\}$, $B \subseteq f^{-1}\{1\}$ and $f^{-1}F \in H$ for each closed $F \subseteq [0, 1]$. If we take $F = [0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ we obtain $D = f^{-1}[0, \frac{1}{2}] \in H$, $C = f^{-1}[\frac{1}{2}, 1] \in H$ such that $A \cap C = \phi$, $B \cap D = \phi$ and $C \cup D = X$, as required.

Remark 3.2. If $\mathbf{H} = \mathbf{F}(X)$ is the lattice closed subsets of a topology on $X$ then 3.1 is just Urysohn's lemma. Another special case is when $\mathbf{H} = \mathbf{Z}(X)$ is the lattice of all zero-sets of a completely regular (Hausdorff) space $X$. In this case we have proved (cf. [5] 1.15) that disjoint zero-sets can be separated by a continuous function.

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Our application of 3.1 is in the following result:

THEOREM 3.3. *Let* **H** *be a normal* $(2,\sigma)$*-ring of subsets of a set* $X$. *Then the following are equivalent for an element* $A \in H$:

1) *There exists a sequence* $\{A_n : n \geqq 1\}$ *of elements of* $H$ *with the following properties*:

(α) $A \cap A_n = \phi$ *for all* $n$;
(β) *if for* $B \in H$ *we have* $B \cap A_n = \phi$ *for all* $n$, *then* $B \subseteq A$.

2) *There exists a function* $f: X \to [0,1]$ *such that*
(α) $f^{-1}F \in H$ *for every closed subset* $F$ *of* $[0,1]$;
(β) $A = f^{-1}\{0\}$.

PROOF: 1) implies 2). By 3.1 there exists $f_n: X \to [0,1]$ satisfying 2) (α) such that $A \subseteq f_n^{-1}\{0\}$ and $A_n \subseteq f_n^{-1}\{1\}$. This uses only 1) (α) and works for all $n$. Now consider the element $\bigcap_{n\geqq 1} f_n^{-1}\{0\}$ of $H$; clearly $A \subseteq \bigcap_{n\geqq 1} f_n^{-1}\{0\}$, and for any $n$,

$$A_n \cap \bigcap_{n\geqq 1} f_n^{-1}\{0\} \subseteq f_n^{-1}\{1\} \cap \bigcap_{n\geqq 1} f_n^{-1}\{0\} = \phi.$$

Thus by 1) (β) $A \supseteq \bigcap_{n\geqq 1} f_n^{-1}\{0\}$ and if we define $f: X \to [0,1]$ by

$$f = \sum_{n\geqq 1} 2^{-n} f_n$$

it is easy to see $f^{-1}\{0\} = \bigcap_{n\geqq 1} f_n^{-1}\{0\} = A$, and Lemma 2.4 of [6] implies that $f$ satisfies 2) (α). This completes the proof of the first implication.

2) implies 1). If $A = f^{-1}\{0\}$ for a function $f$ satisfying 2) (β), then we may define $A_n = f^{-1}[1/n, 1]$ for $n \geqq 1$, With this definition

$$\complement A = \bigcup_{n\geqq 1} A_n$$

and conditions 1) (α), 1) (β) are readily checked.

The following corollary can easily be proved using the two previous results.

COROLLARY 3.4. *Let* **H** *be a* $(2,\sigma)$*-ring of subsets of* $X$. *Then the following are equivalent*:

1) **H** *is a perfectly normal lattice.*
2) *For every* $A \in H$ *there exists a function* $f: X \to [0,1]$ *such that*:
(α) $f^{-1}F \in H$ *for every closed subset* $F$ *of* $[0,1]$;
(β) $A = f^{-1}\{0\}$.

In the terminology we are using, a zero-set space is a pair $(X, \mathscr{Z})$ where $X$ is a set and $\mathscr{Z}$ is a perfectly normal $(2,\sigma)$-ring of subsets of $X$ which separates points of $X$. With any such space Gordon associates the set $S(X, \mathscr{Z})$ of all functions $f: X \to \mathbb{R}$ such that $f^{-1}F \in \mathscr{Z}$ for every closed subset $F$ of $\mathbb{R}$; such functions are called zero-set functions. The lemma ([6] 2.4) used in the previous proof shows that $S(X, \mathscr{Z})$ is a uniformly closed ring of functions on $X$; also $S(X, \mathscr{Z})$ separates points of $X$ and contains the constant functions.

We give a new proof of [6] 3.5 viz:

THEOREM 3.5. *Let $(X, \mathscr{Z})$ be a zero-set space and $S(X, \mathscr{Z})$ the family of all zero-set functions on $X$. Then*

$$\mathscr{Z} = \{Z(f) : f \in S(X, \mathscr{Z})\}.$$

PROOF. By 3.4 every $A \in \mathscr{Z}$ is the zero-set of a suitable function of $S(X, \mathscr{Z})$; if $f \in S(X, \mathscr{Z})$ then $Z(f) = f^{-1}\{0\} \in \mathscr{Z}$ and the proof is complete.

## 4. The lattice $\mathbf{Z}(X)$ for $X$ compact

It is well known that a completely regular space $X$ is compact if, and only if, every $z$-ultrafilter is fixed. However, as in the case of the ring $C(X)$, the notion of fixed (resp. free) is not a lattice-theoretic invariant and so we must proceed slightly differently. At this point also, our treatment begins to differ from that in [6] since we only have the lattice $\mathbf{Z}(X)$ and not $X$ itself.

The main result of this section is given a proof independently of the discussion in the next section, although it can also be derived from results there. We do this because the simplications which occur when $X$ is compact allow quite different techniques to be used.

THEOREM 4.1. *Let* $\mathbf{L}$ *be a lattice. Then the following are equivalent:*

1) $\mathbf{L}$ *is isomorphic to the lattice* $\mathbf{Z}(X)$ *for a compact space* $X$.

2) ($\alpha$) $\mathbf{L}$ *is a* $(2, \sigma)$*-complete lattice;*

($\beta$) *Every minimal prime ideal of* $\mathbf{L}$ *is* $\sigma$*-prime;*

($\gamma$) $\mathbf{L}$ *is perfectly normal.*

*The space* $X$ *of* 1) *is NOT unique up to homeomorphism.*

PROOF. 1) implies 2). We will show that for any compact space $X$ the lattice $\mathbf{Z}(X)$ has properties 2) ($\alpha$), ($\beta$), ($\gamma$); these are obviously lattice invariants and so the implication will be proved. But we have already noted the validity of ($\alpha$), ($\gamma$) for $X$ general, and ($\beta$) follows since every minimal prime ideal of $\mathbf{Z}(X)$ is exactly those elements not belonging to a particular fixed ultrafilter $u_x = \{a \in \mathbf{Z}(X) : x \in a\}$, where $x \in X$ is unique. Clearly such a minimal prime is $\sigma$-prime, completing the proof.

2) implies 1) Suppose we are given a lattice $\mathbf{L}$ satisfying ($\alpha$), ($\beta$), ($\gamma$) of 2). Let $X$ denote the set of all minimal prime ideals of $\mathbf{L}$, and equip $X$ with the topology whose closed sets are intersections of the sets in $L' = \{X_a : a \in L\}$ where for $a \in L$, $X_a = \{x \in X : a \notin x\}$. We will prove that $X$ so defined is a compact

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(Hausdorff) space, and that the family of all zero-sets of $X$ is exactly $\mathbf{L}'$, a lattice which will be shown to be isomorphic to $\mathbf{L}$.

We prove this last remark first. Condition 2) ($\gamma$) together with Lemma 2.5 above implies that $\mathbf{L}$ is disjunctive, and so by a result which is well known (see e.g. [8]) $a \to X_a$ is bijective. It can be readily checked that for $a$, $b \in L$, $X_a \cup X_b = X_{a \vee b}$, and since the ideals in $X$ are all $\sigma$-prime, we find that for $\{a_n : n \geqq 1\}$ $L, \bigcap_{n \geqq 1} X_{a_n} = X_a$ where $a = \bigwedge_{n \geqq 1} a_n$. The latter exists, of course, by 2) ($\alpha$).

The proof that $X$, so topologised, is a compact (Hausdorff) space given (in a dual form) in [4] Theorem 7.3 hence we omit it.

And so it remains to prove that $L'$ is exactly the family of all zero-sets of $X$. Now $X$ is normal and so it is enough to prove that $L'$ is exactly the set of all closed $G_\delta$-subsets of $X$. But every $a \in L$ is a $G_\delta$ and this is easily seen to imply

$$\complement X_a = \bigcup_{n \geqq 1} X_{a_n},$$

proving that $X_a$ is a $G_\delta$-subset, by definition, closed, of $X$. This proves half of what is required, and to complete the proof we take an arbitrary closed $G_\delta$-subset $F$ of $X$. By definition, there is a sequence $M_n$ of subsets of $L$, and a subset $B \subseteq L$ such that

$$\bigcap_{b \in B} X_b = F = \bigcap_{n \geqq 1} \left\{ \bigcup_{a \in M_n} \complement X_a \right\}.$$

We concentrate on the right-hand equality first. Since $F$ is compact, for any $n \geqq 1$ there is a finite subset $m_n \subseteq M_n$ such that

$$F \subseteq \bigcup_{a \in m_n} \complement X_a = \complement X_{a_n},$$

where $a_n = \bigwedge_{a \in m_n} a$. Thus we see that

$$\bigcap_{b \in B} X_b = F = \bigcap_{n \geqq 1} \complement X_{a_n}.$$

Now each $X_{a_n}$ is compact, and so for each $n$ there is a finite subset $B_n \subseteq B$ such that

$$X_{b_n} = \bigcap_{b \in B_n} X_b \subseteq \complement X_{a_n},$$

where $b_n = \bigwedge_{b \in B_n} b$. Putting these results together gives

$$F = \bigcap_{n \geqq 1} \complement X_{a_n} \supseteq \bigcap_{n \geqq 1} X_{b_n} \supseteq \bigcap_{b \in B} X_b = F,$$

whence $F = \bigcap_{n \geqq 1} X_{b_n} = X_b$, where $b = \bigwedge_{n \geqq 1} b_n$, and in this last step we have used the fact that $b \to X_b$ is a $(2, \sigma)$-homomorphism. The proof is now complete.

An interesting byproduct which will be explained in the next section is the following:

COROLLARY 4.2. *Let Y be a pseudocompact topological space. Then there is a compact space X such that* $\mathbf{Z}(Y)$ *and* $\mathbf{Z}(X)$ *are lattice isomorphic.*

PROOF. Putting together 5.8(b) and 5.14 of [5] we find that for $Y$ pseudocompact $\mathbf{Z}(Y)$ satisfies (4.1) 2) ($\beta$) and so the result follows.

In particular, the corollary shows that non-homeomorphic pseudocompact spaces can have isomorphic lattices of zero-sets. We will see that this cannot happen when the spaces are both realcompact. Finally, we note that the space $X$ in 4.2 can be taken to be $\beta Y$, or any space $Y \subseteq X \subseteq \beta Y$.

## 5. The lattice $\mathbf{Z}(X)$ for a general $X$

In this section we characterise the lattice $\mathbf{Z}(X)$ algebraically, for a general topological space $X$. We begin with a reduction, relying heavily upon results from [5].

PROPOSITION 5.1. *For every topological space X there exists a completely regular space Y and a continuous map* $\tau$ *of X onto Y such that the map:*

$$Z_Y(g) \to Z_A(g \circ \tau)$$

*is an isomorphism of* $\mathbf{Z}(Y)$ *onto* $\mathbf{Z}(X)$.

PROOF. See [5] 3.9. The details are easy, and omitted.

The next stage of our reduction is again similar to the ring case.

PROPOSITION 5.2. *For every completely regular space X there exists a realcompact space* $\upsilon X$ *and a continuous map* $\tau$ *of X into* $\upsilon X$ *such that the map:*

$$a \to cl_{\upsilon X^a}$$

*is an isomorphism of* $\mathbf{Z}(X)$ *onto* $\mathbf{Z}(\upsilon X)$).

PROOF. See [5] 8.8.

From now on we will suppose, where appropriate, that $X$ is realcompact. As a first attack on our characterisation problem we abstract the lattice-theoretic properties of a zero-set structure.

DEFINITION 5.3. A lattice $\mathbf{L}$ is a *z-lattice* if

($\alpha$) $\mathbf{L}$ is $(2, \sigma)$-complete;
($\beta$) $\mathbf{L}$ has enough $\sigma$-prime minimal prime ideals;
($\gamma$) $\mathbf{L}$ is perfectly normal.

For any topological space $X$ the lattice $\mathbf{Z}(X)$ is a $z$-lattice. To see this we need only check ($\beta$) as ($\alpha$) and ($\gamma$) of 5.3 have already been noted. Now for any $x \in X$ the family $j_x = \{a \in \mathbf{Z}(X): x \notin a\}$ is easily seen to be a $\sigma$-prime ideal of $\mathbf{Z}(X)$ and there are certainly enough of these ideals to distinguish elements of $\mathbf{Z}(X)$. Thus ($\beta$) will be satisfied if we show that all the $\sigma$-prime ideals $j_x$ are minimal. But this follows from 2.5 above; alternatively a direct proof can be given.

We will see below that although not every $z$-lattice is isomorphic to a lattice $\mathbf{Z}(X)$, such a lattice can be embedded as a sublattice of $\mathbf{Z}(X)$ for a suitable $X$ in a particularly precise manner, which it is convenient to formulate separately. For any $z$-lattice $\mathbf{L}$ (possibly with superscripts) we denote by $X^{\mathbf{L}}$, or just $X$ if no ambiguity is possible, (with the same superscripts) the set of all $\sigma$-prime minimal prime ideals of $\mathbf{L}$; for $a \in L$ we write $X^{\mathbf{L}}_a = X_a = \{x \in X^{\mathbf{L}}: a \notin x\}$.

Definition 5.4. A $z$-lattice $\mathbf{L}$ is said to be a *$z$-sublattice* of the $z$-lattice $\mathbf{L}'$, equivalently, $\mathbf{L}'$ is a *$z$-extension* of $\mathbf{L}$, if

($\alpha$) $\mathbf{L}$ is a $(2,\sigma)$ regular sublattice of $\mathbf{L}'$;
($\beta$) the map $x' \to x' \cap L$ is a bijection from $X'$ onto $X$;
($\gamma$) for any $b \in L'$, $X'_b = \bigcap\{X'_a: a \in L, a \geqq b\}$.

We will see that the property of being a $z$-sublattice is transitive, a fact needed below.

Lemma 5.5. *If $\mathbf{L}$ is a $z$-sublattice of $\mathbf{L}'$, and $\mathbf{L}'$ is a $z$-sublattice of $\mathbf{L}''$, then $\mathbf{L}$ is a $z$-sublattice of $\mathbf{L}''$.*

Proof. Clearly ($\alpha$) and ($\beta$) are true so we need only prove ($\gamma$). Let $b \in L'$. We will show that

$$(*) \qquad X_b = \bigcap\{X_a: a \in L, a \geqq b\}$$

is true, and then the fact that for any $c \in L''$

$$X''_c = \bigcap\{X''_b: b \in L', b \geqq c\}$$

will complete the proof. Now suppose that $x'' \in X''$ is such that $a \notin x'$ for all $a \in L$ with $a \geqq b$. Then $a \notin x'' \cap L' = x'$ say, for all $a \in L$ such that $a \geqq b$, and so $b \notin x'$, since $\mathbf{L}'$ is a $z$-extension of $\mathbf{L}$. Thus we have proved $b \notin x''$ and the equality (*) is proved.

The following results is the main step in our characterisation theorem.

Theorem 5.6. *Let $\mathbf{L}$ be a $z$-lattice. Then $X = X^{\mathbf{L}}$ is a realcompact space, and $\mathbf{L}$ is isomorphic to a $z$-sublattice of the $z$-lattice $\mathbf{Z}(X)$.*

Proof. We give $X$ the topology whose closed subsets are intersections of sets in $L' = \{X_a: a \in L\}$. Exactly as in 4.1 above we can prove that $\mathbf{L}$ is isomorphic to $\mathbf{L}'$ where $\mathbf{L}'$ is the set $L'$ under the operations of finite set-union and countable set-intersection; the isomorphism is a $(2, \sigma)$-homomorphism.

Thus $\mathbf{L}'$ is a $z$-lattice, and so the results of §3 above will apply. We prove that $X$ is completely regular. Take a point $x \in X$ and a closed set $F = \bigcap\{X_a: a \in \mathbf{M}\}$ not containing $x$. Then there is $X_a \subseteq F$ with $x \notin X_a$; since $a$ in $L$ and hence $X_a$ in $L'$ is a $G_\delta$, $x \in X_b$ with $X_a \cap X_b = \phi$ for a suitable $b \in L$. By Theorem 3.1 there is a continuous function $f: X \to [0,1]$ with $f(x) = 1$ and $f^{-1}\{0\} \supseteq F$. Now Corollary 3.4 shows that every element of $L'$ is a zero-set of $X$ and so $L \subseteq Z(X)$. Before we show that $\mathbf{L}'$ is a $z$-sublattice of $\mathbf{Z}(X)$ it will be necessary to prove that $X$ is realcompact. Let $\eta$ be a real $z$-ultrafilter on $X$; then $y = \{a \in Z(X): a \notin \eta\}$ is a $\sigma$-prime minimal prime ideal on $\mathbf{Z}(X)$, and so $y' = y \cap L'$ is a $\sigma$-prime ideal of $\mathbf{L}'$. Lemma 2.5 implies that $y'$ is in fact a minimal prime ideal, and so $y' = x$ for some unique $x \in X$. Now the intersection of all the zero-sets in $\eta$ is, by the definition of the topology, an intersection of all the zero-sets of the form $X_a$ in $\eta$ and this intersection contains $x$; thus $X$ is realcompact by [5] 5.15.

Having now established that $\mathbf{L}$ is isomorphic to the sublattice $\mathbf{L}'$ of the lattice $\mathbf{Z}(X)$ where $X$ is a realcompact space, our proof is completed by proving that $\mathbf{Z}(X)$ is a $z$-extension of $\mathbf{L}'$. This is really quite easy once we observe that the $z$-lattice $\mathbf{Z}(X)$ has a space $X^{Z(X)}$ of $\sigma$-prime minimal prime ideals which is canonically homeomorphic to $X$ under the map $x \to j_x = \{a \in Z(X): x \notin a\}$. Referring to 5.4 we see that ($\alpha$) is valid, ($\beta$) follows from Lemma 2.5 and the preceding remark, and ($\gamma$) simply expresses the fact that every zero-set in $X$ (more precisely, its homeomorph $X^{Z(X)}$) is closed and hence an intersection of the basic closed sets in $L'$ (more precisely, their copies inside $X^{Z(X)}$). Thus the theorem is proved.

It might have been hoped that in the previous construction, $\mathbf{L}'$ actually coincides with $\mathbf{Z}(X)$, but as already observed, this is not generally so. After examining an example which validates this assertion we formulate and prove the maximality property possessed by lattices $\mathbf{Z}(X)$, and our main characterisation theorem quickly follows.

EXAMPLE 5.7. Consider the $z$-lattice $\mathbf{B} = \mathbf{B}[0,1]$ of all Borel subsets of $[0,1]$. Then $\mathbf{B}$ is a $z$-sublattice of the power set $\mathbf{P} = \mathbf{P}[0,1]$.

PROOF. To see this we also need to refer to $\mathbf{F} = \mathbf{F}[0,1]$, the $z$-lattice of all closed sets (= zero-sets) of $[0,1]$ with the usual topology. Before the assertion can be proved we need to describe the $\sigma$-prime minimal prime ideals of each of $\mathbf{F}$, $\mathbf{B}$, and $\mathbf{P}$. Since $[0,1]$ is compact those of $\mathbf{F}$ are all fixed, i.e. of the form $j_x \cap F$ where $j_x = \{a \in P: x \notin a\}$, for $x \in [0,1]$. Also the non-measurability of the cardinal of $[0,1]$ implies that the $\sigma$-prime minimal prime ideals of $\mathbf{P}$ are all of the form $j_x$ for $x \in [0,1]$. Now all three of $\mathbf{F}$, $\mathbf{B}$ and $\mathbf{P}$ are perfectly normal and so Lemma 2.6 implies that every $\sigma$-prime minimal prime ideal of $\mathbf{B}$ is of the form $j_x \cap B$ for some $x \in [0,1]$. This last result is also a consequence of 8.4 [6].

Turning now to proving that $\mathbf{B}$ is a $z$-sublattice of $\mathbf{P}$ we note that 5.4 ($\alpha$) is obviously true, ($\beta$) has already been remarked upon, and so only ($\gamma$) remains. But each singleton $\{x\}$ belongs to $B$ and so ($\gamma$) is easily seen to be equivalent to

$$[0,1]\backslash b = \bigcap_{x \in b} [0,1]\backslash\{x\}, \ b \subseteq [0,1];$$

where the sets in the right-hand intersection are all in $B$. Thus $\mathbf{B}$ is a proper $z$-sublattice of $\mathbf{P}$.

Our next result shows that a zero-set lattice $\mathbf{Z}(X)$ can never be a proper $z$-sublattice of a $z$-lattice, and this is the point where we can see why Gordon's results [6] differ in some respect from the usual topological ones. Simply put, his zero-set structures are more general than those which can arise in the topological context, and so a result such as: a product of pseudo-compact zero-set spaces is pseudo-compact, can be valid in the former while failing in the latter. Put another way, $z$-lattices such as the $\mathbf{B}$ of 5.7 can never arise as $\mathbf{Z}(X)$ for a topological space $X$. We note that if this could happen, results of Mandelker [10] imply that $X$ would be at least a $P$-space!

THEOREM 5.8. *Suppose $X$ and $Y$ are realcompact spaces and that $\mathbf{Z}(Y)$ is isomorphic to a z-sublattice $\mathbf{L}$ of $\mathbf{Z}(X)$. Then $\mathbf{Z}(Y)$ is isomorphic to $\mathbf{Z}(X)$.*

PROOF. We prove that $X$ and $Y$ are homeomorphic under the stated assumptions. It is easy to see that the space of $\sigma$-prime minimal prime ideals of $\mathbf{Z}(X)$ topologised as in 5.6 is canonically homeomorphic to $X$; we denote it $X^*$ with points $j_x = \{a \in Z(X) \colon x \notin a\}$. Similarly for $\mathbf{Z}(Y)$. Thus we have the following diagram, where $L' = \{X_a^{Z(X)} \colon a \in L\}$ and $Z' = \{X_b^{Z(X)} \colon b \in Z(X)\}$:

$$\begin{array}{ccccc} \mathbf{Z}(Y) \overset{h}{\to} & \mathbf{L} & \subseteq & \mathbf{Z}(X) & \\ & \downarrow & & \downarrow & \searrow \\ & \mathbf{L}' & \subseteq & \mathbf{Z}' & \subseteq \ \mathbf{Z}(X^*) \end{array}$$

where all the maps which are not inclusions are isomorphisms; the vertical maps are as in the construction 5.6 and the diagonal map is defined using the homeomorphism $x \to j_x$.

Now (5.4) ($\beta$) states that the map

$$x^* = j_x \to j_x \cap L$$

is a bijection; by construction it is continuous from $X^*$ onto $X^{\mathbf{L}}$. We show that it is a closed map. A typical closed subset $F$ of $X^*$ is of the form

$$F = \bigcap_{b \in B} X_b^*$$

where $B \subseteq \mathbf{Z}(X)$. Condition (5.4) ($\gamma$) states in this context that for each $b \in \mathbf{Z}(X)$, $X_b^* = \bigcap \{X_a^* \colon a \in L, \ a \geqq b\}$ whence

$$F = \bigcap_{a \in A} X_a^*$$

where $A \subseteq L$. But now we may apply the bijection above, and we find that $F \to \bigcap_{a \in A} X_a^{\mathbf{L}}$, a closed subset of $X^{\mathbf{L}}$. Thus $X^*$ is homeomorphic to $X^{\mathbf{L}}$ and so we deduce that $X$ is homeomorphic to $Y$. Finally we complete the proof by noting that the lattice of zero-sets of a topological space is a topological invariant.

We can now finish off with the characterisation theorem.

THEOREM 5.9. *The following are equivalent for a lattice* $\mathbf{L}$.

1) $\mathbf{L}$ *is isomorphic to* $\mathbf{Z}(X)$ *for a topological space* $X$.

2) ($\alpha$) $\mathbf{L}$ *is* $(2, \sigma)$*-complete*;

($\beta$) $\mathbf{L}$ *has enough* $\sigma$*-prime minimal prime ideals*;

($\gamma$) $\mathbf{L}$ *is perfectly normal*;

($\delta$) $\mathbf{L}$ *is isomorphic to every* $z$*-extension* $\mathbf{L}'$ *of* $\mathbf{L}$.

PROOF. 1) implies 2). Properties ($\alpha$), ($\beta$) and ($\gamma$) have already been observed Suppose that $\mathbf{Z}(X)$ is a $z$-sublattice of a $z$-lattice $\mathbf{L}'$. Noting that we may suppose $X$ is realcompact by 5.2, we have the following diagram:

$$\begin{array}{ccccc} \mathbf{Z}(X) & \subseteq & \mathbf{L}' & & \\ \downarrow & & \downarrow & & \\ \mathbf{Z}' & \subseteq & \mathbf{L}'' & \subseteq & \mathbf{Z}(X^{\mathbf{L}'}) \end{array}$$

where the vertical maps are isomorphisms as in the construction of 5.6, and the horizontal maps are inclusions. By the transitivity of the property of being a $z$-sublattice, $\mathbf{Z}'$ is a $z$-sublattice of $\mathbf{Z}(X^{\mathbf{L}'})$ isomorphic to $\mathbf{Z}(X)$, and so $\mathbf{Z}(X) \cong \mathbf{L}'$ follows from 5.8.

2) implies 1). We have already proved that if $\mathbf{L}$ satisfies 2) ($\alpha$), ($\beta$), ($\gamma$), $\mathbf{L}$ is isomorphic to a $z$-sublattice $\mathbf{L}'$ of $\mathbf{Z}(X)$ for a completely regular space $X$, and so by ($\delta$) we may conclude that $\mathbf{L} \cong \mathbf{Z}(X)$.

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