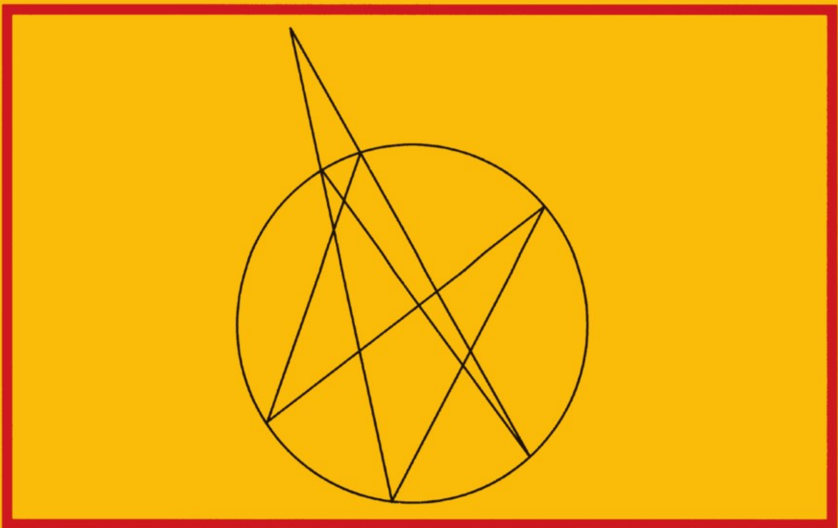
Undergraduate Texts in Mathematics

*Readings in Mathematics*

W.S. Anglin

# Mathematics: A Concise History and Philosophy



Springer

Undergraduate Texts in Mathematics

*Readings in Mathematics*



Springer-Science+Business Media, LLC

## Graduate Texts in Mathematics
*Readings in Mathematics*

Ebbinghaus/Hermes/Hirzebruch/Koecher/Mainzer/Neukirch/Prestel/Remmert: *Numbers*
Fulton/Harris: *Representation Theory: A First Course*
Remmert: *Theory of Complex Functions*
Walter: *Ordinary Differential Equations*

## Undergraduate Texts in Mathematics
*Readings in Mathematics*

Anglin: *Mathematics: A Concise History and Philosophy*
Anglin/Lambek: *The Heritage of Thales*
Bressoud: *Second Year Calculus*
Hairer/Wanner: *Analysis by Its History*
Hämmerlin/Hoffmann: *Numerical Mathematics*
Isaac: *The Pleasures of Probability*
Laubenbacher/Pengelley: *Mathematical Expeditions: Chronicles by the Explorers*
Samuel: *Projective Geometry*
Stillwell: *Numbers and Geometry*
Toth: *Glimpses of Algebra and Geometry*

W.S. Anglin

# Mathematics: A Concise History and Philosophy

With 15 Illustrations

Springer

W.S. Anglin
Keane, Inc.
19 Old King's Highway South
Darien, CT 06820
USA



Mathematics Subject Classification (1991): 01-01, 01A05, 00A30

Library of Congress Cataloging-in-Publication Data
Anglin, W. S.
Mathematics, a concise history and philosophy / W.S. Anglin.
p. cm.—(Undergraduate texts in mathematics. Readings in mathematics)
Includes bibliographical references and index.
ISBN 978-1-4612-6930-4 ISBN 978-1-4612-0875-4 (eBook)
DOI 10.1007/978-1-4612-0875-4
1. Mathematics—History. 2. Mathematics—Philosophy. I. Title. II. Series.
QA21.A54 1994
510′.9—dc20 94-8075

Printed on acid-free paper.



Production managed by Natalie Johnson; manufacturing supervised by Vincent Scelta.
Photocomposed copy prepared from the author's LATEX files.

9 8 7 6 5 4 3

SPIN 10727955

To Jim Lambek

# Preface

This is a concise introductory textbook for a one-semester (40-class) course in the history and philosophy of mathematics. It is written for mathematics majors, philosophy students, history of science students, and (future) secondary school mathematics teachers. The only prerequisite is a solid command of precalculus mathematics.

On the one hand, this book is designed to help mathematics majors acquire a philosophical and cultural understanding of their subject by means of doing actual mathematical problems from different eras. On the other hand, it is designed to help philosophy, history, and education students come to a deeper understanding of the mathematical side of culture by means of writing short essays. The way I myself teach the material, students are given a choice between mathematical assignments, and more historical or philosophical assignments. (Some sample assignments and tests are found in an appendix to this book.)

This book differs from standard textbooks in several ways. First, it is shorter, and thus more accessible to students who have trouble coping with vast amounts of reading. Second, there are many detailed explanations of the important mathematical procedures actually used by famous mathematicians, giving more mathematically talented students a greater opportunity to learn the history and philosophy by way of problem solving. For example, there is a careful treatment of topics such as unit fractions, perfect numbers, linear Diophantine equations, Euclidean construction, Euclidean proofs, the circle area formula, the Pell equation, cubic equations, log table construction, the four square theorem, quaternions, and Cantor's set theory. Third, several important philosophical topics are pursued throughout the text, giving the student an opportunity to come to a full and consistent knowledge of their development. These topics include infinity and Platonism. In the essay questions, students are challenged to address a wide range of important topics. In short, this book offers, in fewer pages, a deep penetration into the key mathematical and philosophical aspects of the history of mathematics.

The research for this book was carried out at McGill University from 1989 to 1992, and I should like to acknowledge the support of the Social Sciences and Humanities Research Council of Canada from 1989 to 1991. I am also greatly indebted to Jim Lambek, whose own course in this subject was the inspiration and basis of this work. The final version of this book was created at the University of Regina from 1992 to 1993, and I should like to thank the university for the opportunity of teaching the material in a liberal arts course entitled 'Mathematical Problems, Ideas and Personalities'. I should also like to thank Andonowati, J. Brown, E. Choueke, J. Denton, D. Hanson, I. Rabinovitch, and D. Zhang for their help and encouragement.

W. S. ANGLIN

# Contents

# 1

# Mathematics for Civil Servants

Aristotle thought that mathematics was begun by the priests in Egypt, 'because there the priestly class was allowed leisure' (*Metaphysics* 981b 23–24). Herodotus, however, believed that geometry was created because the annual flooding of the Nile necessitated surveying, to redetermine land boundaries. Indeed, Democritus called the Egyptian mathematicians 'rope-stretchers'.

From a philosophical point of view, it is interesting that the Egyptians held that mathematics had a divine source. It had been given them by the god Thoth.[1] In this book, we shall encounter a view, called *Aristotelianism,* which sees mathematics ascending from the human animal, and another view, called *Platonism,* which sees mathematics descending from a divine realm.

## The Moscow Papyrus

Our only sources of information on the mathematics of ancient Egypt are the Moscow Mathematical Papyrus and the Rhind Mathematical Papyrus. The Moscow Mathematical Papyrus dates from 1850 B.C., about the time of Abraham. V. S. Golenishchev acquired it in 1893 and brought it to Moscow.

The most interesting problem in the Moscow Mathematical Papyrus is

[1]See Plato's *Phaedrus* 274c-d

Problem 14. This is a computation of the volume of a frustum, using the correct formula. A frustum is a pyramid with a similar pyramid cut off its top. If it has a square base of side $a$, and a square top of side $b$, and if its height is $h$, then, as the ancient Egyptians realised, the volume of the frustum is

$$\frac{h(a^2 + ab + b^2)}{3}$$

Note that if $b = 0$, we get the formula for the volume of a square-base pyramid: $a^2h/3$.

We do not know how the Egyptians arrived at these formulas. Perhaps it was by trial and error.

## The Rhind Papyrus

The Rhind Mathematical Papyrus is a copy of an even earlier work. It was copied by a scribe called Ahmose in 1650 B.C., about the time Joseph was governor of Egypt. Alexander Henry Rhind acquired it in Luxor, Egypt, in 1858, and the British Museum bought it from his estate in 1865.

The Rhind Mathematical Papyrus opens by promising the reader 'a thorough study of all things, insight into all that exists, knowledge of all obscure secrets'. In fact, it is a sequence of solved problems in elementary mathematics, a Schaum's Outline for aspiring scribes. These scribes had to calculate how many bricks were needed to build a ramp of a certain size, how many loaves of bread were required to feed the slave labourers, and so on.

To multiply 70 by 13, the Egyptians would work as follows:

| | | |
|---|---|---|
| 70 | 13 | / |
| 140 | 6 | |
| 280 | 3 | / |
| 560 | 1 | / |
| 910 | | |

In general, the method was to set up two columns, each headed by one of the multipliers. The entries in the first column were doubled, while those in the second column were halved (first subtracting 1 if the number was odd). Finally, those entries in the first column beside odd second column entries (the checked ones) were added. (The method works because the odd-numbered entries in the second column correspond to 1's in the scale 2 expression of the second multiplier.)

The Rhind Mathematical Papyrus shows us how the Egyptians divided, extracted square roots, and solved linear equations. They used the formula $(4/3)^4r^2$ for the area of a circle (giving 3.16 as an approximation for $\pi$), and they did interesting work with arithmetic progressions. Problem 64, for example, was to find an arithmetic progression with 10 terms, with sum 10, and with common difference $1/8$.

## Unit Fractions

From the Rhind Mathematical Papyrus we learn that the ancient Egyptians expressed all fractions (except $2/3$) as sums of distinct unit fractions (that is, fractions of the form $1/n$, with $n$ a positive integer). Thus they wrote $2/9$ as $1/6 + 1/18$, and $8/11$ as

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{22} + \frac{1}{66}$$

In 1880, J. J. Sylvester proved that any proper fraction $a/b$ can be written as a sum of distinct unit fractions. This is certainly true when the numerator $a = 1$. Suppose it true for proper fractions with numerator $< a$ (with $a > 1$). Let $1/q$ be the largest unit fraction less than $a/b$. Then

$$\frac{1}{q} < \frac{a}{b} < \frac{1}{q-1}$$

Hence $0 < aq - b < a$. But

$$\frac{a}{b} = \frac{1}{q} + \frac{aq-b}{bq}$$

By the induction hypothesis, $\frac{aq-b}{bq}$ is a sum of distinct unit fractions. Moreover, none of them is $\frac{1}{q}$, since

$$\frac{1}{q} > \frac{aq-b}{bq}$$

This completes the proof—and gives us a way to find a distinct unit fraction sum equal to a given proper fraction.

For example, to express $3/7$ in the Egyptian manner, we round $7/3$ up to the nearest integer, namely, 3. Then $1/3$ is the largest unit fraction less than $3/7$. We have

$$\frac{3}{7} - \frac{1}{3} = \frac{2}{21}$$

The largest unit fraction less than $2/21$ is $1/11$, and we obtain

$$\frac{2}{21} - \frac{1}{11} = \frac{1}{231}$$

Hence

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$$

Note that this is not the only possibility. For example, we also have

$$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$$

Recently Paul Erdös posed the problem of showing that if $n$ is an integer $> 4$, then $4/n$ is a sum of three distinct unit fractions. This problem has not yet been solved, although there are some partial results:

$$\begin{aligned}
\frac{4}{4m+2} &= \frac{1}{m+1} + \frac{1}{(m+1)(2m+1)} \\
\frac{4}{4m+3} &= \frac{1}{m+2} + \frac{1}{(m+1)(m+2)} + \frac{1}{(m+1)(4m+3)} \\
\frac{4}{8m+5} &= \frac{1}{2(m+1)} + \frac{1}{2(m+1)(3m+2)} + \frac{1}{2(3m+2)(8m+5)} \\
\frac{4}{3m+2} &= \frac{1}{m+1} + \frac{1}{3m+2} + \frac{1}{(m+1)(3m+2)}
\end{aligned}$$

## Great Pyramid Nonsense

Attempts have been made to use the dimensions of the Great Pyramid (built about 2600 B.C.) to draw conclusions about Egyptian mathematics. For example, it is claimed that half the perimeter of the base of the pyramid, divided by its height, equals 3.14. From this it is supposed to follow that the Egyptians of 2600 B.C. knew the value of $\pi$ to two decimal places. Against this idle speculation, we advance the following considerations. (1) Over the centuries, people have taken stone from the pyramid for their own building projects; the original surface of the pyramid has thus disappeared, and we have no way of knowing its original dimensions with two-decimal-place accuracy. (2) There are dozens of ratios one can calculate given the alleged dimensions of a pyramid; it is not surprising if one of them happens to be close to $\pi$. (3) In the Rhind Papyrus, the value used for $\pi$ is about 3.16; if the Egyptians knew a better approximation for $\pi$ in 2600 B.C., they would not have been using a worse one in 1650 B.C.[2]

## Exercises 1

1. Using the fact that the volume of a pyramid is

$$\frac{1}{3} \times \text{base} \times \text{height}$$

[2]Martin Gardner gives a good account of Great Pyramid nonsense in *Fads and Fallacies in the Name of Science.*

show that the Egyptian formula for the volume of a frustum is correct.

2. Find $1359 \times 2578$ in the Egyptian manner.

3. Find an arithmetical progression with 10 terms, sum 10, and common difference $1/8$.

4. If the scribe receives $1+1/3$ out of every 42 portions, while the Illahun Temple Director receives 10 out of every 42 portions, how much does the Director get when the scribe gets $2 + 1/6 + 1/18$ loaves?

5. Express $13/14$ as a sum of distinct unit fractions.

6. Express all the proper fractions with denominator 11 in the Egyptian manner.

7. Prove that $8/11$ cannot be written as a sum of fewer than 4 distinct unit fractions.

8. Express $4/253$ as a sum of three distinct unit fractions.

9. Show that if $p$ is prime, $2/p$ can be expressed as a sum of two distinct unit fractions in exactly 1 way.

10. Show that the expression given above for $4/(3m+2)$ as a sum of three distinct unit fractions is correct.

11. Show that if $p + q = 4ef$, while $p + e = gq$ (with $e$, $f$, $g$, $p$, and $q$ positive integers) then

$$\frac{4}{p} = \frac{1}{ef} + \frac{1}{efg} + \frac{1}{fgp}$$

12. Show that, to solve the Erdös problem, it would suffice to show that if $p$ is a prime of the form $24m + 1$ then $4/p$ is a sum of three distinct unit fractions.

## *Challenges for Experts*

1. Let $a/b$ be a proper fraction, and $x$, $y$, $z$, $w$ distinct positive integers such that $a/b = 1/x + 1/y + 1/z + 1/w$. Prove $w < 288b^8$.

2. Show that
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807}$$
is the largest proper fraction that can be expressed using five distinct unit fractions.

3. Prove that there is no integer $n$ such that every proper fraction can be written as a sum of $n$ or fewer distinct unit fractions.

## *Essay Question*

1. An archaeologist has found an old Egyptian building stone measuring 1 cubit by 1 cubit by 1 cubit. 'This stone dates from 4000 B.C.,' he says. 'And if you add the distance between the opposite corners to the diagonal of one of the sides, you get a sum of 3.15 cubits. This proves that the Egyptians of 4000 B.C. used the value 3.15 for $\pi$.' Make up a reasoned reply to this statement to convince this fine professor that he has lost his wits.

# 2
# The Earliest Number Theory

## The Sumerians and Babylonians

The Sumerians lived in the southern part of Mesopotamia (Iraq). About 2000 B.C., their civilisation was absorbed by the Babylonians, and Babylonian culture reached its peak about 575 B.C., under Nebuchadnezzar. The mathematical achievements we shall discuss in this chapter are recorded on the clay tablets of the Sumerians and Babylonians. Most of these achievements go back as far as 2000 B.C. — about the time when Abraham's father was living in the Sumerian city of Ur. We shall use the word 'Babylonian' for what is perhaps more accurately described as 'Mesopotamian' mathematics.

The Babylonians used a counting scale, not of 10, but of 60, and this scale was taken over into Greek astronomy by Hipparchus of Nicaea (about 150 B.C.). It is thanks to the Babylonians, and Hipparchus, that we have 60 minutes in an hour. According to the prophet Ezekiel (573 B.C.), in the ancient system of weights, scale 60 was endorsed by God himself:

> Lord Yahweh says this: ... Twenty shekels, twenty-five shekels and fifteen shekels are to make one mina (Ezekiel 45:9-12).

The Babylonians could solve linear and quadratic equations. They could even solve the simultaneous equations

$$\begin{aligned} x^8 + x^6y^2 &= (3,200,000)^2 \\ xy &= 1,200 \end{aligned}$$

The Babylonians built pyramid-shaped 'ziggurats'. The first story of a ziggurat might measure $n \times n \times 1$, the second story $(n-1) \times (n-1) \times 1$, and so on — with the top two stories measuring $2 \times 2 \times 1$ and $1 \times 1 \times 1$. The volume of such a ziggurat is

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2$$

and the Babylonians knew that this equals

$$\frac{n(n+1)(2n+1)}{6}$$

a result first proved by Archimedes (287–212 B.C.).

The Bible tells us that there was once an attempt to build a ziggurat 'with its top reaching heaven' (Genesis 11:4). Perhaps the promoters of the Tower of Babel mistakenly believed that the infinite series $1^2 + 2^2 + 3^2 + \cdots$ converges.

The Babylonians knew the formulas for the areas of the triangle, trapezium, and circle. According to a clay tablet found in Susa in 1936, they used the value $3\frac{1}{8}$ for $\pi$.

## Pythagorean Triples

A triple $(x, y, z)$ of positive integers, with $x, \quad y < z$ gives the lengths of the sides of a right angled triangle if and only if $x^2 + y^2 = z^2$. Although such triples are called *Pythagorean triples,* they were studied by the Babylonians, long before Pythagoras (525 B.C.). From a clay tablet called *Plimpton 322,* we know that the Babylonians were interested in a certain kind of Pythagorean triple, which we shall call a *Babylonian triple.* The triple $(x, y, z)$ is a *Babylonian triple* just in case the lengths $x$, $y$, and $z$ can be expressed in the form

$$2uv, \quad u^2 - v^2, \quad \text{and} \quad u^2 + v^2$$

with $u$ and $v$ relatively prime positive integers having no prime factors other than 2, 3, and 5 (the prime divisors of the Babylonian scale 60). The numbers $u$ and $v$ are *generating numbers.* As the Babylonians realised,

$$(2uv)^2 + (u^2 - v^2)^2 = (u^2 + v^2)^2$$

and hence the coordinates of a Babylonian triple are the lengths of the sides of a right-angled triangle, a *Babylonian triangle.*

For example, (56, 90, 106) is a Babylonian triple (with $u = 9$ and $v = 5$), but (28, 45, 53) is not (since we would have $u = 7$ with $u$ having a prime factor other than 2, 3, and 5).

With one exception, the Pythagorean triples listed on Plimpton 322 are Babylonian triples with $v < 60$ and $\left(\frac{u^2+v^2}{2uv}\right)^2 < 2$. They are arranged so that the ratio $\frac{u^2+v^2}{2uv}$ decreases.

The key columns on Plimpton 322 are the second column, which gives the side $u^2 - v^2$ of the Babylonian triangle, and the third column, which gives its hypotenuse $u^2 + v^2$. Translating these two columns into our scale ten numerals, we have the following table.

**Plimpton 322**

| | | |
|---|---|---|
| 119 | 169 | 1 |
| 3367 | 4825 | 2 |
| 4601 | 6649 | 3 |
| 12709 | 18541 | 4 |
| 65 | 97 | 5 |
| 319 | 481 | 6 |
| 2291 | 3541 | 7 |
| 799 | 1249 | 8 |
| 481 | 769 | 9 |
| 4961 | 8161 | 10 |
| 45 | 75 | 11 |
| 1679 | 2929 | 12 |
| 161 | 289 | 13 |
| 1771 | 3229 | 14 |
| 56 | 106 | 15 |

# Square Roots

The Babylonian method of extracting square roots is sometimes incorrectly called 'Heron's method', after Heron of Alexandria (75 A.D.), who included it in his *Metrica*. It is a special case of the iteration method of Isaac Newton (1642–1727). In essence, it goes as follows.

Let $a_1$ be the greatest integer less than $\sqrt{R}$. For $n = 1, 2, 3, \ldots$, calculate $a_{n+1} = \frac{1}{2}(a_n + R/a_n)$. Then $a_1, a_2, a_3, \ldots$ is a sequence of better and better approximations to $\sqrt{R}$.

To find, say, the square root of 2, the Babylonians proceeded as follows.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{2}(1 + 2/1) = \frac{3}{2} \end{aligned}$$

$$a_3 = \frac{\frac{3}{2}+\frac{2}{\frac{3}{2}}}{2} = \frac{17}{12}$$

$$a_4 = \frac{\frac{17}{12}+\frac{2}{\frac{17}{12}}}{2} = \frac{577}{408}$$

$$a_5 = \frac{\frac{577}{408}+\frac{2}{\frac{577}{408}}}{2} = \frac{665,857}{470,832}$$

and so on — to any desired degree of accuracy.

## Exercises 2

1. Solve the simultaneous equations

$$\begin{aligned} x^8 + x^6y^2 &= (3,200,000)^2 \\ xy &= 1,200 \end{aligned}$$

2. Using the theorem of Pythagoras, prove that if a triangle has sides with lengths $x$, $y$, and $z$, with $x^2 + y^2 = z^2$ then that triangle has a right angle opposite the side of length $z$.

3. By finding the generating numbers $u$ and $v$, show that the right-angled triangle with hypotenuse 169 and side 119 is Babylonian.

4. Which row on Plimpton 322 gives, not a Babylonian triangle, but merely one similar to a Babylonian triangle?

5. List the 15 Babylonian triples with $v < 60$ and $\frac{53}{45} \leq \frac{u^2+v^2}{2uv} < \sqrt{2}$.

6. Find all the Babylonian triangles with hypotenuse $\leq 100$.

7. Find all Pythagorean triangles with perimeter 1716. Which of them are Babylonian?

8. Use the Babylonian method to find the square root of 3. Use 5 terms of the sequence.

9. Show that fractions $f/g$ in the sequence of approximations to $\sqrt{2}$ give integer solutions to the equation $x^2 - 2y^2 = 1$.

## *Essay Question*

1. Describe the Babylonian numeral system. Was it purely scale 60 or not? Why?

# 3

# The Dawn of Deductive Mathematics

## Thales

The ancient Greek world was not confined to what we now call 'Greece', but extended to Ionia (west Turkey) in the east and to southern Italy in the west. The first Greek mathematician and philosopher was Thales of Miletus, a contemporary of the prophet Ezekiel (600 B.C.). (Miletus was on the southwestern coast of Turkey.) According to Proclus, Thales visited Egypt and learned geometry there. Thales predicted the solar eclipse that occurred over Greece and Mesopotamia on May 28, 585 BC.

Plato repeats a story about Thales being an absent-minded professor who was so preoccupied with celestial matters that he failed to observe what was in front of his feet and once fell into a well (*Theaetetus* 174a). According to other anecdotes, however, Thales had a practical mind. He constructed an almanac, figured out how to calculate the distance of ships from shore, and he once cornered the market in olive oil.

Thales is associated with a number of theorems in geometry:

(1) a circle is bisected by a diameter;
(2) the base angles of an isosceles triangle are equal;
(3) vertically opposite angles are equal;
(4) two triangles are congruent if their angles and one side are equal;
(5) an angle in a semicircle is right.

Theorem (5) is *Thales's theorem.* What it means is that if $AC$ is a diameter of a circle and $B$ is a point on the circumference of the circle (other than $A$ or $C$), then $\angle ABC$ is a right angle (has 90 degrees).

All these theorems were known by the Egyptians and Mesopotamians. The reason they are associated with Thales is that he was the first person to offer *proofs* for them. This was an essential difference between pre-Greek and Greek mathematics: the Greeks established the logical connections among their results, deducing the theorems from a small set of starting assumptions or *axioms.*

As a philosopher, Thales is famous for his statement that everything is made of water. This statement committed Thales to the following views:

(a) there is more than one thing;

(b) there is only one *kind* of thing (namely, water);

(c) the physical universe should not be understood in terms of unconnected fragments (e.g., quarks), but in terms of a continuous substance (e.g., space).

The fact that contemporary physicists disagree with Thales about (b) and (c) is less important than the fact that it was Thales who first raised these issues.

## Anaximander and the Infinite

Anaximander (610–540 B.C.) was a follower and compatriot of Thales. According to Anaximander, there are infinitely many worlds, all made out of an infinitely extended indeterminate substance that has always existed and will always exist. Earth, air, and fire are not forms of water, but forms of this 'Infinite'.

Throughout the history of philosophy, there has been a debate as to whether there is anything 'actually' infinite in some respect. Everyone agrees that the set of natural numbers is at least 'potentially' infinite, in the sense that, no matter how far you count, you might count further. However, thinkers divide on the question of whether the set of natural numbers exists as a completed totality, as an 'actually' infinite object.

Anaximander opened this discussion by coming out in favour of the infinite: the universe contains infinitely many worlds; the duration of the universe is infinite; the uniform material from which everything (including water) is made is infinite in bulk. The first opponent of infinites was Aristotle (384–322 B.C.). In Book III of the *Physics,* he cites Anaximander and tries to refute his position.

Someone who believed in the existence of natural numbers might argue on Anaximander's behalf, saying that if the set of natural numbers is only potentially infinite, then it is actually finite, and thus it contains a largest natural number, which is absurd. Hence there is an (actually) infinite number of natural numbers, and thus an infinite number of things in the universe.

To this Aristotle might reply that numbers are not things that exist separate from the human mind, and hence a set of numbers need not have an objective status as either actually infinite or actually finite: it could be, somehow, in between.

According to standard logic, a set is either infinite or finite — there is no in between. However, there are nonstandard logics, such as intuitionist logic, in which one denies this.

## The Role of Individuals

Many historians consider it important to tell their readers who discovered what first. In a history of mathematics — as opposed to a history of mathematicians — this is not necessary. For example, our ignorance of the names of individual Egyptian and Mesopotamian mathematicians does not prevent us from relating the history of their mathematics. As a member of an individualistic culture, and as a believer in a personal freedom of the will, I feel a need to praise individuals. However, it should be noted that this practice often obscures the key role that the mathematician's spouse, education, and culture inevitably play in his or her discoveries. Without the help of parents or teachers, Thales would have done nothing. Nor should we forget God. A theist might claim that if God did not create us and protect us, we would never discover anything.

## Exercises 3

1. Let $ABC$ be a triangle, and let $d$ be a straight line through $A$ and parallel to $BC$. Assuming that the 'alternate angles are equal', prove that the sum of the angles of $ABC$ equals two right angles.

2. Prove the theorem of Thales, using the previous exercise, and the fact that the base angles of an isosceles triangle are equal.

3. Prove the converse of Thales's theorem: If $A$, $B$, and $C$ are points on a circumference of a circle, and $\angle ABC$ is right, then $AC$ is a diameter.

4. How might you use the theory of similar triangles to calculate the distance of a ship from shore?

*Essay Questions*

1. How would the laws of arithmetic change if there was a largest natural number?

2. Comment on the following. A person is not just some material with uniform properties (e.g., mass-energy, spirit). Since there are persons, Thales was wrong to say 'all is water'.

3. In discussing the origin of a piece of mathematical knowledge, is it more important to mention the individual discoverer — or the educational, technological, or spiritual background?

# 4
# The Pythagoreans

## Pythagoras and Theano

Pythagoras (570–500 B.C.) was born in Samos, a Greek island off the coast of what is now Turkey. According to Iamblichus, Porphyry, and Diogenes Laertius, Pythagoras studied under the Babylonians, and he may have met the prophet Daniel in Babylon. From the clay tablet Plimpton 322, we know that the Babylonians had a well-worked-out theory of 'Pythagorean triangles', and Pythagoras would have learnt this from them. Pythagoras may have discovered the first proof of the 'theorem of Pythagoras', but he certainly did not discover the theorem itself.

According to Iamblichus, Porphyry, and Diogenes Laertius, Pythagoras also studied under the 'Magi', or Zoroastrians. Indeed, it is not impossible that Pythagoras talked with Zoroaster himself. Nor is it impossible that Pythagoras studied in India. His belief in reincarnation certainly had an Indian origin. Perhaps Pythagoras met Buddha, another of his contemporaries.

About 525 B.C., Pythagoras moved to Croton, a town in southern Italy, and founded the brotherhood of the Pythagoreans. He married a woman Pythagorean called Theano. Theano may have been the first woman mathematician.

## Number Mysticism

Whereas Thales had claimed that 'all is water', Pythagoras taught that 'all is number'. For Pythagoras, this implied that everything could be understood in terms of whole numbers and their ratios. In particular, every line segment was a whole number or ratio of whole numbers. Although the discovery of the irrationality of the diagonal of the square of side 1 was made by followers of Pythagoras, Pythagoras himself was not aware of this.

Pythagoras gave a special place to the number 10. He called it the 'divine number'. He was attracted by it probably for the following reasons. It is the scale in which the ancient Greeks counted. As the sum of the first four positive integers, it represents the three dimensions — with 1 for points, 2 for lines, 3 for planes, and 4 for solids. Finally, there are ten vertices in the five-pointed Pythagorean star.

## Pythagorean Mathematics

The Pythagoreans ascribed all their mathematical discoveries to Pythagoras, but there is not, in fact, a single theorem we can safely credit to the master. Pythagorean accomplishments include the following.

**(1) A Proof of the Theorem of Pythagoras**

The Pythagoreans were responsible for the proof of this theorem found in Euclid. They also found a proof of the converse of this theorem.

**(2) Means**

The Pythagoreans examined the arithmetic mean $(a+b)/2$, the geometric mean $\sqrt{ab}$, the harmonic mean $2ab/(a+b)$, and the relationships among them.

**(3) Perfect and Amicable Numbers**

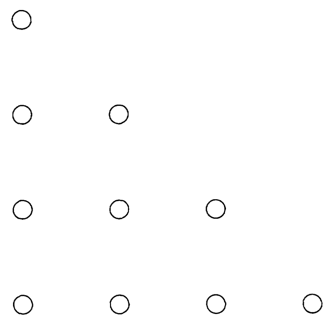
A *perfect number* is a positive integer, such as 6, which equals the sum of its proper divisors: $6 = 1+2+3$. The Pythagoreans found a formula giving even perfect numbers. See Chapter 5. An *amicable pair* is two positive integers, each of which is the sum of the proper divisors of the other. Iamblichus (300 A.D.) credits Pythagoras with a knowledge of the amicable pair 220 and 284.

**(4) Regular Solids**

The Pythagoreans discovered the dodecahedron, and proved that there are just 5 regular polyhedra. This accomplishment was unsurpassed until J. Kepler (1571–1630) discovered the lesser and greater stellated dodecahedra. See Chapter 6.

**(5) The Irrationality of $\sqrt{2}$**

The Pythagoreans discovered that $\sqrt{2}$ is not a ratio of whole numbers. They used integer solutions of $x^2 - 2y^2 = 1$ to find good approximations to it. See Chapter 7.



**Ten as a Triangle**

**(6) Figurative Numbers**
If $m$ is a positive integer, and $t$ is a nonnegative integer, an $(m+2)-$*gonal number* is a natural number of the form

$$m\frac{t^2-t}{2}+t$$

The first few 3-gonal, or triangular, numbers are

$$0, \ 1, \ 3, \ 6, \ 10, \ \ldots$$

The first few 4-gonal, or square, numbers are

$$0, \ 1, \ 4, \ 9, \ 16, \ \ldots$$

The first few 5-gonal, or pentagonal, numbers are

$$0, \ 1, \ 5, \ 12, \ 22, \ \ldots$$

These numbers are called 'figurative' because they can be represented by figures made up of pebbles. For example, the triangular number 10 can be represented in the form of a triangle as in the above Figure.

Looking at the sequence of squares, represented by pebble diagrams, the Pythagoreans noticed that $n^2+(2n+1)=(n+1)^2$, and

$$1+3+5+\cdots+(2n-1)=n^2$$

Fitting two equal triangular numbers together to form a rectangle, the Pythagoreans noticed that twice the $n$th positive triangular number is base

$\times$ height $= n(n+1)$. Since the $n$th positive triangular number is $1+2+\cdots+n$, it follows that

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

The study of figurative numbers has remained a central part of number theory. One of the highlights of the career of C. F. Gauss (1777–1855) was his proof that every positive integer is a sum of 3 triangular numbers. As another example, a 1989 *Journal of Number Theory* paper by N. Tzanakis and B. de Weger showed that there are exactly 6 triangular numbers that are products of three consecutive integers (the largest of these triangular numbers being 258,474,216).

## Exercises 4

1. How might a Pythagorean have derived the fact that the angle at the tip of the points of his star is $36^\circ$ ?

2. What is the ratio of the side to the base in a triangle that is one of the points of the Pythagorean star?

3. What is the ratio of a diagonal of a regular pentagon to its side? (Do not use analytic geometry or trigonometry in your answer; they had not been discovered yet.)

4. Arrange 12 pebbles in such a way as to show that 12 is really pentagonal.

5. Show that 28 is perfect.

6. Show that 220 and 284 are amicable.

7. The number 12,285 is one member of an amicable pair. Who is its friend?

8. Prove that every hexagonal number is triangular.

9. The largest triangular number that is a product of three consecutive integers is 258,474,216. What are the three consecutive integers? How long is the side of the triangle?

10. Find the first three square triangular numbers.

11. Find a triangular number greater than 1 that equals the sum of the cubes of its scale 10 digits.

12. What is the exact area of a Pythagorean star with side 1?

## *Essay Question*

1. How can a glass of water be a number? How might Pythagoras have answered this question?

# 5

# The Pythagoreans and Perfection

## The Proof in Euclid

The Pythagoreans were interested in perfect numbers, that is, numbers, such as 6 and 28, that equal the sum of their proper divisors. If $s(n)$ denotes the sum of all the divisors of a positive integer $n$, including $n$ itself, then $n$ is perfect if and only if $s(n) = 2n$.

The culmination of Book IX of Euclid's *Elements* (300 B.C.), is a proof that any positive integer of the form

$$n = 2^{m-1}(2^m - 1)$$

is perfect, provided $2^m - 1$ is prime. The proof is probably due to the Pythagorean Archytas (428–347 B.C.). It goes as follows.

If $p = 2^m - 1$ is prime, then the divisors of $n = 2^{m-1}p$ are

$$1,\ 2,\ 2^2,\ \ldots,\ 2^{m-1},\ p,\ 2p,\ \ldots,\ 2^{m-1}p$$

Thanks to unique factorisation, we know this list is complete. The sum of these divisors is

$$(1 + 2 + 2^2 + \cdots + 2^{m-1})(1 + p) = (2^m - 1)(1 + p) = p2^m = 2n$$

It should be noted that, although Archytas attempted to give a fully rigorous proof of unique factorisation for numbers of the form $2^{m-1}(2^m - 1)$, he failed to do so. The first fully rigorous demonstration of unique factorisation was given only in 1801, by Carl Friedrich Gauss (1777–1855).

## Mersenne Primes

An integer of the form $2^m - 1$ is prime only if $m$ is prime. For if $m = ab$, with $a$, $b > 1$, we have the following factorisation:

$$2^{ab} - 1 = (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1)$$

The converse is not true. Although 11 is prime, $2^{11} - 1$ is the product of 23 and 89.

Primes of the form $2^m - 1$ give rise, as we have seen, to perfect numbers. Such primes are called *Mersenne primes,* after Father Marin Mersenne (1588–1648). In the preface of his *Cogitata Physico-Mathematica* (1644), Mersenne correctly stated that the first 8 perfect numbers are given by

$$m = 2,\quad 3,\quad 5,\quad 7,\quad 13,\quad 17,\quad 19,\quad 31$$

He also claimed that $2^{67} - 1$ is prime. Here he erred. In 1903, Frank Nelson Cole gave a lecture that consisted of two calculations. First Cole calculated $2^{67} - 1$. Then he worked out the product

$$193{,}707{,}721 \times 761{,}838{,}257{,}287$$

He did not say a single word as he wrote down the numbers. The two calculations agreed, and Cole received a standing ovation. He had factored $2^{67} - 1$, proving Mersenne wrong.

## Lucas's Test

A French artillery officer and schoolteacher, Edouard Lucas (1842–1891), found an efficient way of testing whether $2^m - 1$ is prime. His ideas were refined by Derrick H. Lehmer (1905– ), leading to the following algorithm. Let

$$\begin{aligned} u_1 &= 4 \\ u_{n+1} &= u_n^2 - 2 \end{aligned}$$

Thus $u_2 = 14$, and $u_3 = 194$. If $m > 2$ then $2^m - 1$ is prime just in case $2^m - 1$ is a factor of $u_{m-1}$. For example, since $2^5 - 1$ is a factor of $u_4 = 37{,}634$, it follows that $2^5 - 1$ is prime, and hence

$$2^4(2^5 - 1) = 496$$

is perfect.

Thanks to Lucas's test — and the computer — we know that $2^m - 1$ is prime when $m$ has the 32 values given in the table. The ancient Greeks knew just the first 4 Mersenne primes. Mersenne himself knew the first 8.

Before 1950, we knew just the first 12 Mersenne primes. Then, with the help of the computer, another 20 came to light in the last half of the twentieth century.

**The 32 Exponents Known to Make $2^m - 1$ Prime**

| | | |
|---|---|---|
| 2 | 107 | 9941 |
| 3 | 127 | 11213 |
| 5 | 521 | 19937 |
| 7 | 607 | 21701 |
| 13 | 1279 | 23209 |
| 17 | 2203 | 44497 |
| 19 | 2281 | 86243 |
| 31 | 3217 | 110503 |
| 61 | 4253 | 132049 |
| 89 | 4423 | 216091 |
| | 9689 | 756839 |

We do not know if there are infinitely many Mersenne primes. Nor do we know if there are any odd perfect numbers — although it has been shown that there is none $< 10^{300}$.

# Euler's Proof

Every even perfect number has the form given by Euclid. This was proved by Leonhard Euler (1707–1783), as follows.

Suppose $n$ is perfect. Let $n = 2^{m-1}q$ with $q$ odd and $m,\quad q > 1$. Each divisor of $n$ has the form $2^r d$ where $0 \le r \le m-1$, and $d$ is a divisor of $q$. Thus

$$s(n) = (1 + 2 + \cdots + 2^{m-1})s(q) = (2^m - 1)s(q)$$

Since $n$ is perfect,

$$2^m q = s(n) = (2^m - 1)s(q)$$

and hence $(2^m - 1)(s(q) - q) = q$.

Suppose $s(q) - q > 1$. Then $q$ has distinct factors 1, $s(q) - q$, and $q$. (If $s(q) - q = q$ then $(2^m - 1)q = q$, which is impossible.) Thus $s(q) \ge 1 + s(q) - q + q = s(q) + 1$. Contradiction. Thus $s(q) - q = 1$.

Hence $s(q) = q + 1$, so that $q$ is prime. Moreover, the fact that

$$(2^m - 1)(s(q) - q) = q$$

implies that $2^m - 1 = q$.

## The Fascination of Perfect Numbers

Perfect numbers have always appealed to number mystics. In *De Institutione Arithmetica,* Boethius (475–524) defines a *superfluous* number as one with $s(n) > 2n$, and a *diminished* number as one with $s(n) < n$. He writes:

> Between these two kinds of number, as if between two elements unequal and intemperate, is put a number which holds the middle place between the extremes like one who seeks virtue.

In the *City of God,* Bishop Augustine (354–430) writes:

> Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because this number is perfect, and it would have been perfect even if the work of the six days did not exist.

## Exercises 5

1. Show that 8128 is perfect.

2. Is 672 superfluous, diminished, or perfect? What is special about it?

3. Suppose that perfect number $A$ has divisors $f_1, \ldots, f_t$. Then
$$\frac{2}{n} = \frac{1}{f_1 n} + \frac{1}{f_2 n} + \cdots + \frac{1}{f_t n}$$

4. Prove that every even perfect number ends in 6 or 8.

5. Prove that every even perfect number (except 6) has the form
$$1^3 + 3^3 + 5^3 + \cdots + (2^{n+1} - 1)^3$$

6. Show that if $m$ and $n$ are relatively prime positive integers, then $s(mn) = s(m)s(n)$.

7. Show that if $p$ is prime,

$$s(p^m) = \frac{p^{m+1} - 1}{p - 1}$$

8. Show that $s(n)$ is odd iff $n$ is a square or twice a square.

## *Essay Question*

1. Comment on the quotations from Boethius and Augustine.

# 6
# The Pythagoreans and Polyhedra

## Regular Polyhedra

A *polyhedron* is a solid whose surface consists of polygon faces. A polyhedron is *regular* or *Platonic* if its faces are congruent regular polygons and if its polyhedral angles are all congruent. Five regular polyhedra are the following.

The **cube** is bound by 6 squares, with 3 squares at a vertex.

The **tetrahedron** is bound by 4 equilateral triangles, with 3 triangles at a vertex.

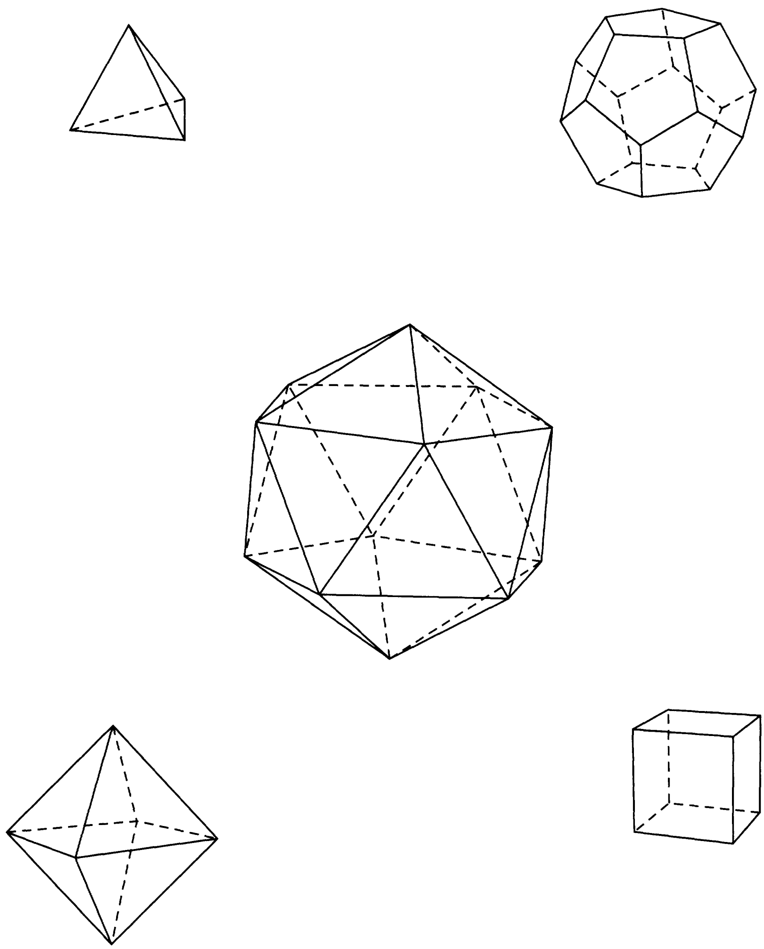
The **octahedron** is bound by 8 equilateral triangles, with 4 triangles at a vertex.

The **icosahedron** is bound by 20 equilateral triangles, with 5 triangles at a vertex.

The **dodecahedron** is bound by 12 regular pentagons, with 3 pentagons at a vertex.

Pythagoras himself knew of the first four of these solids, but it was Hippasus (470 B.C.) who discovered the dodecahedron. On one account, Hippasus was expelled from the Pythagorean order for failing to attribute the discovery to the Master.

A proof that there are only these 5 regular polyhedra is found in Euclid's *Elements* (300 B.C.). This proof is based on the fact that if $q$ regular $p$-gon faces meet at a vertex, then the sum of the $q$ angles in the $q$ faces is less than $360°$. This is proved rigorously in Proposition 21 of Book IX of the *Elements,* but it can be seen intuitively by imagining someone cutting the



**The Five Regular Solids**

$q$ edges and flattening the polyhedral angle. For example, if one cuts the three edges at the vertex of a cube and flattens the solid angle, the three right angles still have a common vertex, and one can see that their sum is less than a complete revolution.

Since one can dissect a polygon with $p$ sides into $p-2$ triangles, the sum of the angles of a polygon with $p$ sides is $(p-2) \times 180°$. Each angle of a regular $p$-gon is thus $\frac{(p-2)\times 180°}{p}$. For a regular polyhedron whose faces are regular $p$-gons, with $q$ $p$-gons meeting at a vertex, we thus have

$$q\frac{(p-2) \times 180°}{p} < 360°$$

or $\frac{1}{2} < \frac{1}{p} + \frac{1}{q}$. Of course, $p$ and $q$ are each at least 3, and, moreover, they cannot both be greater than 3 (lest $\frac{1}{2} > \frac{1}{p} + \frac{1}{q}$). So there are only 5 possibilities for $p$ and $q$, one for each of the 5 regular solids already given.

If $p = 3$ and $q = 3$, we get the tetrahedron.
If $p = 3$ and $q = 4$, we get the octahedron.
If $p = 4$ and $q = 3$, we get the cube.
If $p = 3$ and $q = 5$, we get the icosahedron.
If $p = 5$ and $q = 3$, we get the dodecahedron.

## The *Timaeus*

The material in Plato's *Timaeus* is often attributed to the Pythagoreans. Certainly its theorising is in the spirit of the slogan 'all is number'. Plato explains the composition of the physical universe in terms of the five regular polyhedra. The cube is associated with earth, the tetrahedron with fire, the octahedron with air, the icosahedron with water, and the dodecahedron with the whole cosmos. Plato explains the boiling of water by means of a 'chemical equation' which we might write as follows:

$$F_4 + W_{20} \to 2A_8 + 2F_4$$

That is, fire, with 4 faces, combines with water, with 20 faces, to produce 2 air atoms (each with 8 faces) and 2 fire atoms (each with 4 faces). Note that the numbers of equilateral triangles 'balance':

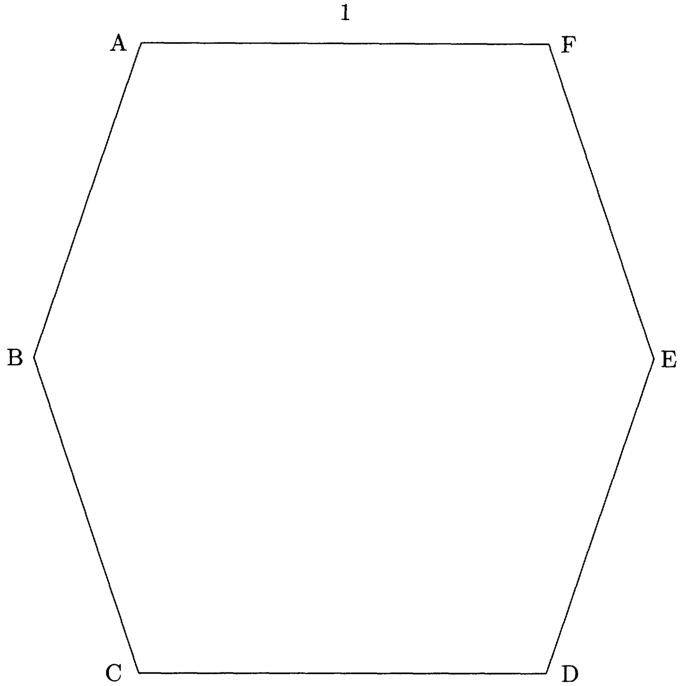
$$4 + 20 = 2 \times 8 + 2 \times 4$$

A modern chemist would not accept Plato's explanations, but he would, like Plato, accept the Pythagorean idea that the physical universe can be understood in terms of whole numbers. For the modern chemist, these whole numbers are the atomic numbers of the elements.

## The Apex of the *Elements*

The ancient Greeks were fascinated by the 5 Platonic solids. Without the help of trigonometry or calculus, they managed to prove all the basic properties of these solids. The final book in Euclid's *Elements* is devoted to them. For each of the five regular polyhedra, Euclid calculates the ratio of its side to the radius of the sphere that circumscribes it.

For example, if one cuts an icosahedron in half, cutting along an edge $AF$ (of length, say, 1), the resulting cross-section is a hexagon $ABCDEF$ with $CD$ the edge of the icosahedron opposite $AF$ (see the figure below). $AC$ and $DF$ are diagonals in regular pentagons formed by the sides of the icosahedron. (You may have to construct an icosahedron out of, say,




**Cross Section of an Icosahedron**

cardboard in order to see this.) Thus if $AF = CD = 1$, $AC = DF = \phi$, where $\phi$ is the 'golden ratio' $\frac{1+\sqrt{5}}{2}$. (See Exercises 4, number 3.)

The diameter of the circumscribing sphere is $CF$, which is the hypotenuse of the right triangle with sides $CD$ and $DF$. Thus $CF^2 = 1^2 + \phi^2$, and hence the radius of the circumscribing sphere is $\frac{\sqrt{1+\phi^2}}{2}$.

Note that $\phi$ is also the ratio of the side to the base in a triangle that is one of the points of the 5-pointed Pythagorean star. For Leonardo da Vinci (1452–1519), this 'golden ratio' was a mark of beauty. The icosahedron or dodecahedron is beautiful partly because it expresses this ratio.

## Exercises 6

1. Make an icosahedron (by, say, taping together 20 identical equilateral triangles cut out of cardboard).

2. Show that the radius of a sphere passing through the vertices of a cube with side 1 is $\frac{\sqrt{3}}{2}$.

3. Show that the volume of an octahedron with side 1 is $\frac{\sqrt{2}}{3}$.

4. What is the surface area of a dodecahedron of side 1?

5. Let $V$ be the number of vertices of a polyhedron, $E$ the number of its edges, and $F$ the number of its faces. For each of the 5 regular solids, calculate $V+F-E$. (Euler noted the rather interesting result.)

6. Show that the radius of a sphere passing through the vertices of a tetrahedron with side 1 is $\frac{\sqrt{6}}{4}$.

### *Challenges for Experts*

1. Show that the radius of a sphere passing through the vertices of a dodecahedron of side 1 is $\phi\sqrt{3}/2$.

2. Show that the volume of a dodecahedron with side 1 is $\frac{15+7\sqrt{5}}{4}$.

3. Show that the volume of an icosahedron of side 1 is $\frac{15+5\sqrt{5}}{12}$.

4. Show that if the same sphere passes through the vertices of an icosahedron and through the vertices of a dodecahedron then

   dodecahedron area / icosahedron area

   = dodecahedron volume / icosahedron volume.

5. If you join the centres of the 12 regular pentagon faces of a dodecahedron you get an icosahedron. If the side of the dodecahedron is 1, prove that the side of the icosahedron is $\frac{5+3\sqrt{5}}{10}$.

6. Show that an icosahedron can be placed inside an octahedron so that the 12 vertices of the icosahedron divide the 12 edges of the octahedron in the golden ratio.

7. Let $R$ be the radius of a sphere circumscribed about a dodecahedron and $r$ the radius of a sphere tangent to its faces. Then

$$\frac{R}{r} = \sqrt{15 - 6\sqrt{5}}$$

Show that the same ratio obtains for the icosahedron.

## *Essay Questions*

1. Read and give a summary of the *Timaeus.*

2. A piece of mathematics is elegant in so far as it is well-written, brief, illuminating, simple, unifying, and exciting. Comment.

# 7

# The Pythagoreans and Irrationality

## Hippasus and the Leak

Two lengths $a$ and $b$ are *commensurable* if there are positive integers $p$ and $q$ such that $a/b = p/q$. When the Pythagoreans claimed that all things are numbers, they meant to imply that all pairs of lengths are commensurable. For the Pythagoreans, 'number' meant 'rational number'.

Unfortunately, they soon discovered that the diagonal of a unit square is not commensurable with its side. A proof of this is found in Aristotle's *Prior Analytics* 41a23-30. Let $ABCD$ be a square whose sides have length 1. By the theorem of Pythagoras, the diagonal $AC$ measures $\sqrt{2}$. Suppose

$$\sqrt{2} = AC/AB = p/q$$

where $p$ and $q$ are positive integers. We may assume that $p$ and $q$ are relatively prime (have no common factor). In particular, we may assume that they are not both even.

Now $p^2 = 2q^2$, so that $p^2$ is even. As the Pythagoreans well knew, the square of an odd number is odd, while the square of an even number is even. Thus, from the fact that $p^2$ is even, it follows that $p$ is even. Suppose $p = 2r$. Then $(2r)^2 = 2q^2$ and hence $q^2 = 2r^2$. But this means that $q$ is even as well. Contradiction. The assumption that $AC$ and $AB$ are commensurable leads to an absurdity.

The Pythagoreans tried, at first, to keep this discovery a secret, as it undermined their philosophy. Some say it was Hippasus (470 B.C.) who leaked the secret and that he drowned as a punishment for having done so.

The Greeks did not know how to handle $\sqrt{2}$ in an arithmetic or algebraic fashion. They did, however, know that it was a length (of a diagonal), and they turned to geometry for an understanding of it. The problem of incommensurables was one reason why they preferred to do algebra in a geometric manner. For example, the ancient Greeks thought of the distributive law $a(b+c) = ab + ac$ as an addition rule for areas of rectangles with the same width $a$.

## Diophantine Equations and Approximations to Irrationals

The Pythagoreans found a way of approximating $\sqrt{2}$, as closely as could be desired, by rational numbers. Their method involved the use of 'Euclid's algorithm', a procedure found in Propositon 2 of Book VII of the *Elements,* and possibly due to the Pythagorean Archytas. In essence, it works as follows.

Recall that if $x$ is any real, then $[x]$ is the greatest integer $\leq x$. Given some real number $X_1$, we form the following three sequences. First we have

$$\begin{aligned} X_2 &= \frac{1}{X_1 - [X_1]} \\ X_3 &= \frac{1}{X_2 - [X_2]} \\ &\vdots \end{aligned}$$

If $X_1$ is rational then so are all the other $X$s, and this sequence will end when we hit a 0 denominator. If $X_1$ is irrational then so are all the other $X$s, and this sequence will never end.

Second, we form the sequence

$$\begin{aligned} f_1 &= [X_1] \\ f_2 &= [X_2]f_1 + 1 \\ f_3 &= [X_3]f_2 + f_1 \\ f_4 &= [X_4]f_3 + f_2 \\ &\vdots \end{aligned}$$

Third, we form the sequence

$$\begin{aligned} g_1 &= 1 \\ g_2 &= [X_2] \\ g_3 &= [X_3]g_2 + g_1 \\ g_4 &= [X_4]g_3 + g_2 \end{aligned}$$

and so on.

If $X_1 = a/b$, a rational, then, for some $n$, we have $X_n - [X_n] = 0$, and

$$ag_{n-2} - bf_{n-2} = \pm\gcd(a, b)$$

Hence we can use Euclid's algorithm to solve

$$ax - by = \pm\gcd(a, b)$$

Moreover, if $X_1 = \sqrt{R}$, an irrational, then

$$|f_n/g_n - \sqrt{R}| < 1/g_n^2$$

so that $f_n/g_n$ gives us an approximation to $\sqrt{R}$. Finally, $p$ and $q$ are integers such that $p^2 - Rq^2 = \pm 1$ just in case, for some $n$ such that $[X_n] = 2[\sqrt{R}]$, $p = f_{n-1}$ and $q = g_{n-1}$.

For example, suppose a Pythagorean wanted to find an integer solution to

$$17x - 19y = 320$$

He would reason in a way we would describe as follows:

$$\begin{aligned}
X_1 &= 17/19 \\
X_2 &= \frac{1}{17/19 - [17/19]} = 19/17 \\
X_3 &= \frac{1}{19/17 - [19/17]} = 17/2 \\
X_4 &= \frac{1}{17/2 - [17/2]} = 2 \\
X_5 &= \frac{1}{2 - [2]} = \text{undefined}
\end{aligned}$$

Also

$$\begin{aligned}
f_1 &= 0 \\
f_2 &= 1 \\
f_3 &= 8
\end{aligned}$$

and, finally,

$$\begin{aligned}
g_1 &= 1 \\
g_2 &= 1 \\
g_3 &= 9
\end{aligned}$$

Hence $17 \times 9 - 19 \times 8 = \pm 1$ so that

$$17 \times (9 \times 320) - 19 \times (8 \times 320) = \pm 320$$

giving us an integer solution to the original equation.

It should be noted that the Pythagoreans did not have the concept of a negative number (since they thought of numbers as collections of pebbles or lengths). It was Brahmagupta (628 A.D.) who first showed how to obtain all the integer solutions, negative as well as positive, to equations such as $17x - 19y = 320$.

To get approximations to $\sqrt{2}$, the Pythagoreans would work as follows.

$$\begin{aligned} X_1 &= \sqrt{2} \\ X_2 &= \frac{1}{\sqrt{2} - [\sqrt{2}]} = \sqrt{2} + 1 \\ X_3 &= \frac{1}{\sqrt{2} + 1 - [\sqrt{2} + 1]} = \sqrt{2} + 1 \\ &\vdots \end{aligned}$$

so that $[X_2] = [X_3] = \ldots = 2$. Hence

$$\begin{aligned} f_1 &= 1 \\ f_2 &= 3 \\ f_3 &= 7 \\ f_4 &= 17 \end{aligned}$$

and so on. Also

$$\begin{aligned} g_1 &= 1 \\ g_2 &= 2 \\ g_3 &= 5 \\ g_4 &= 12 \end{aligned}$$

and so on. The sequence

$$1/1, \ 3/2, \ 7/5, \ 17/12, \ \ldots$$

gives better and better approximations to $\sqrt{2}$. It also provides all the positive integer solutions of $x^2 - 2y^2 = \pm 1$, namely,

$$(1,1), \ (3,2), \ (7,5), \ \ldots$$

The way in which Euclid's algorithm relates to equations such as $x^2 - Ry^2 = 1$ was not fully understood until 1768, when J. L. Lagrange published a definitive paper on the subject. The Pythagoreans had insights that took over 2000 years to comprehend.

As a final example, let us use Euclid's algorithm to find a nontrivial solution to $x^2 - 29y^2 = \pm 1$.

$$
\begin{aligned}
X_1 &= \sqrt{29} \\
[X_1] &= 5 \\
X_2 &= \frac{1}{\sqrt{29}-5}\frac{\sqrt{29}+5}{\sqrt{29}+5} = \frac{\sqrt{29}+5}{4} \\
[X_2] &= 2 \\
X_3 &= \frac{\sqrt{29}+3}{5} \\
[X_3] &= 1 \\
X_4 &= \frac{\sqrt{29}+2}{5} \\
[X_4] &= 1 \\
X_5 &= \frac{\sqrt{29}+3}{4} \\
[X_5] &= 2 \\
X_6 &= \frac{\sqrt{29}+5}{1} \\
[X_6] &= 10
\end{aligned}
$$

Since $[X_6]$ is twice $[X_1]$, we stop here and calculate $f_{6-1}$ and $g_{6-1}$.

$$
\begin{aligned}
f_1 &= 5 \\
f_2 &= [X_2]f_1 + 1 = 11 \\
f_3 &= [X_3]f_2 + f_1 = 16 \\
f_4 &= [X_4]f_3 + f_2 = 27 \\
f_5 &= [X_5]f_4 + f_3 = 70 \\
g_1 &= 1 \\
g_2 &= [X_2] = 2 \\
g_3 &= [X_3]g_2 + g_1 = 3 \\
g_4 &= [X_4]g_3 + g_2 = 5 \\
g_5 &= [X_5]g_4 + g_3 = 13
\end{aligned}
$$

Hence one solution to $x^2 - 29y^2 = \pm 1$ is $x = 70$ and $y = 13$.

# Exercises 7

1. Prove that $\sqrt{3}$ is irrational.

2. Prove that $\sqrt[3]{2}$ is irrational.

3. Let $a$, $b$, $c$, and $d$ be integers. Show that if $a + b\sqrt{2} = c + d\sqrt{2}$ then $a = c$.

4. Use Euclid's algorithm to find an integer solution of $91x + 221y = 1053$.

5. Express $67/120$ as a sum of distinct unit fractions by solving
$$67x - 120y = 1$$
to get $67/120 = 1/120x + y/x$, and then solving
$$yx' - xy' = 1$$
to get $y/x = 1/xx' + y'/x'$ and so on.

6. Show that $ax+by = c$ has no solution in integers unless $c$ is a multiple of $\gcd(a, b)$.

7. Use Euclid's algorithm to find an approximation of $\sqrt{3}$ that is within $10^{-10}$ of the true value.

8. By factoring, find all pairs of integers $x$ and $y$ such that $x^2 - 4y^2 = 1$.

9. Use Euclid's algorithm to find a nontrivial integer solution of $x^2 - 13y^2 = 1$.

10. The Sultana used to divide her maids into two companies, one that would follow her five abreast, and one that would follow her seven abreast — both companies in rectangular formation. These companies, moreover, would consist of different numbers of maids on each of nine consecutive days. What is the smallest number of maidens the Sultana could have had?

11. Kind-hearted Doctor Diana lives on the side of the street with the even-numbered buildings. The sum of the numbers of the buildings to her left equals the sum of the number of the buildings to her right. Show that if her address is number $D$ and there are $B$ buildings on her side of the street then $(2B+1)^2 - 2D^2 = 1$. Find out where she lives if there are fewer than 40 houses on her side of the street.

## *Essay Question*

1. What did Pythagoras mean by 'number'? What did he mean by saying 'all is number'? Why does the irrationality of $\sqrt{2}$ undermine this saying?

# 8

# The Need for the Infinite

## Parmenides

In opposition to Anaximander, Parmenides of Elea, Italy (480 B.C.) was a monist. That is, he held that the universe consists of only one object. The number of things that exists is just one.

The unique thing, according to Parmenides, does not have infinite duration, but exists timelessly, and changelessly: 'nor was it, nor will it be, since now it is, all together, one'. Nor does the one existing object have infinite spatial extension: 'it is completed on all sides, like the bulk of a well-rounded ball'. (The quotations are from J. Barnes, *Early Greek Philosophy,* pages 134-5.)

Parmenides taught that nothing moves, since motion implies the existence of more than one thing, namely, a finishing place and a starting place. Although it may *look* as if something is moving, this is just an illusion.

## Zeno

Zeno (450 B.C.) was a disciple of Parmenides. He produced four arguments for the conclusion that there is no motion — this in support of the claim of his master.

## Zeno's First Argument

Motion is impossible, said Zeno, because a moving object must first go half the total distance it will travel, then half the remaining distance, and so on, forever. If a point moves from position 0 to position 1 on the number line, it first reaches position 1/2, then position 3/4, then position 7/8, and so on. At the $n$th stage, it is at position $1 - \frac{1}{2^n}$. From the fact that there is no $n$ such that $1 - \frac{1}{2^n} = 1$, it follows that that moving point never reaches position 1. It just cannot get through the infinite number of stages necessary to do so. Hence there is no motion, motion from 0 to 1 being typical of any motion whatsoever.

In modern physics, we counter this argument by asserting that, indeed, the point can and does traverse each of the infinite number of intervals from $1 - \frac{1}{2^n}$ to $1 - \frac{1}{2^{n+1}}$ for $n = 1, 2, 3, \ldots$ — *ad infinitum.* There is no $n$ such that the moving point does not cross position $1 - \frac{1}{2^n}$. Starting from the premiss that there is motion, modern physicists invoke the infinite to explain it. Like Zeno, they assume that motion is continuous, but, unlike Zeno, they are willing to say that a moving object does pass over an infinite number of points. Zeno rejected the infinite, and so he rejected motion too. Modern physicists accept motion, and so they accept the infinite too.

## Zeno's Second Argument

The famous runner Achilles and his rival (usually thought to be a tortoise) are racing along the positive number line. Achilles starts at position 0, but the tortoise has a head start, beginning at position 1. Since Achilles runs twice as fast as the tortoise, one might expect him to overtake the tortoise at position 2. However, when Achilles arrives at position 1, the tortoise is already at position $1 + \frac{1}{2}$; when Achilles reaches position $1 + \frac{1}{2}$, the tortoise has raced on to position $1 + \frac{1}{2} + \frac{1}{4}$; and so on. When Achilles finally gets to position $2 - \frac{1}{2^n}$, for large $n$, the tortoise is still ahead, at position $2 - \frac{1}{2^{n+1}}$. Despite the appearances, which lead us to believe there is motion, Achilles will never catch up to the tortoise.

In this second argument, Zeno again assumed, as we do, that space and time are continuous, and that, if there is motion, there is uniform motion. Zeno also assumed, unlike us, that Achilles and the tortoise can never 'get through' the infinite number of stages into which Zeno analysed their motion.

For modern physics, precisely, motion typically consists of the occupation of infinitely many distinct locations at infinitely many distinct instants — all within a finite time interval. Because we accept the infinite, we do not find Zeno's argument troubling. However, if someone rejected the infinite, he or she would, indeed, have to reject the possibility of continuous motion.

## Zeno's Third Argument

At every instant, a flying arrow is in exactly one fixed place. Hence it does not really move.

To this argument we would reply that the fact that the arrow covers 0 distance in an instant does not imply that it covers 0 distance in an interval consisting of an infinite number of instants. As every calculus student learns, there are cases in which

$$0 \times \infty = 1$$

Zeno did not like the infinite, so he did not make this reply.

## Zeno's Fourth Argument

This argument is open to various interpretations. One is the following. There are three rows of people:

$$\begin{array}{cccccc} & A & A & A & A & \\ & B & B & B & B & \rightarrow \\ \leftarrow & C & C & C & C & \end{array}$$

The $A$s are stationary, the $B$s are moving to the right at top speed, and the $C$s are moving to the left at top speed. Relative to each other, however, the $B$s and $C$s are going at twice top speed, which is impossible. So there cannot be any motion.

In answer to this argument we can either challenge Zeno's finitist assumption that there is a top speed, or we can invoke the Theory of Special Relativity, which explains how the $B$s and $C$s can both be going at the speed of light relative to the $A$s and yet *not* be going faster than the speed of light relative to each other.

## The General Form of Zeno's Arguments

Each of Zeno's arguments has the following form:

$$\begin{array}{l} \text{Rejection of the infinite} \\ \underline{+ \text{ other considerations (including the continuity of space)}} \\ \text{No motion} \end{array}$$

This form is logically equivalent to the form:

$$\frac{\begin{array}{l}\text{Motion}\\ +\text{ other considerations (including the continuity of space)}\end{array}}{\text{Acceptance of the infinite}}$$

Most of us accept the existence of motion and would sooner give up finitism than embrace the static reality of Parmenides. The modern physicist, for one, is quite happy to base the analysis of motion on the mathematician's real number system, accepting the existence of infinite sets of numbers.

## Democritus

Democritus of Abdera (in north-east Greece) lived about 420 B.C. He claimed that everything is made up of tiny indestructible atoms. The number of these atoms, he said, is infinite, and the empty space containing them is also infinite.

Democritus was a determinist. He asserted that 'from infinite time back are foreordained by necessity all things that were and are and are to come'. In harmony with this, he also held that everything happens without purpose or design.

Commenting on the circular sections of a cone cut by planes parallel to its base, Democritus asked:

> Are they equal or unequal? For, if they are unequal, they will make the cone irregular as having many indentations, like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal, circles, which is very absurd.

## Exercises 8

1. In the Cartesian plane, let

$$\begin{aligned} A_n &= (\tfrac{1}{2^{4n}},\ 0) \\ B_n &= (0,\ \tfrac{1}{2^{4n+1}}) \\ C_n &= (\tfrac{-1}{2^{4n+2}},\ 0) \\ D_n &= (0,\ \tfrac{-1}{2^{4n+3}}) \end{aligned}$$

Consider the path $A_0B_0C_0D_0A_1B_1C_1D_1A_2B_2C_2\ldots$, where each pair of adjacent points is joined by a straight line. Draw the beginning of this path. Show that this path has length $\sqrt{5}$. If you go along it, all the way, where will you end up? How many turns will you have made by the time you get there?

2. Suppose that, at time $t \geq 0$, Achilles is at point

$$(2 - \frac{1}{2^{t-1}},\ t)$$

— in polar coordinates — and the tortoise is at

$$(2 - \frac{1}{2^{t}},\ t)$$

Where are they when $t = 2\pi$? How far apart are they at time $t$? Will Achilles ever catch the tortoise?

## *Challenge for Experts*

1. Let $f(0) = 0$ and, otherwise, $f(t) = t\sin(1/t)$. Then $f$ is continuous, and, as $t$ goes from 0 to 1, the graph of $f$ is infinitely long.

## *Essay Questions*

1. Does Zeno inadvertently prove that there is an infinite?

2. How would you answer Democritus's question? Is Democritus thinking of a cone as an infinite number of circles, one on top of the other?

# 9

# Mathematics in Athens Before Plato

In 479 B.C., the Greeks drove off the Persians, and Athens emerged as a great centre of civilisation. This position she continued to hold, even after losing the Peloponnesian War to Sparta, in 404 B.C.

Athens had some bad points. For example, she extended few rights to women, slaves, and foreigners. On the other hand, she provided an atmosphere conducive to the arts and sciences. She had some of the greatest playwrights in history: Aeschylus, Sophocles, and Euripides. She helped form some of the world's greatest philosophers: Socrates, Plato, and Aristotle. Finally, she hosted some great mathematicians: Hippias (425 B.C.), Antiphon (425 B.C.), Hippocrates (425 B.C.), Theaetetus (369 B.C.), Eudoxus (408–355 B.C.), and Menaechmus (350 B.C.).

Athens later had a strong intellectual rival in Alexandria, Egypt (founded in 332 B.C.), but she remained a centre of culture until 529 A.D., when Justinian closed the Academy founded by Plato. This was because the Academy had failed to accept the new Christian knowledge.

## Hippias and the Quadratrix

The sophist Hippias (425 B.C.) came from Elis, on the west coast of Greece. In Plato's *Protagoras,* we hear Hippias resolving a dispute between Protagoras and Socrates. Hippias asks them to recall that Athens is 'the centre and shrine of Greek wisdom', so that 'it would be a disgrace if we produced nothing worthy of our fame but fell to bickering like the lowest of mankind'

(*Protagoras* 337d-e). In the *Lesser Hippias* (366c-d), we hear Socrates teasing Hippias about his mathematics:

> **Socrates:** And tell me, Hippias, are you not a skillful calculator and arithmetician?
> **Hippias:** Yes, Socrates, assuredly I am.
> **Socrates:** And if someone were to ask you what is the sum of 3 multiplied by 700, you would tell him the true answer in a moment, if you pleased?
> **Hippias:** Certainly I should.
> **Socrates:** Is not that because you are the wisest and ablest of men in these matters?
> **Hippias:** Yes.

Hippias discovered a curve called the *quadratrix,* which can be used for trisecting an arbitrary angle. Consider a unit square $ABCD$ with $AB$ on top and $DC$ on the bottom. Imagine that side $AB$ moves at a rate of 1 unit per second towards the opposite side $DC$. Imagine also that side $AD$ rotates about $D$ and toward $DC$, at a rate of $90°$ per second, so that, after 1 second, both $AD$ and $AB$ coincide with $DC$. At any time $t$ (with $0 \leq t \leq 1$), the two moving sides meet at a point $P$. The set of these points $P$ is the quadratrix.

In terms of modern analytic geometry and trigonometry, $P$ has coordinates

$$\left( \frac{1-t}{\tan(\frac{\pi}{2}(1-t))},\ 1-t \right)$$

so that the equation of the quadratrix is $y = x\tan(\frac{\pi}{2}y)$, with $0 \leq y \leq 1$.

To trisect an angle of, say, $60°$, we place it so that its vertex is at $D$, one of its arms lies along $DC$, and the other arm meets the quadratrix at a point $Q$. If $d$ is the distance from $Q$ to $DC$, we construct a line parallel to $DC$, at a distance $d/3$ from $BC$. (There is a straightedge and compass construction for this.) If this parallel meets the quadratrix at $P$, then $\angle PDC = 20°$.

Furthermore,

$$\lim_{y\to 0} \frac{y}{\tan(\frac{\pi}{2}y)}$$

$$= \frac{y}{\frac{\sin(\frac{\pi}{2}y)}{\frac{\pi}{2}y}\frac{\pi}{2}y} = \frac{2}{\pi}$$

and hence the quadratrix meets $DC$ at a point $2/\pi$ units from $D$. Hence it can be used to construct a square equal in area to a circle with radius 1.

Plato oppugned the quadratrix on the grounds that it is more elegant to use only straight lines and circles in the solution of mathematical problems. One ought to trisect angles using only a straightedge and compass. In 1837, a French opium addict, Pierre Wantzel (1814–1848), proved that it is not possible to trisect an arbitrary angle using only a straight-edge and compass. One has to use some other device, such as the quadratrix. Hippias was right and Plato wrong — but it took over 2000 years for Hippias to be vindicated.

## Antiphon and the Circle Area

Antiphon (425 B.C.), another sophist, asserted the equality of all human beings.

In mathematics, he was the first person to suggest that the area of a circle be calculated in terms of regular polygons inscribed in it.

An inscribed square takes up more than 1/2 the area of a circle, while an inscribed regular octagon takes up more than 3/4 the area of a circle. As the ancient Greeks realised (see the *Elements* XII 2), one can use what we now call mathematical induction to show that an inscribed regular $2^n$-gon takes up more than $1 - \frac{1}{2^{n-1}}$ of the area of a circle.

If we inscribe a regular $2^n$-gon in a circle, its longest diagonals are diameters of that circle. The ancient Greeks knew that the area of a regular $2^n$-gon is proportionate to the square on its longest diagonal, and from this it follows that, *in so far as a circle is like a regular $2^n$-gon,* its area is proportionate to the square on its diameter:

$$\text{area of circle } = k(2r)^2 = (4k)r^2$$

where $r$ is the radius of the circle.

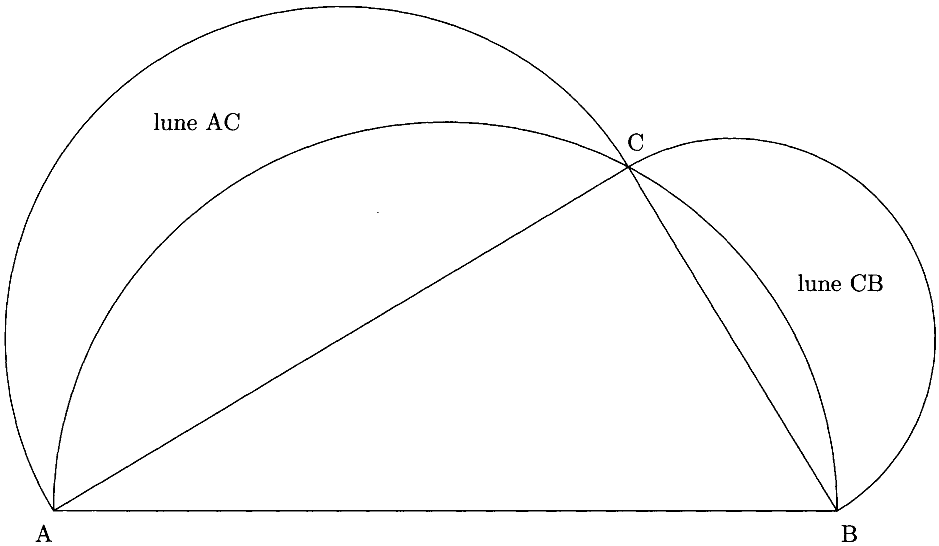
Antiphon argued for this (correct) conclusion by boldly asserting that a circle simply *is* a regular polygon with an infinite number of sides.

## Hippocrates and the Lunes

Hippocrates came from the Greek island of Chios, near present-day Turkey. (He was not the famous doctor.) Hippocrates had been swindled in business, and he went to Athens, about 430 B.C., in order to recover his property through legal action. The case dragged on, and Hippocrates spent the time studying and teaching geometry.

Hippocrates was responsible for much of the material on circles and regular polygons in Books III and IV of the *Elements.* He was also the first person to find the precise area of a region bound by curves.

Hippocrates constructed semicircles on the three sides of a right triangle, so that the semicircle on the hypotenuse $AB$ went through the vertex $C$ of the right angle (in harmony with the theorem of Thales), and so that the semicircles on the two 'legs' $AC$ and $BC$ lay outside the triangle. The areas included in the two smaller semicircles but outside the larger semicircle




**The Lunes of Hippocrates**

are *lunes* (so named after the crescent moon).

Hippocrates discovered and proved the fact that the sum of the areas of the lunes equals the area of the right triangle.

**The proof of Hippocrates:** Since, by the theorem of Pythagoras,

$$AC^2 + BC^2 = AB^2$$

and since, as Antiphon had pointed out, the area of a circle, or semicircle, is proportionate to the square on its diameter, it follows that the sum of the areas of the semicircles on $AC$ and $BC$ equals the area of the semicircle on $AB$. Subtracting the areas where the semicircles overlap, it follows that, indeed, the sum of the areas of the lunes equals the area of the right triangle.

Hippocrates also discovered and proved the following. Let $ABCD$ be half a regular hexagon inscribed in a semicircle with diameter $AD$. Construct a lune by drawing, outside the hexagon, a semicircle on $AB$ as diameter. Do the same, using diameters $BC$ and $CD$. Then the area of the semihexagon

$ABCD$ equals the sum of the areas of the three lunes, plus the area of a semicircle on $BC$.

## Duplicating the Cube

Legend has it that, during a typhoid plague in 430 B.C., the Athenians consulted the oracle at Delos for help, and the oracle answered that they must double the volume of the cubical altar of Apollo. The Greeks were thus faced with the problem of constructing a length $x$ such that $x^3 = 2$.

Hippocrates noted that one could do this if one could construct lengths $y$ and $z$ such that $1/y = y/z = z/2$. For then $y$ would be the required cube root of 2. Of course, the Greeks knew how to find approximations to the cube root of 2. The problem was to find a geometrical construction that, theoretically, would give a length *exactly* equal to the cube root of 2.

## Socrates

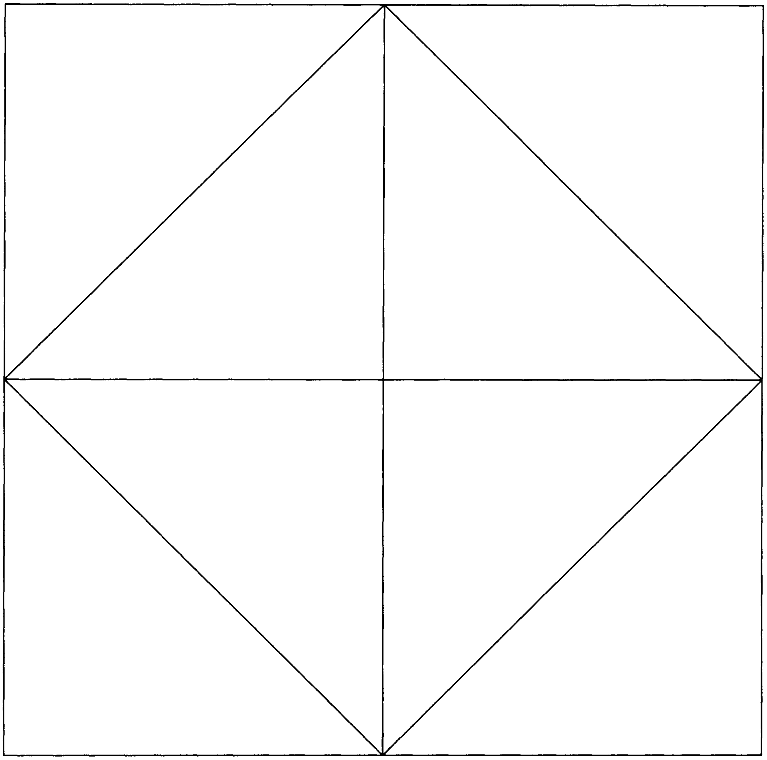
Socrates (469–399 B.C.) was Plato's mentor. He was not a mathematician but, as Plato portrays him in the *Meno,* he made use of mathematics in philosophy. In the course of a conversation about virtue with Meno, Socrates has one of Meno's uneducated slave boys 'double the square'. The boy at first thinks that one doubles the area of a square by doubling its side, but Socrates soon leads him to see his mistake. Then Socrates shows him the figure of a square with the midpoints of its four sides joined to form a smaller square. Socrates then gets the boy to 'remember' that it is the square on the diagonal of the original square that has double its area (see the figure).

The mathematics was not very revolutionary, but the rest of Socrates's message was. Here was a boy quite capable of learning mathematics, and Meno was failing to educate him.

## Exercises 9

1. Do you think Hippias could have told Socrates 'the sum of 3 multiplied by 700'? If so, what is the answer?

2. Prove that an inscribed regular $2^n$-gon takes up more than $1 - \frac{1}{2^{n-1}}$ of the area of a circle.



**The Diagram of the *Meno***

3. Plimpton 322 refers to a right triangle with side 4961 and hypotenuse 8161. What is the area of the lunes on this triangle?

4. Prove the second of the lune theorems of Hippocrates.

5. Show that if $1/y = y/z = z/2$ then $y = \sqrt[3]{2}$.

6. Prove that $\sqrt[3]{2}$ is irrational.

## *Challenge for Experts*

1. Let $s_k$ be the side of a regular $k$-gon inscribed in a circle of diameter 1. Let $S_k$ be the side of a regular $k$-gon circumscribed about a circle of diameter 1. Show that

$$S_{2k} = \frac{s_k S_k}{s_k + S_k} \quad \text{and } s_{2k} = \sqrt{\frac{s_k S_k}{2}}$$

## *Essay Question*

1. Protagoras was a relativist, holding that any given thing 'is to me such as it appears to me, and is to you such as it appears to you'. In *Theaetetus* 169a, Plato suggests a mathematical argument against relativism. What is it?

# 10
# Plato

Plato (427–349 B.C.) was a student of Socrates. After Socrates's execution in 399 B.C., Plato travelled to North Africa, where he studied with Theodorus of Cyrene (Lybia), a mathematician who proved the irrationality of the square roots of 3, 5, 6, 7, 10, 11, 13, 14, 15, and 17.

At age 40, Plato visited Italy, and spent some time with the Pythagorean mathematician Archytas (428–347 B.C.). It was Archytas who first found a construction for the $\sqrt[3]{2}$. Plato's trip to Italy came to a sudden end when his enemy Dionysius I, ruler of Syracuse, sold him into slavery! Happily, one of Plato's friends ransomed him, and he returned to Athens.

About 380 B.C., Plato found the Academy. At the entrance of this research institute was the inscription:

**LET NO ONE IGNORANT OF GEOMETRY ENTER HERE!**

Plato was a realist: he held that reality exists independently of the human mind. He was also a correspondence theorist: he held that a statement is true just in case it correctly describes the actual state of affairs (in mind-independent reality). Not surprisingly, Plato attacked the relativism of Protagoras, according to which anything 'is to me such as it appears to me, and is to you such as it appears to you'.

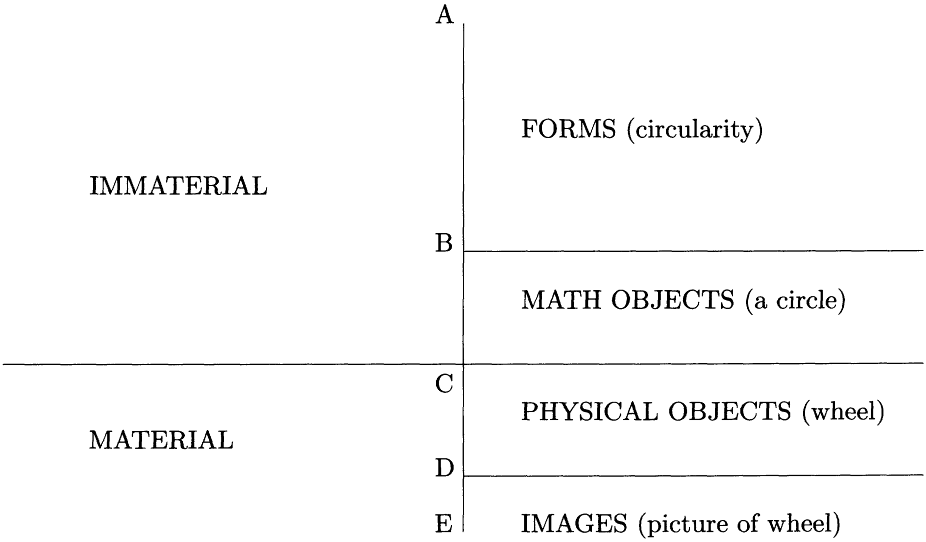
Plato believed that the objects in the universe fall into two very different classes, the material and the immaterial. Objects such as the sun, that bed, and Diana's body belong to the class of material things. Objects such as goodness, that circle, and Diana's soul belong to the class of immaterial things. A drawing of a square belongs to the material realm, but the square itself belongs to the immaterial realm. Plato says of geometry students that they

> make use of the visible forms and talk about them, though they are not thinking of them but of those things of which they are a likeness, pursuing their inquiry for the sake of the square as such and the diagonal as such, and not for the sake of the image of it which they draw (*Republic* 510d).

For Plato, the class of material things is characterised by contingency, change, uncertainty, ignorance, and imperfection. The drawing of the square can be erased. Its angles are not exactly right angles. Its sides are not absolutely straight. On the other hand, the class of immaterial objects is characterised by necessity, permanence, certainty, knowledge, and perfection. Real squares have sides that are infinitely thin and absolutely straight. Their properties can be deduced with infallible rigour. We can know with certainty that every square has two equal diagonals. For Plato, these squares are no mere abstractions or mental concepts. On the contrary, they are necessarily existing particulars. Just as the eye sees visible objects, which exist independently of the human body, so the 'eye of the soul' intuits immaterial objects, which exist independently of the human soul.

For Plato immaterial things are good, but material things are bad. For example, 'Platonic friendship' is good, but sex is bad. (Even within marriage, it should be minimised.)

At the end of Book VI of the *Republic*, Plato discusses the two classes of things in terms of a line segment $AE$. This segment is divided at $C$, which




represents the boundary between the material ($CE$) and the immaterial

($AC$). Segment $AC$ is subdivided at $B$, and $CE$ at $D$. The segment $DE$ represents pictures, reflections, or shadows of physical objects. The segment $CD$ represents the physical objects themselves. In the immaterial realm, segment $BC$ represents mathematical objects, and segment $AB$ represents Plato's *forms*. These *forms* are qualities such as goodness, beauty, oneness, circularity, squareness, humanity, and so on. According to Plato, a physical object is, say, circular just in case it 'participates' in the form of circularity. The mathematical objects have many 'instances': we can add two 1s, or compare two circles. The forms, however, are unique: there is only one oneness and only one circularity. (See Aristotle, *Metaphysics* 987b14–17.)

Plato tells us that
$$\frac{AC}{CE} = \frac{AB}{BC} = \frac{CD}{DE}$$
from which one can deduce that $BC = CD$ (see *Republic* 509d and 534a). Thus we might have $AB = 4$, $BC = CD = 2$, and $DE = 1$.

For Plato, the best way to get acquainted with the wonderful immaterial realm is to do mathematics. One should study number theory 'for facilitating the conversion of the soul itself from the world of generation to essence and truth'. One should study geometry 'to facilitate the apprehension of the idea of good' (see *Republic* 525c, 526e). Having turned his or her mind to visible geometric diagrams, the student then raises it to the circles themselves and finally 'sees' the form of circularity and the form of goodness that illuminates all the other forms.

It is not clear where Plato thought the number 1 should go on the divided line. On the one hand, it is a mathematical object with various 'instances'. I can ask for the sum of four 1s or warn the students not to confuse the subscript 1 with the 1 in the data. On the other hand, the ancient Greeks tended to identify the number 1 with oneness (the form). For example, in his proofs, Euclid uses the word 'monad' to refer to the number 1, but, in his definitions, he describes it as 'that by virtue of which each of the things that exist is called one' — which is exactly how Plato characterises the form of oneness.

The importance of Plato in mathematics is due not to any mathematical contribution, but to the influence he exerted on others. It was Plato who insisted that a 'proper' solution of a geometry problem involves no curves other than the circle (see *Timaeus* 34a). It was Plato who emphasised the importance of clear definitions and postulates. Finally, it was Plato who encouraged the study of mathematics as a way of becoming virtuous.

# Exercises 10

1. Prove the results of Theodorus.

2. Prove that, in the divided line, $BC = CD$.

## *Essay Question*

1. Argue in favour of the view that mathematical objects exist independently of the human mind.

# 11
# Aristotle

Aristotle (384–322 B.C.) was a student of Plato for twenty years, but he disagreed with Plato about the nature of mathematics. For Aristotle, the word 'two' was not a noun referring to an abstract object existing independently of physical objects, but an adjective describing a physical object (e.g., that two-metre ladder). The 'two' of the length is 'in' the ladder (*Metaphysics* 1077a).

## Aristotle's Logic

Aristotle's work on logical validity is found in the *Prior Analytics.* He distinguished four basic types of statements.

**PaS**
Each thing having property P also has property S.
(All conservatives are cowards.)

**PeS**
Nothing has both property P and property S.
(No Canadian is a billionaire.)

**PiS**
At least one thing has properties P and S.
(Some professors are clever.)

**PoS**
At least one thing has property P but lacks property S.
(Some professors are not clever.)

Note that Aristotle understood PaS to imply PiS. Thus for Aristotle the statement

All unicorns have horns

is false because there are no unicorns.

A *syllogism* is a list of three of these statements. The first two are *premisses.* They must share exactly one *property letter* (the P or S). The third statement is the *conclusion.* It must contain the two property letters not shared by the premisses. For example, the following are syllogisms.

| | |
|---|---|
| CaN | All clever things are neurotic. |
| PaC | All professors are clever. |
| PaN | All professors are neurotic. |

| | |
|---|---|
| CeW | No conservative is a coward. |
| BaC | All businessmen are conservatives. |
| BeW | No businessman is a coward. |

| | |
|---|---|
| SaP | All students are poor. |
| WiS | Some women are students. |
| WiP | Some women are poor. |

| | |
|---|---|
| CeB | No Canadian is all bad. |
| MiC | Some businessmen are Canadians. |
| MoB | Some businessmen are not all bad. |

| | |
|---|---|
| DaB | All dogs bark. |
| DaS | All dogs sleep. |
| BiS | Some barking things sleep. |

| | |
|---|---|
| CiY | Some cowards are young. |
| YiS | Some young things are sweet. |
| CiS | Some cowards are sweet. |

A syllogism is either *valid* or *invalid.* This has nothing to do with the truth of the statements in it. It has to do only with whether or not the conclusion follows logically from the premisses. The first four syllogisms given above are valid, even though some of the statements involved are false, because, in each case, the premisses could not be true without the conclusion also being true. The fifth syllogism is also valid, if we grant Aristotle's view that PaS implies PiS. The sixth syllogism, however, is invalid — even though all the statements in it are true — because the conclusion might be false, even though the premisses are true. The fact that some cowards are young and the fact that some young things are sweet do not rule out the possibility that all the sweet young things are brave.

In the *Posterior Analytics,* Aristotle formulates what we call the deductive method. It was adopted by Euclid and has always been an essential characteristic of mathematics. This method consists of starting with propositions called axioms and then proving propositions called theorems. Each statement in a proof has to be justified either by an axiom or by a previously proved theorem or by a principle of logic.

For Aristotle, the axioms of mathematics are truths, and hence the theorems are also truths. Aristotle does not say, 'if there is a triangle with such and such a property, then the sum of its angles is two right angles'. Rather, he says, 'the triangle in virtue of its own nature contains two right angles' (*Metaphysics* IV 3).

Aristotle also did pioneering work in modal logic. In *De Interpretatione* 12 and 13, he notes the following implications.

(1) If it is possible that $p$ is not the case, then it is not necessary that $p$ be the case.

(2) If it is not possible that $p$ be the case, then, necessarily, $p$ is not the case.

For example, if it is not possible to pass the exam then, necessarily, you will fail it.

On Aristotle's scheme, every statement falls into exactly one of the following three categories.

(A) It is necessarily true.
For example: $2 + 5 = 7$.
(B) It is necessarily false (or impossible).
For example: This dog is actually a telephone number.
(C) It is contingent.
For example: Hitler invaded Russia. The French never fought the British.

## Aristotle and the Infinite

Aristotle was a staunch finitist. He rejected infinite sets and infinite lines (*Physics* 206b, 266b, 207; *Metaphysics* 1084a). He rejected infinitesimals (*Physics* 266b). For Aristotle, the geometer can have arbitrarily long segments, but not a line that 'goes to infinity'.

Aristotle had a number of reasons for rejecting the infinite.

(1) The infinite is too big to be beautiful. In *De Poetica* 1450b–51a, Aristotle writes:

> to be beautiful, a living creature, and every whole made up of parts, must not only present a certain order in its arrangement of parts, but also be of a certain definite magnitude. Beauty is a matter of size and order, and therefore impossible ... in a creature of vast size — one, say, 1000 miles long — as in that case, instead of the object being seen all at once, the unity and wholeness of it is lost to the beholder.

(2) Infinite lines lead to contradictions in kinematics. Suppose there were an infinite straight line $AB$. Let $C$ be a point not on $AB$, and let $XCY$ be another infinite straight line that rotates with $C$ as its axis, cutting $AB$ at a variable point $P$. Suppose that at 3 P.M., $XCY$ is parallel to $AB$, and suppose that $XCY$ rotates clockwise about $C$, at a constant rate of half a revolution per hour. Then $XCY$ is parallel to $AB$ at 4 P.M., 5 P.M., 6 P.M., and so on — every hour on the hour. At all other times, $XCY$ cuts $AB$ at a point $P$, and, as each hour goes by, $P$ travels the whole length of $AB$. However, said Aristotle, no distance is infinite if it can be traversed in a finite time. Thus $AB$ is not infinite. Contradiction. (See *On the Heavens* 271b26–272a20.)

(3) Infinite sets lead to contradictions in mathematics. If there is an infinite collection of objects, then it has a proper subset that is also infinite. For example, the set of natural numbers contains the set of evens as a proper part, and the set of evens is infinite. However, said Aristotle, since the proper part is bounded by the whole and less than it, the proper part is not infinite. Contradiction. (See *Physics* 204a20–29 and *Metaphysics* 1066b11–17.)

(4) Aristotle also had a version of the 'Thomson lamp paradox'. Elaborating a bit on Aristotle, let us imagine a lamp that comes on at time $t = 1 - \frac{1}{2^n}$ if $n$ is even, but goes off at time $t = 1 - \frac{1}{2^n}$ if $n$ is odd. If, indeed, we can divide an interval of time into an actually infinite number of instants, then this lamp is theoretically possible, and, theoretically, it would turn on and off an actually infinite number of times in the time interval from $t = 0$ to $t = 1$. However, at time $t = 1$ the lamp would be neither on nor off — because the infinite is neither even nor odd. But this

is impossible. Hence we cannot divide an interval of time into an actually infinite number of instants. (See *Metaphysics* 1083b37–1084a6.)

As a replacement for the infinite, Aristotle put forward the idea of the *potentially infinite*. Imagine that Aristotle, using ruler and compass, is actually constructing the subintervals of a given segment, at a rate of one a minute. Imagine, moreover, that he will continue doing so for an indefinite period of time, so that, for any given whole number $n$, he will eventually construct more than $n$ subintervals. Then, on the one hand, the set of constructed subintervals is never at any time infinite, but, on the other hand, its size is not bounded by some predetermined, fixed number. It is in this sense potentially infinite. (See *Physics* 206a18–26 and *Metaphysics* 1048b10–18.)

Of course, we could press Aristotle, insisting that he say something about the size of the *atemporal* set of all the subintervals that will *ever* be constructed, but, in that case, he might only reply that, like the unicorn, it is neither finite nor infinite — because it does not exist.

## Exercises 11

1. Express each of the following syllogisms in the P–S notation. Then say whether it is valid or not. If it is not valid, give a syllogism of the same form with true premisses and a false conclusion.

   (a) All mammals are camels. Some mammals do not swim. Therefore some camels do not swim.

   (b) All people called 'Socrates' are mortal. All mortals die. Therefore all people called 'Socrates' die.

   (c) Those who did not study did poorly. The boys did poorly. Therefore the boys did not study.

   (d) No insects are birds. No birds are mammals. Therefore no insects are mammals.

   (e) All Nazis are cowards. All cowards are damned. Therefore there are some damned Nazis.

   (f) Some cake eaters are fat. No fat person is healthy. Therefore some cake eaters are not healthy.

   (g) All stupid people are victims of propaganda. No logic student is stupid. Therefore no logic student is a victim of propaganda.

   (h) No swans are black. Some black things are dogs. Therefore no swans are dogs.

(i) All statesmen are honourable. Some politicians are honourable. Therefore some statesmen are politicians.

(j) Some animals are furry. Some furry things are cats. Therefore some animals are cats.

(k) All old men are Pharisees. All old men are rich. Therefore all rich people are Pharisees.

(l) All hippies smoke pot. No student smokes pot. Therefore some students are not hippies.

(m) All hippies smoke pot. Some students do not smoke pot. Therefore some students are not hippies.

(n) No fossils can be crossed in love. An oyster may be crossed in love. Thus no fossil is an oyster.

(o) Some poetry is original. No original work is producible at will. Thus some poetry is not producible at will.

(p) Some pillows are soft. No pokers are soft. Hence some pillows are not pokers.

(q) No misers are unselfish. None but misers save eggshells. Therefore everyone who saves eggshells is unselfish.

(r) All my cousins are unjust. All judges are just. Therefore none of my cousins is a judge.

(s) Some buns are rich. All buns are nice. Therefore some rich things are nice.

(t) Pigs cannot fly. Pigs are greedy. Thus some greedy things cannot fly.

2. Classify each of the following statements as necessarily true, impossible, or contingent.

(a) $2 + 2 = 50$

(b) Anyone who is someone's sister is female.

(c) Aristotle was Plato's student.

(d) Marilyn will go to Hollywood next week.

(e) If there are 5 balls in 4 boxes, one of the boxes is empty.

(f) If there are 5 balls in 4 boxes, one of the boxes contains at least two balls.

*Essay Questions*

1. What trouble might a mathematician run into if he or she tried to base mathematics on a set of axioms that includes the axiom: there is no infinite set?

2. How might Aristotle answer Zeno's arguments against motion?

3. What might Anaximander say about Thomson's lamp?

# 12
# In the Time of Eudoxus

## Theaetetus

Plato had a brilliant student called Theaetetus, who died in battle in 369 B.C. It was Theaetetus who showed that the square root of a natural number is irrational if and only if the natural number is not a square. Theaetetus was responsible for the material in Books X and XIII of Euclid's *Elements.*

## Eudoxus

Eudoxus (405–355 B.C.) came from Cnidus, a small Greek island near present-day Turkey. He distinguished himself in astronomy, medicine, geography, philosophy, and, of course, mathematics.

As a young man, Eudoxus studied at Plato's Academy, commuting on foot from Piraeus, the harbour district. Later he engaged in a philosophical controversy with Plato. Eudoxus was a hedonist, but Plato put wisdom above pleasure. (See Plato's *Philebus* and Aristotle's *Nicomachaean Ethics* 1101b27 and 1172b9 for an account of this debate.)

Eudoxus was responsible for the material in Books V and XII of Euclid's *Elements.* Book V is a theory of proportion. We would define '$a$ is to $b$ as $c$ is to $d$' (written $a : b :: c : d$) to mean $a/b = c/d$. However, this definition presupposes our real number field. It presupposes that we already have some way of understanding what it is to multiply or divide arbitrarily given irrational numbers. Eudoxus, however, was starting from scratch. He could not use multiplication or division to define proportion because it was part of his program to define multiplication and division in terms of proportion. The definition of proportion on which he based his presentation of the number system was the following:

$$a : b :: c : d$$

iff, for any positive integers $p$ and $q$, (1) $pa > qb$ iff $pc > qd$, and (2) $pa = qb$ iff $pc = qd$, and (3) $pa < qb$ iff $pc < qd$.

(Eudoxus assumed that all the numbers were positive.)

# Eudoxus and the Circle

Eudoxus gave the following proof that the area of a circle is proportionate to its diameter squared. Suppose $k$ is the area of the circle with diameter 1. (Hence $k$ is the number we today call $\pi/4$.) Let $c$ be a circle with diameter $d$. To obtain a contradiction, suppose

$$kd^2 < \text{ area } c$$

Let regular $2^n$-gons be inscribed in both circles, where $n$ is so large that

$$\frac{1}{2^{n-1}} \text{ area } c < \text{ area } c - kd^2$$

(This is possible according to the 'axiom of Archimedes', which, in fact, shows up in Aristotle and goes back at least to Eudoxus himself.) Then

$$\left(1 - \frac{1}{2^{n-1}}\right) \text{ area } c > kd^2$$

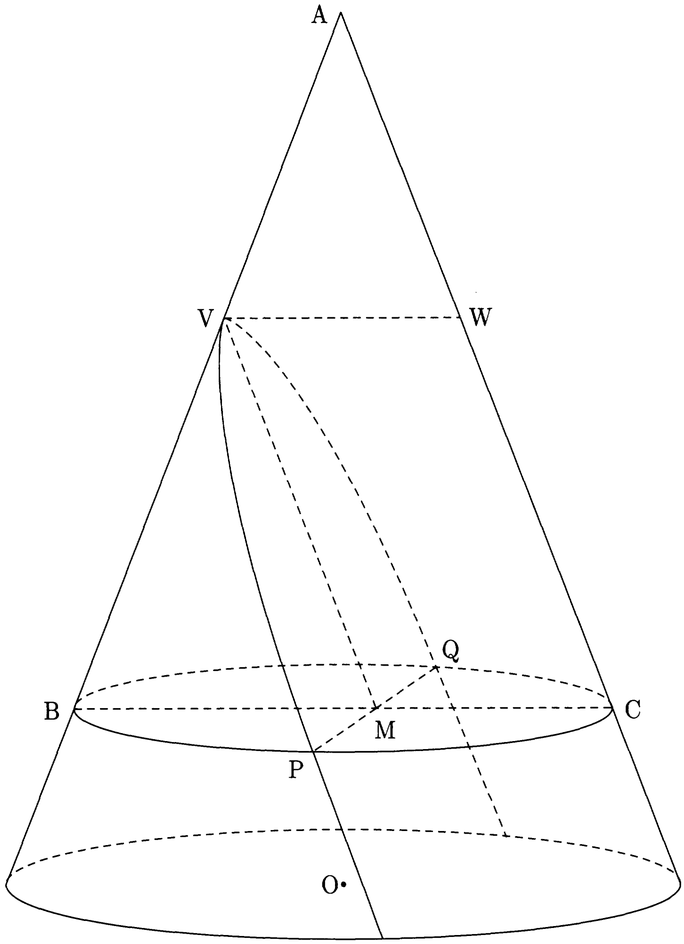
Now, as Antiphon realised, an inscribed regular $2^n$-gon takes up more than $1 - 1/2^{n-1}$ of the area of a circle. Hence

$$\text{area of } 2^n\text{-gon in } c > kd^2$$

Antiphon also knew that the area of the $2^n$-gon inscribed in $c$ is $d^2$ times bigger than the one inscribed in the circle with unit diameter. Thus

$$kd^2 > \text{ area of } 2^n\text{-gon in unit diameter circle} \times d^2 > kd^2$$

Contradiction.




**The Cone and the Parabola**

Similarly, we get a contradiction if we assume that $kd^2 >$ area $c$. From this it follows that $kd^2 =$ area $c$.

# Menaechmus

Menaechmus (350 B.C.) studied under Plato and Eudoxus. It was Menaechmus who told Alexander the Great:

> O King, through the country there are private roads and royal roads, but in geometry there is only one road for all.

Menaechmus discovered the conics, defining them as 'sections' of a cone and deriving equivalents of their analytic geometry formulas. For example, he defined a parabola as the intersection of a right circular cone and a plane parallel to a straight line in the (surface of) the cone.

Suppose the plane cuts the cone at points $V$, $P$, and $Q$, where $V$ is the vertex of the parabola, and $P$ and $Q$ are points opposite each other on the cone. There is a circle $BPCQ$ that is in the (surface of) the cone, at right angles to the cone's axis, and is such that its diameter $BC$ is perpendicular to $PQ$. If $BC$ meets $PQ$ at $M$, then $VM$ is the axis of symmetry of the parabola.

Let $A$ be the vertex of the cone, and let $W$ be the point on the cone opposite $V$. Then triangle $VMB$ is similar to triangle $AWV$ (since $VM$ and $AW$ are parallel). Hence $VM/BM = AW/VW$, this being a constant independent of $P$. By the theorem of Thales, $\angle BPC$ is right, and thus $PM^2 = BM \times MC$. Since $VMCW$ is a parallelogram, we have $MC = VW$. Hence

$$\frac{VM}{PM^2} = \frac{VM}{BM \times MC} = \frac{VM}{BM \times VW} = \frac{AW}{VW^2}$$

which is a constant $k$, independent of the choice of $P$. This yields $VM = kPM^2$, which is essentially the same as the analytic geometry formula for the parabola.

Menaechmus used the conics to 'double the cube'. To do this, he may have used the fact we express as follows: the parabolas $y = \frac{1}{2}x^2$ and $x = y^2$ meet at a point whose $y$-coordinate is $\sqrt[3]{2}$.

Plato was unhappy that Menaechmus did not stick to straight lines and circles, but, in 1837, a French opium addict, Pierre Wantzel, proved that it is not possible to construct a segment equal to $\sqrt[3]{2}$ using only straight lines and circles. Menaechmus was right to introduce new curves.

# Exercises 12

1. Prove the theorem of Theaetetus.

2. Using the definition of proportion given by Eudoxus, show that

$$a : b :: c : d \text{ iff } d : c :: b : a$$

3. In Eudoxus's proof that the area of a circle is $kd^2$, how exactly do we get a contradiction from the assumption that $kd^2 >$ area $c$?

4. What is the intersection point of $y = \frac{1}{2}x^2$ and $x = y^2$?

5. What is the analytic geometry equation of the parabola if, in the above notation, $AV = 4$ and $VW = 3$?

## *Essay Question*

1. Summarise the *Philebus.*

# 13

# Ruler and Compass Constructions

The ancient Greeks searched for a way of using a straightedge and a compass to trisect an arbitrary angle and draw a segment of length $\sqrt[3]{2}$. They also tried to 'square the circle', that is, construct a segment of length $\sqrt{\pi}$. Finally, they struggled to find straightedge and compass constructions for regular polygons with 7, 9, 11, 13, and 17 sides. In all this they failed, but it was not proved until the nineteenth century that the reason for their failure was that all these problems are impossible — except one. In 1796 Gauss discovered a straightedge and compass construction for the regular 17–sided polygon. It was this discovery, the first advance on Greek construction problems in 2000 years, that motivated Gauss to devote himself to mathematics.

Sadly, it is now possible to obtain a Ph.D. in mathematics and not know that Euclid lived in Alexandria, Egypt, about 300 B.C., and wrote a book called the *Elements*. When we do geometry today, we usually start with a plane that already contains a point corresponding to every ordered pair of reals. Euclid was more parsimonious. He started with just two points (corresponding to (0, 0) and (1, 0)) and then constructed, one by one, just enough extra points to meet his immediate needs.

The rules for construction were strict.

(1) If $A$ and $B$ are previously given or constructed points, you can 'join $AB$', constructing the line segment $AB$; if this segment intersects any previously constructed line segments or circles, you have thereby constructed the points of intersection.

(2) If $AB$ is a previously constructed segment and $O$ is a previously given

or constructed point, you can draw a circle (that is, a circumference) with centre $O$ and radius $AB$; if this circle intersects any previously constructed line segments or circles, you have thereby constructed the points of intersection.
(3) If $AB$ is a previously constructed segment, you can lengthen, or 'produce', it in either direction to meet a previously constructed segment or circle (assuming that that segment or circle lies 'in its way'), and thereby construct a point.
(4) The only way to construct anything is to apply the above rules a finite number of times.

As examples, we give the following 8 constructions.

**C1. To bisect an angle**
Let $ABC$ be an angle, with previously constructed 'arms' $AB$ and $BC$. With centre $B$ and radius $BA$, cut $BC$ in $E$. That is, construct a circle with centre $B$ and radius $BA$. If the circumference meets $BC$ in a point, call that point $E$. Otherwise, produce $BC$, in the direction going from $B$ to $C$, until it meets the circumference in a point, which we shall call $E$. With centres $A$ and $E$, construct two circles each with radius $AE$. These circles meet in two points. Let $F$ be the meeting point that is on the side of $AE$ away from $B$. Note that $AEF$ is an equilateral triangle. Join $BF$. Then $BF$ is the required bisector. This can be proved using the 'side-side-side' congruence theorem to show that triangles $BAF$ and $BEF$ are congruent.

If $\angle ABC = 180^\circ$ then $BF$ is perpendicular to $AC$. Thus construction C1 is also a construction for drawing a perpendicular to a given segment through a given point in that segment.

**C2. To construct the right bisector of a segment**
Let $AB$ be a previously constructed segment. With centres $A$ and $B$, draw two circles, each with radius $AB$. These circles meet in exactly two points $C$ and $D$. Join $CD$. Then $CD$ is the required right bisector.

Note that $CD$ meets $AB$ in its midpoint, and hence this construction also works as a construction of the midpoint of a given segment.

**C3. To construct a segment through a given point and parallel to a given segment**
Let $A$ be the point and $BC$ the segment. It is assumed that $A$ is not on the line $BC$. With centre $C$ and radius $AB$, draw a circle. With centre $A$ and radius $BC$, draw a second circle to cut the first circle in point $D$, where $D$ and $B$ are on opposite sides of $AC$. Then $AD$ is the required parallel.

**C4. To add two segments**
Let $AB$ and $CD$ be two previously constructed segments. With centre $B$ and radius $CD$, draw a circle. Produce $AB$ (in the direction from $A$ to $B$)

so that it meets this circle at $E$. The segment $AE$ is the required sum.

**C5. To multiply two segments**

Let $AB$ and $CD$ be previously constructed segments. With centres $C$ and $D$ and radius $CD$, construct two circles meeting in $E$ and $E'$. With centre $C$ and radius $AB$, cut $CE$ (or $CE$ produced in the direction from $C$ to $E$) in $F$. If $O$ and $X$ are the two points with which Euclid started, so that $OX$ is a unit segment, then, with centre $C$ and radius $OX$, cut $CD$ (or $CD$ produced in the direction from $C$ to $D$) in $G$. Join $FG$. Using C3, draw a segment through $D$ parallel to $FG$, to meet $CE$ (or $CE$ produced) in $H$. Then $CH$ is the required product.

This is proved by using the theory of similar triangles. Since $CH : CF :: CD : 1$, it follows that $CH = CF \times CD = AB \times CD$.

**C6. To draw the multiplicative inverse of a segment**

Let $AB$ be a previously constructed segment. With centres $A$ and $B$, construct circles with radius $AB$, to meet in $C$ and $C'$. With centre $A$ and radius $OX$ (the unit segment), cut $AC$ (or $AC$ produced in the direction from $A$ to $C$) in $D$. With centre $A$ and radius $OX$, cut $AB$ (or $AB$ produced in the direction from $A$ to $B$) in $E$. Draw a line through $E$ that is parallel to $BD$ to meet $AC$ in $F$. Then $AF$ is the required segment.

**C7. To construct the square root of a segment**

Let $AB$ be a previously constructed segment. Add the unit segment $OX$ to it, drawing a segment $AC = AB + 1$, with $B$ between $A$ and $C$. Using C1, erect a perpendicular to $AC$ through $B$. Using C2, construct the midpoint $D$ of $AC$. With centre $D$ and radius $DC$, draw a circle to cut the perpendicular at $E$. Then $BE$ is the required square root.

This is proved by noting that $\angle AEC$, being an angle in a semicircle, is right. Hence triangles $ABE$ and $EBC$ are similar. This gives $AB : BE :: BE : BC$, so that $AB \times BC = BE^2$. But $BC = OX = 1$.

**C8. To construct a Pythagorean star**

With centre $O$ and radius $OX$ draw a circle. Join $XO$ and produce it to meet the circle in $Y$. Construct the midpoint $C$ of $OX$. Construct the right bisector of $YX$, meeting the circle in $E$. With centre $C$ and radius $CE$, cut $OY$ in $F$. With centre $E$ and radius $EF$, cut the original circle in $G$ and $H$. With centre $G$, and the same radius, cut the original circle again at $J$. With centre $H$, and the same radius, cut the original circle again at $K$. Join $EJ$, $EK$, $GK$, $GH$, and $HJ$.

From the above, it is clear that, starting with the unit segment $OX$, Euclid could construct segments of any positive rational length. He could also construct segments equal in length to numbers like

$$3+\sqrt{5\sqrt{3}+\frac{7}{\sqrt{10}+2}}$$

The reason that the Greeks failed to 'duplicate the cube' is simply that $\sqrt[3]{2}$ is not a number of this type. For example, if we had

$$\sqrt[3]{2}=a+b\sqrt{c}$$

with $a$ and $b$ rational and $\sqrt{c}$ irrational, and $3a^2b+c$ nonzero, then we would have

$$2=a^3+3a^2b\sqrt{c}+3ab^2c+c\sqrt{c}$$

or

$$\sqrt{c}=\frac{2-a^3-3ab^2c}{3a^2b+c}$$

which is rational. Contradiction.

The first rigorous proof of the fact that the duplication of the cube is impossible was given in 1837 by Pierre Wantzel (1814–1848). At the same time Wantzel showed that $\cos 20°$ is not a constructible length, and hence one cannot trisect an angle of $60°$ using only straightedge and compass.

Drawing on the work of Carl Friedrich Gauss (1777–1855), Wantzel also proved that if $p$ is an odd prime, a regular $p$-gon is constructible just in case $p$ has the form $2^n+1$. Odd primes of this form are named *Fermat primes,* after Pierre de Fermat (1601–1665), who mistakenly thought that $2^{2^k}+1$ is prime for any natural number $k$. (Thanks to Leonhard Euler (1707–1783), we know that $2^{2^5}$ has factor 641.) It is not known whether there are any Fermat primes greater than 65,537.

The problem of squaring the circle held out until 1882, when C. L. F. Lindemann (1852–1939) proved that $\pi$ is not constructible.

# Exercises 13

1. Get a straightedge and compass and actually construct a regular hexagon.

2. Give a straightedge and compass construction for a line through a given point not on a given line and perpendicular to the given line.

3. Give a Euclidean construction for an angle of $3°$.

4. Prove that the above construction for the five-pointed star works.

5. Prove that if a regular polygon with $n$ sides is constructible, then so is a regular polygon with $2n$ sides.

6. Construct a common tangent to two given circles. You must apply Euclid's rules and not just 'move the ruler round til it touches both circles'.

7. Use Wantzel's results to prove that a regular polygon with 771 sides is constructible.

## *Challenges for Experts*

1. Show that a regular 17-gon is constructible by proving the following:
(a) let $p = \frac{1}{4}(-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}})$; then $p$ is a root of

$$2w^2 + w - 2 = -\sqrt{17}$$

and hence a root of

$$(2w^2 + w - 2)^2 = 17w^2$$

or

$$w^4 + w^3 - 6w^2 - 2 + 1 = 0$$

(b) $3/10 < p < 4/10$;
(c) let $q = (p-1)/(p+1)$; then $pq = p - q - 1$ and $-6/10 < q < -4/10$;
(d) let

$$x = \frac{p}{4} + \sqrt{\frac{p^2}{16} - \frac{q}{4}}$$

and

$$y = \frac{p}{4} - \sqrt{\frac{p^2}{16} - \frac{q}{4}};$$

then $x + y = p/2$, $xy = q/4$, $x^2 = px/2 - q/4$, and $y^2 = py/2 - q/4$;
(e) $(\sqrt{5} - 1)/4 < x < 1/2$ and $-1 < y < 0$;
(f) let $A$ be the acute angle whose cosine is $x$, and let $B$ be the obtuse angle whose cosine is $y$; then $60^\circ < A < 72^\circ$;
(g) $p^3 = 2pq + 4p - 1$ and $p^2q = q^2 + 4q - p + 2$;
(h) since $x^2 - px/2 + q/4 = 0$,

$$2p^2x^2 - (2pq + 4p - 1)x + (q^2 + 4q - p + 2)/2 = 0$$

(using (g) above); hence

$$2(px - q/2 - 1)^2 - 1 = p/2 - x \text{ and } 2(2x^2 - 1)^2 - 1 = y;$$

similarly, $2(2y^2 - 1)^2 - 1 = x$;
(i) since $x = \cos A$ and $y = \cos B$ we have $\cos 4A = \cos B$ and $\cos 4B = \cos A$;
(j) hence $\cos 16A = 2(2\cos^2 4A - 1)^2 - 1 = 2(2\cos^2 B - 1)^2 - 1 = \cos 4B = \cos A$, so that $16A \pm A$ is a multiple of $360°$;
(k) $900° < 15A < 1080°$, so that $17A$ is a multiple of $360°$, and hence $A = (12/17)90°$;
(l) $A$ is constructible using only straightedge and compass, and hence so is $A/4$;
(m) there is a straightedge and compass construction for a regular 17-gon.

2. Use the preceding exercise to construct a regular heptadecagon.

## *Essay Question*

1. Does Plato's insistence that there be only two drawing instruments reflect the virtue of parsimony? What is parsimony, and is it a virtue?

# 14
# The Oldest Surviving Math Book

The city of Alexandria (on the northern coast of Egypt) was founded by Alexander the Great in 332 B.C. Ptolemy I made Alexandria his capital and opened a university there, about 300 B.C. This university, called the 'Museum', soon had a library with more than 600,000 papyrus rolls. This library was destroyed by the Arabs in 641 A.D.

The first chair of mathematics at the Museum was occupied by Euclid. He wrote books on optics, music, and astronomy, but his fame rests on the *Elements,* a collection of 13 small books that present the 'elements' or introductory parts of the mathematics studied in Alexandria.

None of the theorems in the 13 books can be ascribed to Euclid himself. The Pythagoreans, including Archytas, were responsible for the contents of Books I, II, VI, VII, VIII, IX, and XI. Hippocrates was the genius behind Books III and IV. For Books V and XII we can thank Eudoxus. Books X and XIII are based on the work of Theaetetus.

Euclid's contribution was the logical organisation of the *Elements* — its axiomatic structure in which everything is carefully deduced from a small number of definitions and assumptions. This structure served as a model for Aquinas's *Summa Contra Gentiles,* for Newton's *Principia,* and for Spinoza's *Ethics.* The *Elements* has been the most influential textbook in history.

Our contemporary axiomatic approach is to take sets or natural numbers as basic, but Euclid, perhaps because he did not know how to give a set theoretical or arithmetic treatment of irrationals, started with points and lines. Euclid expressed the laws of arithmetic geometrically. For Euclid, a number is a line segment.

Another difference between us and Euclid is that Euclid regarded his starting assumptions not as mere hypotheses, but as truths. He intended to instantiate the ideal described by Aristotle at the beginning of the *Posterior Analytics:* sure knowledge is obtained by the rigorous deduction of the consequences of basic truths. These truths are either definitions or existence assertions.

## A Synopsis of the *Elements*

Euclid begins the *Elements* with a list of 23 definitions. The first is that 'a point is that which has no part'. These are followed by 5 postulates governing what can be constructed and hence what has mathematical 'existence'. For example, the first postulate says: 'to draw a straight line from any point to any point'. In other words, given any two points, there is a straight line that passes through both of them.

Euclid's fifth postulate is the famous Parallel Postulate:

> if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

In other words, if $A$ and $D$ are points on the same side of line $BC$, and $\angle ABC + \angle DCB < 180°$ then there is a point on that same side of $BC$ where $BA$ (suitably lengthened if necessary) meets $CD$ (suitably lengthened if necessary).

For a long time mathematicians tried to prove the fifth postulate from Euclid's other starting assumptions (including some tacit ones), but in the nineteenth century, Eugenio Beltrami (1835–1900) and others showed that this cannot be done. The geometry obtained by adding the negation of the fifth postulate to Euclid's other axioms is a weird, but consistent, geometry called *hyperbolic geometry.* In hyperbolic geometry there are no squares, and not every triangle has a circumcircle.

Following Euclid's 5 postulates are his 5 common notions, or logical truths. The first is 'things which are equal to the same thing are also equal to one another.' The fifth common notion is 'the whole is greater than the part'. Note that in modern set theory there is a sense in which the whole set of natural numbers is not greater than its part the set of evens, since the two sets can be placed in one-to-one correspondence.

Following the 5 common notions are the 48 propositions of Book I, culminating in the theorem of Pythagoras and its converse. Before deducing its properties, Euclid is careful to show that it is indeed possible to construct a square on the hypotenuse.

Book II gives a geometric treatment of some basic algebraic identities, such as the distributive law. It also includes the Law of Cosines.

Book III derives the basic properties of the circle. Euclid attempts to give rigorous proofs of his assertions, not merely relying on the diagrams. For example, he offers a proof of the fact that a point on a chord lies in the interior of the circle.

Book IV gives constructions for the regular pentagon and the regular 15-gon. This achievement was not surpassed until 1796, when Carl Friedrich Gauss (1777–1855) found a construction for the regular 17-gon.

In Book V, Euclid uses Eudoxus's definition of proportion, together with the 'axiom of Archimedes', to deduce a basic field arithmetic for line segments. The commutativity of multiplication is proved in Proposition V 16.

In Book VI Euclid studies similar (equiangular) triangles and proves that the length of a circular arc is proportionate to the angle it subtends at the centre of the circle.

Books VII to IX are on number theory. Included are proofs for Euclid's algorithm (VII 2), the unique factorisation of square-free integers (IX 14), the infinitude of primes (IX 20), and the formula for even perfect numbers (IX 36).

Book X investigates expressions like

$$\sqrt{7 + 2\sqrt{6}}$$

reducing them, if possible, to expressions with fewer square root signs (e.g., $1 + \sqrt{6}$).

Book XI presents the basic theorems of solid geometry. Euclid constructs a cone by rotating a right triangle about one of its sides. This takes him beyond straightedge and compass constructions. Indeed, by intersecting his cones with planes, he could have constructed the parabolas that Menaechmus used to duplicate the cube. Straightedge and compass constructions are only a proper subset of the constructions found in Euclid.

Book XII is the masterpiece of Eudoxus. Without the aid of calculus, he manages to give a rigorous treatment of the volumes of the pyramid, cone, and sphere.

Book XIII is the apex of the *Elements.* For each of the 5 regular polyhedra, Euclid derives the ratio of its side to the radius of the sphere in which it can be inscribed. Euclid also proves that there are no other regular polyhedra than the 5 known to the Pythagoreans.

In the remainder of this chapter, and in the next, we shall summarise the material in Books I, III, and VI of the *Elements.* This will give the reader what he or she needs to know to do some 'Euclidean geometry'. Sadly, this is now almost a forgotten art. It used to be taught in secondary schools, but it required imagination and insight, and so the average student (not to mention the average teacher) refused to do it, and it had to be replaced by a subject called 'Memorisation of Algebraic Formulas for Rote Application on the Test'.

# Book I

Euclid's first proposition is 'to construct an equilateral triangle'. Whereas today we think in terms of a plane that is an infinite set of points and already contains an infinite number of equilateral triangles, Euclid's plane is at first empty (except for a couple of 'starting points'). Later it contains points and lines and triangles — but only those that have been constructed, one by one, using straightedge and compass.

To construct an equilateral triangle on starting segment $AB$, Euclid constructs two circumferences, one with centre $A$ and radius $AB$, and the other with centre $B$ and radius $BA$. He assumes that they do not just pass through each other without touching but meet in a point $C$. He then joins $C$ to $A$ and to $B$. Of course, it is easy to show that $ABC$ is equilateral: $AC = AB = BA = BC$.

In Proposition 4, Euclid shows that if $ABC$ and $DEF$ are triangles such that $\angle ABC = \angle DEF$, $AB = DE$, and $BC = EF$, then $ABC$ is congruent to $DEF$. (That is, each side or angle of $ABC$ has the same size as the corresponding side or angle of $DEF$, and their areas are equal too.) This is the side–angle–side or SAS theorem. Its proof is not rigorous — it involves motion — and modern geometers prefer to take the SAS theorem as another postulate. (Note that in saying $ABC$ and $DEF$ are congruent, we arrange the letters so that the angles at the corresponding vertices are equal: $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.)

In Proposition 5, Euclid applies SAS twice in order to prove that the base angles of an isosceles triangle are equal. That is, if, in triangle $ABC$, $AB = AC$, then $\angle ABC = \angle ACB$. Euclid's proof has been nicknamed the 'Pons Asinorum' or 'Bridge of Asses'. John Denton suggests that this is because the diagram looks like a bridge with piers, and the Arabic for 'bridge piers' is 'bigalu al-qantara', an expression whose literal translation is 'the asses of the bridge'.

To prove Proposition 5, Euclid produces $AB$ to $F$ and produces $AC$ the same length to $G$. By SAS, triangles $FAC$ and $GAB$ are congruent. Hence $FC = GB$ and $\angle F = \angle G$. Since $FB = GC$, it now follows that triangles $BFC$ and $CGB$ are congruent (SAS). Thus $\angle FBC = \angle GCB$ and $\angle BCF = \angle CBG$. Since $\angle ABG = \angle ACF$ (by the first congruence), it follows that

$$\angle ABG - \angle CBG = \angle ACF - \angle BCF$$

or $\angle ABC = \angle ACB$.

Proposition 6 is the converse of Proposition 5: If $\angle ABC = \angle ACB$ then $AB = AC$. For suppose $AB > AC$ and let $D$ be in $AB$ so that $DB = AC$. Join $DC$. Then triangles $DBC$ and $ACB$ are congruent (by SAS). But $DBC$ is only a part of $ACB$ and has less area. Contradiction. Similarly, we get a contradiction if we assume that $AB < AC$. So $AB = AC$.

Proposition 8 shows that if $AB = DE$, $BC = EF$, and $CA = FD$ then triangles $ABC$ and $DEF$ are congruent. For if they are not, there is a triangle $A'EF$, with $A'$ and $D$ on the same side of $EF$, which is congruent to $ABC$ but distinct from $DEF$. Since $A'E = AB = DE$, it follows that $\angle A'DE = \angle DA'E$. Similarly, $\angle A'DF = \angle DA'F$. But $\angle A'DE > \angle A'DF$ and hence $\angle DA'E > \angle DA'F$. Contradiction. Proposition 8 is the 'side-side-side' or SSS congruence theorem.

Proposition 15 tells us that if the straight line $AEB$ meets the straight line $DEC$ at $E$ then $\angle AED = \angle BEC$.

Proposition 16 states that if $ABC$ is a triangle and $D$ lies on $BC$ produced (so that $C$ is between $B$ and $D$) then $\angle ACD > \angle BAC$ and $\angle ACD > \angle ABC$. Euclid proves this by bisecting $AC$ at $E$ and producing $BE$ to $F$, so that $EF = BE$. Then, by SAS, triangles $ABE$ and $CFE$ are congruent, and hence

$$\angle BAC = \angle FCE < \angle ACD$$

Hence, if $G$ is on $AC$ produced,

$$\angle ABC < \angle BCG = \angle ACD$$

the equality following from Proposition 15.

# Exercises 14

Prove the following, using only the propositions given thus far. Do not use the fifth postulate or the fact that the sum of the angles of a triangle is two right angles.

1. Prove Proposition I 15.

2. Prove Proposition I 18: in any triangle the greater side subtends (is opposite to) the greater angle.

3. Prove Proposition I 19, which is the converse of I 18.

4. Prove Proposition I 20: in any triangle two sides taken (added) together are greater than the remaining one.

5. Prove the angle-angle-side congruence theorem (AAS): if $\angle ABC = \angle DEF$ and $BC = EF$ and $\angle ACB = \angle DFE$ then $ABC$ and $DEF$ are congruent. Also if $\angle ABC = \angle DEF$ and $AB = DE$ and $\angle ACB = \angle DFE$ then $ABC$ and $DEF$ are congruent.

## *Essay Question*

1. Are secondary school students today better off not having to learn Euclidean geometry?

# 15

# Euclid's Geometry Continued

## Book I Continued

Proposition 27 of Book I of the *Elements* tells us that if the 'alternate angles' are equal then the lines are parallel: if $\angle AGH = \angle GHD$ then $AB$ and $CD$ are parallel (see the Figure on the next page). Proof: if $AB$ (produced) does meet $CD$ (produced) at $X$ then $GHX$ is a triangle in which the 'exterior angle' $\angle AGH$ is not greater than the 'interior and opposite angle' $\angle GHD$ — against Proposition 16.

In Proposition 29, Euclid uses the fifth postulate for the first time. This is to prove the converse of Proposition 27. Suppose $AB$ is parallel to $CD$. To obtain a contradiction, suppose $\angle AGH \neq \angle GHD$, but, say, $\angle AGH > \angle GHD$. Then

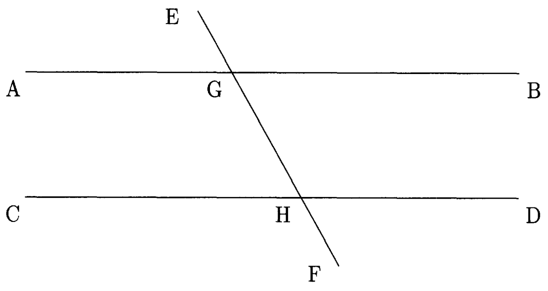
$$\angle GHD + \angle HGB < \angle AGH + \angle HGB = 180^\circ$$

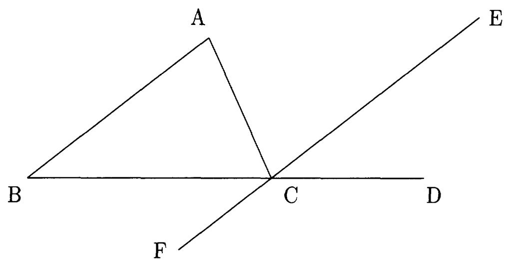
Thus, by the fifth postulate, $AB$ meets $CD$. Contradiction. Thus $\angle AGH = \angle GHD$.

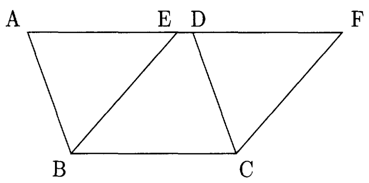
Proposition 32 shows that the sum of the angles in any triangle is two right angles. Indeed, if $FCE$ is parallel to $AB$ then $\angle ABC = \angle BCF = \angle ECD$ (Prop. 29, 15). Also $\angle BAC = \angle ACE$. Thus

$$\angle ABC + \angle BAC + \angle ACB = \angle ECD + \angle ACE + \angle ACB = 180^\circ$$

In non-Euclidean geometry, the sum of the angles of a triangle is not two right angles.




**Diagrams for Propositions I 27, 32, and 35**

Proposition 33 states that if two sides of a quadrilateral are equal and parallel, then so are the other two sides. Proposition 34 states that the opposite sides and angles of a parallelogram are equal, and the diameter bisects the area. The proofs are left to the reader.

In Proposition 35, Euclid broaches the topic of area, proving that 'parallelograms which are on the same base and in the same parallels are equal to one another' (see the Figure above). The proof is as follows. By Proposition 34, $AB = DC$ and $EB = FC$. Also $AD = BC = EF$ Thus

$$AE = AD - ED = EF - ED = DF$$

Hence by SSS, triangles $ABE$ and $DCF$ are congruent. Hence

$$\text{area } ABCD = \text{ area } ABCF - \text{ area } DCF$$

$$= \text{ area } ABCF - \text{ area } ABE = \text{ area } EBCF$$

Furthermore, by the last part of Proposition 34,

$$\text{area } ABC = \frac{1}{2} \text{ area } ABCD = \frac{1}{2} \text{ area } EBCF = \text{ area } EBC$$

In other words, 'triangles which are on the same base and in the same parallels are equal to one another' — which is Proposition 37. Similarly, we have Proposition 41: ' if a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle'.

Proposition 47 is the theorem of Pythagoras. The diagram is sometimes called the 'Bride's Chair' (see the Figure on the next page). To prove that if triangle $ABC$ has a right angle at $A$ then $BC^2 = AB^2 + AC^2$, Euclid reasons as follows. By SAS, triangles $FBC$ and $ABD$ are congruent. Thus, using Proposition 41 twice,

$$AB^2 = \text{ area } AGFB = 2 \times \text{ area } FBC = 2 \times \text{ area } ABD = \text{ area } BDLJ$$

Similarly, $AC^2 = \text{ area } JLEC$. Thus

$$AB^2 + AC^2 = \text{ area } BDLJ + \text{ area } JLEC = \text{ area } BDEC = BC^2$$

Proposition 48 is the converse of Proposition 47. Suppose $AB^2 + AC^2 = BC^2$. Let $\angle DEF$ be right, with $ED = AB$ and $EF = AC$. By the theorem of Pythagoras,

$$DF^2 = ED^2 + EF^2 = AB^2 + AC^2 = BC^2$$

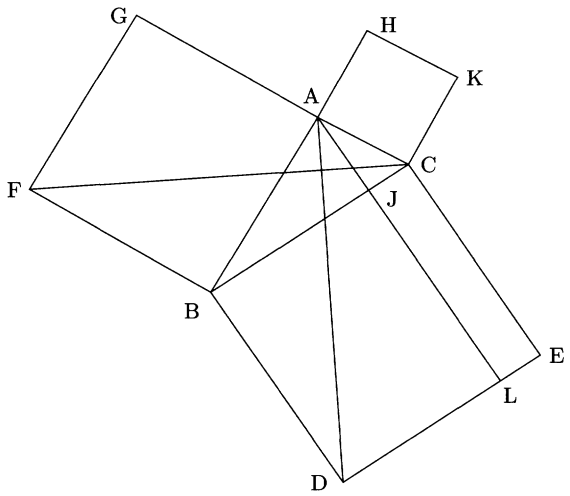
Hence $DF = BC$, and, by SSS, triangles $BAC$ and $DEF$ are congruent. Hence $\angle BAC = \angle DEF = 90°$.

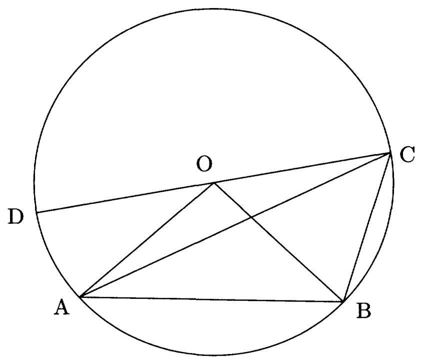
# Book III

In Book III of the *Elements,* Euclid derives the basic properties of the circle.

Proposition III 3 states that if $O$ is the centre of a circle and $AB$ a chord, with point $F$ on $AB$ (between $A$ and $B$), then $OF$ is perpendicular to $AB$ just in case $F$ is the midpoint of $AB$. This is proved using congruences.

Proposition III 12 states that if two circles touch one another externally then the straight line joining their centres passes through the 'point of




**Diagrams for Propositions I 47 and III 20**

contact'. Euclid's proof relies on Proposition I 20: 'in any triangle two sides taken together are greater than the remaining one'.

Proposition III 20 notes that if $O$ is the centre of a circle, $AB$ a chord, and $C$ a point on the circumference (on the same side of $AB$ as $O$) then

$\angle AOB = 2 \times \angle ACB$ (see the Figure above). For

$$\begin{aligned}\angle AOB &= \angle DOB - \angle DOA \\ &= \angle OCB + \angle OBC - \angle DOA \quad \text{(Prop. I 32)} \\ &= 2 \times \angle OCB - \angle DOA \qquad \text{(Prop. I 5)} \\ &= 2 \times \angle OCB - 2 \times \angle OCA \quad \text{(Prop. 32, 5)} \\ &= 2 \times \angle ACB\end{aligned}$$

This leads immediately to Proposition III 21: 'in a circle the angles in the same segment are equal'. That is, if $C$ and $D$ are on the circumference of a circle with chord $AB$, and if $C$ and $D$ are on the same side of $AB$, then $\angle ACB = \angle ADB$ (since each equals half the angle at the centre).

Proposition III 22 states that if a quadrilateral's vertices lie on the circumference of a circle then the opposite angles sum to two right angles. The converse of this is also true, but Euclid omits it.

Proposition III 35 states that if $AEC$ and $BED$ are chords of the same circle meeting at E, then $AE \times EC = BE \times ED$. This is proved using the theorem of Pythagoras.

Proposition III 36 states that if point $D$ is outside a circle and $DB$ is a tangent to that circle, while $DCA$ is a straight line such that $CA$ is a chord of that circle, then $DB^2 = DC \times DA$. Again, the theorem of Pythagoras is used in the proof.

## Book VI

Book VI concerns similar (equiangular) triangles. The key result is Proposition VI 2: if a straight line be drawn parallel to one of the sides of a triangle, it cuts the sides of the triangle proportionately — and vice versa. In other words, if $D$ is in side $AB$ of triangle $ABC$, and $DE$ is parallel to $BC$ with $E$ in $AC$, then $BD/DA = CE/EA$. Conversely, if $D$ and $E$ are in sides $AB$ and $AC$, respectively, and $BD/DA = CE/EA$, then $DE$ is parallel to $BC$.

To prove this, Euclid uses what is, in effect, the formula for the area of a triangle (derived in Proposition VI 1). If $DE$ is parallel to $BC$ then

$$BD/DA = BDE/DAE = CED/DAE = CE/EA$$

Conversely, if $BD/DA = CE/EA$, construct $DE'$ parallel to $BC$ with $E'$ in $AC$. To obtain a contradiction, suppose that $E'$ is not $E$, but that, say, $CE' > CE$. Then $CE'/E'A > CE/EA$. But, by the above, $CE'/E'A = BD/DA$, so that $BD/DA > CE/EA$. Contradiction.

Proposition VI 3 states that if $D$ is in side $BC$ of triangle $ABC$, and $AD$ bisects $\angle BAC$, then $BD/DC = BA/AC$. This is proved by constructing

$CE$ parallel to $DA$, with $E$ in $BA$ produced, and then applying Proposition VI 2.

In Proposition VI 5, Euclid shows that if $ABC$ and $DEF$ are triangles such that $AB/BC = DE/EF$, and $BC/CA = EF/FD$, and $BA/AC = ED/DF$, then they are similar.

Proposition VI 8 says that if $\angle BAC$ is right, and $D$ is in $BC$ so that $AD$ is perpendicular to $BC$, then the three triangles so formed are similar. Hence $AD^2 = BD \times DC$. That is, $AD$ is the 'mean proportional' between $BD$ and $DC$.

## Exercises 15

Give Euclidean proofs of the following. Remember that later proofs are built on earlier propositions, and you cannot prove, say, Proposition I 33 using Proposition VI 2 (which comes later). You are not allowed to use analytic geometry or trigonometry.

1. I 33.

2. I 34.

3. I 41.

4. III 3.

5. III 12.

6. III 22.

7. The converse of III 22.

8. III 35.

9. III 36.

10. VI 3.

11. VI 5.

12. VI 8.

13. Let $DBCE$ be a straight line and $A$ a point not on it. Suppose that the bisectors $BF$ of $\angle ABD$ and $CF$ of $\angle ACD$ meet at $F$. Suppose that the bisectors $BG$ of $\angle ABE$ and $CG$ of $\angle ACE$ meet at $G$. Prove that $A$ lies on the line $FG$.

14. Prove that any triangle with two equal angle bisectors is isosceles. That is, show that if $ABC$ is a triangle and $BM$ bisects $\angle ABC$, with $M$ in $AC$, and $CN$ bisects $\angle ACB$ with $N$ in $AB$, and if $BM = CN$ then $AC = AB$.

    Hint: Suppose $\angle ABC < \angle ACB$, so that $\angle ABM < \angle ACN$. Suppose $BM$ and $CN$ meet in $J$, and construct $M'$ in $JM$ so that $\angle M'CN = \angle ABM$. Then $M'$, $N$, $B$, and $C$ are all on the circumference of the same circle. (Why?) Since $\angle M'CB > \angle CBN$ (why?) and since the greater angle stands on the greater chord, it follows that $BM' > CN$. Hence $BM > CN$.

15. Let $ABC$ be any triangle. Let $A'BC$ be an equilateral triangle on $BC$, with $A'$ and $A$ on different sides of $BC$. Let $B'AC$ be equilateral with $B'$, $B$ on different sides of $AC$. Let $C'AB$ be equilateral with $C'$, $C$ on different sides of $AB$. Let $A''$ be the centre of $A'BC$, $B''$ the centre of $B'AC$, and $C''$ the centre of $C'AB$. Prove that $A''B''C''$ is equilateral.

    Hint: $AA'$, $BB'$, and $CC'$ share a common point $F$ on the circumcircles of the three original equilateral triangles.

16. An old map reads:

    > Start from the gallows and walk to the white rock, counting your paces. At that rock, turn left and walk the same number of paces. Then leave your knife in the ground. Return to the gallows. Count your paces to the black rock, turn right and walk the same number of paces. The treasure is then midway between you and your knife.

    You have the map, you have found the rocks — but the gallows are gone! How do you find the treasure?

Hint: Drop perpendiculars to the line joining the rocks. There are two possible locations for the treasure.

## *Essay Question*

1. The theory of regular polyhedra was a centerpiece of Greek mathematics. Should people be given Ph.D.s in mathematics if they have no idea how it went? Support your answer with reasons related to the purpose of education.

# 16
# Alexandria and Archimedes

The school established by Euclid in Alexandria produced some first-rate mathematicians: Aristarchus, Archimedes, Apollonius, and Eratosthenes.

## Aristarchus

Aristarchus (310–250 B.C.) came from Samos, as had Pythagoras. He gave an important application of mathematics to astronomy. Let $SEM$ be the triangle whose vertices are the sun, earth, and moon (respectively). Aristarchus reasoned that when the moon is at its first quarter, $\angle SME = 90°$. That is why we see exactly half the part of the moon's surface that faces the earth. When the moon is at its first quarter, one can see the sun and moon together in the sky, at the same time. Thus Aristarchus was able to measure $\angle SEM$. Using a drawing of a right-angled triangle with that same angle, Aristarchus found the ratio $ES/EM$. Without a telescope or space ship, he discovered that the sun is $ES/EM$ times further from the earth than the moon is. (Aristarchus thought that $ES/EM = 20$, because he failed to measure $\angle SEM$ with sufficient accuracy. Actually, $ES/EM = 300$.)

Thanks to observations of solar eclipses, Aristarchus knew that the apparent diameter of the moon is equal to that of the sun. From this he deduced that the ratio of the sun's diameter to the moon's diameter is also $ES/EM$.

By observing the shadow of the earth on the moon during lunar eclipses, one can calculate the relative sizes of earth and moon. Aristarchus used this technique and his previous results, to calculate the ratio of the sun's diameter to the earth's diameter. As we shall see, Eratosthenes had a way of determining the size of the earth. The ancient Greeks were thus able to arrive at a knowledge of the size of the sun.

Aristarchus based his work on the following correct assumptions about the solar system:
(1) The sun, moon, and earth are spheres;
(2) The earth goes around the sun, and the moon around the earth;
(3) Light travels in straight lines;
(4) The moon's light is a reflection of the sun's light;
(5) Solar eclipses are caused by the moon's blocking the sun's rays to the earth, and lunar eclipses are caused by the earth's blocking the sun's rays to the moon.

## Archimedes

Archimedes of Syracuse (287–212 B.C.) was the greatest mathematician and physicist before Isaac Newton. Many stories are told about Archimedes. One story relates that, while he was bathing, Archimedes suddenly discovered a simple way of determining the ratio of gold to silver in a gold-silver alloy. Elated by his insight, he leapt from the bath, and ran through the streets of Syracuse, shouting *Eureka!,* which means *I found it!* Archimedes, however, had forgotten to put on his clothes!

It was no accident that Archimedes made his discovery in the bath. Suppose you have an $m$ kg crown made of gold and silver. Suppose you wish to determine the number $x$ of kilograms of gold that the smith has put in it. If $g$ is the density of gold and $s$ is the density of silver, the volume of the crown is

$$v = \frac{x}{g} + \frac{m-x}{s}$$

What Archimedes realised was that, by immersing the crown in a rectangular bath tub, and observing the increase in the water level, you can determine its volume $v$. Then, solving the equation for $x$, you can obtain the mass of gold in the crown.

Thanks to his discovery, Archimedes was able to tell his friend, King Hieron, that the smith had cheated the king by charging him for pure gold, while in fact using a certain percentage of silver in the royal crown.

When Syracuse was besieged by the Romans, Archimedes constructed some machines to help defend his city. In addition to catapults and crossbows, Archimedes designed a crane that lifted the Roman ships from the water and dropped them back in, stern first. When Syracuse finally fell, the Roman general Marcellus gave orders to bring Archimedes to him unharmed. These orders were not obeyed. Archimedes was slain by an unknown soldier. There are various accounts of why this happened. Perhaps it was simply because the soldier had watched his best friend being killed by one of Archimedes's machines. Those who take the sword die by the sword (Matthew 26:52).

Archimedes wrote on many subjects, often solving problems by using what we would call calculus. He is thus, in a sense, one of the creators of that branch of mathematics. For example, he used calculus-like techniques to give the first proofs of many of our basic formulas, such as the formula for the area of a circle. As another example, he used calculus-like techniques to show that if two cylinders, each of radius 1, intersect each other at right angles, then their common volume is 16/3. (Note that a cross-section of the common volume, taken parallel to the plane containing the two axes of the cylinders, is a square. One 'adds up' these squares to get the volume. See the second edition of Schaum's *Calculus,* page 181.)

In number theory, Archimedes posed a problem that took 2200 years to solve. This is the problem of the sun god's herd of cattle, which is equivalent to the problem of finding positive integers $s$ and $t$ such that

$$s^2 - (8 \times 2471 \times 957 \times 4657^2)t^2 = 1$$

This was not done until 1965, when H. C. Williams, R. A. German, and C. R. Zarnke used a computer to generate the 206,545 digit number that is the number of cattle in the sun god's herd. (See H. L. Nelson, 'A Solution to Archimedes' Cattle Problem', *Journal of Recreational Mathematics,* Volume 13, pages 164–176.)

In geometry, Archimedes studied an area called the 'arbelos'. Let $B$ be a point in straight line $AC$. Construct three semicircles with diameters $AB$, $BC$, and $AC$, all on the same side of $AC$. The area that is in the semicircle on $AC$ but not in either of the two smaller semicircles is the *arbelos* (or 'shoemaker's knife'). Archimedes found the area of the arbelos:

> At $B$ erect a perpendicular to $AC$, to meet the large semicircle on $AC$ at $W$. Then the area of the arbelos equals the area of the circle with diameter $BW$.

## Archimedes and the Circle

As an example of Archimedes' mathematics, let us show how he proved that the area of a circle is $\pi r^2$. (The first person to give the name $\pi$ to the circle area constant was not Archimedes, but William Jones, in 1706.) Archimedes started with the following assumptions and theorems.

(1) Circles and circle segments have areas.
(2) The area of a set of pairwise disjoint triangles and circle segments equals the sum of the areas of those triangles and circle segments. If we dissect a circle into triangles and circle segments, the area of the circle is the sum

of the areas of the triangles and circle segments into which it has been dissected. Also, the area of the circle is greater than the sum of the areas of any proper subset of those triangles and circle segments.
(3) Given any circle, there is a straight line segment that is longer than the perimeter of any convex polygon inscribed in the circle and shorter than the perimeter of any polygon circumscribing the circle. Any other such segment is equal in length to this one. And this segment is equal in length to the circumference of the circle.
(4) Given any areas $e$ and $f$, there is a natural number $m$, such that $me > f$. (See Aristotle's *Physics* 266b.)
(5) A regular $2^n$-gon inscribed in a circle takes up more than $1 - \frac{1}{2^{n-1}}$ of its area. A regular $2^n$-gon circumscribed about a circle has an area less than $1 + \frac{1}{2^{n-2}}$ times that of the circle.
(6) The area of a circle is proportionate to its diameter squared (as proved by Eudoxus; see the *Elements* XII 2).

Using these assumptions and theorems, Archimedes derived the circle area formula by deriving two contradictions.

(A) Suppose the circle has area $x$ greater than the area $t$ of a right triangle whose legs equal the radius and circumference of the circle. By (4) and (5), we can find a natural number $n$ such that

$$x - \text{ inscribed regular } 2^n\text{-gon area } < x - t$$

and hence $t < 2^n$-gon area.

Let $AB$ be a side of the inscribed regular $2^n$-gon and $ON$ a perpendicular from the centre $O$ of the circle to $AB$ (with $N$ being the midpoint of $AB$). Then $ON$ is less than the radius of the circle. Using (3), we have

$$\begin{aligned} 2^n\text{-gon area} &= 2^n(\frac{1}{2}AB \times ON) \\ &= \frac{1}{2}(2^n AB)ON \\ &< \frac{1}{2} \text{ circumference } \times \text{ radius} \\ &= t \end{aligned}$$

Contradiction. Thus (A) must be rejected.

(B) Suppose now that $x < t$. By (4) and (5) there is a natural number $n$ such that

$$t > \text{ circumscribed regular } 2^n\text{-gon area}$$

However, if $AB$ is a side of the circumscribed regular $2^n$-gon, then, by (3),

$$\begin{aligned} 2^n\text{-gon area} &= 2^n(\tfrac{1}{2}AB) \times \text{ radius} \\ &= \frac{1}{2}(2^n AB) \times \text{ radius} \\ &> \frac{1}{2} \text{ circumference } \times \text{ radius} \\ &= t \end{aligned}$$

Contradiction. Thus (B) must be rejected.

Since assumptions (A) and (B) must be rejected, it follows from the Law of the Excluded Third that
(C) The area of the circle equals that of a right triangle whose legs equal the radius and circumference of that circle. That is,

$$x = \frac{1}{2} \text{ circumference } \times \text{ radius}$$

Let $k$ be the length of the line segment equal to the circumference of the circle with diameter 1. (Thus $k$ is what we call $\pi$.) Then the area of this circle is $k/4$. Let $c$ be a circle with radius $r$. Then, by (6),

$$\frac{\text{area of } c}{k/4} = \frac{(2r)^2}{1^2}$$

and hence the area of $c$ is $kr^2$.

Archimedes' proof of this formula was the culmination of 200 years of work, beginning with Antiphon (425 B.C.).

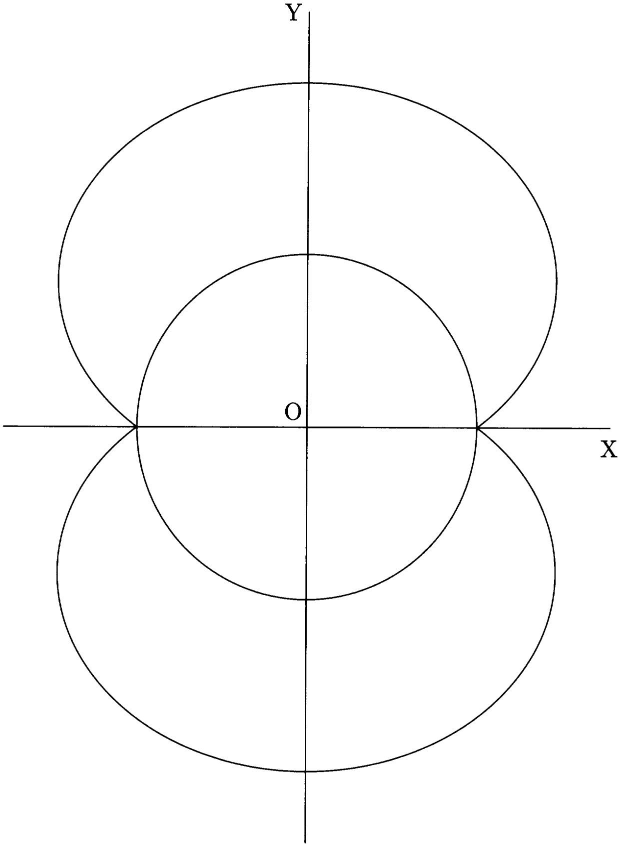
# Apollonius

Apollonius (260–190 B.C.) came from Perga in the south of what is now Turkey. He wrote a book on conics that contained 400 theorems. Three of the things he is famous for are the following.

(1) Apollonius discovered that if the same sphere passes through the vertices of an icosahedron and the vertices of a dodecahedron, then

$$\frac{\text{surface area of dodecahedron}}{\text{surface area of icosahedron}} = \frac{\text{volume of dodecahedron}}{\text{volume of icosahedron}}$$

(2) Apollonius was the first to suggest that the moon and planets move in epicycloids. This was an incorrect but very influential theory.




**The Nephroid or Kidney**

Imagine a circle of radius $r \leq 1$ rolling around the outside of the circle $x^2+y^2 = 4$. Let $P$ be a point on the rolling circle, and suppose that at time $t = 0$, $P$ touches the circle $x^2+y^2 = 4$ at $(2, 0)$. If the circle of radius $r$ rolls in the counterclockwise direction, travelling at a uniform rate and returning to its starting position in $2\pi$ seconds, then at time $t$, the coordinates of $P$ are

$$\left(\, (2+r)\cos t - r\cos\left(\left(1+\frac{2}{r}\right)t\right),\;\; (2+r)\sin t - r\sin\left(\left(1+\frac{2}{r}\right)t\right)\,\right)$$

The path traced out by $P$ is an *epicycloid.* When $r = 2$ the epicycloid is called a *cardioid;* with $r = 1$, it is a *nephroid.*

(3) Apollonius discovered the *inversion transformation.* Let $r$ be a given

positive number. Let $P$ be a point in the straight line segment $OP'$. If the product

$$OP \times OP' = r^2$$

then $P$ and $P'$ are *inverses,* each of the other, with respect to the circle with centre $O$ and radius $r$. A point in the circumference of this circle is its own inverse. It is not hard to prove the following.

(a) The inverse of a straight line through $O$ — that is, the set of points that are inverses of points in the line — is that same straight line.
(b) The inverse of a straight line that does not pass through $O$ is a circle whose circumference does pass through $O$. If $A$ is its centre, then $OA$ is perpendicular to the given line.
(c) The inverse of a circle whose circumference passes through $O$ is a straight line that does not pass through $O$. Again, if $A$ is the centre of the given circle, the straight line is perpendicular to $OA$.
(d) The inverse of a circle whose circumference does not pass through $O$ is another such circle.
(e) If a straight line is tangent to a circle at a point other than $O$, then the inverse of that line is tangent to the inverse of that circle.

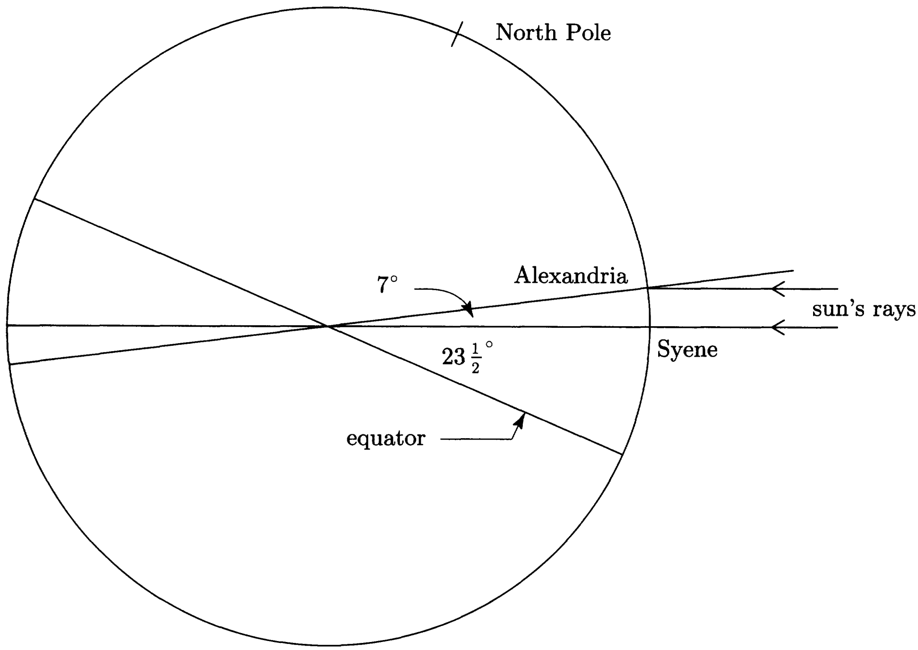
Apollonius wrote a lost treatise on 'Tangencies' in which he gave a straightedge and compass construction for a circle tangent to three given circles. Using inversion, there is an easy way to do this, and this may have been the way actually used by Apollonius.

## Eratosthenes

Eratosthenes of Cyrene (in North Africa) (275–195 B.C.) was chief librarian at Alexandria. He was interested in philosophy, poetry, history, philology, geography, astronomy, and mathematics.

Eratosthenes suggested a method for making a list of all prime numbers. This method, called the *Sieve of Eratosthenes,* works as follows. Start with the sequence of positive integers $\geq 2$. Underline the 2, and cross out all the other multiples of 2. Go to the smallest positive integer $n$ (on this list) which is neither underlined, nor crossed out, underline $n$, and cross out all the other multiples of $n$. Keep repeating the preceding step. In the end, the underlined numbers form a complete list of primes.

To measure the earth, Eratosthenes correctly assumed that since the sun is so far from the earth, those of its rays that hit the earth can be regarded as parallel. (Here he used the result of Aristarchus.) Eratosthenes knew that Syene (present-day Aswan) is on the Tropic of Cancer. That is, at noon on midsummer's day (June 21), the sun is directly overhead.




**The World on Midsummer's Day**

At Alexandria, however, at noon on midsummer's day, the sun is 7° away from the point directly overhead. Since Alexandria is due north of Syene, the arc on the earth's surface between Alexandria and Syene subtends an angle of 7° at the earth's centre (see the Figure above). Eratosthenes knew the distance from Alexandria to Syene. In our metric units, it is 800 km. Using Euclid's theorem (VI 33) that the length of an arc is proportionate to the angle it subtends at the centre of the circle, Eratosthenes concluded that the circumference of the earth is, in effect,

$$\frac{360^\circ}{7^\circ} \times 800 = 41,000 \text{ km}$$

Hence its radius is 6500 km.

Using the work of Aristarchus, Eratosthenes was able to calculate the size of the moon and the sun. Comparing their apparent sizes to their actual sizes, he was able to discover the distance to the moon and the distance to the sun — all without using any of our technology.

# Exercises 16

1. Suppose the moon is represented by the circle $x^2 + y^2 = 1$ and the earth's shadow on the moon is represented by a circle that passes through the three points (0, 0), and $(-0.1364, \pm 0.9907)$. What is the ratio of the earth's diameter to that of the moon?

2. Suppose you know the actual size of the moon. What is a simple way of finding its distance from the earth — without using anything Eratosthenes could not have used?

3. The density of gold is 19300 kg/m$^3$, while that of silver is 10,500 kg/m$^3$. Suppose that a 5 kg crown is made of gold and silver. What formula gives the mass of gold it contains in terms of its volume?

4. Give a proof for Archimedes's result about the area of the arbelos. Also show that if the circle with diameter $BW$ meets the semicircle on $AB$ at $U$ and the semicircle on $BC$ at $V$ then (1) $W$, $U$, and $A$ are collinear; (2) $UV$ is the direct common tangent to the semicircles on $AB$ and $BC$; and (3) $BW$ and $UV$ are equal in length and bisect each other.

5. Prove that a regular $2^n$-gon circumscribing a circle has an area less than $1 + \frac{1}{2^{n-2}}$ that of the circle.

6. Prove the following theorem, found in Book I of the *Conic Sections* of Apollonius. Suppose a straight line cuts a parabola in $A$ and $B$. Suppose another straight line, parallel to $AB$, cuts the parabola in $C$ and $D$. Let $M$ be the midpoint of $AB$ and $N$ the midpoint of $CD$. Produce the segment $MN$ until it cuts the parabola at $X$ and keep producing it to $Y$, so that $XY = MX$. Then $YA$ and $YB$ are tangents to the parabola. (You may use analytic geometry.)

7. Draw a cardioid.

8. Show that the equation of the nephroid is $(x^2 + y^2 - 4)^3 = 108y^2$.

9. Let $E$ be the centre of the circle $x^2 + y^2 = 4$, and let $S$ be the centre of the rolling circle of radius 1 (for the nephroid). Show that $P$ goes

around $S$ 3 times for every time $E$ goes around $S$ — taking the point of view of someone sitting on $S$.

## *Challenges for Experts*

1. Show that the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ have a common volume of 16/3. (Do not try this problem if you do not know some calculus.)

2. Prove Properties (a) to (e) of inversion.

3. Draw two nonoverlapping circles of different radii and a point $P$ outside both of them. With centre $P$, draw a circle with radius so big that it encloses the first two circles. Take this big circle as a circle for inversion. Find a Euclidean method for constructing the inverses of the first two circles. These inverses will be circles. Construct a tangent common to these two circles. Construct the inverse to this tangent. This final inverse is a circle tangent to the two original circles and passing through $P$.

4. Given three nonoverlapping circles, give a straightedge and compass construction for a circle tangent to all three.

## *Essay Question*

1. If Archimedes were alive today, would he have a moral obligation *not* to help his country design weapons? Support your answer with reasons related to the role of science in history.

# 17
# The End of Greek Mathematics

## Hipparchus

Hipparchus (180–125 B.C.) came from Nicaea, the town near present-day Istanbul, which was to be the site of the great pro-monotheistic council of 325 A.D. Hipparchus was an astronomer. He calculated the duration of the year to within 6 minutes.

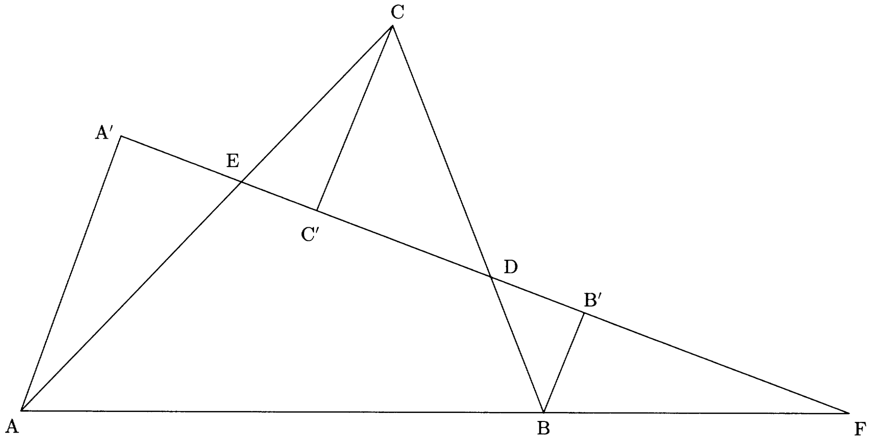
Hipparchus was the father of trigonometry. He drew up a table giving, for each whole number angle with vertex at the centre of a circle of radius 60, the length of the chord it cuts off that circle. For example, suppose $\angle AOB = 30°$, with $O$ the centre of the circle and $OA = OB = 60$. Then the chord in question is the segment $AB$. This has length 31.06, so that in Hipparchus's table, we find

$$\text{chord}(30°) = 31.06$$

In modern terms, $\text{chord}(x) = 120 \times \sin(\frac{x}{2})$. To construct his table, Hipparchus used formulas we would express as

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$2\sin^2(\frac{x}{2}) = 1 - \cos x$$




**Menelaus's Theorem**

# Menelaus

Menelaus of Alexandria (100 A.D.) founded spherical trigonometry. He is known for the following theorem, found, for the first time, in his *Spherica.*

> Let $ABC$ be a triangle. Suppose $D$ is on the line through $B$ and $C$, $E$ is on the line through $A$ and $C$, and $F$ is on the line through $A$ and $B$. If exactly none or two of the points $D$, $E$, and $F$ are on the sides of the triangle then: $D$, $E$, and $F$ are collinear if and only if $BD \times CE \times AF = CD \times AE \times BF$.

**Proof:** Suppose $D$, $E$, and $F$ are collinear in line $z$. We may suppose $z$ does not pass through $A$, $B$, or $C$. Let $A'$, $B'$, and $C'$ be points in $z$ such that $AA'$, $BB'$, and $CC'$ are all perpendicular to $z$. Then

$$\frac{BD}{CD} = \frac{BB'}{CC'}$$

$$\frac{CE}{AE} = \frac{CC'}{AA'}$$
$$\frac{AF}{BF} = \frac{AA'}{BB'}$$

Multiplying, we obtain the result. The converse now follows.

## Nicomachus

Nicomachus of Gerasa (near Jerusalem) lived about 100 A.D. He was a neo-Pythagorean, and he is known for being one of the first thinkers to locate the natural numbers in the mind of God. In his number theory book, the *Introductio arithmeticae,* he considers the following infinite triangle, noting that the sum of the numbers in the $n$th row is $n^3$.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 5 & & \\ & 7 & & 9 & & 11 & \\ 13 & & 15 & & 17 & & 19 \end{array}$$

## Diophantus

In 250 A.D., Emperor Decius was executing Christians who refused to sacrifice to pagan gods. In Rome, Plotinus was teaching his version of Platonism. In Alexandria, Diophantus was working on his *Arithmetica.*

In *The History of the Church,* Eusebius tells us that Bishop Anatolius (260 A.D.) wrote a book called *Elements of Arithmetic.* According to Michael Psellus (in the eleventh century), Anatolius dedicated a tract on Egyptian computation to Diophantus. Diophantus himself dedicated his *Arithmetica* to Dionysius, Bishop of Alexandria from 247 to 264. We do not know if Diophantus himself was a Christian, but it is not impossible.

There were originally 13 books in the *Arithmetica.* Until 1973, we had only 6 of them. Then 3 more were discovered in an Arabic translation going back to the ninth century. (See Jacques Sesiano, *Books IV to VII of Diophantus' Arithmetica.*)

The *Arithmetica* consists of solutions to algebraic problems. The solutions are all rational numbers. Some of the problems are indeterminate and have more than one rational number solution. Diophantus is usually content to give just one solution, but in connection with Problem VI 15, he mentions an equation 'one can solve in an infinite number of ways'.

As an example, let us consider Problem 9 of Book II:

> to divide a given number which is the sum of two squares into two other squares

That is, given rationals $a$ and $b$, find a nontrivial rational solution of

$$x^2 + y^2 = a^2 + b^2$$

Diophantus takes the special case where $a = 2$ and $b = 3$, but his solution is easily generalised. He writes:

> take $(x + 2)^2$ as the first square and $(mx - 3)^2$ as the second, say $(2x - 3)^2$. Therefore
>
> $$(x^2 + 4x + 4) + (4x^2 + 9 - 12x) = 13$$
>
> or $5x^2 + 13 - 8x = 13$. Therefore $x = 8/5$, and the required squares are $324/25$ and $1/25$.

Note that with $a = 1$ and $b = 0$, we get an analysis of a typical Pythagorean triple.

In connection with problem III 19, Diophantus notes that 65 is a sum of two squares in two ways since 65 'is the product of 13 and 5, each of which numbers is the sum of two squares'. This remark led T. L. Heath to speculate that Diophantus was aware of the relations

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$$

The person who discovered these identities was actually al-Khazin (950 A.D.).

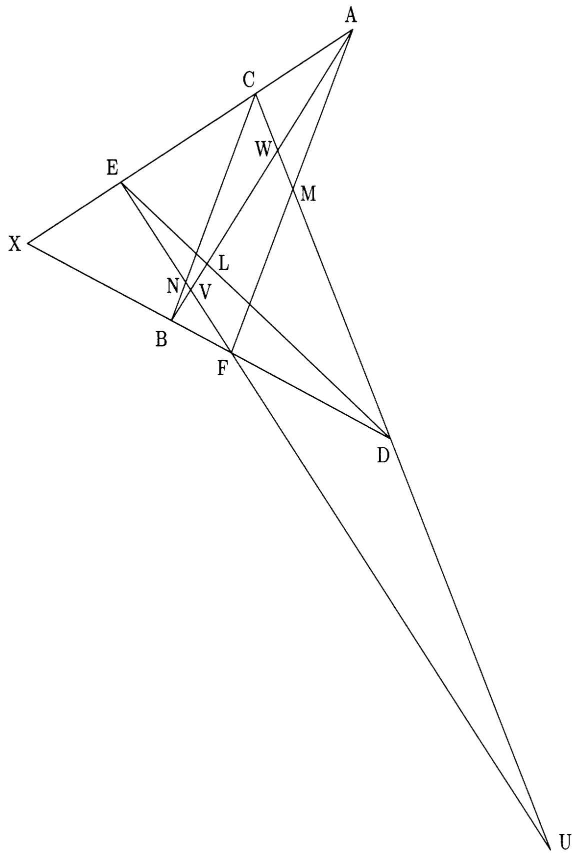
Diophantus was the 'father of algebra' in the sense that he was the first to make systematic use of a symbolic notation for algebraic expressions. He denoted + by juxtaposition, and $-$ by the symbol $\pitchfork$ .

He wrote $K^{\Upsilon}$ for $x^3$, $\Delta^{\Upsilon}$ for $x^2$, and $\varsigma$ for $x$.

## Pappus

In 320 A.D., the Roman Empire had its first Christian emperor, Constantine. In Alexandria, Athanasius was defending the divinity of Jesus against Arius, who thought that Jesus was merely a special kind of human. Also in Alexandria, Pappus was writing his encyclopaedic *Collection.* The school of mathematics had declined, and Pappus was its last lone genius.

The following 'Theorem of Pappus' (actually due to Euclid) is Proposition 139 in Book VII of the *Collection.* It is more important than Pappus




**Theorem of Papus**

realised. It expresses the commutativity of multiplication. It is fundamental in projective geometry. Hilbert used it as a key theorem in his presentation of Euclidean geometry.

**The Theorem of Pappus**

Suppose we have a straight line with points $X$, $E$, $C$, and $A$ on it (in that order) and another straight line with points $X$, $B$, $F$, and $D$ on it (in that order), meeting the first line in $X$. Suppose $ED$ meets $AB$ in $L$, and $EF$ meets $BC$ in $N$, and $CD$ meets $AF$ in $M$. Then $L$, $M$, and $N$ are collinear.

**Proof:** Suppose $CD$ and $EF$ meet in $U$. (If they are parallel, the proof is slightly different.) Suppose $AB$ meets $CD$ in $W$ and $EF$ in $V$. (See the figure above.)

We apply the theorem of Menelaus 6 times, each time taking $UVW$ as the triangle.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| With | $L$, | $D$, | $E$, | we have | $VL \times WD \times UE$ | $=$ | $LW \times DU \times EV$ |
| With | $A$, | $M$, | $F$, | we have | $VA \times WM \times UF$ | $=$ | $AW \times MU \times FV$ |
| With | $B$, | $C$, | $N$, | we have | $VB \times WC \times UN$ | $=$ | $BW \times CU \times NV$ |
| With | $A$, | $C$, | $E$, | we have | $VA \times WC \times UE$ | $=$ | $AW \times CU \times EV$ |
| With | $B$, | $D$, | $F$, | we have | $VB \times WD \times UF$ | $=$ | $BW \times DU \times FV$ |

Multiplying the first three equations and dividing by the product of the last two equations, we obtain

$$VL \times WM \times UN = LW \times MU \times NV$$

and the result follows.

## Hypatia

Hypatia (d. 415) was the daughter of Theon of Alexandria, who put out an edition of Euclid's *Elements.* Hypatia wrote commentaries on Apollonius and Diophantus.

According to Socrates Scholasticus (380–450 A.D.), in Chapter 15 of Book VII of his *History of the Church,* Hypatia was murdered by a mob of 'Christians', led by one 'Peter'. This tragedy is sometimes blamed on the Christian bishop, Cyril, but there is no evidence to support this accusation. Cyril was a zealous leader, but we have no reason to think he 'incited' the crowd to make a physical attack on the pagan mathematician. Indeed, we have no reason to think that the murder had anything to do with religion and science. For all we know, the mob killed Hypatia simply because they were poor and unemployed, while Hypatia had a permanent well-paid job.

## Conclusion

Towards the end of this era, Greek mathematics degenerated into mere commentaries and riddle solving. In 529 Emperor Justinian closed Plato's Academy, apparently because it opposed the Christian revelation, and the few remaining scholars went to Persia. In 641 A.D., Alexandria fell to the Arabs, who burned the famous library. This event may be taken to mark the final end of ancient Greek mathematics.

It is sad that there were so few mathematicians in the early Christian church. Anatolius, and possibly Diophantus, were exceptions. Most of the mathematicians at the Academy and the Museum rejected the new truths of Christ's revelation. This was unfortunate because the split between the old scientific learning and the vibrant new faith weakened the Roman Empire, which was the bulwark of civilisation in the West. If the mathematicians had joined the Christians, the Dark Ages would have been brightened by a dialogue between reason and faith. As it was, this dialogue was postponed to the later Middle Ages, when thinkers like Thomas Aquinas (1225–1275) advanced philosophies that were influenced as much by the *Elements* as by the Bible.

# Exercises 17

1. What is the chord of $17°$?

2. Give the details of the proof of the converse in the theorem of Menelaus.

3. Prove the theorem of Nicomachus on the triangle of odd numbers.

4. Problem XI of Book IV of the *Arithmetica* of Diophantus is to find a rational solution of $x^3 - y^3 = x - y$. Show that if $a$ and $b$ are relatively prime integers, one solution is

$$x = \frac{\pm(b^2 - 3a^2) - 2ab}{3a^2 + b^2} \quad \text{and} \quad y = \frac{4ab}{3a^2 + b^2}$$

5. Prove the theorem of Pappus in the case in which $CD$ and $EF$ are parallel.

## *Essay Questions*

1. Is it morally permissible to spend your days doing pure mathematics while your government is murdering members of a minority religion? What if you are yourself a member of that minority religion?

2. Look up the accounts of Hypatia's death in several history of math books. Do they reflect any bias, going beyond the facts, to glorify Hypatia for some ideological purpose?

# 18
# Early Medieval Number Theory

## Sun Tsu

Sun Tsu (400 A.D.) was one of the first mathematicians to work on the 'Chinese Remainder Problem'. He gave a way of calculating the solutions to the following problem:

> divide by 3, the remainder is 2;
> divide by 5, the remainder is 3;
> divide by 7, the remainder is 2;
> what is the number?

Sun Tsu also gave a formula for determining the sex of a foetus:

> Take 49; add the number of the month in which the woman will give birth; subtract her age. From what now remains, subtract the heaven 1, subtract the earth 2, subtract the man 3, subtract the four seasons 4, subtract the five elements 5, subtract the six laws 6, subtract the seven stars 7, subtract the eight winds 8, subtract the nine provinces 9. If then the remainder be odd, the child shall be a son; and if even, a daughter.

Note that in equating the odd with the masculine, Sun Tsu is in agreement with Pythagoras.

## Thabit Ibn-Qurra

Thabit (836–901) lived in Baghdad and was an active member of a neo-Pythagorean group called the Sabians. He wrote on politics, grammar, Plato's *Republic* , smallpox, bird anatomy, beam balances, seawater salinity, sundials, the Parallel Postulate, cubic equations, and the new crescent moon. He also did work in spherical trigonometry and what we would call calculus.

Unlike Aristotle, Thabit believed there is an actual infinity.

In his book *Book on the Determination of Amicable Numbers,* he gave a new rule:

> Let $n$ be a positive integer $> 1$.
> Let $p = 3 \times 2^n - 1$, and $q = 3 \times 2^{n-1} - 1$, and $r = 9 \times 2^{2n-1} - 1$.
> If $p$, $q$, and $r$ are primes, then $2^n pq$ and $2^n r$ are amicable
> (that is, each is equal to the sum of the proper divisors of the other).

When $n = 2$, we have $p = 11$, $q = 5$, and $r = 71$, and we get the amicable pair 220 and 284. When $n = 3$ (or any multiple of 3), $r$ is divisible by 7, and hence not prime. However, when $n = 4$, we obtain the amicable pair 17,296 and 18,416.

It is not known if Thabit's rule generates infinitely many amicable pairs, but it is known that there are some amicable pairs it does not generate, such as the pair 1184 and 1210, which was first discovered in 1866, by the sixteen-year-old B. N. I. Paganini.

As we shall see in Chapter 20, Thabit also gave a generalisation of the theorem of Pythagoras.

## Brahmagupta

Brahmagupta (628 A.D.) was the first person to give a systematic presentation of rules for working with negative numbers. He wrote:

> Positive, divided by positive, or negative by negative, is affirmative. Cipher [zero] divided by cipher, is nought. Positive divided by negative, is negative. Negative, divided by affirmative, is negative.

Brahmagupta had trouble with zero, but the rest is correct.

Thanks to his theory of negative numbers, Brahmagupta was able to give a complete solution of the linear Diophantine equation $ax + by = F$. His solution was essentially the following.

Since there is no solution unless $\gcd(a, b)$ divide $F$, we may, without loss of generality, take it that $\gcd(a, b) = 1$. For example, to solve

$$4x + 6y = 2$$

it suffices to solve the equation we get by dividing out the gcd of 4 and 6, namely, $2x + 3y = 1$. In this second equation, the coefficients of $x$ and $y$ are relatively prime.

Using Euclid's algorithm (see Chapter 7), we can find a single solution $x = g$ and $y = h$. Moreover, for any integer $k$,

$$a(g + bk) + b(h - ak) = F$$

Thus all the numbers contained in the formulas $x = g + bk$ and $y = h - ak$ are solutions to the equation.

Brahmagupta was no doubt aware of the fact that there are no other solutions. This can be shown as follows.

If $am + bn = F$ then

$$a(g + m - g) + b(h + n - h) = F$$

and hence $a(m - g) = -b(n - h)$. Thus $a|(-b(n - h))$. Since $\gcd(a, b) = 1$, it follows that $a|h - n$. Say $ak = h - n$. Then $a(m - g) = abk$, so that $m - g = bk$. Hence $m = g + bk$ and $n = h - ak$.

Brahmagupta also discovered a formula for the area of a cyclic quadrilateral, as we shall see in Chapter 20.

## Bhaskara

Like other early Indian mathematicians, Bhaskara (1114–1185) liked to write mathematics in poetry:

> The square root of half the number of bees in a swarm
> Has flown out upon a jasmine bush;
> Eight ninths of the swarm has remained behind;
> A female bee flies about a male who is buzzing inside a lotus flower;
> In the night, allured by the flower's sweet odour, he went inside it
> And now he is trapped!
> Tell me, most enchanting lady, the number of bees.

This is certainly a romantic way of asking for the solution of

$$\sqrt{x/2} + (8/9)x + 2 = x$$

One of Bhaskara's books was named after, and addressed to, his daughter, Lilavati. She is the 'enchanting lady' mentioned above. According to an anecdote passed on by a Persian translator (Fyzi), astrologers had foretold that there was but one lucky moment at which Lilavati might marry. Unfortunately, one of Lilavati's pearls fell into the water clock, and it stopped without anyone noticing. The single lucky moment passed, and Bhaskara had to cancel the wedding. Since he could no longer give her a husband, Bhaskara decided to give his daughter a math book instead.

Probably this story is false. A clever mathematician would have enough sense not to sacrifice his daughter's marriage to a foolish superstition.

One of Bhaskara's feats in number theory consisted in finding the smallest positive integer solution of

$$x^2 - 61y^2 = 1$$

namely, $x = 1,766,319,049$ and $y = 226,153,980$. To do this he used the *cakravala* or 'cyclic process'. This process is equivalent to the simple continued fraction method, foreshadowed in Euclid's algorithm (see Chapter 7), but not fully explained until J. L. Lagrange (1736–1813) wrote a paper on the subject, which appeared in 1768.

Bhaskara believed in an actual infinity in mathematics. In his *Vija-Ganita* or *Basic Arithmetic,* he writes:

> Quotient the fraction $\frac{3}{0}$. This fraction, of which the denominator is cipher [zero], is termed an infinite quantity. In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth.

# Exercises 18

1. Find all the solutions to Sun Tsu's Chinese Remainder Problem.

2. Solve the following problem of Sun Tsu: 'A pregnant woman, who is 29 years of age, is expected to give birth to a child in the 9th month of the year. Which should be her child, a son or a daughter?'

3. When does Sun Tsu's rule for sex determination give rise to a negative number?

4. Pursued by a lion, a tourist and her guide are dashing up the steps of a pyramid. The tourist takes 5 steps at a time, the guide 6, and the lion 7. Towards the end of this tale, the tourist is 1 step from the top, the guide 9, and the lion 19. How many steps are in the pyramid?

5. Show that 1184 and 1210 are amicable.

6. Show that 1184 and 1210 are not given by Thabit's rule.

7. Prove that if $n$ is a multiple of 3, then $r$ (in Thabit's rule) is a multiple of 7.

8. Prove that Thabit's rule works.

9. Give all the solutions to the Diophantine equation $101x + 753y = 100,000$.

10. How many bees were there?

11. Solve $x^2 - 13y^2 = 1$ in positive integers.

12. Find the smallest 4 square triangular numbers.

13. 'Of a flock of geese, ten times the square root of the number departed for the *Manasa* lake, on the approach of a cloud. An eighth part went to a forest of *St'halapadminis.* Three couples were seen engaged in sport, on the water abounding with delicate fibres of the lotus. Tell, dear girl, the whole number of the flock.'

## *Essay Question*

1. What does Bhaskara's theological explanation of the fact that $\infty \pm x = \infty$ imply about his conception of God and the universe?

# 19
# Algebra in the Early Middle Ages

## Wang Hs'iao-t'ung

Wang served on the Astronomical Board of the T'ang government, about 625 A.D. One of the problems he solved was the following:

> There is a right-angled triangle, the product of whose legs is 706.02 and the hypotenuse of which exceeds one side by 36.9. Find it.

This leads to a cubic equation that can be solved by inspection.

## Al-Khwarizmi

At the beginning of the ninth century, Caliph al Mamun established a 'House of Wisdom' at Baghdad. One of the first mathematicians associated with this House was Muhammed ibn-Musa al-Khwarizmi (825 A.D.), who came from the area south of the Aral Sea in central Asia. Our word 'algorithm' comes from a book he wrote on the use of Indian numerals. The book began, 'Spoken has al-Khwarizmi ...', or, in the Latin translation, 'Spoken has Algoritmi ...'.

Al-Khwarizmi's most important work was the *Hisab al-jabr w'al-muqabalah,* from which we get the word 'algebra'. The word 'al-jabr' means 'combining', as in 'combining like terms' to solve an equation.

Al-Khwarizmi's 'Algebra' contains nothing that was not known to the ancient Greeks. There are few proofs, and one of them is woefully inadequate. This is al-Khwarizmi's 'proof' of the theorem of Pythagoras, which only works if the right triangle is isosceles!

Al-Khwarizmi gives three approximations for $\pi$. None of them is supported by any reasoning, and Al-Khwarizmi does not seem to care which one is used. Al-Khwarizmi was a transmitter of ancient Greek knowledge, not an original mathematician.

# Exercises 19

1. Solve Wang's problem.

2. One of the problems transmitted by Al-Khwarizmi was the following problem of Heron (75 A.D.): find the side of a square inscribed in a triangle with sides 10, 10, and 12.

3. Abu Ja'far al-Khazin (950 A.D.) gave the following identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac \mp bd)^2 + (bc \pm ad)^2$$

Prove that it is correct.

## *Challenges for Experts*

1. In the *Nine Sections of Mathematics* (1247 A.D.), Ch'in Chiu Shao found a root of

$$x^4 - 763,200x^2 + 40,642,560,000$$

using what is, in effect, Horner's method (rediscovered by William Horner in 1819). Show that Shao's polynomial has four linear factors.

2. Horner's method for solving, say, $f(x) = x^3 + ax^2 + bx + c = 0$, works as follows. Without loss of generality, suppose $f(x)$ has a single root $r$ between 0 and 10. Let $d$ be its first scale ten digit. Let

$$\begin{aligned} a' &= 10(3d + a) \\ b' &= 100(3d^2 + 2da + b) \\ c' &= 1000f(d) \\ g(x) &= x^3 + a'x^2 + b'x + c' \end{aligned}$$

Then $g(x)$ has a single root $r'$ between 0 and 10. Moreover, if $d'$ is the first digit of $r'$, then $d'$ is the second digit of $r$. For $f(d+(r-d)) = 0$ iff $g(10(r-d)) = 0$. Use this to find the first 3 digits in the cube root of 2.

3. Use Horner's method to find a root of Shao's polynomial.

## *Essay Question*

1. Who has a better right to the title 'Father of Algebra', and why: Diophantus or al-Khwarizmi?

# 20
# Geometry in the Early Middle Ages

## Thabit Again

Thabit Ibn-Qurra (836–901) gave a generalisation of the theorem of Pythagoras:

> Let $ABC$ be a triangle with $\angle A$ obtuse. Let $B'$ and $C'$ be in $BC$ such that $\angle AB'B = \angle AC'C = \angle BAC$. Then $AB^2 + AC^2 = BC(BB' + CC')$.

Apart from the above contribution, the only important original work in geometry in the early Middle Ages was done by Brahmagupta (628 A.D.).

## Brahmagupta Again

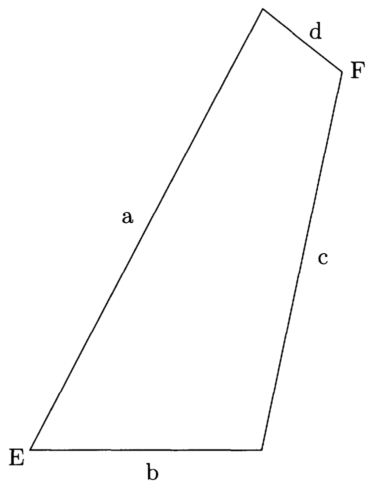
Brahmagupta's most striking achievement was his discovery of the following formula for the area of a cyclic quadrilateral with sides $a$, $b$, $c$, and $d$:

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = \frac{1}{2}(a+b+c+d)$ is the semiperimeter.

Brahmagupta did not give a proof that this is, indeed, the formula for the area of the quadrilateral. If he had, he might have remembered to say that the quadrilateral does have to be cyclic (that is, inscribed in a circle). This he omitted to do.

We can prove Brahmagupta's formula as follows. Suppose that a convex quadrilateral has sides $a$, $b$, $c$, and $d$, with $\angle E$ between sides $a$ and $b$ and $\angle F$ between sides $c$ and $d$.




Then its area squared is

$$(\frac{1}{2}ab\sin E + \frac{1}{2}cd\sin F)^2$$

Now

$$(s-a)(s-b)(s-c)(s-d) = \frac{1}{4}(ab+cd)^2 - \frac{1}{16}(a^2+b^2-c^2-d^2)^2$$

so that, by the Law of Cosines,

$$4(s-a)(s-b)(s-c)(s-d) = (ab+cd)^2 - \frac{1}{4}(2ab\cos E - 2cd\cos F)^2$$

Thus

$$4 \times \text{area squared} = 4(s-a)(s-b)(s-c)(s-d) - 2abcd(\cos(E+F)+1)$$

iff

$$\begin{aligned} &a^2b^2\sin^2 E + 2abcd\sin E\sin F + c^2d^2\sin^2 F \\ &= (ab+cd)^2 - (a^2b^2\cos^2 E - 2abcd\cos E\cos F + c^2d^2\cos^2 F) \\ &\quad -2abcd\cos(E+F) - 2abcd \end{aligned}$$

Collecting the terms with $a^2b^2$, those with $c^2d^2$, and those with $abcd$, we see that this is true. Hence we have the following theorem.

**Theorem:** A convex quadrilateral with sides $a$, $b$, $c$, and $d$, with $\angle E$ between $a$ and $b$, and $\angle F$ between sides $c$ and $d$ has area

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(\cos(E+F)+1)}$$

where $s = (a+b+c+d)/2$.

Hence if the quadrilateral is cyclic, so that $E + F = 180°$, the area is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Note that with $d = 0$ we obtain Archimedes' formula for the area of a triangle.

Brahmagupta also gave a formula for the diagonal of a cyclic quadrilateral. Suppose $g$ is the diagonal opposite angles $E$ and $F$. Then

$$a^2 + b^2 - 2ab\cos E = g^2 = c^2 + d^2 - 2cd\cos F$$

But $\cos F = -\cos E$ (since the quadrilateral is cyclic) and hence

$$\cos E = (a^2 + b^2 - c^2 - d^2)/2(ab + cd)$$

Thus

$$g^2 = a^2 + b^2 - ab(a^2 + b^2 - c^2 - d^2)/(ab + cd)$$

or

$$g^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}$$

Similarly, the square of the other diagonal is

$$h^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}$$

Using these formulas, Brahmagupta found quadrilaterals whose sides, diagonals, and areas are all rational. One of these was the cyclic quadrilateral with sides 52, 25, 39, and 60.

# Exercises 20

1. Prove Thabit's generalisation of the theorem of Pythagoras.

2. Prove that if a triangle has sides of lengths $a$ and $b$ enclosing an angle $E$ then its area is $\frac{1}{2}ab\sin E$.

3. Prove that if $s = (a+b+c+d)/2$ then

$$(s-a)(s-b)(s-c)(s-d) = (ab+cd)^2/4 - (a^2+b^2-c^2-d^2)^2/16$$

4. Prove that a quadrilateral is cyclic iff its opposite angles are supplementary (that is, add up to 180°).

5. If you know the sides of a cyclic quadrilateral, how can you determine the radius of the circumscribing circle?

6. Prove that there is a cyclic quadrilateral with sides 52, 25, 39, and 60. Prove that its diagonals and area are all integers.

7. Show that there is a cyclic quadrilateral with sides 25, 25, 25, and 39. What are the lengths of its diagonals and its area?

## *Essay Question*

1. Why was so little mathematics done between the years 500 and 1000 ?

# 21

# Khayyam and the Cubic

## The Three Friends

When Omar Khayyam (1050–1123) was young, he and two fellow students, Nizam and Hassan, promised that if one of them became rich, he would share his wealth with the other two. Nizam did become rich, and it was only thanks to Nizam's money that Omar could give up tentmaking for mathematics. The word 'khayyam' means tentmaker.

## The Positive Roots of the Cubic Equation

Omar spent some time writing poetry and some time working on a reform of the calendar, but much of his life was devoted to the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

There are 8 possibilities for the signs of $a$, $b$, and $c$, and Omar treated each one separately, finding positive solutions by means of intersecting conics. In some cases, Omar found the solution as the abscissa of an intersection of a hyperbola and a circle. In others, it was the abscissa of the intersection of two hyperbolas. Sometimes he used a parabola.

Thanks to our modern notation and our system of negative numbers, we can summarise Omar's work in the following trivial theorem:

**Theorem**

$$x^3 + ax^2 + bx + c = 0 \text{ and } y = x^2$$

iff

$$(x+a)(y+b) = ab - c \text{ and } y = x^2$$

Hence $r$ is a real solution of $x^3 + ax^2 + bx + c = 0$
iff $r$ is the abscissa of a point where the hyperbola (or straight lines) $(x+a)(y+b) = ab - c$ meets the parabola $y = x^2$.

The hyperbola in question has asymptotes $x = -a$ and $y = -b$. It passes through the point $(0, -c/a)$, assuming $a \neq 0$.

For example, consider the equation

$$(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6 = 0$$

The hyperbola has asymptotes $x = 6$ and $y = -11$. Its upper left branch meets the parabola $y = x^2$ at $(1,1)$, $(2,4)$, and $(3,9)$.

## The Rubaiyat

In the introduction to his book on cubic equations (the *Al-jabr W'al Muqabalah),* Omar comes across as a good Muslim. He writes:

> Praise be to God, lord of all worlds, a happy end to those who are pious, and ill-will to none but the merciless. May blessings repose upon the prophets, especially upon Mohammed.

Omar complains that

> Most of our contemporaries are pseudo-scientists who mingle truth with falsehood, who are not above deceit and pedantry, and who use the little that they know of the sciences for base material purposes only. When they see a distinguished man intent on seeking truth, one who prefers honesty and does his best to reject falsehood and lies, avoiding hypocrisy and treachery, they despise him and make fun of him. In all circumstances we seek refuge in God, the Helper.

Omar was less religious in his poetry. The *Rubaiyat* (or Quatrains) is a pessimistic work in which Omar claims that the only important thing is wine, and the only certain thing is endless death:

> Oh, threats of Hell and Hopes of Paradise!
> One things at least is certain — *This* Life flies;
> One thing is certain and the rest is Lies;
> The Flower that once has blown for ever dies.

Needless to say, Omar was not popular with the Muslim authorities.

# Exercises 21

1. Prove the trivial theorem.

2. On the same graph, draw $(x-6)(y+11) = -60$ and $y = x^2$, showing the intersection points.

3. Find an approximate solution to $x^3 + 2x^2 + 10x = 20$ by drawing a careful graph of $(x+2)(y+10) = 40$ and $y = x^2$. (As we shall see, this equation was important in the life of Fibonacci (1180–1250).)

4. One of the problems in Khayyam's *Al-jabr W'al Muqabalah* is to 'divide ten into two parts [summands] so that the sum of the squares of both parts plus the quotient obtained by dividing the greater [summand] by the smaller is equal to seventy-two'. Using algebra, find the exact solutions.

5. One of Omar's theorems, expressed in terms of modern analytic geometry, is the following. Suppose $b > 0$, $x$ is real, and $x \neq -c/b$. Then $x^3 + ax^2 + bx + c = 0$ iff $x$ is the first coordinate of a point where the circle
$$\left(x + \frac{a + c/b}{2}\right)^2 + y^2 = \left(\frac{a - c/b}{2}\right)^2$$
meets the hyperbola $x(y - \sqrt{b}) = c/\sqrt{b}$. Prove this.

## *Essay Questions*

1. Would Omar have condemned those 'who use the little that they know of the sciences for base material purposes only' if he had not been independently wealthy? Why, or why not?

2. Omar's life was a failure. His mathematics was trivial, and he failed to find God. Do you agree with this evaluation? Why, or why not?

# 22
# The Later Middle Ages

## Fibonacci and the Rabbits

In the thousand years from 300 to 1300, there was only one outstanding European mathematician, namely, Leonardo of Pisa (1180–1250), who was known as Fibonacci. He learned his mathematics in Algeria, where his father was a custom-house officer.

In 1202, Fibonacci published his *Liber Abaci* in which he explained the Arabic system of numerals we now use and gave the Rabbit Problem:

> Suppose that rabbit pregnancy lasts one month, and that every female rabbit gets pregnant at the beginning of every month, from the time she is one month old on. Suppose that female rabbits always give birth to two bunnies, one male and one female. How many pairs of rabbits will you have on January 2, 1203 if you start with a newborn pair on January 1, 1202?

The number of rabbit pairs increases as follows:

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad \ldots$$

If $F_n$ is the $n$th *Fibonacci number,* we have

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}$$

## De Moivre's Formula

The following theorem was discovered by Abraham de Moivre, about 1730.

**Theorem** Let $s = \frac{1+\sqrt{5}}{2}$ — this is the 'golden ratio'. Then the $n$th Fibonacci number is the natural number nearest $s^n/\sqrt{5}$.
**Proof:** Let $r = \frac{1-\sqrt{5}}{2}$. Then $r + s = 1$ and $rs = -1$. Thus

$$\begin{aligned} F_{n+1} - rF_n &= s(F_n - rF_{n-1}) \\ &= s^2(F_{n-1} - rF_{n-2}) \\ &= \ldots \\ &= s^{n-1}(F_2 - F_1) \\ &= s^{n-1}(1-r) \\ &= s^n \end{aligned}$$

Similarly, $F_{n+1} - sF_n = r^n$. Subtracting, we obtain $(s-r)F_n = s^n - r^n$, so that

$$F_n = s^n/\sqrt{5} - r^n/\sqrt{5}$$

Furthermore, $r^n/\sqrt{5}$ is close to 0, and it is not hard to show now that $F_n$ is the natural number closest to $s^n/\sqrt{5}$.

For example, the 10th Fibonacci number is the integer nearest 55.0036, in other words, 55.

## The *Liber Quadratorum*

In 1225, Emperor Frederick II organised a mathematics contest. Leonardo answered all the questions correctly, winning easily. Two of the problems were the following:

(1) solve $x^3 + 2x^2 + 10x = 20$
(2) find a rational $a/b$ such that $(a/b)^2 \pm 5$ are both squares of rationals.

If $k$ is a positive integer such that, for some rational $a/b$ both $(a/b)^2 \pm k$ are squares of rationals, then $k$ is *congruent*. Note that if $k$ and $x$ are positive integers, then $k$ is congruent if and only if $kx^2$ is congruent. Problem (2) of the contest was the problem of showing that 5 is congruent.

In solving Problem 19 of Book III of the *Arithmetica,* Diophantus notes that in a right triangle,

$$(\frac{1}{2}\text{ hypotenuse})^2 \pm \text{ area } = \text{ a square}$$

From this it follows that if there is a Pythagorean triangle with area $kx^2$, then $k$ is congruent. For example, the triangle with sides 9, 40, and 41 is a right triangle (since $9^2 + 40^2 = 41^2$). Hence

$$(\frac{1}{2} \times 41)^2 \pm 180 = \text{ a square}$$

Since $180 = 5 \times 6^2$, it follows that 5 is congruent.

Conversely, if $k$ is congruent, then there is a Pythagorean triangle with area $kx^2$, for some integer $x$. Indeed, if

$$(a/b)^2 + k = (c/d)^2 \quad \text{and} \quad (a/b)^2 - k = (e/f)^2$$

then $(bcf + bde)^2 + (bcf - bde)^2 = (2adf)^2$ and

$$\frac{1}{2}(bcf + bde)(bcf - bde) = kb^2d^2f^2$$

Fibonacci gave an account of his solution to Problem (2) in the *Liber Quadratorum,* or *Book of Squares* (1225). He did not make use of Diophantus' identity, but followed a more complicated method.

The *Liber Quadratorum* also contains the first proof of the formula

$$(a^2 + b^2)(c^2 + d^2) = (ac \mp bd)^2 + (bc \pm ad)^2$$

This formula had been given by Abu Ja'far al-Khazin (950 A.D.), who mentioned it in connection with the same Problem 19 in Book III of the *Arithmetica,* where Diophantus gives a numeral instance of it.

The *Liber Quadratorum* concludes with a treatment of

> The question proposed to me by Master Theodore, Philosopher to the Emperor
> I wish to find three numbers [positive integers] which added together with the square of the first number make a square number. Moreover, this square, if added to the square of the second number, yields thence a square number. To this square, if the square of the third number is added, a square number similarly results.

In other words, the problem is to solve the simultaneous Diophantine equations

$$\begin{aligned} x + y + z + x^2 &= w^2 \\ w^2 + y^2 &= u^2 \\ u^2 + z^2 &= v^2 \end{aligned}$$

Fibonacci notes that $(7k)^2 + (24k)^2 = (25k)^2$ and $(25k)^2 + (60k)^2 = (65k)^2$. He takes $w = 7k$, $y = 24k$, and $z = 60k$, and then looks for $x$:

$$x + 24k + 60k + x^2 = (7k)^2$$

If $a = 7k - x$, this gives

$$k = \frac{a(a-1)}{7(2a-13)}$$

With $a = 7$, we obtain $k = 6$ and $x = 35$. Hence $y = 144$ and $z = 360$.

## The Infinite

Medieval Europe also produced some good work on the infinite. This was partly due to the fact that, believing in an actually infinite God, Medieval thinkers were not limited to Aristotle's potential infinity.

**Gregory of Rimini** (1300–1358) maintained, against Aristotle, that God could create an actually infinite stone. Gregory explained that God could do this by creating equal-sized bits of the stone at each of the times $t = 0,\ 1/2,\ 3/4,\ 7/8,\ \ldots$.

**Albert of Saxony** (1350 A.D.) showed that one can take a proper subset of an infinite set and rearrange its elements so that it shows itself to be just as big and unbounded as the infinite set of which it is a proper part. Specifically, he noted that if one has an infinitely long beam of wood, with equal width and depth, one can saw it up into equal-sized cubic blocks with which one can fill the whole of what we call Euclidean 3-space. (Surround the first block with $3^3 - 1$ more blocks, making a cube of side 3; then surround that cube with $5^3 - 3^3$ more blocks, making a cube of side 5; and so on.) In modern terminology, what Albert proved is that there is a one-to-one correspondence between the set of triples $(n, 1, 1)$, with $n$ a positive integer and the set of triples $(a, b, c)$, with $a$, $b$, and $c$ any integers.

**Nicole Oresme** (1350 A.D.) was the first mathematician to prove the divergence of the harmonic series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Oresme also found the (finite) sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} + \cdots$$

## Exercises 22

1. What is the answer to the Rabbit Problem?

2. Give a detailed account of the fact that $F_n$ is the integer nearest $s^n/\sqrt{5}$, showing that, even for small $n$, the $r^n/\sqrt{5}$ does not throw things off.

3. Show that $41/12$ is a solution to Problem (2) on Frederick's math contest.

4. Prove that $(\frac{1}{2}$ hypotenuse$)^2 \pm$ area $=$ a square.

5. Show that 30 is congruent.

6. Show that if $x$ is any positive integer, $6(1^2+2^2+\cdots+x^2)$ is congruent. Hint: look at

$$(2x^2+2x+1)^2 \pm 4x(x+1)(2x+1)$$

7. Show that if $x$ is any positive integer, $8x^3-2x$ is congruent. Hint: look at

$$(4x^2+1)^2 \pm 4(8x^3-2x)$$

8. Show that 14 is congruent.

9. Prove that the harmonic series diverges, using a method that does not involve calculus.

10. Find the finite sum of Oresme's infinite series.

## *Challenges for Experts*

1. Prove that $F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$ where $k$ is a positive integer less than $n$.

2. Prove that $\gcd(F_n, F_m) = F_{\gcd(n,m)}$.

## *Essay Questions*

1. Aristotle argued that there is no infinite collection, since it would have a proper part that was bounded by it and smaller than it, and

yet infinite too, which is absurd. How does Albert's one-to-one correspondence answer this argument?

2. In Volume II of his *History of Mathematics,* D. E. Smith notes that

> In the same years and in the same region in which Leonardo was bringing new light into the science of mathematics, St. Francis, humblest of the followers of Christ, was bringing new light into the souls of men.

Comment on this quote, and, in particular, say whether you think Fibonacci and Francis brought 'new light' or merely old light from pre-Medieval times.

# 23
# Modern Mathematical Notation

Aside from the invention of our Indo-Arabic numerals (and the algorithms used to perform basic arithmetical operations with them) and the work of a few talented persons, such as Ibn-Qurra and Fibonacci, few significant advances in mathematics occurred in the thousand years after Diophantus and Pappus. In the fifteenth and sixteenth centuries, however, there was a spurt of activity, aided by the invention of printing (1450), which carried mathematics beyond the achievements of the ancients.

Closely related to the invention of printing were improvements in mathematical notation. A powerful new symbolism emerged, the one we still use today.

**Johannes Regiomontanus** of Königsberg, Germany (1436–1476) gave the first systematic exposition of plane and spherical trigonometry. He wrote 'res' for $x$, and 'census' for $x^2$. Columbus took a copy of Regiomontanus's *Ephemerides* on his fourth voyage and used its prediction of the lunar eclipse of February 29, 1504 to intimidate some hostile Indians in Jamaica.

**Johannes Widman** of Eger (now in Czechoslovakia) (1462–1500) introduced the symbols $+$ and $-$ in his *Mercantile Arithmetic,* published in 1489.

**Luca Pacioli** of Italy (1445–1517) was a Franciscan. He used the 'res' notation of Regiomontanus, sometimes abbreviating 'res' as $R$. In 1509 he published the *Divina proportione,* a book about the five regular polyhedra. It was lavishly illustrated by none other than Leonardo da Vinci (1452–1519). There is a famous painting of Pacioli by Jacopo de' Barbari, which now hangs in the National Museum at Naples, Italy. It shows the friar with his friend Guidebaldo and a model of a dodecahedron.

**Robert Recorde** of England (1510–1558) was the first person to use the symbol $=$ for equality, asserting that 'noe 2 thynges can be moare equalle' (*Whetstone of witte,* 1557). Recorde worked as a royal physician. He got into a tangle with the Earl of Pembroke and died in gaol.

**Christoff Rudolff** of Germany introduced our square root sign in 1525.

**Michael Stifel** (1487–1567) was a monk who became a follower of Luther. He introduced the symbols $1A$, $1AA$, and $1AAA$ for $A$, $A^2$, and $A^3$. Stifel had a way of applying mathematics to the Bible that led him to conclude that Pope Leo X was the Beast of the Book of Revelation and to prophecy the end of the world for October 18, 1533. The peasants of Holzdorf, where Stifel was pastor, spent their money accordingly, and, when the world failed to end, Stifel found himself, not in heaven, but in a gaol in Wittenberg.

**Thomas Harriot** of England (1560–1621) wrote $aa$, and $aaa$ for $a^2$ and $a^3$. He introduced the signs $>$ and $<$ for strict inequalities.

In 1585 Harriot went to America where be became addicted to tobacco smoke. In 1605, after the discovery of the Gunpowder Plot, Harriot was briefly imprisoned on suspicion of having cast the horoscope of King James I. Harriot died of cancer in 1621.

In 1603, Harriot proved the following formula for the area of a spherical triangle:

$$\left(\frac{\angle A + \angle B + \angle C - 180^\circ}{360^\circ}\right) 2\pi r^2$$

Proof: The great arcs $AB$ and $AC$ meet again at a point $A'$, which is opposite $A$ on the sphere. They cut out two slices of the surface, each with area

$$\frac{\angle A}{360^\circ} 4\pi r^2$$

(It was Archimedes who first proved that the area of the whole surface is $4\pi r^2$.) Similarly, the great arcs $BA$ and $BC$ meet again in a point $B'$ and cut out two slices each with area

$$\frac{\angle B}{360^\circ} 4\pi r^2$$

and the great arcs $CA$ and $CB$ meet again in a point $C'$ and cut out two slices each with area

$$\frac{\angle C}{360^\circ} 4\pi r^2$$

These 6 slices cover the whole area of the sphere, and they cover the triangles $ABC$ and $A'B'C'$ three times. Thus

$$2\left(\frac{\angle A + \angle B + \angle C}{360^\circ}\right) 4\pi r^2 = 4\pi r^2 + 4(\text{ area } ABC)$$

(since $A'B'C'$ is identical to $ABC$) and the result follows.

# Exercises 23

1. Write the following in words, using no mathematical symbolism:

$$2a^2 + \sqrt{a^3 - 7} = 4a + 12$$

2. Regiomontanus solved the following problem: find the angles of a cyclic quadrilateral with given sides $a$, $b$, $c$, and $d$. Do the same.

3. Solve the following problem given by Pacioli: The radius of the inscribed circle of a triangle is 4, and the segments into which one side is divided by the point of contact (of the circle and the triangle) are 6 and 8. Determine the other sides.

4. What is the area of a spherical triangle with angles of $50^\circ$, $60^\circ$, and $90^\circ$, if the radius of the sphere is 12?

5. A spherical triangle with angles $45^\circ$, $60^\circ$, and $90^\circ$ has area 1. What is the area of the sphere?

# 24

# The Secret of the Cubic

Cubic equations were studied in ancient times. Archimedes, for example, worked on the problem of cutting a sphere with a plane so that one of the resulting pieces has twice the volume of the other. The problem of where the sphere should be cut leads to the cubic equation

$$x^3 - 3x + 2/3 = 0$$

(see *De Sphaera et Cylindro,* Lib. II).

Prior to the Renaissance, mathematicians found solutions of such equations either by arithmetical approximation or by geometrical methods. (Recall Menaechmus's solution of $x^3 - 2 = 0$ as the $y$-coordinate of a point of intersection of the parabolas $y = \frac{1}{2}x^2$ and $x = y^2$.) With the exception of some special cases, however, mathematicians were not able to give algebraic solutions to cubic equations. Indeed, in 1494 (just two years after Columbus discovered America), Luca Pacioli (1445–1509) asserted that there is no general algebraic solution to the cubic equation.

As an example, mathematicians would say of Fibonacci's equation

$$x^3 + 2x^2 + 10x = 20$$

that the solution was approximately 1.3688, or they would say that it was the abscissa of the point where the hyperbola we call $(x+2)(y+10) = 40$ meets the parabola we call $y = x^2$. They did not know that the root is exactly

$$\frac{\sqrt[3]{352 + 6\sqrt{3930}} + \sqrt[3]{352 - 6\sqrt{3930}} - 2}{3}$$

Nor did they know that there are also two nonreal roots.

The person who first found the algebraic solution of one type of cubic equation was Scipio del Ferro (1465–1526), a professor at the University of Bologna, Italy. Ferro kept his result secret — so that he would have an advantage over other mathematicians in contests — but, just before dying, he passed it on to Antonio Fior.

## Tartaglia

Niccolo Tartaglia was born in Brescia, Italy, in 1499. In 1512, the French sacked Brescia, and a French soldier split Tartaglia's jaws with a sword. It was thus that Tartaglia acquired his name, which means 'stammerer'.

As a child, Tartaglia would go to the graveyard and write his mathematics on tombstones, since his family was so poor that they could not afford more ordinary writing material.

In 1535, Fior challenged Tartaglia to a contest. Tartaglia, suspecting that Fior would ask him to give algebraic solutions of cubic equations, quickly worked out a general method for doing this. Both Fior and Tartaglia were able to solve equations of the form

$$x^3 + bx = c$$

(with $b$ and $c$ given positive reals) but only Tartaglia was able to solve equations of the form

$$x^3 + ax^2 = c$$

(with $a$ and $c$ given positive reals). Tartaglia had the victory. Ferro's method did not apply to all types of cubic equations.

The latter part of Tartaglia's life was embittered by a quarrel with Girolamo Cardano (1501–1576).

## Cardano

In his autobiography, Cardano tells us that in spite of attempts to abort him, he was born on September 24, 1501. Fascinated by signs and wonders, he studied medicine, mathematics, and astrology.

When he was young, Cardano concluded from his astrological studies that he would not live to be 45 (*De Vita Propria Liber,* ch. X). However, he was still alive in 1570 when he was imprisoned for heresy, on account of having cast the horoscope of Jesus Christ. Cardano recanted and was released. In 1575, in chapter 39 of his autobiography, Cardano confessed:

> That branch of astrology which teaches the revealing of the future I studied diligently, and much more, indeed, than I should; and I also trusted in it to my own hurt.

Basing himself on astrology, Cardano had also predicted long life for King Edward VI of England. Edward died at age 16.

In 1531, Cardano married Lucia Bandarini, a woman who had first appeared to him in a dream. She was only 15 when they married. Cardano gambled, and he relates that

> In a turn of ill-luck at dicing, I put to pawn my wife's jewelry and some of the furniture.

The gambling was not good for his family life, but it did spur him on to study probability, so that Cardano is now considered to be one of the fathers of that subject. As for Lucia, she died in 1546, at age 31.

In 1539, in Cardano's house in Milan, Italy, Cardano cajoled Tartaglia into telling him the secret for solving cubic equations. This Tartaglia did, but only on the condition that Cardano would never reveal it. Indeed, Cardano swore 'by the Sacred Gospel' never to publish Tartaglia's discovery. In 1543, Cardano learned that part of the secret could be found in Ferro's posthumous papers, and he decided to give a complete treatment of the cubic equation in his *Ars Magna,* published in 1545. Why should he keep a secret whose 'key component' was already in Ferro's papers, where anyone could go and read it? It is in the *Ars Magna* that we find the first use of imaginary numbers.

Cardano gave due credit to Tartaglia, but Tartaglia was annoyed that Cardano had broken the promise — and thereby deprived Tartaglia of his advantage in mathematics contests.

In 1548, Tartaglia went to Milan to have a contest with Cardano. Cardano left town, but was represented by his student, Ludovico Ferrari (1522–65). Tartaglia lost the contest and died in poverty 9 years later.

In 1552, Cardano travelled to Scotland where he cured Archbishop John Hamilton of asthma. It was on the way back to Italy that Cardano met King Edward VI of England and made the ill-fated astrological prediction that His Majesty would have a long life. (Edward died in 1553.)

Cardano preferred animals and angels over human beings. In chapter 13 of his autobiography we read:

> I become the owner of all sorts of little animals that get attached to me: kids, lambs, hares, rabbits, and storks. They litter up the whole house.

In chapter 53, we find:

> I am never more with those I love than when I am alone. For I love God and my good angel.

There was an exception. In 1560, Cardano's son, Giambattista, was beheaded for murder. Giambattista, tired of his wife's infidelities, had resorted to the not-uncommon Renaissance solution of poison. Cardano was distraught, and he wrote a lament:

> Who has snatched thee away from me —
> O, my son, my sweetest son?
> Who had the power to bring my age
> Sorrows more than I can count?
> ...
> Prince and Senate and ancient law
> Ordered thy doom whilst thou in rash haste,
> Brought an adultress the wage of her crime.

Cardano's other son, Aldo, was a burglar who spent much of his life in gaol.

In chapter 45 of his autobiography, Cardano assures us that

> all wisdom is from God, the Lord, and, as the Platonists think, our intellect is united thereto through the agency of Eternal Good.

In his *History of Western Philosophy,* Frederick Copleston reports that Cardano believed that all matter is alive. Also Cardano disagreed with the orthodox Christian view that God created the world freely, rather than out of necessity.

Cardano died in Rome, in 1576.

## Ferrari

Ludovico Ferrari (1522–1565) came from Bologna, Italy. His solution to the quartic equation appears in the *Ars Magna.* Ferrari became rich in the service of Cardinal Ferrando Gonzago, but ill health forced him to retire to Bologna in 1565 to teach mathematics. According to W. Ball, in *A Short Account of the History of Mathematics,* Ferrari 'was poisoned the same year either by his sister, who seems to have been the only person for whom he had any affection, or by her paramour'.

## Viète

François Viète (1540–1603) was a French lawyer and member of parliament. He helped Henry IV in a war against Spain by deciphering the Spanish code. Henry IV challenged Viète to solve an equation of degree 45, and Viète gave the answer almost immediately. (This problem had been posed by Adraen van Roomen in 1593.)

Viète's other accomplishments include the following:

1. He used trigonometry to help solve cubic equations (see Chapter 25);
2. He showed how to construct a circle tangent to three given circles, using only straight-edge and compass;
3. He discovered the identity $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ (and the

three identities similar to it);
4. He noted that if $a_1 = \sqrt{1/2}$ and $a_{n+1} = \sqrt{(1+a_n)/2}$ then the product $a_1 a_2 a_3 \ldots = 2/\pi$.

Viète described his mathematics as 'not alchemists' gold, soon to go up in smoke, but the true metal, dug out from the mines where dragons are standing watch'.

# Exercises 24

1. Tartaglia wrote a number theory book that contains the following puzzle: Three couples wish to cross a wide river using a boat that holds only two people. How can they do this if no woman is to be with a man unless her husband is present?

2. Tartaglia's book also contains the following: Three people wish to divide 24 ounces of oil equally. The oil is in a single jar, and the three people have irregularly shaped measuring jars of capacity 5, 11, and 13 ounces. How can they divide the oil?

3. In his *Liber de Ludo Aleae (Book on Games of Chance),* Cardano gives the correct answer to the following: if you throw three dice three times, what is the probability of getting at least one one each time?

4. Cardano thought the following had answer 2/5: if you throw two dice three times, what is the probability that you will get at least one one at least two times? Show that the correct answer is 5203/23328.

## *Challenges for Experts*

1. Show that

$$\frac{\sin x}{2^n \sin \frac{x}{2^n}} = (\cos\frac{x}{2})(\cos\frac{x}{2^2})(\cos\frac{x}{2^3})\ldots(\cos\frac{x}{2^n})$$

   whence

$$\frac{\sin x}{x} = (\cos\frac{x}{2})(\cos\frac{x}{2^2})(\cos\frac{x}{2^3})\ldots$$

   (Recall that $\lim_{h\to 0}(\sin h)/h = 1$.)

2. Prove Viète's formula for $2/\pi$. (Recall that $\cos A/2 = \sqrt{(\cos A + 1)/2}$.)

## *Essay Questions*

1. Was Cardano wrong to break his promise to Tartaglia? Why or why not?

2. Is it ever right to keep a scientific discovery secret, rather than sharing it with the world? Why?

3. The sun, moon, and planets are unthinking chunks of matter. Their movements are determined by well-understood physical laws. We, on the other hand, have free will. We can choose to mould our characters in any way we decide is good. We make our own future. Our future is not implicit in any horoscope. Identical twins, for example, can, and often do, choose different careers. Thus astrology has no truth in it. Comment on this.

4. Astrology fusses about the moment of birth. Yet this moment is less important than the moment of conception, or the moment when a person first has brainwaves, or the moment when a person utters their first sentence. Birth is just a geographical change relative to the mother. Thus there is no reason to think there is any truth in astrology. Comment on this.

5. In helping Henry IV fight the Spanish, Viète was using his God-given mathematical talent in the service of hatred and violence. This was wrong. Comment on this evaluation.

# 25
# The Secret Revealed

In this chapter we give the solution to the cubic equation that is essentially that of Ferro, Tartaglia, and Viète. We also give the solution to the quartic equation, due to Ferrari.

## The Cubic

First recall that any complex number $x + yi$ can be written in 'polar form' $r(\cos A + i \sin A)$ where $r$ is a nonnegative real and $A$ is a real. Since (as Viète first showed)

$$\cos 3A = \cos^3 A - 3\cos A \sin^2 A$$

$$\sin 3A = 3\cos^2 A \sin A - \sin^2 A$$

it follows that

$$(r(\cos A + i \sin A))^3 = r^3(\cos 3A + i \sin 3A)$$

Now let $\omega = (-1+\sqrt{-3})/2$. A calculation shows that $\omega^2 = (-1-\sqrt{-3})/2$. We have $\omega + \omega^2 = -1$, $\omega - \omega^2 = \sqrt{-3}$, and, finally, $\omega^3 = 1$.

Let $s$ be any real and let $t$ be its real cube root. (This number is not, in general, 'constructible' using straightedge and compass, but Cardano had no doubts about its existence.) Then

$$x^3 - s = (x - t)(x - t\omega)(x - t\omega^2)$$

so that $x^3 - s = 0$ has exactly 3 solutions, namely, $t$, $t\omega$, and $t\omega^2$.

The general cubic (after division by the leading coefficient) has the form

$$x^3 + ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are reals. Putting $x = y - a/3$ (a trick discovered by Tartaglia), this equation becomes

$$y^3 - 3py - 2q = 0$$

where

$$p = \frac{a^2 - 3b}{9} \text{ and } q = \frac{-2a^3 + 9ab - 27c}{54}$$

If $p$ or $q$ is 0, the solution is trivial. Suppose this is not so.

Let $u$ and $v$ be the (possibly nonreal) roots of $z^2 - yz + p$ (where $y^3 - 3py - 2q = 0$). Then $u + v = y$ and $uv = p$. Substituting $u + v$ for $y$ in $y^3 - 3py - 2q = 0$, we obtain

$$u^3 + v^3 + 3(uv - p)(u + v) - 2q = 0$$

or

$$u^3 + v^3 = 2q$$

Since, moreover, $u^3v^3 = p^3$, it follows that $u^3$ and $v^3$ are the solutions of

$$w^2 - 2qw + p^3 = 0$$

Therefore we have, say,

$$u^3 = q + \sqrt{q^2 - p^3} \text{ and } v^3 = q - \sqrt{q^2 - p^3}$$

Thus $x = y - a/3 = u + p/u - a/3$, where $u$ is a cube root of

$$q + \sqrt{q^2 - p^3}$$

If one of these cube roots is $u_1$, then the others are $u_1\omega$ and $u_1\omega^2$. Let $v_1 = p/u_1$. Then

$$u_1\omega + p/(u_1\omega) = u_1\omega + v_1\omega^2$$
$$u_1\omega^2 + p/(u_1\omega^2) = u_1\omega^2 + v_1\omega$$

Hence the cubic has the following three solutions only:

$u_1 + v_1 - a/3$
$u_1\omega + v_1\omega^2 - a/3$
$u_1\omega^2 + v_1\omega - a/3$

For practical cubic equation solving, we consider 3 cases:

Case 1. $q^2 - p^3 > 0$. Let $r = \sqrt{q^2 - p^3}$. Let $u_1$ be the real cube root of

$q + r$. Then $v_1 = p/u_1$ is real, and hence $v_1$ is the real cube root of $q - r$. The solutions are the real number $u_1 + v_1 - a/3$ and the nonreal numbers

$$u_1\omega + v_1\omega^2 - a/3$$

$$u_1\omega^2 + v_1\omega - a/3$$

(These numbers are nonreal, lest $u_1 = v_1$ and $r = 0$.)

**Case 2.** $q^2 - p^3 = 0$. Let $u_1 = v_1 = \sqrt{p}$. The cubic has exactly two solutions: $2u_1 - a/3$ and $-u_1 - a/3$ (since $\omega + \omega^2 = -1$).

**Case 3.** $q^2 - p^3 < 0$. Let $r = \sqrt{p^3 - q^2}$. Since $r^2 + q^2 = p^3$, it follows that $q + ir$ can be written in polar form as $p^{3/2}(\cos\alpha + i\sin\alpha)$ for some real angle $\alpha$. ($\alpha = \arctan(r/q)$ if $q > 0$ and $\alpha = \arctan(r/q) + \pi$ if $q < 0$.) Let

$$e = \sqrt{p}\cos(\alpha/3)$$

$$f = \sqrt{p}\sin(\alpha/3)$$

Then $(e + fi)^3 = q + ir$. Hence we can take $u_1 = e + fi$. Since $e^2 + f^2 = p$, it follows that $v_1 = p/u_1 = e - fi$. Hence the cubic has the following solutions:

$2e - a/3$
$u_1\omega + v_1\omega^2 - a/3 = -e - f\sqrt{3} - a/3$
$u_1\omega^2 + v_1\omega - a/3 = -e + f\sqrt{3} - a/3$

In summary, we have the following solution for the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

1. Compute

$$p = \frac{a^2 - 3b}{9} \quad \text{and} \quad q = \frac{-2a^3 + 9ab - 27c}{54}$$

If either of these is 0, the problem is trivial.
2. Compute $q^2 - p^3$ and go into the case in question.
3a. If $q^2 - p^3 > 0$ let

$$\begin{aligned} u_1 &= \sqrt[3]{q + \sqrt{q^2 - p^3}} \\ v_1 &= \sqrt[3]{q - \sqrt{q^2 - p^3}} \end{aligned}$$

(where the cube root is the real cube root). The solutions are

$$\begin{aligned} &u_1 + v_1 - a/3 \\ &u_1\omega + v_1\omega^2 - a/3 \\ &u_1\omega^2 + v_1\omega - a/3 \end{aligned}$$

3b. If $q^2 - p^3 = 0$ the solutions are

$$\begin{array}{l} 2\sqrt[3]{q} - a/3 \\ -\sqrt[3]{q} - a/3 \end{array}$$

3c. If $q^2 - p^3 < 0$, let $r = \sqrt{p^3 - q^2}$ and let $\alpha = \arctan(r/q)$ if $q > 0$ — or let $\alpha = \arctan(r/q) + \pi$ if $q < 0$. The equation has three real solutions:

$$\begin{array}{l} 2\sqrt{p}\cos\frac{\alpha}{3} - a/3 \\ -\sqrt{p}\cos\frac{\alpha}{3} \pm \sqrt{3p}\sin\frac{\alpha}{3} - a/3 \end{array}$$

# The Quartic

The general quartic equation has the form

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

It was treated by Ferrari as follows. Let $y$ be a real root of

$$y^3 - by^2 + (ac - 4d)y - a^2d + 4bd - c^2$$

Let

$$\begin{aligned} A &= \frac{a^2}{4} - b + y \\ B &= -c + \frac{ay}{2} \\ C &= -d + \frac{y^2}{4} \end{aligned}$$

Then $4AC = B^2$.

The quartic equation is equivalent to

$$(x^2 + \frac{1}{2}ax + \frac{1}{2}y)^2 = Ax^2 + Bx + C$$

or

$$(x^2 + \frac{1}{2}ax + \frac{1}{2}y)^2 = A(x + \frac{B}{2A})^2$$

provided $A \neq 0$. (If $A = 0$ the solution is trivial.) Then the quartic is equivalent to

$$x^2 + (\frac{a}{2} \pm \sqrt{A})x + \frac{y}{2} \pm \frac{B}{2\sqrt{A}} = 0$$

with the signs corresponding. This leads to at most 4 solutions.

In the *Ars Magna* Cardano solves the equation

$$x^4 - 10x^2 + 4x + 8 = 0$$

The equation

$$y^3 + 10y^2 - 32y - 336 = 0$$

has root $-6$. (This can be found by checking the divisors of 336: there is no need to use the full cubic apparatus.) Here $A = 4$ and $B = -4$. The quartic is thus equivalent to

$$x^2 \pm 2x - 3 \mp 1 = 0$$

This has solutions $1 \pm \sqrt{3}$ and $-1 \pm \sqrt{5}$.

## The Quintic?

For over 200 years people tried to find similar methods for solving equations of degree 5. They failed. It was only in 1799 that Paolo Ruffini (1765–1822), an Italian physician, proved that there is no algebraic solution for the general equation of degree 5 (or higher). His proof was not clear and rigorous, but it was soon supplemented by the work of Niels Abel (1802–1829) and Evariste Galois (1811–1832). It is not possible to go beyond Ferro, Tartaglia, Viète, and Ferrari.

## A Recent Application of the Cubic Formula

In 1955, Roy E. Wild used the above solution to the cubic equation to sharpen a result of J. Lambek and L. Moser. Wild showed that the number of primitive Pythagorean triangles with area less than $n$ is approximately

$$.531\sqrt{n} - .297\sqrt[3]{n}$$

(see the *Pacific Journal of Mathematics,* 5 (1955), 85-91).

## Exercises 25

In the following problems, do **not** use decimal approximations.

1. Give the exact solutions of Fibonacci's equation $x^3 + 2x^2 + 10x = 20$.

2. Give the exact solutions of the equation sent by Zuanne da Coi to Tartaglia in 1530: $x^3 + 6x^2 + 8x = 1000$.

3. An oracle ordered a prince to build a sacred building whose volume should be 400 cubits, the length being 6 cubits more than the width

and the width 3 cubits more than the height. What was its height supposed to be? (This problem comes from the *Ars Magna.*)

4. Divide 10 into two summands the product of which is 40. (This problem comes from the *Ars Magna.*)

5. A merchant bought 1 pound of saffron, 2 of cinnamon, and 5 of pepper (all these being sold by weight). He paid 6 aurei. The price of the 5 pounds of pepper was to the price of the 2 pounds of cinnamon as the price of the 2 pounds of cinnamon was to the price of the saffron. Again, at the same prices per pound, he bought 30 pounds of saffron, 50 pounds of cinnamon, and 40 pounds of pepper, all for 100 aurei. How much was a pound of saffron? (This problem comes from the *Ars Magna.*)

6. Solve Archimedes's sphere problem.

7. Where exactly does the parabola $y = -x^2 + 3x + 4$ meet the circle $x^2 + y^2 = 100$?

8. Two ladders, 20 and 30 metres long, cross at a point 8 metres above a lane bordered by two rows of high buildings. Each ladder reaches from the base of one wall to some point on the opposite wall. Exactly how wide is the lane?

9. Show that $x^3 - 3cx + c^3 = 0$ has 3 distinct roots iff $0 < c < 2^{2/3}$.

10. What are the 4 primitive Pythagorean triangles with area less than 100?

## *Essay Question*

1. Since there are efficient ways of getting good approximations to the roots of cubics, there is no point having an algebraic formula giving the exact roots. Ferro's work was a waste of time. Comment.

# 26
# A New Calculating Device

John Napier (1550–1617) was a Scottish Baron who gave 20 years of his life to the construction of logarithms. Like Stifel, he was interested in proving that the Pope was Antichrist, and in 1594 he published 'A plaine discovery of the whole Revelation of St. John'. Part of Napier's discovery was that the world would end before 1700.

Once one of Napier's servants was stealing from him. In order to expose the thief, Napier told his servants to go, one by one, into a darkened room where they must pat his psychic rooster. He assured them that the rooster would know from the patting who had been stealing and would then tell Napier. Unknown to the servants, Napier had smeared soot on the bird. The thief, who had not dared to touch it, was the only one with clean hands!

Napier describes his technique for calculating logs in his *Mirifici Logarithmorum Canonis Constructio,* which was published in 1619, two years after his death. This *Construction of the Wonderful Canon of Logarithms* is tricky to read because what Napier calls the 'logarithm' of $x$ is actually

$$10^7 \log_{1/e} \frac{x}{10^7}$$

The book is also tedious, because Napier pays minute attention to error bounds. However, if we simplify and modernise Napier's presentation somewhat — as we do in this chapter — we obtain a lucid piece of mathematics.

Napier's basic ideas are these. Suppose there is a particle on the negative half of the real number line moving towards the origin at a speed proportionate to its distance from the origin. At time 0, it is at $-1$. For any distance $d$, there is a time when the particle is that distance from the origin. Call this time the 'logarithm' of $d$.

Assume that the constant of proportionality is 1. Then $dx/dt = -x$ and $x(0) = -1$. Hence $x(t) = -e^{-t}$, and so

$$t = -\log_e d = \log_{1/e} d$$

In this section '$\log x$' shall mean $\log_{1/e} x$. Note that $\log_{1/e} d$ decreases as $d$ increases from $\frac{1}{2}$ to 1.

Napier used the following three ideas to calculate his tables. (1) When $z$ is very close to 0, $\log(1-z) \approx z$. (2) The natural number powers of $s = 1 - 10^{-m}$ — where $m$ is a natural number — are easy to calculate. For $s^2 = s(1-10^{-m}) = s - s/10^m$. Similarly, $s^3 = s^2 - s^2/10^m$. It is always just a matter of shifting the decimal point $m$ places and subtracting. Thus Napier has

$$\begin{aligned}
1 - 10^{-5} &= 0.999,990,000,000,000 \\
(1-10^{-5})^2 &= 0.999,990,000,000,000 \\
&\quad -.000,009,999,900,000 \\
&= 0.999,980,000,100,000 \\
(1-10^{-5})^3 &= 0.999,980,000,100,000 \\
&\quad -.000,009,999,800,001 \\
&= 0.999,970,000,299,999
\end{aligned}$$

(3) Near 1, the log curve is smooth, and linear interpolations give excellent approximations to it.

Near $-1$, Napier's particle is moving at about 1 unit/second, and we can take it that $\log d = 1 - d$ for $d = 1 - 10^{-5} = 0.99999$. As in (2), Napier calculates $d$, $d^2$, ..., $d^{50}$. He finds that $d^{50} = 0.999,500,122,5$. Hence $\log 0.999,500,122,5 = 50 \times \log d = 50 \times 0.99999$. With linear interpolation, Napier can thus obtain a very accurate value for $\log u$ where $u = 0.9995$.

Next, using ideas similar to those in (2), Napier quickly and accurately calculates $u^2$, $u^3$, ..., $u^{20}$. He obtains $u^{20} = 0.990,047,358$. Using interpolation, he then gets a very accurate value for $\log w$ where $w = 0.99$. (A pocket calculator will not be more accurate.)

For $a = 1, 2, \ldots, 20$, and $b = 0, 1, \ldots, 68$, Napier calculates $u^a w^b$. This gives him 1380 points between $u = 0.9995$ and $u^{20}w^{68} = 0.499,860,940$ for calculating logarithms in that range.

For example, $u^{19}w^{68} = 0.500,110,996 > \frac{1}{2} > 0.499,860,940 = u^{20}w^{68}$. Where

$$k = \frac{\log(u^{19}w^{68}) - \log(u^{20}w^{68})}{u^{19}w^{68} - u^{20}w^{68}}$$

we have

$$\log\frac{1}{2} = \log(u^{20}w^{68}) + (\frac{1}{2} - u^{20}w^{68})k$$

$$= 20\log u + 68\log w - 0.000,278 = 0.693,147$$

This value of $\log\frac{1}{2}$ is accurate to 6 decimal places.

It is now easy to calculate other logs. For example, if $0 < t < \frac{1}{2}$, we can find $\log t$ by finding a positive integer $m$ such that $\frac{1}{2} < 2^m t < 1$. With such an $m$ we have

$$\log t = \log 2^m t + m\log\frac{1}{2}$$

As another example, the log of 2 (to our base $1/e$) is just $-\log\frac{1}{2}$.

Henry Briggs (1561–1631) travelled to Edinburgh in 1615 to discuss logarithms with Napier. They agreed that there were many advantages to having logs to the base 10. In 1617 Napier died, but Briggs continued the work, publishing tables for logs to the base 10 in 1624.

# Exercises 26

1. Solve $\log_2 3x^2 = \log_3 4x^2$.

2. Graph $y = \log_x 2$.

3. Show that to calculate the logarithm of any positive real number, it suffices to have a table of logarithms of primes.

4. Verify the following:

$$\begin{aligned}\log_e 9601 &= \log_e(1 + 1/9600) + \log_e 96 + 2\log_e 10\\ &= \log_e\frac{1+\frac{1}{19201}}{1-\frac{1}{19201}} + 7\log_e 2 + \log_e 3 + 2\log_e 5\end{aligned}$$

5. Show that

$$\log_2\log_2\sqrt{\sqrt{\sqrt{2}}} = -3$$

*Challenge for Experts*

1. How should Napier have had his particle moving to get logs to base $e$?

# 27

# Mathematics and Astronomy

## Galileo

Thanks to Einstein, we are aware that motion is relative. One can choose any heavenly body one likes as a fixed frame of reference for studying the motion of other heavenly bodies. One can say that the earth goes around the sun or one can say that the sun goes around the earth. From a mathematical point of view, it is best to take as a fixed frame of reference neither the earth nor the sun, but the centre of mass of the solar system — because this choice makes the mathematics of solar system motion simpler. From Einstein's point of view, then, it seems silly that Galileo Galilei (1564–1642) and the Inquisition fought over whether the earth goes around the sun or vice versa.

Galileo was born in Pisa, Italy, on the day Michelangelo died. He did not make any original contributions to mathematics. His much tooted result that the set of square integers can be placed in one-to-one correspondence with the set of natural numbers was well known to Medieval thinkers, such as Albert of Saxony. So if he did not contribute to mathematics, what is Galileo doing here?

Galileo is sometimes included in histories of mathematics because the anti-Catholic historian wants the chance to tell everyone how badly the Catholic Church treated Galileo. David M. Burton, for example, tells us that

> Although 70 years old and seriously ill, the author of the *Dialogue* [Galileo] was summoned to Rome to stand trial before

> a tribunal of the dreaded Inquisition (p. 333 in *The History of Mathematics*).

Burton also tells us that Galileo's book was placed on the Index of Prohibited Books, 'where it remained until 1822'. Writers who do not dislike the Catholic Church sometimes include Galileo 'to set the record straight' and show that there is no contradiction between religion and science: if the Bible presupposes a Babylonian cosmology, it does so, not to teach us physics, but to express the truth that a unique personal God created the stars and planets. In other words, Galileo is like Hypatia. Neither did much for mathematics, but both are tools for those who want to make philosophical points.

## Kepler

Johann Kepler (1571–1630) studied at the University of Tübingen. He originally wanted to be a Lutheran minister but his interest in astronomy led him to change his plans. In 1594, he obtained a lectureship at the University of Grätz, Austria, but he lost it when the city fell to the Catholics.

Kepler discovered two of the star-polyhedra, and in his work on conics, he introduced the notion of a 'point at infinity'. Kepler gave us the word 'focus', which is Latin for 'hearthside'. He also worked out the volume of the 'apple', this being a solid obtained by rotating a major segment of a circle around its boundary chord.

Kepler's first marriage was not happy. The woman went mad and died. Howard Eves reports that

> his second marriage was even less fortunate than the first, although he took the precaution to analyze carefully the merits and demerits of eleven girls before choosing the wrong one.

Kepler's primary interest was astronomy. Seeking a geometric explanation for the distances of the various planets from the sun, he began by constructing an equilateral triangle with its vertices on the orbit of Saturn (which he assumed was circular), and then inscribing a circle in this triangle to represent the orbit of Jupiter. Kepler's construction did, more or less, give the correct ratio for the distances of Saturn and Jupiter from the sun. Kepler extended his idea by constructing a square with its vertices on the orbit of Jupiter and inscribing a circle in this square to represent the orbit of Mars. Here, however, the data did not conform to the theoretical model.

Undiscouraged, Kepler replaced the circles by spheres and the regular polygons by regular polyhedra. There were five planets (or so Kepler thought) and five regular polyhedra. This could not be an accident! Kepler succeeded in making this new model fit the observational data, more or less, and he published his theory in *Cosmic Mystery* (1596).

In 1601, Tycho Brahe died, and Kepler, who was his assistant, inherited his job (as 'Imperial Mathematician' of the Holy Roman Empire) and his very accurate astronomical data. These data convinced Kepler that he needed to revise his theory once more. In 1609, after an intense study of the orbit of Mars, Kepler came to the following conclusions:
(1) Each planet moves in an ellipse with the sun at one focus;
(2) The line joining the sun to a planet sweeps out equal areas (bounded by the ellipse) in equal times.

In 1619, Kepler added his third law:
(3) The square of the period of revolution of a planet (its 'year') is proportionate to the cube of the length of the major axis of its orbit.

This time Kepler was right. Applying Apollonius' previously 'useless' work on conics, Kepler found a beautiful mathematical explanation of the motion of the planets. In everything he did, Kepler was driven by a fervent Pythagorean belief that there is no phenomenon that does not have a mathematical structure.

# Exercises 27

1. According to Kepler's initial model, what is the ratio of the radius of Saturn's orbit to that of Jupiter's orbit?

2. Kepler saw a circle as a polygon with infinitesimal sides. It was made up of 'little triangles', each with height $r$, the radius of the circle. Show how Kepler used this conception of the circle, together with the formula for the circumference of the circle, to derive the correct formula for the area of the circle.

## *Challenge for Experts*

1. What is the volume of the apple with diameter 2 and chord 1 ? (This question requires calculus.)

*Essay Questions*

1. Psalm 104 praises God 'who laid the foundations of the earth, so that it should not be moved forever'. Write a short essay showing that this verse can be interpreted in a way that respects both the truth of God's revelation and the truth of Science.

2. Comment on the following. Socrates and Jesus were willing to die for what they believed was right, but Galileo recanted because he was a coward.

# 28
# The Seventeenth Century

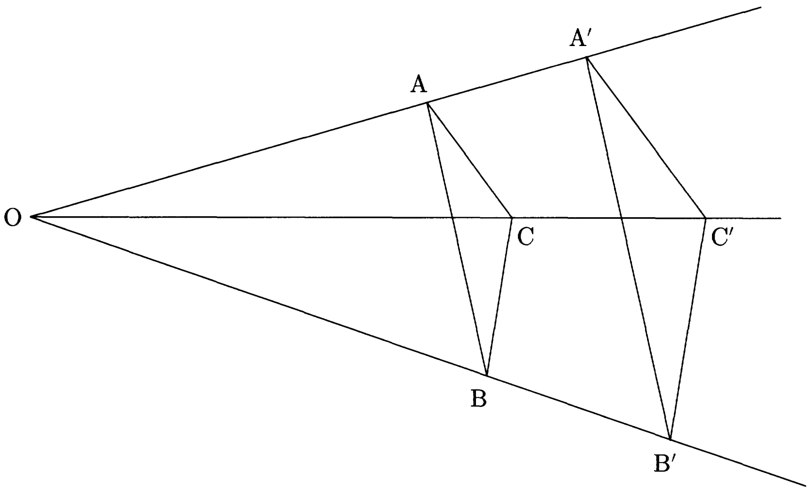
The seventeenth century saw an explosion of mathematical activity. A few of the important mathematicians were Desargues, Descartes, Fermat, Pascal, Torricelli, Wallis, Newton, and Leibniz. We shall report on their work in this and the next three chapters.

## Desargues

Girard Desargues (1591–1661) founded projective geometry. Partly because of his obscure style, his achievement was not recognised until 1845, when Michel Chasles chanced upon a copy of his book and realised its importance. Desargues's famous theorem is this:

> If two triangles, in the same plane or not, are such that the lines joining pairs of corresponding vertices are concurrent, then the points of intersection of pairs of corresponding sides are collinear (if only on the 'line at infinity'), and conversely.

The Figure on the next page shows the 'parallel case' for two triangles in the same plane. According to the theorem, if the lines joining the corresponding vertices meet at $O$, and if $AB$ is parallel to $A'B'$, and if $AC$ is parallel to $A'C'$, then $BC$ is parallel to $B'C'$. In his *Foundations of Geometry,* David Hilbert (1862–1943) showed that, even in the absence of the axiom of Archimedes, the theorem of Desargues can be used for defining multiplication within Euclidean plane geometry, and for proving, within that geometry, that multiplication is associative.




**Desargues's Theorem**

## Descartes

René Descartes (1596–1650) was born the year before his mother died. He attended school at the Jesuit college of La Flèche, where, on account of his delicate health, he was allowed to sleep in. It was during his morning hours in bed that Descartes thought out many of his ideas.

In 1618, Descartes joined the army of Prince Maurice of Nassau. Ten years later, he settled in Holland, where he devoted himself to writing philosophy. In 1635, he had a daughter by a servant girl, Helen. In 1649, Descartes moved to Stockholm to tutor Queen Christina. The queen insisted on having her lesson at 5:00 A.M. Descartes, a late riser from his days at La Flèche, did not survive the winter.

It was Descartes who wrote:

> no opinion, however absurd and incredible, can be imagined, which has not been maintained by some one of the philosophers.

This quotation is found in Part II of the *Discours de la Méthode pour bien conduire sa Raison et chercher la Vérité dans les Sciences* (1637). This 'Discourse' explained Descartes's program of systematically doubting accepted knowledge and then carefully building an edifice of true knowledge on clear and certain principles — such as 'I think, therefore I exist'.

Descartes's epistemological edifice included a geometrical proof of the existence of God. This proof was a version of the Ontological argument, which goes back to Augustine:

> recurring to the examination of the idea of a Perfect Being, I found that the existence of the Being was comprised in the idea in the same way that the equality of its three angles to two right angles is comprised in the idea of a triangle, or as in the idea of a sphere, the equidistance of all points on its surface from the centre, or even still more clearly; and that consequently it is at least as certain that God, who is this Perfect Being, is, or exists, as any demonstration of geometry can be.

Unfortunately for Descartes, there are triangles — in hyperbolic geometry — whose three angles do not add up to two right angles. We need to repair Descartes' argument by adding the word 'Euclidean' before the word 'triangle'.

Part of Descartes' re-examination of accepted knowledge was a re-examination of geometry. In his search for precision and logic, he related all the components of a geometry figure to two straight lines: what we call the $x$-axis and the $y$-axis. Descartes' work on geometry was published as an appendix to the 'Discourse'. It was both a fruit and a vindication of his way of doing philosophy.

The geometrical appendix is divided into three 'books'. The first book gives the basic rules of what we call analytic geometry and addresses a problem that defeated Pappus. One version of this problem, expressed in modern notation, is the following:

$$\text{graph} \quad y = \frac{x^3 - 2x^2 - x + 2}{x}$$

This was beyond the ancient Greeks, but it was well within the range of Descartes' new geometry.

In the second book of the geometrical appendix, Descartes considers equations of the form

$$F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

He shows that if (1) $a \geq 0$, and (2) the quadratic does not factor into linear factors, and (3) at least two points satisfy the relation, then the equation represents an ellipse if $ab > h^2$, a hyperbola if $ab < h^2$, and a parabola if $ab = h^2$. Descartes also gives a way of finding a tangent to a given conic, through a given point — without using calculus.

The third book of the appendix on geometry contains observations on algebraic equations, including the 'Rule of Signs', which goes back to Thomas Harriot in 1631.

Descartes was pro-infinity. He asserted that our concept of the infinite is logically and epistemologically prior to our idea of the finite:

> I clearly perceive that there is more reality in the infinite substance than in the finite, and therefore that in some way I possess the perception (notion) of the infinite before that of the finite.

In his appendix on geometry, Descartes states that a 'locus' is an infinite set of points. For example, a straight line, or circle, is an infinite collection of ordered pairs.

## Fermat

Pierre de Fermat (1601–1665) was a councillor for the parliament of Toulouse, a town in the south of France, and he only did mathematics in his spare time. He published only one mathematical article during his lifetime. His contributions to mathematics are contained in his correspondence and in papers found after his death.

Fermat gave a presentation of analytic geometry in his *Ad Locas Planos et Solidos Isagoge.* As Boyer notes in his *History of Analytic Geometry:*

> Analytic geometry was the independent invention of two men, neither one of whom was a professional mathematician. Pierre de Fermat ... was a lawyer with a deep interest in the geometrical works of classical antiquity. René Descartes ... was a philosopher who found in mathematics a basis for rational thought.

It should be noted that Descartes believed he was finding a basis for mathematics in rational thought, rather than the other way round.

In his correspondence with Pascal, Fermat helped develop the theory of probability. He also found a way of constructing a plane tangent to four given spheres, and he came close to developing calculus.

Today Fermat is best known for his discoveries in number theory. It was Fermat who first proved that 1 is not a congruent number, and it was Fermat who first conjectured that every natural number is a sum of $n$ $n$-gonal numbers. He also discovered what is now called Fermat's little theorem.

> **Fermat's Little Theorem:**
> If $p$ is a prime, and $a$ is any integer, then $p$ is a factor of $a^p - a$.

**Proof:** The theorem is true when $a = 0$. Suppose it true for $a$. By the binomial theorem (known to the Arabs and Chinese long before Fermat),

$$(a+1)^p = a^p + \binom{p}{1} a^{p-1} + \cdots + \binom{p}{p-1} a + 1$$

But if $0 < k < p$ then $p$ divides $\binom{p}{k}$, and hence, for some integer $m$,

$$(a+1)^p = a^p + mp + 1$$

On the induction hypothesis, $p$ divides $a^p - a$. Hence $p$ divides

$$(a+1)^p - (a+1) = a^p - a + mp$$

The result now follows by mathematical induction.

In reading a translation of Diophantus's *Arithmetica,* Fermat came across the equation $x^2 + y^2 = z^2$ (see Book II, Problem 8). In the margin of this translation, Fermat wrote a note to the effect that if $n > 2$ then there are no positive integers $x$, $y$, and $z$ such that $x^n + y^n = z^n$:

> To divide a cube into two other cubes, a fourth power, or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.

Fermat's assertion is called his 'Big Theorem' or 'Last Theorem' because, for a long time, it was the only one of Fermat's conjectures we could neither prove nor disprove. Finally, in 1993, Andrew Wiles presented work which, soon afterwards, led him to a proof that Fermat was right.

Fermat himself may have had a proof for the case in which $n = 3$. He certainly had the proof for the case with $n = 4$. In 1823, Legendre disposed of the case with $n = 5$, and, in 1832, Dirichlet handled the case with $n = 7$. In 1849, Kummer vindicated Fermat's claim for all $n < 100$, except 37, 59, and 67.

## Exercises 28

1. Assuming the usual theory of similar triangles, give Euclidean proofs of the parallel case of Desargue's theorem, and its converse. (The converse is: if $AB$, $A'B'$, and $AC$, $A'C'$, and $BC$, $B'C'$ are three pairs of parallels, then $AA'$, $BB'$, and $CC'$ are concurrent.)

2. Is the following argument logically valid?

   > If you are using the word 'God' correctly, then God is as great a being as possible. If God merely exists in the human imagination then he is not as great as he would be if he really existed. If God is not as great as he would be if he really existed, then God is not as great a being as possible. Hence, if you are using the word 'God' correctly, God does not merely exist in the human imagination.

3. Graph
   $$y = \frac{x^3 - 2x^2 - x + 2}{x}$$
   Why did Newton call this curve the 'trident'?

4. Graph the 'folium of Descartes': $x^3 + y^3 = 3xy$.

5. $2x^2+3xy+4y^2+7x = 1$ is the equation for which sort of conic? Why?

6. Let $A(x, y)$ be a point on the parabola $y = x^2/4p$. What is the equation of a line through $A$ with slope $m$? Descartes showed that $m$ must be $x/2p$ if that line is to meet the parabola only at $A$ (and hence be a tangent). Do the same.

7. What is the smallest positive integer that quadruples when its final (scale 10) digit is shifted to the front?

## *Essay Question*

1. Give reasons for and against the thesis that the best mathematics is done by nonmathematicians.

# 29
# Pascal

## A Madman?

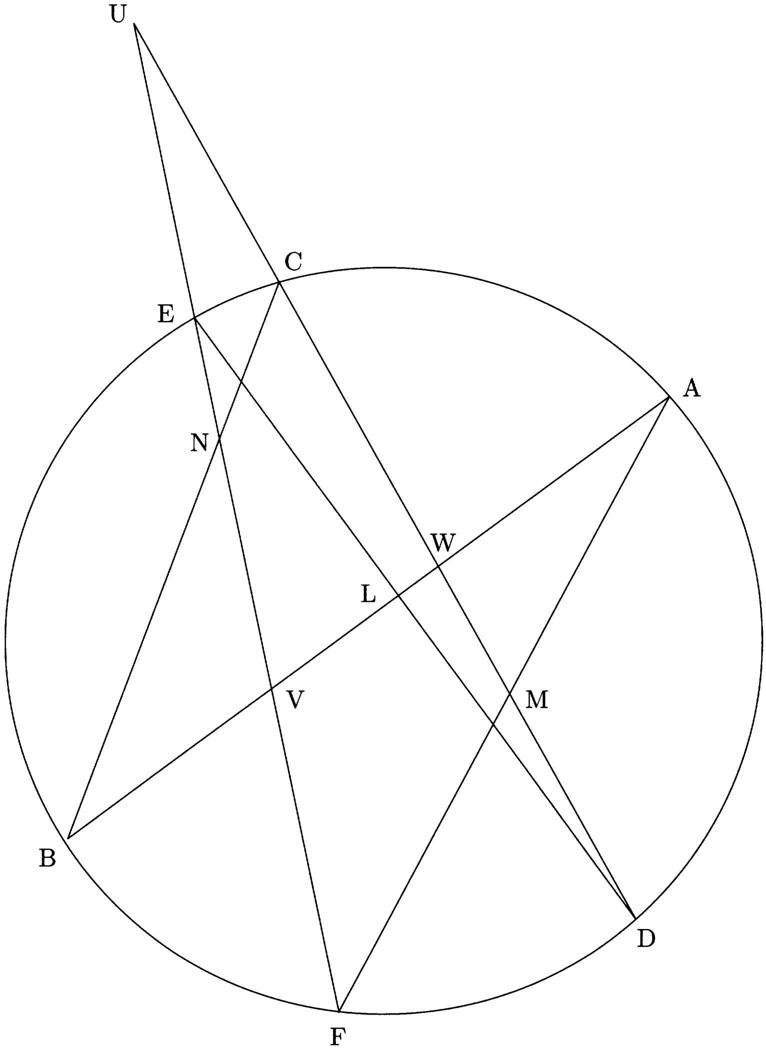
Some historians feel that to be 'scientific' they must do their work on the assumption that there is no God. This assumption leads to curious results when the history involves religious persons, such as Blaise Pascal (1623–1662). Throughout his life, Pascal was an excellent writer, philosopher, and mathematician. Yet, because of his religious priorities, historians such as E. T. Bell have branded him a madman. In *Men of Mathematics,* Bell writes:

> we shall consider Pascal primarily as a highly gifted mathematician who let his masochistic proclivities for self-torturing and profitless speculations on the sectarian controversies of his day degrade him to what would now be called a religious neurotic.

A few pages later, Bell laments:

> If only the man could have been human enough to let himself go when his whole nature told him to cut loose, he might have lived out everything that was in him, instead of smothering the better half of it under a mass of meaningless mysticism and platitudinous observations on the misery and dignity of man.

S. Hollingdale follows suit. In *Makers of Mathematics,* he asserts that Pascal's




**Pascal's Mystic Hexagram**

outstanding intellectual powers were exercised mainly on sterile theological speculations occasioned by the sectarian religious controversies of his day.

What did Pascal do so that he is at once hailed as a genius at reasoning and condemned as mentally ill?

# Before November 23, 1654

Pascal's early work was in geometry. At about age 16, he discovered, and proved, the theorem that the opposite sides of a 'mystic hexagram' inscribed in a conic meet in 3 points that are collinear (if only on the 'line at infinity'). For example, suppose we have the points $A$, $D$, $F$, $B$, $E$, and $C$, in that order, on the circumference of a circle. Suppose $AB$ and $DE$ meet at $L$, and $BC$ and $EF$ meet at $N$, and, finally, $CD$ and $FA$ meet at $M$. Then $L$, $M$, and $N$ lie in a straight line.

At about age 18, Pascal built the world's first computer. Within a few years, he had built, and sold, about fifty machines. The computer language PASCAL is named in his honour.

During his early twenties, Pascal studied atmospheric pressure. In 1651, he published an article in which he proved that the earth's atmosphere weighs $8.2 \times 10^{18}$ pounds. Pascal had to argue against Aristotle's false belief that there is no such thing as a vacuum. The metric unit for pressure is named after Pascal:

$$1 \text{ pascal} = 1 \text{ newton per square metre}$$

During his late twenties, Pascal worked with Fermat in the development of probability theory. Pascal solved the following problem, proposed by a gambler, Chevalier de Méré:

> How many times must you throw two dice in order to have at least half a chance of getting double sixes?

Pascal also solved the 'Problem of the Points':

> Two players are flipping a coin. Every time a head comes up Player 1 gets 1 point. Every time a tail comes up Player 2 gets 1 point. It has been agreed that the first player to get 100 points wins \$1000. Suppose Player 1 has $100 - m$ points, and Player 2 has $100 - n$ points. What is the probability $f(m, n)$ that Player 1 will win?

Pascal showed that, if $r = m + n - 1$, then

$$f(m,n) = \frac{\binom{r}{0} + \binom{r}{1} + \cdots + \binom{r}{n-1}}{2^r}$$

This can be proved from the fact that

$$f(m+1,n) = \frac{1}{2}f(m,n) + \frac{1}{2}f(m+1,n-1)$$

Pascal also worked on the concept of mathematical expectation. There is a distribution in probability named after him.

In August 1654, in connection with his studies in probability, Pascal wrote a treatise on what is now called 'Pascal's triangle' (although it had previously been given by Chu Shih-chieh (Zhu Shijie) in 1303). The first 6 rows of this infinite triangle are as follows.

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Note that each number is the sum of the two above it. As Shih-chieh had realised, the $n$th row of the triangle gives the coefficients of the powers of $x$ in the expansion of $(x+1)^n$. Pascal proved such things as the fact that the $k$th entry in row $n$ is

$$\binom{n}{k-1}$$

and the fact that the sum of the entries in the $n$th row is $2^n$.

## November 23, 1654

On November 23, 1654, Pascal's horses went wild, and his carriage nearly fell into the Seine. That night Pascal had a religious experience. He reported it as follows:

> God of Abraham, God of Isaac, God of Jacob, not of the philosophers and the professors. Certainty. Certainty. Sentiment, Joy, Peace. God of Jesus Christ.

From that moment on, Pascal lived in harmony with the words of Saint Paul:

> while the Jews demand miracles and the Greeks look for wisdom, here are we preaching a crucified Christ; to the Jews an obstacle that they cannot get over, to the pagans *madness,* but to those who have been called, whether they are Jews or Greeks, a Christ who is the power and wisdom of God. For God's foolishness is wiser than human wisdom (1 Corinthians 1:22–25).

As a result of his conversion, Pascal got involved, not in the 'sectarian controversies of his day', but in the age-old discussion about the way in which God helps people freely choose to do good. The fruit of this was the beautifully written *Provincial Letters,* which continues, to this day, to draw the praise of theologians and philosophers.

In 1658, Pascal interrupted his work in the philosophy of religion to produce a treatise on the cycloid. This is the curve traced out by a point on the rim of a wheel as the wheel moves over level ground. The techniques Pascal used in this treatise were close to those of calculus.

Also in 1658, Pascal wrote a lucid article about mathematical reasoning. In this article, he discussed the nature of definition, the nature of numbers, and the concept of infinity.

Pascal's last and unfinished work is the *Pensées* — this means 'Thoughts'. This is his defence of the truth of the Christian faith. It is in this work that we find 'Pascal's wager', a mathematical proof that it is wiser to believe in God. Pascal's wager goes as follows.

Suppose there is a nonzero number $\epsilon$ (perhaps extremely small) that is the probability of there being a God. If there is no God, and you believe in God anyway, then, although you will be deluded, and although you may suffer the mockery of atheists (for nothing), your loss will not be enormous. Let us say that it will not exceed 1 'utile' of happiness. If, however, there is a God, and you believe in him, he will make you very happy, giving you at least, say, $\frac{2}{\epsilon}$ utiles of happiness. Hence the mathematical expectation of believing in God is at least

$$\epsilon \times \frac{2}{\epsilon} \;-\; (1-\epsilon)\times 1 = 1+\epsilon$$

On the other hand, if there is no God and you do not believe in God, you may gain a little, but not more than, say, 1 utile of happiness. However, if there is a God, and you do not choose to believe in him, then you will not get anything good out of your not believing in him. The mathematical expectation of not believing in God is thus less than

$$(1-\epsilon)\times 1 \;+\; \epsilon\times 0 = 1-\epsilon$$

Since $1+\epsilon > 1-\epsilon$, and since, other things being equal, a wise person will act in such a way as to maximise the expectation of their happiness, it follows that, other things being equal, a wise person will choose to believe in God.

## The Real Madman

What about E. T. Bell? Given Bell's view, we have the following conclusions:
(1) a person who lacks reason is an expert at mathematical reasoning;
(2) a person who lacks reason writes a lucid philosophical article on reason;
(3) a person with 'masochistic proclivities' delights in the joy and peace he finds in God;
(4) a writer of 'platitudinous observations' is a brilliant author;
(5) a writer of 'profitless speculations' is a world-famous philosopher.
These conclusions are insane, and one might well raise some questions about Bell's mental state.

# Exercises 29

1. Make your own drawing of Pascal's mystic hexagram, and show, from the drawing, that the three meeting points are collinear.

2. Solve the problem of the two dice.

3. Using mathematical induction (a technique employed in Euclid), show that the $(n+1)$-th Fibonacci number is

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

4. Two jacks and an ace are placed face down on a table. You get to pick one of them (without turning it over). If you pick the ace, you get to marry the charming royal heir. If you pick a jack, you get nothing. So you choose a card, without yet turning it over. At this point, the organiser of the game, who knows exactly where the ace is, chooses a jack from one of the two cards you have not picked, and turns it over. He says, 'if you want, you can stick to your original choice. However, if you give me \$300, I'll let you change your mind and pick the other face-down card.' You desperately want to marry the royal heir, but should you take this offer? Why or why not?

5. Some crazy math teacher tells you that the equation of the cycloid is

$$y = \arccos(1-x) - \sqrt{2x - x^2}$$

but she does not tell you the radius of the wheel or the starting position of the point on the rim, or the direction in which the wheel is rolling! Supply this missing information.

## *Challenges for Experts*

1. Show that Pascal's solution of the Problem of the Points is correct.

2. Find the area under the cycloid.

3. Check the following proof of Pascal's mystic hexagram theorem. By the theorem of Menelaus, applied to triangle $UVW$ in connection with points $L$, $D$, $E$ (see the Figure above) we obtain

$$VL \times WD \times UE = LW \times DU \times EV$$

With the same triangle and points $A$, $M$, $F$, we obtain

$$VA \times WM \times UF = AW \times MU \times FV$$

With the same triangle and points $B$, $C$, $N$, we obtain

$$VB \times WC \times UN = WB \times UC \times NV$$

Thus

$$VL \times WD \times UE \times VA \times WM \times UF \times VB \times WC \times UN$$

$$= LW \times DU \times EV \times AW \times MU \times FV \times WB \times UC \times NV$$

But

$$\begin{aligned} UE \times UF &= UC \times UD \\ VA \times VB &= VE \times VF \\ WC \times WD &= WA \times WB \end{aligned}$$

(Why?) Hence, cancelling, we have

$$VL \times WM \times UN = WL \times UM \times VN$$

By another application of Menelaus's theorem, $L$, $M$, and $N$ are collinear.

## *Essay Questions*

1. Is it psychologically possible to choose one's beliefs in order, say, to maximise one's expectation of happiness? Why or why not?

2. Was Pascal right to spend more time on philosophy and prayer than on mathematics? Why?

# 30
# The Seventeenth Century II

## Torricelli

Evangelista Torricelli (1608–1647) was the person who invented the barometer. He found the length of the logarithmic spiral $r = e^\theta$ as $\theta$ goes from 0 to $-\infty$, and he showed that, although the area under the curve $y = 1/x$ from $x = 1$ to $\infty$ is infinite, the solid obtained by revolving this area about the $x$-axis has a finite volume. In 1644, Torricelli published an original proof of the fact that the area under the cycloid is 3 times the area of the wheel that generates it. This led to a priority dispute with Gilles Persone de Roberval (1602–1675) who had previously solved the same problem but not published his solution.

Torricelli also solved a problem posed by Fermat:

> Let $ABC$ be a triangle, each of whose angles is $< 120°$. Find the point $T$ such that $TA + TB + TC$ is minimised.

Torricelli found the point by constructing equilateral triangles $ABC'$, $BCA'$, and $CAB'$ on the sides of $ABC$ and outside it. He showed that $AA'$, $BB'$, and $CC'$ are concurrent at the desired point $T$.

## Wallis' Expression for $\pi$

John Wallis (1616–1703) held the Savilian chair of geometry at Oxford. He was a royalist, and he ended up as a chaplain to Charles II. He also invented a system for teaching deaf mutes.

Wallis discovered that

$$\lim_{n\to\infty} 2\,\frac{2^2 4^2 \dots (2n)^2}{3^2 5^2 \dots (2n-1)^2(2n+1)} = \pi$$

This can be proved as follows. Let $f(n)$ be the fraction in the expression. Let $k$ be a positive integer $> 1$. Using integration by parts, we have

$$I(k) = \int_0^{\pi/2} \sin^k x dx = \frac{k-1}{k} \int_0^{\pi/2} \sin^{k-2} x dx$$

Thus

$$I(2n) = ((2n+1)f(n))^{-1/2}\frac{\pi}{2}$$

while

$$I(2n+1) = (f(n)/(2n+1))^{1/2}$$

Since $\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x$, then, taking integrals, we obtain

$$(f(n)/(2n+1))^{1/2} \le ((2n+1)f(n))^{-1/2}\frac{\pi}{2} \le (f(n)(2n+1))^{1/2}/(2n)$$

so that

$$2f(n) \le \pi \le 2f(n)(2n+1)/(2n)$$

and the result follows.

Wallis also gave a proof of the Parallel Postulate based on the assumption that two triangles can be similar but not congruent.

## Newton

Isaac Newton (1642–1727) was born on Christmas Day, about three months after his father died. His mother abandoned him when he was 3. In 1661, he went to Cambridge University, where he met Isaac Barrow (1630–1677), who was the first mathematician to realise that

$$\int_a^b f'(x)dx = f(b) - f(a)$$

This is the fundamental theorem of calculus. During 1665 and 1666, Newton did much of the work for which he is acclaimed as one of the founders of calculus (the other being Leibniz). He also analysed the nature of colour, discovered the generalised binomial theorem, and produced a theory of gravitation.

The generalised binomial theorem states that if $x$ is any real between $-1$ and 1, and $m$ is any real number, then

$$(1+x)^m = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where

$$a_k = \frac{m(m-1)(m-2)\dots(m-(k-1))}{k!}$$

For example,

$$(1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \cdots$$

When $m$ is a positive integer, the terms are all 0 after a certain point, and the formula is just the 'nongeneralised' binomial theorem known to the Arabs and Chinese long before. However, when $m$ is not a positive integer, we have something wholly new.

## The Law of Gravity

Newton's Law of Universal Gravitation states that, for some constant $G$, the force of gravitational attraction between two objects $A$ and $B$, with masses $m_A$ and $m_B$, respectively, and separated by distance $r$ is

$$F = G\frac{m_A m_B}{r^2}$$

More precisely, suppose $A$ is a sphere and suppose that the density of the material in $A$ is a function of the distance of that material from the centre of $A$. Suppose that $B$ is a similar sphere, wholly outside $A$. Then if $r$ is the distance between the centres of $A$ and $B$, the gravitational attraction between them is given by the above formula. It turns out that

$$G = 6.67 \times 10^{-11}\frac{Nm^2}{kg^2}$$

Newton also realised that if $A$ is as above and $C$ is an object much smaller than $A$ and inside $A$, then the gravitational attraction that $A$ exerts on $C$ is

$$G\frac{m'_A m_C}{r^2}$$

where $m'_A$ is the mass of material in $A$ that is closer to the centre of $A$ than $C$ is, $m_C$ is the mass of $C$, and $r$ is the distance between $C$ and the centre of $A$. This has some interesting consequences.

For example, suppose $A$ is a planet with uniform density $d$, with no atmosphere and with a narrow hole along one of its diameters. Suppose that you drop a small object $C$ down this hole. Then $C$ will come to rest on the other side of the planet, in a time that is independent of the radius of the planet.

As another example, suppose $A$ is a planet with a spherical inner core of radius $R$ and uniform density 5. Suppose that the radius of the whole planet is $5R$ and that the material not in the inner core has uniform density 1. Suppose there is a narrow straight hole, going through the centre of this planet. Then, if a ball is dropped down this hole, its weight at first decreases, then increases, and finally decreases to zero. Indeed, if $R < x < 5R$, then the weight of the ball at distance $x$ from the centre of the planet is proportionate to

$$\frac{4R^3 + x^3}{x^2}$$

and this expression has a minimum at $x = 2R$.

# The Cows in the Meadow

In 1669, Barrow resigned from his job at Cambridge University so that Newton could have it. For almost thirty years, Newton worked as a professor. His lectures did not always attract many students. In *Journey through Genius,* William Dunham reports that

> Newton's lectures would last for half an hour except when there was no one at all in the audience; in that case he would stay only 15 minutes.

In 1696, Newton quit his job as a professor, to work as the Master of the Mint. However, he did not entirely give up mathematics. About 1772, he posed the following problem:

> Suppose that grass grows at a constant rate. For $i = 1, 2, 3$, suppose it takes $x_i$ oxen $t_i$ weeks to eat all the grass on $a_i$ acres. Prove that
>
> $$a_1a_2x_3t_3(t_2 - t_1) + a_2a_3x_1t_1(t_3 - t_2) + a_1a_3x_2t_2(t_1 - t_3) = 0$$

Newton lived on to age 84.

# Newton's Modesty

Newton believed that if he had seen farther than others, it was only because he 'stood on the shoulders of giants'. Newton likened himself to a child:

> I seem to have been only a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

# Exercises 30

1. Let $a_n = 1 - 1/(2n)^2$. Show that $a_1 a_2 a_3 \ldots = 2/\pi$.

2. Show that the absolute values of the coefficients in the binomial expansion of $(1+x)^{-3}$ are the triangular numbers.

3. Let $C(m,n)$ be the coefficient of $x^n$ in the binomial expansion of $(1+x)^{-m}$ where $m$ is a positive integer. Prove that
$$C(m+1, n+1) = C(m, n+1) - C(m+1, n)$$

4. What is the gravitational attraction between two 40 kg persons, separated by $10^7\sqrt{G}$ metres?

5. Let $h$ be the starting height of the grass and $g$ the height added every week through growth. Show that in 1 week 1 ox eats
$$\frac{ha_1 + gt_1a_1}{x_1t_1}$$
units of grass. Then solve Newton's cow problem.

## *Challenges for Experts*

1. Show that the volume of Torricelli's solid of revolution is $\pi$.

2. Let $f(x) = 1 + a_1x + a_2x^2 + \cdots$, with $a_k$ as defined in the text above. Differentiating term by term, show that $(1+x)f'(x) = mf(x)$. From this prove that
$$\frac{d}{dx}\frac{f(x)}{(1+x)^m} = 0$$
so that, for some constant $C$, $f(x) = C(1+x)^m$. Finally, show $C = 1$.

3. Solve the problem Fermat gave to Torricelli.

## *Essay Question*

1. Was Newton being unduly modest, or just accurate, in his self-evaluation?

# 31
# Leibniz

Gottfried Wilhelm Leibniz (1646–1716) was born in Leipzig, Germany. His father died when the boy was only 6. Leibniz educated himself, using his late father's library, and entered university in Leipzig when he was only 15. In 1666, he was refused his degree of Doctor of Law on the grounds that he was too young. In the same year, Leibniz conceived the idea of symbolic logic, a universal language in which all rational thinking could be expressed.

Leibniz worked as a diplomat for the Elector of Mainz. It was in this capacity that he went to Paris in 1672 to convince Louis XIV to attack Egypt (rather than some European country). This diplomatic mission failed, but Leibniz had a chance to meet many of the leading intellectuals. For example, it was in Paris that he met Christian Huygens, who introduced him to geometry and physics.

Huygens challenged Leibniz to sum the series

$$1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{\frac{1}{2}n(n+1)}+\cdots$$

Leibniz solved the problem thus:

$$\frac{1}{\frac{1}{2}n(n+1)}=2\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

so the series equals

$$2(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\cdots)=2(1+0)=2$$

In the twentieth century, we would object to this on the grounds that Leibniz might equally well have said

$$\frac{1}{\frac{1}{2}n(n+1)} = 2\left(\frac{n+1}{n} - \frac{n+2}{n+1}\right)$$

so the series equals

$$2(2 - \frac{3}{2} + \frac{3}{2} - \frac{4}{3} + \frac{4}{3} - \frac{5}{4} + \frac{5}{4} - \cdots) = 2(2+0) = 4$$

We now prefer to solve this problem by first showing that the $m$th partial sum of the series is

$$2\left(1 - \frac{1}{m+1}\right)$$

and then taking the limit as $m$ goes to infinity.

In 1673 Leibniz visited the Royal Society in England. In 1675, he was back in Paris, developing the calculus and using almost the same notation we use today. It was at this time that Leibniz derived the rule for the derivative of a product of two functions. Unknown to Leibniz, Newton had done the same work ten years earlier. However, it was Leibniz who published it first, in 1684. Some British mathematicians unjustly accused Leibniz of plagiarising Newton's work, and a bitter priority battle was fought.

Leibniz thought of the '$dy$' and '$dx$' in $dy/dx$ as 'infinitesimal' quantities. Thus $dx$ was an infinitely small nonzero increment in $x$ and $dy$, defined as

$$dy = f(x+dx) - f(x)$$

was also (usually) different from 0. For example, if $y = f(x) = x^2$, then

$$dy = (x+dx)^2 - x^2 = 2x(dx) + (dx)^2$$

This represented the rise of the function $f$ corresponding to a run of $dx$. Hence the slope of the tangent at $x$ was

$$\frac{\text{rise}}{\text{run}} = \frac{dy}{dx} = 2x + dx$$

and hence, now equating the $dx$ to 0, the tangent at $x$ had slope $2x$.

The concept of the infinitesimal, which was also found in Newton's work, was criticised by philosopher and bishop George Berkeley (1685–1753). How, he asked, can we divide by $dx$ if it is 0? How can we get the slope of the tangent to be $2x$, rather than $2x + dx$, if it is not 0? Either $dx$ is 0 or it is not, and either way there is a problem.

Nineteenth-century mathematicians Augustin Cauchy and Karl Weierstrass agreed there were problems and responded by putting calculus on the firm footing it has today. In the twentieth century, $dy/dx$ is seen not as a quotient, but as a limit of a quotient: $dy/dx$ is the number $L(x)$ such that, for any given $\epsilon > 0$, there is a number $\delta$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - L(x) \right| < \epsilon \quad \text{if} \quad |t - x| < \delta$$

There is also, today, a rigorous version of the infinitesimal itself. In 1966, Abraham Robinson introduced 'nonstandard' real numbers, some of which are less than any rational but greater than 0. Whereas Leibniz's infinitesimals are simple but not rigorous, those of Robinson are rigorous but not simple. Nonstandard analysis is a complicated and bizarre system. It seems too ugly to be true.

Leibniz himself did not think that there really were any infinitesimals. He considered them to be 'fictious' — useful but not part of the universe. He held the same view of imaginary numbers and infinite numbers (*New Essays* II 17). On the other hand, in a letter to Simon Foucher in 1693, Leibniz claimed that 'the smallest particle should be considered as a world full of an infinity of creatures'. Although there was, for Leibniz, no mathematical number corresponding to them, there were infinitely many objects in nature.

Leibniz is famous for his assertion that this is the 'best of all possible universes'. (Given that God created the universe, could he have failed to create the best?) Voltaire ridiculed this view in his *Candide,* but it is unlikely that Leibniz intended 'best' to mean 'most pleasant' — as it did in the simple-minded interpretation of Voltaire.

Leibniz's most original contribution to philosophy was the system of the *Monadology.* In this work he proposed the idea that the universe is made up of simple substances, called 'monads', which are capable of perception. Human souls are monads with memory and reason. The monads 'have no windows' in the sense that they have no direct interaction with the rest of the universe. They are related to each other only by a 'pre-established harmony', set up by God.

Leibniz was a theist. In Section XXIII of the *Discourse on Metaphysics,* he writes:

> This is in fact an excellent privelege of the divine nature, to have need only of a possibility or an essence in order to actually exist.

In other words, the statement 'God exists' is not contingent: it is either impossible that God exist, or necessary that God exist. Leibniz argued that it is possible that God exist, and he concluded that it is necessary that God exist. At the end of the *Discourse on Metaphysics,* Leibniz claims that God wants a personal relationship with human beings.

# Exercises 31

1. Give a proof of the product rule for derivatives.

2. Show that the $n$th derivative of $fg$ is

$$\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

3. In 1716, Leibniz found a side 3 magic cube, with its 27 rows and 4 long diagonals each adding up to 42. Show that the following represents such a cube.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 15 | 19 | \| | 24 | 1 | 17 | \| | 10 | 26 | 6 |
| 12 | 25 | 5 | \| | 7 | 14 | 21 | \| | 23 | 3 | 16 |
| 22 | 2 | 18 | \| | 11 | 27 | 4 | \| | 9 | 13 | 20 |

## *Challenge for Experts*

1. Suppose a right circular cone has a base of diameter $d$ and a height $h$. Suppose it is cut by a plane parallel to its axis at a distance $x$ from its axis. Leibniz used his calculus to derive the volumes of the two parts (in terms of $d$, $h$, and $x$). Do the same.

## *Essay Question*

1. Write a paragraph on the following. Since there is no largest cardinal number, there is no 'best' universe. God simply had to pick some very good universe and create it. Thus we cannot blame God if this is not the best of all possible universes.

# 32
# The Eighteenth Century

## De Moivre

Abraham de Moivre (1667–1754) was French but he was a sincere Protestant and he had to leave France for England in 1688, not long after the Edict of Nantes was revoked (in 1685).

De Moivre is famous for his formula

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

He is also well known for his work in probability. In 1718 he published the *Doctrine of Chances,* which contained a series of solved problems, such as the following:

> Suppose that three tickets will be given prizes in a lottery having 40,000 tickets. What is the chance of winning at least one prize if you buy 8000 of those tickets?

It was de Moivre who first found the formula for the $n$th Fibonacci number, and it was de Moivre (not James Stirling) who first discovered 'Stirling's formula':

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

For example, $10! = 3,628,800$ and the formula gives $3,598,695.6$.

Society failed de Moivre. In spite of several publications and a letter of recommendation from Isaac Newton, de Moivre was never given a proper job in mathematics. He had to earn his living tutoring and answering probability questions for gamblers.

As de Moivre approached the end of his life, he slept an extra fifteen minutes a day. When he reached the full twenty-four hours, he died.

# Euler

Leonhard Euler (1707–1783) was Swiss. He spent part of his life in Berlin but died in Saint Petersburg, Russia. In 1766, at age 60, Euler became blind, but this did not slow his flood of publications.

Euler's collected works run to about 75 large volumes. Included are articles on shipbuilding and a reasoned defense of the divine origin of the Bible. Some of the many results in mathematics are the following.

(1) If a convex polyhedron has $V$ vertices, $F$ faces, and $E$ edges, then

$$V+F-E=2$$

For example, a cube has 8 vertices, 6 faces, and 12 edges, and $8+6-12=2$.

(2)

$$e^{i\pi}=-1$$

(3)

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

(4) Every even perfect number has the form $2^{n-1}(2^n-1)$, with $2^n-1$ prime.

(5) If $\phi(n)$ is the number of positive integers not greater than the positive integer $n$ and relatively prime to it, and if $a$ is a positive integer relatively prime to $n$, then $n|a^{\phi(n)}-1$.

(6) Fermat was wrong when he conjectured that all numbers of the form $2^{2^n}+1$ are prime, since $641|2^{2^5}+1$.

(7) The circumcentre, orthocentre, and centroid of any nonequilateral triangle are collinear. (The line that passes through these three points is the *Euler line.* )

The last result can be proved as follows. Let $ABC$ be the triangle, with $A'$ the midpoint of $BC$. Let $H$ be its orthocentre, $O$ its circumcentre, and $G$ its centroid. Let $H'$ be such that $G$ is between $O$ and $H'$ and $H'G/GO=2$. Since $GA/GA'=2$, it follows that $AGH'$ and $A'GO$ are similar. Hence $AH'$ and $OA'$ are parallel. Thus $AH'$ is perpendicular to $BC$. Similarly, $CH'$ is perpendicular to $AB$. Thus $H'$ is $H$.

Euler tried to prove that every natural number is a sum of four natural number squares. He failed to do this, but he provided some partial results that helped J. L. Lagrange in a successful attempt on that problem. Euler's partial results were the following.

**Lemma A:** If

$$\begin{aligned} p &= ae + bf + cg + dh \\ q &= af - be + ch - dg \\ r &= ag - bh - ce + df \\ s &= ah + bg - cf - de \end{aligned}$$

then

$$(a^2 + b^2 + c^2 + d^2)(e^2 + f^2 + g^2 + h^2) = p^2 + q^2 + r^2 + s^2$$

Hence if every prime is a sum of four squares, so is every natural number.

**Lemma B:** For every odd prime $p$ there is an integer $m$ such that $0 < m < p$ and $mp$ is a sum of four squares.
**Proof:** The squares $0^2$, $1^2$, $2^2$, ..., $\left(\frac{p-1}{2}\right)^2$ all leave different remainders when divided by $p$. For suppose $A^2 = ap + r$ and $B^2 = bp + r$, with $A > B$. Then $p$ is a factor of

$$A^2 - B^2 = (A - B)(A + B)$$

However, $0 < A - B,\ \ A + B < p$, so $p$ is a factor of neither $A - B$ nor $A + B$. Contradiction.

Similarly,

$$-1 - 0^2,\ \ -1 - 1^2,\ \ -1 - 2^2, \ldots, -1 - \left(\frac{p-1}{2}\right)^2$$

all leave different remainders when divided by $p$.

Each of the above two lists has $\frac{p+1}{2}$ members. Together, they contain $p + 1$ integers. Since there are only $p$ possible remainders when one divides by $p$, there is some $x^2$ from the first list and some $-1 - y^2$ from the second list that leave the same remainder when divided by $p$. Hence $p$ divides their difference, $x^2 + y^2 + 1$. That is, for some integer $m$, we have

$$mp = x^2 + y^2 + 1^2 + 0^2$$

Moreover, since $0 \leq x,\ \ y < \frac{p-1}{2}$ it follows that $0 < m < p$.

Nicolas Condorcet's *Elogium of Euler* describes Euler's death:

> he dined with Mr Lexell and his family, talked of Herschell's planet, and of the calculations which determine its orbit. A little after he called his grand-child, and fell a playing with him as he drank tea, when suddenly, the pipe, which he held in his hand, dropped from it, and he ceased to calculate and to breathe.

## Laplace

Pierre Simon Laplace (1749–1833) was born of poor parents, but ended up as a marquis under the restored Bourbons. Politically, he was an opportunist.

Napoleon once told Laplace, 'you have written a big book on the universe without mentioning its creator' — to which Laplace replied, 'I do not need that hypothesis'. The book in question was the *Mécanique Céleste.* In it, Laplace argued that, gravitational perturbations notwithstanding, the solar system is stable — and hence does not need occasional adjustments by a God.

Laplacian mechanics is sometimes associated with an impersonal materialistic determinism, according to which our every thought and choice can be predicted by a clever 'demon'. The demon starts with a full and exact description of the universe at some moment in the distant past, and then, using the laws of physics, deduces exactly where, say, the atoms in the President of the United States will be on December 1, 2020 A.D. This idea is not consistent with what we now know about the laws of physics, namely, that they are statistical in nature.

Laplace's great contribution to mathematics was his phrase, 'it is easy to see'. Nathaniel Bowditch (1773–1838), who translated much of the *Mécanique Céleste* into English, wrote:

> I never came across one of Laplace's 'Thus it plainly appears' without feeling sure that I had hours of hard work before me to fill up the chasm and find out and show how it plainly appears.

## Legendre

Adrien Marie Legendre (1752–1833) contributed to geometry, number theory, and the theory of elliptic functions. His *Eléments de Géométrie* was a famous reworking of Euclid. In one of the editions, Legendre showed that the parallel postulate follows from the axiom of Archimedes and the assumption that there is a square.

In number theory, Legendre was the first person to prove, in 1825, that the equation $x^5 + y^5 = z^5$ has no solution in positive integers.

In 1792, when he was 40, Legendre married a woman not quite half his age. As we shall see in the next chapter, Lagrange did the same thing in the same year.

# Exercises 32

1. Prove de Moivre's formula for positive integers $n$.

2. Use de Moivre's formula to find the cube roots of $1 + \sqrt{-2}$.

3. Solve de Moivre's lottery problem.

4. Show that the triangle with vertices $(0,0)$, $(12,0)$, and $(16,8)$ has circumcentre $(6,8)$, orthocentre $(16,-8)$, centroid $(28/3, 8/3)$, and Euler line $8x + 5y = 88$.

5. Euler conjectured that $a^5 + b^5 + c^5 + d^5 = e^5$ has no positive integer solutions. In 1966, L. Lander and T. Parkin found a counterexample with $a = 27$, $b = 84$, $c = 110$, $d = 135$, and $e = 144$. Verify that this is a counterexample.

## *Challenges for Experts*

1. Let

$$g(n) = \frac{e^n n!}{n^n \sqrt{n}}$$

Then, where $f$ is defined as in the section on Wallis in Chapter 30,

$$\frac{(g(n))^2}{g(2n)} = \sqrt{\frac{(4n+2)f(n)}{n}}$$

2. Hence, taking limits as $n$ goes to infinity and assuming that $g(n)$ has a finite nonzero limit as $n$ goes to infinity,

$$\lim_{n \to \infty} g(n) = \sqrt{2\pi}$$

and Stirling's formula follows.

3. Prove that $g(n)$ does have a finite nonzero limit as $n$ goes to infinity.

*Essay Questions*

1. Every math professor should be fired every five years, so that people like de Moivre could get good jobs and the deadwood would be cleared away. Comment.

2. In his article 'Rettung der Göttlichen Offenbahrung' (1747), Euler claims that the Bible offers us a way to be happy. Comment on this claim.

3. Even if the solar system is gravitationally stable, it still needs God to keep it in existence. Comment.

4. Is it ethical for mathematicians to use the expression 'it is easy to see' when (1) this is not scientific language, (2) it is boastful, and (3) it causes many students humiliation and frustration when they do not see?

# 33
# Lagrange

## Joseph and Renée

Joseph Louis Lagrange (1736–1813) was born in Italy. His father lost the family fortune through speculation, but Lagrange later commented that if it had not been for this bad luck, he might never have turned to mathematics.

In 1764, Lagrange went to Paris and met the French mathematicians. Clairaut described him as

> a young man, no less remarkable for his talents than for his modesty; his temperament is mild and melancholic; he knows no other pleasure than study.

From 1766 to 1787, Lagrange worked in Berlin, but after the death of Frederick II, Lagrange moved to Paris, where he became a favourite of Marie Antoinette — just in time for the Revolution!

Lagrange was involved in the introduction of the metric system. When people pleaded the advantages of base 12, he would ironically defend base 11.

About 1790, Lagrange became subject to fits of depression and loneliness. He no longer wanted to do mathematics. He was rescued from this state by the love of a teenaged girl, Renée Lemonnier, who insisted on marrying him. They married in 1792, and, for the remaining twenty years of his life, Lagrange was happy and mathematically productive.

## Some of Lagrange's Triumphs

Among Lagrange's many achievements are the following:
(1) an explanation of why the moon always shows the same face to the earth;
(2) the first proof (given in 1771) of John Wilson's theorem that if $p$ is a prime, then it factors $(p-1)!+1$, and conversely;
(3) the first proof (given in 1766) that if $R$ is any given positive nonsquare integer, then $x^2 - Ry^2 = 1$ has a positive integer solution;
(4) the first complete solution of the Diophantine equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

(5) the first proof (given in 1770) that every natural number is a sum of four natural number squares;
(6) a systematic theory of differential equations; and
(7) his book *Mécanique analytique* (1788).

## Four Square Theorem

Euler had shown that if $p$ is an odd prime, then there is an integer $m$ with $0 < m < p$ such that $mp = a^2 + b^2 + c^2 + d^2$ (for some natural numbers $a$, $b$, $c$, and $d$). Consider the least such number $m$. As we saw in Chapter 32, in order to establish the theorem that every natural number is a sum of four squares, it suffices to show that $m = 1$.

**Lemma C:** If $p$ is an odd prime and $m$ is the least integer such that $0 < m < p$ and $mp = a^2 + b^2 + c^2 + d^2$ for some natural numbers $a$, $b$, $c$, and $d$, then $m$ is odd.

**Proof:** If $m$ is even, then 0, 2, or 4 of $a$, $b$, $c$, and $d$ are odd. Pairing the odd numbers, we get, say,

$$\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 + \left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2 = \frac{mp}{2}$$

which is an expression of $\frac{mp}{2}$ as a sum of four natural number squares. Since $\frac{m}{2} < m$, this is impossible — by $m$'s minimality. So $m$ is odd.

Using Lemmas A and B (see Chapter 32) and Lemma C, Lagrange proved the following in 1770.

Every natural number is a sum of four natural number squares.

**Proof:** Since $2 = 1^2 + 1^2 + 0^2 + 0^2$, Lemma A implies that it suffices to prove that every odd prime $p$ is a sum of four squares.

Let $p$ be any odd prime and let $m$ be the least integer between 0 and $p$ such that $mp$ is a sum of four squares. (That there is such an $m$ follows from Euler's Lemma B.) By Lemma C, $m$ is odd.

To obtain a contradiction, suppose $m \geq 3$.

Suppose $mp = a^2 + b^2 + c^2 + d^2$ and let $x$ be the integer closest to $a/m$. Then $|a/m - x| < \frac{1}{2}$ and $x' = a - mx$ is between $-\frac{m}{2}$ and $\frac{m}{2}$. Let $y$, $z$, and $w$ be the integers closest to $b/m$, $c/m$, and $d/m$, respectively. Then

$$\begin{aligned} y' &= b - my \\ z' &= c - mz \\ w' &= d - mw \end{aligned}$$

are each between $-\frac{m}{2}$ and $\frac{m}{2}$.

Let $Z' = x'^2 + y'^2 + z'^2 + w'^2$. Then $Z' < 4(\frac{m}{2})^2 = m^2$. Also $Z' \neq 0$, lest $m$ divide each of $a$, $b$, $c$, and $d$, with the result that $m^2$ divide $a^2+b^2+c^2+d^2 = mp$. This is impossible because $p$ is prime, and $1 < m < p$.

Let $Z = x^2 + y^2 + z^2 + w^2$. Let $T = xx' + yy' + zz' + ww'$. Then the fact that $mp = a^2 + b^2 + c^2 + d^2$ implies that

$$mp = m^2 Z + 2mT + Z'$$

(since $a = x' + mx$, and so on).

Let $M = Z'/m = p - mZ - 2T$, an integer. Since $Z' \neq 0$, it follows that $M \neq 0$. Also, since $Z' < m^2$, it follows that $M < m$.

Now

$$\begin{aligned} Mp &= (M/m)mp \\ &= (M/m)(m^2 Z + 2mT + Z') \\ &= ZMm + 2MT + MZ'/m \\ &= ZZ' - T^2 + (T + M)^2 \end{aligned}$$

(since $M = Z'/m$).

By Lemma A, $ZZ' = T^2 + q^2 + r^2 + s^2$ for some natural numbers $q$, $r$, and $s$. Thus $Mp = q^2 + r^2 + s^2 + (T + M)^2$, a sum of four squares. But $M < m$. Contradiction.

# Exercises 33

1. Verify Wilson's theorem for $p = 7$.

2. Use factorisation to find all the integer solutions of

$$14x^2 + 53xy + 14y^2 - 13x - 23y + 3 = 0$$

3. Write 99 as a sum of four squares.

4. Prove that a natural number of the form $8n+7$ cannot be written as a sum of three squares.

5. Let $x = (m-m^3)/6$, where $m$ is an integer. Show that $x$ is an integer. Then show that

$$m = m^3 + (x+1)^3 + (x-1)^3 + (-x)^3 + (-x)^3$$

a sum of 5 cubes.

6. Write 239 as a sum of 9 nonnegative cubes, and show that it cannot be written as a sum of 8 nonnegative cubes.

# 34
# Nineteenth-Century Algebra

It was in the nineteenth century that algebra acquired the very abstract nature it has today. The key event in its development was the overthrow of the law of commutativity for multiplication (that is, for all $x$ and $y$, $xy = yx$). This was accomplished by William Rowan Hamilton (1805–1865) who, incidentally, was the first person to conceive of complex numbers as ordered pairs. Just as many people before Lobachevsky thought that Euclid's parallel postulate was a kind of sacred truth, so many people before Hamilton thought that the law of commutativity for multiplication was ineluctable. For us it is a commonplace that this law need not hold, since we have a ready example of noncommutativity in matrix multiplication. Hamilton, however, made his discovery about fifteen years before matrix algebra had been discovered.

The system Hamilton discovered is the 'quaternions'. This system not only shows that one can 'break' fundamental 'laws' of arithmetic, but also serves as a kind of theory of three-dimensional vectors. Hamilton's noncommutative multiplication is closely related to the cross product for 3-component vectors (which was investigated after Hamilton's work on quaternions).

Hamilton's idea dawned on him as he crossed Broome Bridge in Dublin, Ireland, while walking with his wife, on October 16, 1843. He wrote about his discovery to his son Archibald:

> An electric circuit seemed to close; and a spark flashed forth ... Nor could I resist the impulse — unphilosophical as it may have been — to cut with a knife on a stone of Brougham Bridge,

as we passed it, the fundamental formula with the symbols, $i$, $j$, $k$; namely,

$$i^2 = j^2 = k^2 = ijk = -1$$

A *quaternion* is a 'number' of the form $a + bi + cj + dk$ where $a$, $b$, $c$, and $d$ are reals, and $i$, $j$, and $k$ are special numbers subject to the rules cut on the bridge.

If quaternion multiplication were commutative, we would get a contradiction. For $ijk = -1$ implies that $ijk^2 = -k$, whence $-ij = -k$. Also $jiij = 1$, so that $jiijk = k$, whence $-ji = k$. With commutative multiplication we get $-k = k$ and hence $k = 0$ — although $k^2 = -1$. To avoid this contradiction, we must have $ij \neq ji$.

Although they do not obey the law of commutativity for multiplication, the quaternions do obey the other laws for fields. For example, the nonzero quaternion $a + bi + cj + dk$ has a multiplicative inverse, namely,

$$\frac{a}{N} - \frac{b}{N}i - \frac{c}{N}j - \frac{d}{N}k$$

where $N = a^2 + b^2 + c^2 + d^2$. Indeed, a short calculation shows that the product of a quaternion $a + bi + cj + dk$ and its *conjugate* $a - bi - cj - dk$ does equal its *norm* $a^2 + b^2 + c^2 + d^2$. The conjugate of a quaternion $q$ is written $\overline{q}$, and its norm is written $Nq$.

The nineteenth century saw quite a bit of algebra. It was proved that there is no general algebraic solution for fifth-degree equations. Matrix algebra was developed. Hamilton's quaternions, however, were the first instance of the otherworldly abstraction that characterises modern algebra and which was its hallmark in the twentieth century.

# Exercises 34

1. If $q$ and $q'$ are quaternions, then $\overline{qq'} = \overline{q}\overline{q'}$.

2. If $q$ is a quaternion, then $N\overline{q} = Nq$.

3. If $q$ and $q'$ are quaternions, then $N(qq') = (Nq)(Nq')$.

4. Show that quaternion multiplication is associative.

5. Find all the quaternions whose square is $-1$.

6. Show that Euler's 'Lemma A' in Chapter 32 is an instance of the quaternion theorem $(Nq)(Nq') = N(\overline{qq'})$.

7. Solve the following problem posed by Hamilton. Find a route along the edges of a dodecahedron that passes exactly once through each vertex.

## *Essay Question*

1. What is abstraction in mathematics? Is a more abstract piece of mathematics necessarily more general? Is a more abstract piece of mathematics necessarily better?

# 35
# Nineteenth-Century Analysis

A lack of mathematical rigour has to do with significant gaps in arguments. Either the mathematician is being careless or is relying on intuitions that cannot be easily translated into deductive reasoning. Prior to the nineteenth century, many arguments in calculus lacked rigour. Words like 'small', 'infinity', 'approaches', and 'limit' were used without ever being precisely defined. Infinite series were treated by methods analogous to those used for finite series, and no justification for doing this was offered.

Two of the mathematicians responsible for finally putting calculus on a rigorous basis were Cauchy and Weierstrass. Their work, with its emphasis on rigour, is typical of nineteenth-century work in analysis.

## Cauchy

Augustin-Louis Cauchy (1789–1857) was born in Paris the year of the French Revolution. His career as a mathematician was well underway when, in 1818, he married Aloise de Bure. He wrote her a poem that ends:

> I shall love you, my tender friend,
> Until the end of my days;
> And since there is another life
> Your Louis will love you always.

The noble Christian sentiment expressed in this poem was typical of Cauchy's life. He was a determined idealist. He supported the ancient line of French kings, consistently refusing to support the new French rulers. He supported the unpopular Jesuits. He supported a project to help the poor in Ireland and another project to set up Christian schools in the Middle East. People did not appreciate Cauchy's determined idealism, and he lost several jobs and job opportunities on account of it. For example, in 1843, it was Guglielmo Libri (1803–1869), not Cauchy, who was given the chair of mathematics at the Collège de France — not because anyone thought Libri was a better mathematician, but because Libri attacked the Jesuits, whereas Cauchy defended them. (It was this same Libri who later fled France when it was discovered that he was stealing library books.)

Rigour was one of Cauchy's mathematical ideals, and Cauchy had to suffer for this ideal too. His students hated rigour. For them, it meant extra lectures and longer hours of study. One year Cauchy started his calculus course with thirty students and all but one dropped out. Cauchy was painfully aware of his bad reputation with the students, but he nonetheless clung to the ideal of giving complete and careful proofs of each and every theorem.

Cauchy made enormous contributions to analysis. A few of them are

(1) the first rigorous definition of continuity,
(2) the Cauchy criterion for sequence convergence,
(3) the extended mean value theorem, and
(4) the Cauchy residue theorem of complex analysis.

## Weierstrass

A second great analyst and proponent of rigour was Karl Weierstrass (1815–1897). He studied at the University of Bonn, Germany, but spent most of his time drinking and fencing and had to leave without his degree. At age 26, he got a job as a high school teacher. He did mathematics at night, but it was not until he was about 40 that one of his articles brought him a university position.

In 1861, Weierstrass gave his famous example of a continuous, nowhere differentiable function, something that had been thought impossible. In 1874, he gave proofs of basic calculus theorems, using even more rigour than Cauchy had. Weierstrass's students included famous mathematicians such as Georg Cantor, Sonya Kovalevsky, and David Hilbert.

Calculus students know Weierstrass as the mathematician who noticed that the substitution $t = \tan(x/2)$ converts any rational function of $\sin x$ and $\cos x$ to an ordinary rational function (which can then be integrated using partial fractions). We have

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

# Exercises 35

1. Prove the following result of Cauchy: if

$$a_1 \geq a_2 \geq a_3 \geq \cdots$$

then the series

$$a_1 + a_2 + a_3 + \cdots$$

converges if and only if the series

$$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

2. Use the previous result to show that

$$\frac{1}{2(\ln 2)^p} + \frac{1}{3(\ln 3)^p} + \frac{1}{4(\ln 4)^p} + \cdots$$

converges if and only if $p > 1$.

## *Challenge for Experts*

1. Use the Weierstrass substitution to find

$$\int \frac{1}{3\sin x - 4\cos x}\, dx$$

## *Essay Question*

1. Define 'rigour' in mathematics and give some examples.

# 36
# Nineteenth-Century Geometry

The nineteenth century saw so many advances in geometry that a full volume would not do them justice. The most startling discovery was that of the (relative) consistency of non-Euclidean geometry, for this challenged the idea that there is only one self-evidently true way to view space.

## The Golden Age of Euclidean Geometry

At age 18, when he was trying to decide whether he should be a mathematician or a philologist, Carl Friedrich Gauss (1777–1855) discovered a ruler and compass construction for a regular 17-gon. This had been overlooked by the Greeks, and Gauss was so excited about his discovery that there was no longer any question about his becoming a mathematician. Other mathematicians were excited too, and many were led to ask if there were not other theorems in Euclidean geometry, unknown in antiquity, but now ripe for the harvest. There were.

In 1809, Louis Poinsot (1777-1859) discovered the great dodecahedron. This is the polyhedron whose 60 faces are congruent $36-36-108^\circ$ triangles, meeting in 3's at the $108^\circ$ vertex, and meeting in 10's at the $36^\circ$ vertex. It is like a regular icosahedron but with an indentation in each of the 20 faces. Centred on the 12 non-indented vertices, there are 12 three-dimensional Pythagorean stars. It has to be built to be believed. Plato would have drooled.

In 1820, Charles-Julien Brianchon (1783–1864) and Jean Victor Poncelet

(1788–1867) published the first proof of the fact that, in any triangle, the 3 midpoints of the sides, the 3 feet of the altitudes, and the 3 midpoints of the segments connecting the vertices to the orthocentre all lie on a single '9 point' circle.

In 1822, Pierre Germinal Dandelin (1794–1847) showed that if you cut a cone with a plane, forming an ellipse, and if you wedge a sphere in the cone so that it just touches that plane, then it will touch it at a focus of the ellipse. If the sphere is inserted between the vertex of the cone and the plane, it touches one focus; if it is inserted on the other side of the ellipse, so that it is now a larger sphere, it touches the other focus.

Also in 1822, Karl Wilhelm Feuerbach (1800–1834) published the monograph in which he proved that the centre of the 9 point circle lies on the Euler line, midway between the orthocentre and the circumcentre. He also showed that the 9 point circle is tangent to the incircle of the given triangle. Unfortunately, these beautiful revelations did not prevent Feuerbach from becoming insane.

In 1837, Pierre Wantzel (1814–1848) showed that one cannot duplicate the cube or trisect an angle of $60°$ using only straight-edge and compass. Wantzel also proved that a regular $n$-gon with a prime number of sides is constructible with straight-edge and compass if and only if that prime has the form $2^m + 1$. With $m = 4$, we obtain the fact that the regular heptadecagon is constructible — the discovery that launched the young Gauss on his mathematical career. Wantzel died young on account of drinking too much coffee and smoking too much opium.

## Hyperbolic Geometry

The parallel postulate seems less natural than Euclid's other assumptions — whose conjunction we shall call '$A$'. $A$ includes the first four postulates, along with the tacit betweenness and continuity assumptions made by Euclid. Noteworthy attempts to derive the parallel postulate from $A$ were made by Proclus (410–485) and Gerolamo Saccheri (1667–1733). Gauss may have been the first person to suspect the truth. In a letter to Franz Taurinus, written in 1824, Gauss says he is sure that the parallel postulate cannot be proved (from $A$). Let $P$ be the parallel postulate, and $H$ its negation. What Gauss believed is that $A\&H$ is consistent.

The geometry based on $A\&H$ is called *hyperbolic geometry.* Nikolai Ivanovitch Lobachevsky (1793–1856) published some results in hyperbolic geometry in 1829. The same year, the same results were discovered independently by Janos Bolyai (1802–1860). Neither Lobachevsky nor Bolyai received much acclaim for their pains. The world was not interested in the new geometry. Lobachevsky carried on with his mathematical career only to die in poverty, while Bolyai gave the whole thing up in disgust. Abandoning mathematics, Bolyai set up house with his mistress, Rosalie von

Orban, in 1834, and had three children.

The reason the world rejected hyperbolic geometry is that it is so weird. Its squares do not have 4 right angles. The theorem of Pythagoras and the analytic geometry distance formula are false. Two triangles are congruent if their corresponding angles are equal. Not every triangle has a circumcircle. There is a circle larger in area than any triangle. And, of course, a straight line can get nearer and nearer to another straight line without ever meeting it.

In was not until 1868 that it was proved that this weird geometry is consistent. In that year, Eugenio Beltrami (1835-1900) gave a Euclidean model for hyperbolic geometry. This showed that if hyperbolic geometry contained any logical contradiction, then that contradiction could be translated into a contradiction in Euclidean geometry. Since, presumably, there is no inconsistency in Euclidean geometry, there is none in hyperbolic geometry either. (Lobachevsky had previously shown the converse: any contradiction in Euclidean geometry can be translated into a contradiction in hyperbolic geometry. Hence if there is no contradiction in hyperbolic geometry there is none in Euclidean geometry either.)

In 1882, in the first article ever published in *Acta Mathematica,* Henri Poincaré (1854–1912) gave a sketch of a second Euclidean model for hyperbolic geometry. To show that this really is a Euclidean model, one uses the inversion transformation of Apollonius. The reader can find the details in volume 1 of *A Survey of Geometry* by Howard Eves. (Note that one can drop the logarithms in Eves's treatment without affecting its validity.)

## Elliptic Geometry

In elliptic geometry, developed by Bernhard Riemann (1826–1866) in 1854, the straight lines are finite, and there are *no* parallels. A 'point' is like a pair of points opposite each other on a sphere, and a 'line' is like a great circle on a sphere.

The mathematician Richard Dedekind describes Riemann's death from tuberculosis as follows:

> On the day before his death he lay beneath a fig tree, filled with joy at the glorious landscape [in Selasca, Italy], writing his last work, unfortunately left incomplete. His end came gently, without struggle or death agony; it seemed as though he followed with interest the parting of the soul from the body; his wife had to give him bread and wine, he asked her to convey his love to those at home, saying "Kiss our child." She said the Lord's prayer with him, he could no longer speak; at the words "Forgive us our trespasses" he raised his eyes devoutly, she felt his hand in hers becoming colder.

## Which Geometry is True?

In contemporary mathematics, one can vary the postulates of Euclid at will, constructing as many geometries as one wishes. In the nineteenth century, this was a radical idea. People thought of Euclid's axioms as self-evident and necessary truths about space, truths that underlie the whole of astronomy and physics.

If Euclidean and hyperbolic geometry are both consistent, how should one respond to the question, 'which is true?'

**A Formalist Response**
Although certain interpretations of either geometry may be true, the geometries themselves are neither true nor false. They are just games in which one manipulates mathematical symbols.

**A Pragmatist Response**
A set of statements is true if it is useful for us humans to believe they are true. If we can base the simplest, most accurate theory of space-time on Euclidean geometry, then we have a reason to take Euclidean geometry as true. Otherwise we should take it that some other geometry is true.

**An Intuitionist Response**
We cannot say of two paintings that one is true and the other false. We talk instead about their coherence, energy, or beauty. Geometries are free-will creations, and it is not logical to ask whether they are true. Just as there are many art forms, so there are many ways to perceive space and time.

**A Platonist Response**
As we consider a Cartesian grid, we are turning our mind's eye on an independently existing abstract reality. What we see there is a lot of squares, each with four right angles. This grid is thus not the hyperbolic plane, but the Euclidean plane. So Euclidean geometry is, in this way, 'seen' to be true. On the other hand, hyperbolic geometry is false, since we know what straightness is, and we know that straight lines cannot forever get closer and closer without actually meeting.

## Exercises 36

1. Build a model of a great dodecahedron.

2. Prove that the 9 points of the 9 point circle really do lie on a single circle.
Hint: Suppose triangle $ABC$ has medians $AA'$, $BB'$, $CC'$, and altitudes $AD$, $BE$, $CF$. Let $H$ be the orthocentre, and $K$, $L$, $M$ the midpoints of $AH$, $BH$, $CH$, respectively. Then $B'C'LM$ is a rectangle. So is $A'B'KL$. Hence $A'K$, $B'L$, and $C'M$ are 3 diameters of a circle. Since $\angle A'DK$ is right, this circle passes through $D$.

3. Prove Dandelin's theorem.

## *Challenges for Experts*

1. If $A'$ is the midpoint of $BC$ in triangle $ABC$, and $A''$ is the point in $BC$ such that $\angle BAA'' = \angle CAA'$, then $AA''$ is a *symmedian* of the triangle. In 1873, E. M. H. Lemoine (1840–1912) proved that the 3 symmedians are concurrent. Do the same.

2. In 1893, James Joseph Sylvester (1814–1897) posed the following problem. Let $S$ be a finite set of points in the Euclidean plane. Let $T$ be the set of all lines $AB$ such that $A$ and $B$ are distinct points in $S$. Assume that $T$ contains more than one line, and prove that $T$ contains a line containing only two points in $S$.

## *Essay Questions*

1. 'The true geometry is the simplest and most beautiful'. Comment.

2. 'The possibility of hyperbolic geometry undermines absolute truth: there is no absolute truth whatsoever'. Comment, giving, if possible, examples of absolute truths outside mathematics.

# 37

# Nineteenth-Century Number Theory

## Gauss and Cauchy

Pierre de Fermat (1601–1665) had conjectured that every positive integer is a sum of 3 triangular numbers, 4 square numbers, 5 pentagonal numbers, and so on. If $m$ is a positive integer and $t$ a nonnegative integer, an $(m+2)-$*gonal number* is a number of the form

$$m\frac{t^2-t}{2}+t$$

The first few 3-gonal, or triangular, numbers are 0, 1, 3, 6, 10, 15, .... The first few 4-gonal, or square, numbers are 0, 1, 4, 9, .... The first few 5-gonal, or pentagonal, numbers are 0, 1, 5, 12, 22, .... What Fermat conjectured is that every positive integer is a sum of $m+2$ $(m+2)-$gonal numbers. As an example of this conjecture,

$$19 = 1+3+15 = 1+1+1+16 = 0+1+1+5+12$$

One of the major achievements of nineteenth-century number theory was a proof of this conjecture. Lagrange had proved it in the case $m = 2$, but it was left for Gauss to prove it for triangular numbers and for Cauchy to prove it in general. The proof for $m = 1$ was given in 1801, in Gauss' *Disquisitiones Arithmeticae,* while a general proof, based on Gauss' result, was given by Cauchy in a series of three memoirs from 1813 to 1815.

## Dirichlet

In 1837, Peter Dirichlet (1805–1859), a student of Gauss, used analysis to prove that if $a$ and $b$ are relatively prime positive integers, then the arithmetic progression

$$a, \quad a+b, \quad a+2b, \quad a+3b, \quad \ldots$$

contains infinitely many primes. Dirichlet's brain is preserved in the Department of Physiology at Göttingen University in Germany.

## Sylvester

James Joseph Sylvester (1814-1897) was a model of perseverance. In 1842 he lost his job at the University of Virginia (over a row with a student), and he had to do actuarial and legal work for thirteen years until, finally, he obtained another professorship, at the Royal Military Academy at Woolwich. In 1870 he lost his job at the Royal Military Academy (because of their early retirement policy), and he had to wait another six years before he obtained another professorship, this time at Johns Hopkins University in Baltimore. In 1878 he founded the *American Journal of Mathematics* and, in 1884 (at age seventy), he was given a prestigious job at Oxford University. It was while he was at Oxford University that he gave his proof of the fact that every fraction can be expressed as a sum of unit, or Egyptian, fractions (see Chapter 1). Sylvester once asked:

> May not Music be described as the Mathematic of Sense, Mathematic as Music of the reason? the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music.

One of Sylvester's students was Florence Nightingale, the reformer of hospital nursing.

## Hadamard and de la Vallée-Poussin

Using analysis, Jacques Hadamard (1865–1963) and Charles-Jean de la Vallée-Poussin (1866–1962), working independently, each proved the prime number theorem in 1896: if $f(n)$ is the number of primes $\leq n$ then

$$\lim_{n\to\infty} \frac{f(n)}{n/\log_e n} = 1$$

This had been conjectured by Gauss. For example, when $n = 10^6$, we have $f(n) = 78,498$ and $n/\log_e n = 72,382.4$; the fraction is 1.08. When

$n = 10^9$, we have $f(n) = 50,847,478$, while $n/\log_e n = 48,254,942.4$; the fraction is 1.05.

## Lucas

Another important nineteenth-century number theorist was Edouard Lucas (1842–1891). In 1875, Lucas challenged the readers of the *Nouvelles Annales de Mathématiques* to prove that

> A square pyramid of cannon-balls contains a square number of cannon-balls only when it has 24 cannon-balls along its base.

The choice of cannon-balls was not fortuitous. Lucas was French, and France had just lost Alsace-Lorraine to the Germans in 1871. In an 1885 High School Prize Day Speech, Lucas quoted 'la Marseillaise des petits soldats' by Victor de Laprade. The closing stanza says of France that

> She has suffered a great insult
> But God wants her to get up again
> Our schoolchildren will avenge her
> By their minds, and by their swords

It is interesting that Lucas' square pyramid problem was not solved until 1918, the year in which France won World War I against the Germans. A simple elementary proof of the assertion was first given in 1990 by W. S. Anglin (1949– ).

## Exercises 37

1. Express 42 as a sum of 3 triangular numbers in 4 different ways.

2. Find a number that cannot be expressed as a sum of fewer than 4 squares and prove that this is, indeed, the case.

3. List the first 6 hexagonal numbers.

4. Prove that all hexagonal numbers are triangular numbers.

5. List the first 6 primes of the form $7n + 3$.

6. Express $14/15$ as a sum of unit fractions.

7. How many primes are there less than 100, and what is $100/\log_e 100$? What is the fraction relevant to the prime number theorem?

8. How many cannonballs are in Lucas's pyramid?

9. What is the smallest number that has 2 distinct expressions as a sum of 5 pentagonal numbers?

10. For which $m$ is 100 an $(m+2)$–gonal number?

## *Essay Questions*

1. There was no point in keeping Dirichlet's brain, since it was his soul that did the work. Comment on this.

2. Do you agree with Sylvester's thoughts about music. Why or why not?

3. How might Lucas have expressed his problem if he had been a pacifist?

# 38
# Cantor

Empiricist philosophers, such as Hobbes, Locke, and Hume, had convinced some mathematicians, such as Gauss, that there is no infinite in mathematics. Thanks to Georg Cantor (1845–1918), however, almost every mathematician now accepts the infinite. Georg Cantor single-handedly produced a clear and complete theory of the infinite that answers all the objections previously raised by anti-infinity philosophers, and which has become the basis of contemporary mathematics. Thanks to Cantor, we have a new and deeper understanding of real numbers and of the many branches of mathematics such as calculus, which presuppose them.

Because of the anti-infinity attitude of L. Kronecker and others, Cantor never obtained a position at a good university. In his own day many people rejected his theory, and, in 1884, he had a mental breakdown from which he never fully recovered, dying in a psychiatric clinic in Halle, Germany.

History, however, has judged Cantor to be one of the most original and important mathematicians of all time. The opening sentence in Michael Hallett's *Cantorian Set Theory and Limitation of Size* is not exaggeration:

> Cantor was the founder of the mathematical theory of the infinite, and so one might with justice call him the founder of modern mathematics.

Cantor believed in an infinite God and in infinite sets of numbers. For Cantor, the latter belief was justified by the former. Cantor wrote:

> Since God is of the highest perfection one can conclude that it is possible for Him to create a *transfinitum ordinatum* [realm

> of the infinite]. Therefore, in virtue of His pure goodness and majesty we can conclude that there actually is a created *transfinitum.*

Here Cantor invoked Augustine's principle of plenitude, which states that God creates every possible good thing.

Cantor's belief in God led him to the correct belief that not every collection of abstract objects is itself eligible as a member of a collection. Cantor believed this because he believed that the collection consisting of everything is divine, and divine in such a way that it is overqualified to be a member in what would be a higher collection. Some of the early workers in set theory, such as atheist Bertrand Russell (1872–1970), originally thought no such restriction was necessary. They espoused what is now called the 'naive' view that any collection is eligible for membership in any collection. A collection could even be a member of itself. This led them into a contradiction, discovered by Russell, called the *Russell paradox:*

> Let $C$ be the collection such that, for any collection $X$,
> $X$ is a member of $C$ just in case $X$ is not a member of $X$.
> Then $C$ is a member of $C$ just in case $C$ is not a member of $C$.

A similar thing happened in the case of the well ordering principle, an axiom equivalent to the axiom of choice. Cantor adopted it because he believed there is a God who can arrange the elements of any set so that they are well-ordered. As it was discovered later, the well ordering principle plays a key role in many branches of mathematics. Cantor's faith in God guided him in the right direction.

One of Cantor's striking results is that there is an infinite hierarchy of distinct infinites, each infinitely greater than those below it. The medievals had noted that the number of points in a large circle is the same as that in a small concentric circle, in the sense that each radius of the large circle passes through exactly one point of each circle. Similar observations led Bernhard Bolzano (1781–1848) and others to the conclusion that any two infinite sets are 'equal' because they can be linked by a one-to-one correspondence. In 1873 Cantor discovered that this is wrong. One of his proofs goes as follows.

Let $A$ be an infinite set (that is, one containing infinitely many members). Let $P(A)$ be the set of subsets of $A$. Suppose that $A$ and $P(A)$ are linked by a one-to-one correspondence $f : A \to P(A)$. Let $S$ be the set of members $x$ of $A$ such that $x$ is not a member of $f(x)$. Then $S$ is in $P(A)$, and there is some $y$ in $A$ such that $S = f(y)$.

If $y \in S = f(y)$ then, by the defining property of $S$, $y$ is not a member of $f(y)$. However, if $y$ is not a member of $f(y) = S$, then, by the definition of $S$, it is a member of $S$. Contradiction. Hence $A$ and $P(A)$ are not linked by a one-to-one correspondence.

Since, for every member $x$ of $A$, $P(A)$ has $\{x\}$ as a member, there is a 'copy' of $A$ that is a subset of $P(A)$. Hence $A$ is smaller than $P(A)$, and we can write $A < P(A)$. Similarly, $P(A) < P(P(A))$. Indeed, we have an infinite hierarchy of infinite sets, each more infinite than the previous ones:

$$A < P(A) < P(P(A)) < P(P(P(A))) < P(P(P(P(A)))) < \cdots$$

Results like this incurred the scorn of Kronecker.

Cantor raised the following question. If $A$ is the set of positive integers, we know that $A < P(A)$, but is there some set $B$ such that $A < B$ and $B < P(A)$? Cantor conjectured that the answer is 'no', and it is this conjecture that is called the *continuum hypothesis.* It has been proved, by Kurt Gödel and Paul Cohen, that neither the continuum hypothesis nor its negation follows from the 'basic' axioms of set theory, and no one has yet been able to produce a not-so-basic axiom that would yield a convincing answer to Cantor's question. The continuum hypothesis is the parallel postulate of set theory.

Cantor was a Christian. In a letter to C. Hermite, Cantor writes about his failure to get a decent job:

> I thank God, the all-wise and all-good, that He always denied me the fulfillment of this wish [for a good position], for He thereby constrained me, through a deeper penetration into theology, to serve Him and His Holy Roman Catholic Church better than I have been able with my exclusive preoccupation with mathematics (p. 147 of Dauben's *Georg Cantor*).

One of Cantor's last compositions was a love poem to his wife, Vally Guttmann. After forty years of marriage, Cantor talks about

> The love you gave me my good wife,
> You cared for me so well.

# Exercises 38

1. Show that there is a one-to-one correspondence between the positive integers and all the integers.

2. Show that there is a one-to-one correspondence between the positive integers and the set of ordered triples of integers.

3. By viewing them as base 11 integers, show that there are, in a sense, no more fractions than integers.

4. Suppose the set of reals can be linked by a one-to-one correspondence with the positive integers, so that they can be listed as $r_1, r_2, r_3, \ldots$.

For each $n$, $r_n$ is contained in the interval

$$[r_n - \frac{1}{2^n},\ r_n + \frac{1}{2^n}]$$

Thus all the reals are contained in intervals the sum of whose lengths is 2. Explain this last sentence, and then draw a conclusion.

5. Show, as Cantor did, that the points inside a square are linked by a one-to-one correspondence to the points in one side of that square.

6. Show that there are more functions with domain the set of reals and codomain the set $\{0,\ 1\}$ than there are reals. Hint: such functions determine subsets of reals.

# 39
# Foundations

In the twentieth century, there was a great deal of concrete, practical mathematics. Statistics flourished, the computer proved the Four Colour theorem, and number theorists factored 100 digit integers. On the other hand, much twentieth century mathematics was characterised by a degree of abstraction never seen before. It was not the Euclidean plane that was studied, but the vector spaces and topological spaces which are abstractions of it. It was not particular groups that were studied so much as the whole 'category' of groups. Much twentieth century mathematics can be classified as philosophical. Set theorists attempted to find the ultimate basis for all mathematics. Set theorists also probed the infinite. Workers in foundations examined the limits of human reason itself, with Kurt Gödel (1906–1978) showing that some mathematical statements are subject neither to proof nor to counterexample. Various logics were put forward in an attempt to elucidate the nature of valid, human thought.

Twentieth century workers in foundations showed that real numbers can be defined in terms of rationals, rationals in terms of natural numbers, and natural numbers in terms of sets. But what are sets? Are they God-given? Are they products of human minds, subject to some 'free creation'? In response to these questions, foundation workers divided into at least three schools: Platonism, formalism, and intuitionism.

## Platonism

Platonists, such as Kurt Gödel, hold that numbers are abstract, necessarily existing objects, independent of the human mind. Because numbers have an independent, objective existence, any statement $p$ about numbers is either true or false — because it either correctly describes these abstract entities, or it does not.

It is as if the infinite totality of numbers are somehow 'there' to be inspected by a God who thinks infinitely quickly. This God simply checks each number to see how a statement $p$ fares in connection with it. After a complete inspection, God can then either report, 'yes, $p$' — or else, 'no, not $p$'. For example, if $p$ is 'every even natural number greater than 2 is a sum of two primes' then God, having checked through all the evens greater than 2, will know if $p$ is true or not, and thus $p$ will have one of the two truth values 'true' or 'false'.

Because of his philosophy, a Platonist is quite ready to accept the Axiom of Choice, that is, the assumption that, for every infinite class of pairwise disjoint sets, there is a class containing exactly one member from each of these sets, for, on a Platonist view, the sets are all 'there', and one can imagine a God going through them and choosing one element from each.

There are various objections to Platonism: (1) Platonism does not mean very much unless abstract objects can have an effect on minds, but it is not easy to see how this might happen; (2) Platonism presupposes that we (or God) can identify abstract objects, recognising such and such a line segment or set as, say, the number two, and, again, it is not clear how this might work. To these objections, the Platonist can reply: (1) the fact that we do not understand how physical objects affect minds does not prevent us from believing in physical objects, and (2) the fact that our concepts often relate to physical objects in a loose and open-ended way does not mean that we cannot identify physical objects.

## Formalism

Formalists, such as David Hilbert (1862–1943), hold that mathematics is no more or less than mathematical language. It is simply a series of games played with strings of linguistic signs, such as the letters of the English alphabet. The number two is just a collection of physical marks, such as 2, II, or $SS0$. It is true that we sometimes read meaning into mathematical terms, but, really, mathematical terms do not have any exterior meaning or reference.

There are various objections to formalism: (1) formalism understands mathematical objects, such as circles, in terms of concrete, material signs, but circles, precisely, are not contingent, physical objects; (2) formalism offers no guarantee that the games of mathematics are consistent. To these objections the formalist can reply: (1) circles, like everything else, are material objects, and (2) although some of the games of mathematics are indeed inconsistent, and hence trivial, others are not.

## Intuitionism

Intuitionists, such as L. E. J. Brouwer (1882–1966), hold that mathematics is a creation of the human mind. Numbers, like fairy tale characters, are merely mental entities, which would not exist if there were never any human minds to think about them.

Intuitionism is a philosophy in the tradition of Kantian subjectivism where, at least for all practical purposes, there are no externally existing objects at all: everything, including mathematics, is just in our minds. Since, in this tradition, a statement $p$ does not acquire its truth or falsity from a correspondence or noncorrespondence with an objective reality, it may fail to be 'true or false'. Thus intuitionists can, and do, deny that, for any mathematical statement $p$, it is a logical truth that 'either $p$ or not $p$'.

Since intuitionists reject objective existence in mathematics, they are not necessarily convinced by reasoning of the form:

> If there is no mathematical object $A$, then there is a contradiction; hence there is an $A$.

If the details of the reasoning provide a way of imagining or conceiving an $A$, in a way open to ordinary human beings (by, say, calculating it in a finite number of steps), then the intuitionist will agree that there is, indeed, an $A$. However, if the details of the reasoning do not provide this, the intuitionist will remain skeptical. For a Platonist, it is of interest that there is an $A$ even if it is only God who can conceive it. For an intuitionist, however, a mathematical object is meaningless unless it can be somehow 'constructed' and 'intuited' by a human being.

For the intuitionist, the human mind is basically finite, and Cantor's hierarchy of infinites is just so much fantasy. Intuitionists thus reject any mathematics which is based on it, including most of calculus and most of topology.

There are various objections to intuitionism: (1) intuitionism cannot account for the feeling that mathematical objects are noncontingent, that, even if there were no human beings, 2 and 2 would still make 4; (2) intuitionist mathematicians are so badly crippled by their rejection of the logical law 'either $p$ or not $p$', and by their rejection of the infinite, that they only have a small fraction of contemporary mathematics. To these objections, the intuitionist can reply: (1) it does not make sense for human minds to try to conceive a world without human minds, and (2) it is better to have a small amount of mathematics all of which is solid and reliable than to have a large amount of mathematics, most of which is nonsense.

# Exercises 39

## *Essay Questions*

1. Unless he or she thinks that mathematics is beneath God's notice, a believer in God has to be a Platonist about mathematics. Comment.

2. Intuitionism is a negative doctrine, rejecting large parts of mathematics and refusing to accept a reality external to human beings. It is a doctrine for people who want to deify humanity by pretending there is nothing outside it. Comment.

# 40
# Twentieth-Century Number Theory

Much of what went under the name 'number theory' in the twentieth century had little to do with the natural numbers. There was an obsession with results concerning abstract structures used to prove results concerning abstract structures. It was as if carpenters were using their tools to make new tools to make new tools — without ever using any of these tools to build a house. Happily, there were exceptions. A few number theorists escaped the obsession with abstraction and produced the meaningful concrete results listed below.

## The Bachet Equation

The *Bachet equation* is the Diophantine equation $x^2 + k = y^3$, where $k$ is a given nonzero integer. It is named after Claude-Gaspar Bachet (1581–1638), who studied it in the seventeenth century, but it goes back to Diophantus himself (see Problem 17 of Book VI of the *Arithmetica*). Many special cases of the Bachet equation had been solved before, but it was only in 1968 that Alan Baker, a Cambridge mathematician, found a completely general solution, working for any given $k$. At first, Baker's solution was merely an enormous bound $M(k)$ on the sizes of $x$ and $y$. However, soon after, Baker and other mathematicians, such as H. Davenport, transformed Baker's insights into a practical method for actually obtaining a solution set for any given $k$. W. J. Ellison used Baker's ideas to show, for the first time, that when $k = 28$, the Bachet equation has only three solutions in positive integers (with $x = 6$, 22, and 225). Ray P. Steiner used a version of Baker's result, due to M. Waldschmidt, to show, again for the first time, that when $k = 999$, the Bachet equation has only 6 solutions in positive integers (with $x = 1$, 27, 251, 1782, 2295, and 3,370,501). In his *Algebraic Numbers and Diophantine Approximation,* K. B. Stolarsky had claimed that $x^2 + 999 = y^3$ could not be solved by 'a thousand wise men'. Alan Baker was wise man a thousand and one.

## Hilbert's Tenth Problem

At the second International Congress of Mathematicians (in Paris, 1900), David Hilbert (1862–1943) presented a list of problems he hoped would be settled in the twentieth century. Some of these problems were the following:
(1) prove or disprove the continuum hypothesis;
(2) show that arithmetic is consistent;
(8) show that all the nontrivial zeros of the Riemann zeta function lie on the line $x = \frac{1}{2}$;
(10) find an algorithm (computer program) that will tell you whether or not a given polynomial Diophantine equation (with known integer coefficients and known exponents) has a solution.

Today we know that several of Hilbert's problems cannot be solved in the way he intended. It was proved, for example, that, from the usual axioms of set theory (assuming they are consistent), there is no proof of the continuum hypothesis, and no proof of its negation. Hilbert's tenth problem falls into this category. In 1970, Yuri V. Matijasevich showed that the desired computer program cannot exist. This is because, as Matijasevich proved, almost any mathematical problem can be translated into a problem about solving a Diophantine equation. The procedure Hilbert was looking for would have been so powerful that it could have solved problems that cannot be settled in any way whatsoever, using only our present axioms (assuming they are consistent). Matijasevich based his proof on work done by a woman mathematician, Julia Robinson. In a 1992 *Mathematical Intelligencer* article, Matijasevich reveals some of the personal history behind his solution of Hilbert's tenth problem.

Incidentally, Hilbert's eighth problem is still unresolved and is considered to be the most important outstanding problem in contemporary number theory.

## Computational Advances

Thanks to the computer, twentieth-century number theorists succeeded in finding twenty new perfect numbers and hundreds of new amicable pairs. They also produced programs capable of factoring 100 digit integers in just a few hours.

Particularly noteworthy was the computer solution of Archimedes's cattle problem (200 B.C.), which is equivalent to the Diophantine equation

$$x^2 - (8 \times 2471 \times 957 \times 4657^2)y^2 = 1$$

This was achieved, for the first time, in 1965, by H. C. Williams, R. A. German, and C. R. Zarnke.

## Congruent Numbers

In 1983, using the 'theory of modular forms of weight 3/2', J. B. Tunnell advanced the knowledge of congruent numbers by showing that if $n$ is a square-free odd congruent number then the number of ways of writing $n$ in the form

$$2x^2 + y^2 + 8z^2$$

with $x$, $y$, and $z$ integers and $z$ odd, equals the number of ways of writing $n$ in the same form, but with $z$ even. For example, with $z$ odd, 11 has exactly 8 decompositions into the above form, namely,

$$2(\pm 1)^2 + (\pm 1)^2 + 8(\pm 1)^2$$

If $z$ is even, 11 has exactly 4 such decompositions:

$$2(\pm 1)^2 + (\pm 3)^2 + 8(0)^2$$

Since $8 \neq 4$, it follows that 11 is not congruent.

Tunnell conjectured that the converse of this theorem is also true, but that remains to be proved.

## Fermat's Last Theorem

About 1637, Fermat had conjectured that the equation

$$x^{p+2} + y^{p+2} = z^{p+2}$$

has no solution in positive integers. This conjecture, known as 'Fermat's last theorem', was studied by G. Frey, K. Ribet, and J.-P. Serre. Finally, in 1994, it was proved by A. Wiles, with help from R. Taylor.

## Angles in Pythagorean Triangles

Elementary, recreational number theory was still going strong. In his 1988 *American Mathematical Monthly* article, W. S. Anglin proved the following. Let $B$ be any angle in degrees, with $0 < B < 90$. Let $\epsilon$ be any real number such that $0 < \epsilon < 1$, and $\epsilon < B$, and $\epsilon < 90 - B$. Let

$$\begin{aligned} X &= \tan(B - \epsilon) + \sec(B - \epsilon) \\ Y &= \tan(B + \epsilon) + \sec(B + \epsilon) \end{aligned}$$

Suppose $u$ and $v$ are relatively prime positive integers such that

$$X < \frac{u}{v} < Y$$

Then the Pythagorean triangle with sides $2uv$, $u^2 - v^2$, and $u^2 + v^2$ has an angle of $A$ degrees (the one opposite the side $u^2 - v^2$) such that $|A - B| < \epsilon$.

## Partitions

A 'partition' of a positive integer is a way of writing it as a sum of nonincreasing positive integers. For example, 5 has 7 partitions, namely,

$$5,\ \ 4+1,\ \ 3+2,\ \ 3+1+1,\ \ 2+2+1,\ \ 2+1+1+1,\ \ 1+1+1+1+1$$

The number $p(n)$ is the number of partitions $n$ has. For example, $p(5) = 7$. In 1918, Ramanujan (1887-1920) and Godfrey Harold Hardy (1877–1947) gave the first known fast way of calculating $p(n)$ for any $n$, and in 1937, Hans Rademacher refined their work into the first known formula for $p(n)$. It is

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\, \sqrt{k}\, \frac{d}{dn} \left( \frac{\sinh\left(\frac{\pi}{k}\sqrt{(2/3)(n - 1/24)}\right)}{\sqrt{n - 1/24}} \right)$$

where

$$A_k(n) = \sum_{0 \le h \le [k/2],\ \gcd(h,k)=1} 2\cos\left(\pi\, s(h,k) - \frac{2\pi n h}{k}\right)$$

with the 'Dedekind sum' $s(h, k)$ defined as

$$s(h,k) = \sum_{r=1}^{k-1} (r/k)(hr/k - [hr/k] - 1/2)$$

(Note that $s(0, 1)$ is taken to equal 0.)

Hardy and Rademacher made substantial contributions to the discovery of this formula, but the spark of insight came from Ramanujan, an extraordinary genius born near Madras, India. Perhaps the greatest number theorist of the twentieth century, Ramanujan sometimes credited his discoveries to providence. He once said:

> An equation for me has no meaning unless it expresses a thought of God. [1]

# Exercises 40

1. Show that 3,370,501 is one of the values of $x$ solving $x^2 + 999 = y^3$.

2. It is a corollary of Matijasevich's work that if $x$ and $y$ are positive integers and
$$z = y(2 - (x^2 + xy - y^2)^2)$$
then $z > 0$ iff $z$ is a Fibonacci number. Find a Fibonacci number $> 1$ expressed in the above form.

3. Use Tunnell's theorem to show that 417 is not congruent.

4. Find a Pythagorean triangle that has an angle within 0.001 of $12°$.

5. In 1971, R. Finkelstein and H. London published a paper showing that $x^3 + 5 = 117y^3$ has no integer solutions. Prove this using the fact that 9 divides 117.

## *Essay Question*

1. Because they must 'publish or perish', second-rate mathematicians fill the journals with useless abstractions, calling their work 'number theory' when it is merely jejune generalisation. Can you suggest some replacement for the 'publish or perish' system that is currently cluttering our libraries with junk?

---

[1] R. Kanigel, *The Man who Knew Infinity* (New York: Charles Scribner's Sons, 1990), pages 7 and 282.

# References


[1] Archimedes. *The Works of Archimedes.* Trans. T. L. Heath. New York: Dover, 1897.

[2] Boyer, Carl B., and Uta C. Merzbach. *A History of Mathematics.* 2d ed. New York: John Wiley, 1989.

[3] Brahmegupta and Bhascara. *Algebra.* Trans. H. T. Colebrooke. London: John Murray, 1817.

[4] Cantor, Georg. *Transfinite Numbers.* New York: Dover, 1955.

[5] Cardano, Girolamo. *The Book of My Life.* Trans. Jean Stoner. New York: Dover, 1962.

[6] Cardano, Girolamo. *The Great Art.* Trans. T. Richard Witmer. Cambridge, Mass.: MIT Press, 1968.

[7] Descartes, René. *The Geometry of René Descartes.* Trans. D. E. Smith and M. L. Latham. New York: Dover, 1954.

[8] Diophantus. *Arithmetica.* 2d ed. Trans. T. L. Heath. New York: Dover, 1964.

[9] Diophantus. *Books IV to VII of Diophantus' Arithmetica.* Trans. Jacques Sesiano. New York: Springer-Verlag, 1982.

[10] Euclid. *Elements.* Trans. T. L. Heath. 2d ed. New York: Dover, 1956.

[11] Euler, Leonhard. *Opera Omnia.* Geneva: 1944.

[12] Eves, Howard. *An Introduction to the History of Mathematics.* 5th ed. New York: Holt, Rinehart and Winston, 1983.

[13] Fermat, Pierre. *Oeuvres de Fermat.* Paris: Gauthier-Villars, 1891.

[14] Fibonacci. *The Book of Squares.* Trans. L. E. Sigler. Boston: Academic Press, 1987.

[15] Gillings, Richard J. *Mathematics in the Time of the Pharaohs.* New York: Dover, 1972.

[16] Khayyam, Omar. *The Algebra of Omar Khayyam.* Ed. D. S. Kasir. New York: AMS Press, 1972.

[17] Khwarizmi, M. *The Algebra.* Trans. Frederic Rosen. London: Oriental Translation Fund, 1831.

[18] Lagrange, J. *Oeuvres de Lagrange.* Paris: Gauthier-Villars, 1869.

[19] Napier, John. *The Construction of the Wonderful Canon of Logarithms.* Trans. W. R. MacDonald. Edinburgh: William Blackwood and Sons, 1889.

[20] Neugebauer, O. *The Exact Sciences in Antiquity.* 2d ed. New York: Dover, 1969.

[21] Ore, Oystein. *Cardano: The Gambling Scholar.* New York: Dover, 1965.

[22] Pappus of Alexandria. *La Collection Mathématique.* Trans. Paul ver Eecke. Paris: Bruges, 1933.

[23] Pascal, Blaise. *Oeuvres Complètes.* Tours: Gallimard, 1954.

[24] Smith, David E. *History of Mathematics.* New York: Dover, 1958.

[25] Stillwell, John. *Mathematics and Its History.* New York: Springer-Verlag, 1989.

[26] Van der Waerden. *Science Awakening.* Vol. 1. 4th ed. Trans. Arnold Dresden. Dordrecht: Kluwer, 1975.

# Appendix A
# Sample Assignments and Tests

## Assignment 1

This assignment is based on Chapters 1 to 17. Do exactly 4 of the following questions, attempting each part of the 4 questions you choose.

1.
(a) Multiply 201 by 3330 Egyptian style.
(b) What is the height of a frustum of volume 100, base side 5, and top side 2?
(c) Express 2/89 as a sum of four distinct unit fractions in two different ways.
(d) Find an arithmetical progression with 5 terms, sum 10, and common difference 1/2.
(e) Using the formula for the volume of a pyramid, derive the Moscow Papyrus formula for the volume of a frustum.
2.
(a) An archeologist has found an old Egyptian brick measuring 1 cubit by 1 cubit by 1 cubit. 'This stone dates from 4000 B.C.,' he says, 'and the sum of the distance between the opposite corners and the side diagonal is 3.15 cubits. This proves the Egyptians of 4000 B.C. used the value of 3.15 for $\pi$.' Show that the archeologist's math is correct but that his reasoning is wrong. Convince him that he has lost his wits.
(b) Does the British Museum have a moral obligation to give, or sell, the Rhind Papyrus back to the Egyptians? Why or why not?

3.
(a) In row 14 of Plimpton 322, we find the numbers 1771 and 3229? What is the other side of the right triangle in question?
(b) For row 14, what are the $u$ and $v$ (the generating numbers for the triangle)?
(c) What triangle is generated by $u = 27$ and $v = 10$?
(d) Use the Babylonian method to find an approximation to $\sqrt{7}$, using 4 terms of the relevant sequence. (Start with $a_1 = 3$.)
(e) Solve the simultaneous equations $x^8 + x^6y^2 = 3200000^2$ and $xy = 1200$.
4.
(a) Is it possible that the author of Plimpton 322 was a woman? Why or why not?
(b) Write a short essay on life in Mesopotamia 4000 years ago.
5.
(a) Show that 33,550,336 is perfect.
(b) Show that 30,240 is superfluous.
(c) Prove that every even perfect number ends in 6 or 8.
(d) 10,744 is amicable. Who is its friend?
(e) Show that $2^{23} - 1$ is not a Mersenne prime.
6.
(a) Who was Augustine, what did he say about perfect numbers, and was he right?
(b) What did Pythagoras mean when he said, 'all is number'? Was he right? Why or why not?
7.
(a) Make and hand in an octahedron, constructed out of, say, cardboard.
(b) Make and hand in a dodecahedron, constructed out of, say, cardboard.
(c) What is the volume of an octahedron of side 1?
(d) What is the surface area of an octahedron of side 1?
(e) What is the radius of a sphere passing through the vertices of an octahedron of side 1?
8.
(a) What did the ancient Greeks think about the infinite? In your answer, refer especially to Anaximander, Zeno, and Aristotle.
(b) What were Zeno's paradoxes? What did they prove?
9.
(a) Prove that $\sqrt{5}$ is irrational.
(b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.
(c) Using Euclid's algorithm, find an integer solution of $119x - 32y = 1$.
(d) Using Euclid's algorithm, find a positive integer solution of $x^2 - 87y^2 = 1$.
(e) Use your answer in (d) to give an approximation to $\sqrt{87}$.
10.
(a) What was Plato's philosophy of mathematics? Do you agree with it? Why or why not?

(b) How did Aristotle's philosophy of mathematics differ from that of Plato? How might one decide which of them was right?

11.

(a) Give a construction for a line of length $2\sqrt{5}+1$.

(b) Is there a straightedge and compass construction for a regular polygon with 771 sides? Why?

(c) Prove III 36.

(d) Prove VI 5.

(e) Prove VI 8.

12.

(a) Who was the last great ancient Greek mathematician and why?

(b) Discuss possible reasons for the death of Hypatia.

# Assignment 2

This assignment is based on Chapters 18 to 27. Do exactly 4 of the following questions, attempting each part of the 4 questions you choose.

1.
(a) Divide by 7 the remainder is 4; divide by 13 the remainder is 6; what is the number?
(b) Show that 17,296 is one of an amicable pair.
(c) Does Thabit's formula give amicable numbers when $n = 5$?
(d) Give a complete solution of $17x - 13y = 1$.
(e) Give the smallest positive integer solution of $x^2 - 96y^2 = 1$.
2.
(a) What was Baghdad like in the days of Thabit?
(b) In what ways did Islam encourage people to do mathematics?
3. Problems from *Lilavati*
(a) Pretty girl with tremulous eyes, ... tell me, what is the number, which multiplied by 3, [and then multiplied by 7/4], and [then] divided by 7, and [then multiplied by 2/3], and then multiplied into itself, and having 52 subtracted from the product, and the square root of the remainder extracted, and 8 added, and the sum divided by 10, yields 2 ?
(b) Say, mathematician, how many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt and bitter, taking them by ... threes.
(c) In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen half a cubit above the surface of the water. Forced by the wind, it gradually advanced, and was submerged at the distance of two cubits [from the point where it originally broke through the surface of the water]. Compute quickly, mathematician, the depth of water.
(d) Intelligent friend, if thou know well the spotless Lilavati, say what is the area of a circle, the diameter of which is measured by 7 ?
(e) Tell the quantity of the excavation in a [frustum-shaped] well, of which the length and breadth are equal to twelve and ten cubits at its mouth, and half as much at the bottom, and of which the depth, friend, is seven cubits.
4.
(a) Write a short essay about the life of Bhaskara and his daughter.
(b) What does Bhaskara's theological explanation of the fact that $\infty \pm x = \infty$ imply about his conception of God and the universe?
5.
(a) What is the 20-th Fibonacci number?
(b) Why are two adjacent Fibonacci numbers always relatively prime?
(c) Show that a triangle with sides 65, 72, and 97 is right.
(d) Does this triangle tell us that some number is congruent? Which number? Why?

(e) What is $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$ ?
6.
(a) D. E. Smith writes:

> In the same years and in the same region in which Leonardo was bringing new light into the science of mathematics, St. Francis, humblest of the followers of Christ, was bringing new light into the souls of men.

Was this really a 'new' light, or rather an old light, brought back from ancient Greek times?
(b) How did their belief in God help Medieval mathematicians? Give an example.
7.
(a) Solve $x^3 - 2x^2 - 14x - 5 = 0$ by factoring and using the quadratic formula.
(b) In 1535 Antonio Maria Fior challenged Tartaglia to solve the following. A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is this profit? (No calculators allowed!)
(c) One of the problems in the *Ars Magna* is the following.

$$x^3 = 6x^2 + 72x + 729$$

Use Cardano's method to solve it.
(d) Zuanne da Coi claimed the following could not be solved.

$$60x = x^4 + 6x^2 + 36$$

Solve it, using the method of Ferrari. (No calculator approximations!)
(e) Solve $x^6 + 20x^3 + 1 = 0$.
8.
(a) Give a summary of the life of Cardano.
(b) Did Cardano act rightly in publishing the secret of the cubic? Why or why not?
9.
(a) Evaluate $\log_{10} 1.25 + \log_{10} 80$.
(b) Simplify $\ln 10 + \frac{1}{2}\ln 9$. ('ln' means log to the base $e$.)
(c) Graph $y = \log_2 x$.
(d) Solve $\ln(x+6) + \ln(x-3) = \ln 5 + \ln 2$.
(e) Solve $2^{3^x} = 5$.
10.
(a) Summarise the life of Napier.
(b) Why was the creation of log tables an important step forward in science?

# Assignment 3

This assignment is based on Chapters 28 to 40. Do exactly 4 of the following questions, attempting each part of the 4 questions you choose.

1.
(a) Graph $y = x^2 - 2x - 1 + 2/x$.
(b) What kind of curve represents the equation $x^2 + 7xy - 3y^2 + x + y = 2$? Why?
(c) Without using calculus, find the equation of the tangents through $(5, -6)$ to the parabola $y = x^2$.
(d) What is the smallest positive integer that quadruples when its final digit is shifted to the front?
(e) State Fermat's last theorem and say why it is so named.
2.
(a) Explain Pascal's wager. Do you think it is a good argument? Why?
(b) Write a short essay on Pascal, discussing especially his interest in both mathematics and religion.
3.
(a) Evaluate

$$\int_0^{\frac{\pi}{2}} \sin^{300} x \, dx$$

(b) According to Newton's generalised binomial theorem, what is $(1+x)^{-3}$? Give the first few terms and the $n$th term of the expansion.
(c) If $R$ is a positive constant and $R < x < 5R$, find the max and min values of $y = (4R^3 + x^3)/x^2$ (using calculus).
(d) Suppose two stars are $10^{15}$ metres apart and have the same mass $m$. If they are attracted by each other with a gravitational force of $6.67 \times 10^{19}$ newtons, what is $m$?
(e) Prove Newton's theorem about the oxen and the grass.
4.
(a) What was Leibniz's philosophy of the infinite?
(b) What was Leibniz's philosophy of religion? In particular, what was he thinking about when he said that the divine nature needs only a possibility in order to exist?
5.
(a) Suppose that three tickets will be given prizes in a lottery having 40,000 tickets. What is the chance of winning at least one prize if you buy 8000 of those tickets?
(b) Use de Moivre's formula to find $(5 + 5\sqrt{3}\sqrt{-1})^{68}$.
(c) What is the equation of the Euler line for the triangle with vertices $(0, 0)$, $(12, 0)$, and $(16, 12)$?
(d) How would you express 45 as a sum of four squares?
(e) Find the least positive integer solution of $x^2 - 13y^2 = 1$.

6.
(a) Who was William Hamilton, and what did he do?
(b) Modern algebra is characterised by 'free creation' and 'abstraction'. What might this statement mean, and how is it illustrated in the work of Hamilton?
7.
(a) Construct a great dodecahedron (out of, say, cardboard).
(b) Suppose a triangle has vertices $(0,0)$, $(12,0)$, and $(8,6)$. What are the coordinates of its three midpoints?
(c) What are the coordinates of the three feet of the altitudes of the above triangle?
(d) What are the coordinates of the 3 midpoints of the segments connecting the orthocentre to the vertices in the above triangle?
(e) What is the equation of the 9 point circle of the above triangle?
8.
(a) What is non-Euclidean geometry and how does it threaten the truth of mathematics?
(b) How might different philosophical schools respond to the question 'which geometry is true?'
9.
(a) Is 160 a triangular number? Why?
(b) Find three ways in which 100 is a polygonal number.
(c) Prove that all hexagonal numbers are triangular numbers.
(d) What is the smallest number that has two essentially distinct expressions as a sum of 4 squares?
(e) What was Hilbert's tenth problem and who solved it?
10.
(a) Describe Cantor's views on the infinite.
(b) What are the positions of the different contemporary schools in the philosophy of mathematics?
11.
(a) What is a one-to-one correspondence?
(b) Show that the set of fractions can be placed in one-to-one correspondence with the set of positive integers.
(c) Show that the set of positive integers and its power set (set of sets of positive integers) cannot be placed in one-to-one correspondence.
(d) Show that the points in the real line can be placed in one-to-one correspondence with the points in a semicircle of radius 1.
(e) State the Russell paradox.
12.
(a) Define 'concrete' and 'abstract' and illustrate your definitions in terms of twentieth-century work in mathematics.
(b) Some mathematics say that certain pieces of mathematics are 'elegant'. What does this mean? Illustrate your answer with examples of elegant and inelegant mathematics.

# History of Math Midterm 1

Do exactly 2 of the following questions, attempting each part of the 2 questions you choose.

1.
(a) Multiply 567 by 330 Egyptian style.
(b) What is the height of a frustum of volume 100, base side 5, and top side 2?
(c) Express 67/120 as a sum of distinct unit fractions.
(d) Find an arithmetical progression with 5 terms, sum 11, and common difference 1/2.
(e) How old is the Rhind Papyrus? Who discovered it? When? Where is it now?

2.
(a) On Plimpton 322, we find the numbers 65 and 97? What is the other side of the right triangle in question?
(b) For this row, what are the $u$ and $v$ (the generating numbers for the triangle)?
(c) What triangle is generated by $u = 16$ and $v = 9$?
(d) Use the Babylonian method to find an approximation to $\sqrt{13}$, using 3 terms of the relevant sequence.
(e) Solve the simultaneous equations $x^8 + x^6y^2 = 3200000^2$ and $xy = 1200$.

3.
(a) Is it possible that the author of Plimpton 322 was a woman? Why or why not?
(b) Write a short essay on life in Mesopotamia 4000 years ago.

4.
(a) Show that 8128 is perfect.
(b) Is 1000 diminished or superfluous?
(c) Show that 153 is triangular.
(d) 5564 is amicable. Who is its friend?
(e) Show that $2^{10} - 1$ is not a Mersenne prime.

5.
(a) Who was Augustine, what did he say about perfect numbers, and was he right?
(b) What did Pythagoras mean when he said, 'all is number'? Was he right? Why or why not?

6.
(a) Show that the ratio of side to base in one of the points of the Pythagorean star is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.
(b) What is the height of a tetrahedron of side 1?
(c) What is the volume of an octahedron of side 1?
(d) What is the surface area of an octahedron of side 1?
(e) What is the radius of a sphere passing through the vertices of a cube of side 1?

7.
(a) What did the ancient Greeks think about the infinite? In your answer, refer especially to Anaximander, Zeno, and Aristotle.
(b) What were Zeno's paradoxes? What did they prove?

8.
(a) Prove that $\sqrt{15}$ is irrational.
(b) Prove that $\sqrt[3]{2}$ is irrational.
(c) Using Euclid's algorithm, find an integer solution of $13x + 7y = 79$.
(d) Using Euclid's algorithm, find a positive integer solution of $x^2 - 84y^2 = 1$.
(e) Use your answer in (d) to give a good approximation to $\sqrt{84}$.

9.
Express each of the following syllogisms in the P–S notation. Then say whether it is valid or not.
(a) All people called 'A' are deans; all deans die; therefore all people called 'A' die.
(b) No insects are birds; no birds are mammals; therefore no insects are mammals.
(c) All Nazis are cowards; all cowards are damned; therefore there are some damned Nazis.
(d) No swans are black; some black things are dogs; therefore no swans are dogs.
(e) Some pizza-eaters are fat; no fat person is healthy; therefore some pizza-eaters are unhealthy.

10.
(a) Give a ruler and compass construction for a segment of length $2\sqrt{5}+1$.
(b) Construct an equilateral triangle, giving a detailed description of what you are doing.
(c) Is a regular polygon with 771 sides constructible? Why or why not?
(d) Construct a Pythagorean star, using ruler and compass only, and describing in detail what you are doing.
(e) Is an angle of $51^\circ$ constructible or not? Why?

11.
(a) Give Euclid's proof of the theorem of Pythagoras, including the diagram and all the reasoning.
(b) Give Euclid's proof of the converse of this theorem, including the diagram and all the reasoning.

12.
(a) What was the main difference between Babylonian and Greek mathematics? Illustrate your answer with some examples.
(b) In what sense of 'mathematics' did 'mathematics start with the Greeks'? Or did it?

# History of Math Midterm 2

Do exactly 2 of the following questions, attempting each part of the 2 questions you choose.

1.
(a) Divide by 3 the remainder is 1; divide by 29 the remainder is 6; what is the number?
(b) State Thabit's rule for finding amicable pairs.
(c) 1210 is amicable. Who is its friend?
(d) Give a complete solution of $5x + 7y = 1$.
(e) Give the smallest positive integer solution of $x^2 - 96y^2 = 1$.

2.
(a) Who were Thabit and al-Khwarizmi? Which of them was the greater mathematician, and why?
(b) What was Baghdad like in the days of Thabit?

3.
(a) What is the area of a cyclic quadrilateral with sides 52, 25, 39, and 60?
(b) What is the area of a triangle with sides 6, 8 and 10 ?
(c) What is the radius of the circumcircle of a triangle with sides 8, 15, and 17 ?
(d) What is the area of a triangle with sides 12, 14, and 16?
(e) Who first worked out the formula for the area of a cyclic quadrilateral? When and where did he live?

4.
(a) Write a short essay on the life of Bhaskara and his daughter.
(b) What was Bhaskara's theological explanation of the fact that $\infty \pm 5 = \infty$? In what sense was it a Hindu explanation?

5.
(a) Write the first 10 terms of the Fibonacci sequence.
(b) What is the formula for the $n$th Fibonacci number?
(c) Show that 5 is congruent by showing that it is the area of a right triangle whose sides are rational lengths.
(d) Write $(a^2 + b^2)(c^2 + d^2)$ as a sum of two squares.
(e) Let $F_n$ be the $n$th Fibonacci number. Show that $\gcd(F_{12}, F_{15}) = F_{\gcd(12,15)}$.

6.
(a) Give a summary of the life of Cardano.
(b) Did Cardano act rightly in publishing the secret of the cubic? Why or why not?

7.
(a) Solve Omar's problem $2x^3 - 20x^2 + 27x + 10 = 0$.
(b) Find an exact algebraic expression for a real number solution of $y^3 + y - 500 = 0$.
(c) Use Cardano's method to solve $x^3 = 6x^2 + 72x + 729$.
(d) Find the exact solutions of $x^4 + 3 = 12x$.
(e) Find the exact solutions of $x^3 + 6x^2 + 12x = 22$.

8.
(a) Summarise the life of Napier.
(b) Why was the creation of log tables an important step forward in science?

9.
(a) Without using a calculator, evaluate $\log_{10} .7^3 + 3\log_{10} \frac{1}{7}$.
(b) Graph $y = \log_{10} x$.
(c) Solve $\log x^2 = 4\log 2 + 2\log 3$ .
(d) Solve $\log(x-5) + \log(x+1) = \log 7 + 4\log 2$.
(e) Solve $3^{5k} = 8$.

10.
(a) Summarise the life of Galileo, saying what, if any, were his original contributions to mathematics.
(b) Summarise the life of Kepler, mentioning some of his mathematical as well as his astronomical discoveries.

# History of Mathematics Final Exam

Do at least 5 of the following questions, attempting each part of the questions you choose.

1.
(a) Multiply 67 by 1000 Egyptian style.
(b) What is the volume of a frustum of height $4 + 1/7$, base side $5 + 1/2$, and top side 2? Express the answer Egyptian style.
(c) Express $8/11$ as a sum of distinct unit fractions.
(d) Divide

$$\frac{1}{5} + \frac{1}{35} + \frac{1}{700}$$

by 2, expressing the answer Egyptian style.
(e) An arithmetical progression with 6 terms and common difference $1/3$ has sum 40. What is it? Express the answer Egyptian style.

2.
(a) In row 3 of Plimpton 322 we find the numbers 4601 and 6649. What do these numbers mean?
(b) Is the triangle with sides 65, 70, and 97 a right-angled triangle? Why?
(c) What are the generating numbers for the right triangle with side 1771 and hypotenuse 3229?
(d) Use the Babylonian method to find an approximation to $\sqrt{23}$, using 3 terms of the relevant sequence.
(e) A Babylonian problem reads: One leg of a right triangle is 50. [A 'leg' is a side that is not the hypotenuse.] A line parallel to the other leg and at a distance 20 from that leg cuts off a right trapezoid of area 320. Find the lengths of the parallel sides of the trapezoid.

3.
(a) Prove that $\sqrt{11}$ is irrational.
(b) Prove that $\sqrt{2} + \sqrt{5}$ is irrational.
(c) Find all the integer solutions of $101x - 97y = 1$.
(d) Find a positive integer solution of $x^2 - 78y^2 = 1$.
(e) Prove that $\sqrt[3]{2}$ is irrational.

4. Find any exact algebraic solutions of the following.
(a) $x^3 + 14x^2 + 49x = 0$.
(b) $x^6 + x^3 + 1 = 0$.
(c) $x^4 + 50x^2 + 25 = 0$.
(d) $x^3 + 3x = 10$ (from page 99 of *The Great Art*).
(e) $x^4 = x^3 + 1$.

5.
(a) What is the volume of an octahedron of side 3 ?
(b) What is the area of a cyclic quadrilateral with sides 10, 12, 14, and 16?
(c) What sort of curve is represented by the equation $x^2 + 5y^2 = 1$?
(d) What is the equation of the Euler line in the triangle with vertices $(0,0)$, $(10,0)$, and $(10,10)$ ?
(e) In hyperbolic geometry, how many lines passing through a point $A$ are parallel to a given line $BC$ (with $A$ not on $BC$)?

6.
(a) Is 80,200 triangular? Why?
(b) Write 20 as a sum of 3 triangular numbers.
(c) What are the smallest 3 prime numbers that are also polygonal numbers?
(d) Prove that all hexagonal numbers are triangular numbers.
(e) What are the first 5 square triangular numbers?

7.
(a) Write a short essay on life in Mesopotamia 4000 years ago.
(b) What role, if any, did women play in ancient Mesopotamian mathematics?

8.
(a) What did the ancient Greeks think about the infinite? In your answer refer to Anaximander, Zeno, and Aristotle.
(b) Write a short essay on Zeno's four paradoxes.

9.
(a) What was Pythagoras's philosophy of mathematics?
(b) What did Plato say about mathematics?

10.
(a) Write a short essay on the infinite in Medieval mathematics.
(b) Write a short essay on Cantor and the infinite.

11.
(a) Who were del Ferro, Fior, Tartaglia, Cardano, and Ferrari?
(b) Did Cardano act rightly in publishing his book? Why?

12.
(a) What are Platonism, formalism, and intuitionism?
(b) Which of these three schools is right, and why?

13.
What is the area of the Pythagorean star each of whose 5 points has a side of length 1? You get part marks for a good decimal approximation to the answer and full marks for the exact answer.

14.
(a) Give an account of the infinite in the history of mathematics, mentioning Anaximander, Zeno, Democritus, Aristotle, Bhaskara, Gregory of Rimini, Albert of Saxony, Descartes, Leibniz, Cauchy, Bolzano, Cantor, and Russell.
(b) What do Platonism, formalism, and intuitionism have to say about the infinite?

# Appendix B

# Answers to Selected Exercises

## Exercises 1

1. Volume $= \frac{1}{3}(a^2h_a - b^2h_b)$ but $h_b/h_a = b/a$ (by similar pyramids) and $h_a = h_b + h$. Hence $h_a = ah/(a-b)$ and $h_b = bh/(a-b)$ and the Volume $= \frac{1}{3}(a^3-b^3)h/(b-a)$ and the result follows since $a^3-b^3 = (a-b)(a^2+ab+b^2)$.
5. Using the method of Sylvester, we round 14/13 up to 2 and subtract 1/2 from 13/14, getting 3/7. Then we round 7/3 up to 3 and subtract 1/3 from 3/7, getting 2/21. Then we round 21/2 up to 11 and subtract 1/11 from 2/21, getting 1/231. Hence

$$\frac{13}{14} = \frac{1}{2} + \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$$

9. If $2/p = 1/a + 1/b$ then $(2a-p)(2b-p) = p^2$. Now $p^2$ can be factored into distinct factors in only one way: $p^2 \times 1$. Hence if $a < b$ we have $a = (1+p)/2$ and $b = p(1+p)/2$.

## Exercises 2

1. Let $x = 10s$ and $y = 10t$. Then $s^8 + s^4(12)^2 = 320^2$. Let $s^4 = u$. Then $u^2 + 144u - 320^2 = 0$. Let $u = 16v$. Then $16^2v^2 + 144(16v) - 320^2 = 0$ or $v^2 + 9v - 400 = 0$. Hence $(v-16)(v+25) = 0$. Thus $v = 16$, so that $u = 256$ and $s = 4$. As a result, $x = 40$ and $y = 30$.

5. With one exception, the answers are on Plimpton 322.
7. If $x + y + z = 1716$ and $x^2 + y^2 = z^2$ then

$$x^2 + y^2 = (1716 - x - y)^2$$

$$0 = 1716^2 - 2(1716)x - 2(1716)y + 2xy$$

$$(1716 - x)(1716 - y) = (1716^2)/2 = 2(2 \times 3 \times 11 \times 13)^2$$

The various factorisations of the number on the right give various Pythagorean triangles with perimeter 1716. Moreover, both factors have to be less than 1716. If $x < y$, this means $1716 - y$ is one of 936, 968, 1014, 1089, and 1144, and hence $y$ is one of 780, 748, 702, 627, 572, while $x$ is, respectively, one of 143, 195, 264, 364, 429.

None of these gives a Babylonian solution.

# Exercises 4

1. Let $A$, $B$, $C$, $D$, $E$ be the points of the star in counterclockwise order. Let $x$ be the number of degrees in a tip angle (such as $\angle EBD$). Since $BE$ is parallel to $CD$, $\angle BDC = x$. Similarly, $\angle ECD = x$. Now triangle $ACD$ consists of the two angles just mentioned together with 3 tips. Hence $5x = 180°$.
2. With the notation of the previous answer, suppose $AC$ meets $BD$ at $F$.

$$r = \frac{AC}{CD} = \frac{AF + FC}{CD} = \frac{FD + FC}{CD} = 1 + \frac{1}{r}$$

so that $r^2 - r - 1 = 0$ and the quadratic formula (in effect known to the Babylonians) gives $r = (1 + \sqrt{5})/2$, which is called the 'golden ratio'.
3. With the notation of the previous answer, what we want is $AC/CD$ (since $ABCDE$ is a regular pentagon) and this is the golden ratio.
8. Hexagonal numbers have the form

$$4\frac{t^2 - t}{2} + t = \frac{(2t-1)^2 - (2t-1)}{2} + (2t - 1)$$

10. The answer is 0, 1, 36. Suppose $x^2 - 2y^2 = 1$. Then $x$ is odd, and, by considerations mod 4, $y$ is even. If $x = 2t+1$ and $y = 2s$ then $(t^2+t)/2 = s^2$. And all triangular squares can be obtained in this way. We shall see how to solve $x^2 - 2y^2 = 1$ in Chapter 7.
11. Hint: see *John* 21:11.
12. Using the notation of Answer 1, the area is the area of $AFD$ plus three times the area $a$ of a tip. Now if $FC = 1$ then $AF = r$ and

$$\frac{AFD}{CFD} = r$$

But $CFD/a = r^2$ so that the total area is $r^3a + 3a = 2(r+2)a$. And

$$a = \frac{1}{2r}\sqrt{1 - \left(\frac{r-1}{2}\right)^2} = \frac{1}{4r}\sqrt{r+2}$$

The total is thus

$$\frac{\sqrt{5(r+2)}}{2} = 2.126627$$

# Exercises 5

1. Take $m = 7$ in the formula. Thanks to programs like *Mathematica,* it is not hard to find the first 12 perfect numbers using only a personal computer. What previously took thousands of years now takes only a few seconds.
4. Every even perfect number has the form $2^{m-1}(2^m - 1)$ with the latter factor prime. Starting with $m = 2$ the last digits of $2^{m-1}$ are

$$2,\ 4,\ 8,\ 6,\ 2,\ 4,\ 8,\ 6,\ 2,\ \ldots$$

while the last digits of $2^m - 1$ are

$$3,\ 7,\ 5,\ 1,\ 3,\ 7,\ 5,\ 1,\ 3,\ \ldots$$

The products of the corresponding terms thus end in 6, 8, or 0. The latter is impossible for perfect numbers since none of them is divisible by 5.
5. By mathematical induction,

$$1^3 + 3^3 + \cdots + (2t-1)^3 = t^2(2t^2 - 1)$$

Take $t = 2^{(m-1)/2}$ and we are done.
8. If $n$ is a square or twice a square and $p$ is an odd prime dividing $n$ then the exponent $m$ on $p$ in the prime factorisation of $n$ is even. Now $s(p^m) = 1 + p + p^2 + \cdots + p^m$, which is odd if $m$ is even. Since $s(2^q)$ is always odd, it follows from Exercise 6 that $s(n)$ is odd. The converse follows in the same way.

# Exercises 6

3. An octahedron is two pyramids. The height of either pyramid is half the diagonal of the square base. Thus the volume is $2 \times 1/3 \times 1^2 \times (1/2)\sqrt{2}$.
6. The tetrahedron can be dissected into four congruent pyramids, each with its top at the centre of the tetrahedron. Suppose that the height of

the tetrahedron is $h$ and the height of one of these smaller pyramids is $h'$. Then, where $b$ is the area of the base of the tetrahedron,

$$\frac{1}{3}bh = \frac{4}{3}bh'$$

so that the centre $C$ of the tetrahedron is $1/4$ of the way up an altitude of the tetrahedron. Let $F$ be the foot of an altitude and $V$ a vertex of the tetrahedron. Then $FV = (2/3)(\sqrt{3}/2)$ while $FC = (1/4)\sqrt{2/3}$. Hence, by the theorem of Pythagoras, the required distance is $\sqrt{6}/4$.

# Exercises 7

5. Using Euclid's algorithm,

$$\begin{aligned} X_1 &= \frac{67}{120} \\ X_2 &= \frac{120}{67} \\ X_3 &= \frac{67}{53} \\ X_4 &= \frac{53}{14} \\ X_5 &= \frac{14}{11} \\ X_6 &= \frac{11}{3} \\ X_7 &= \frac{3}{2} \\ X_8 &= \frac{2}{1} \\ X_9 &= \frac{1}{0} \end{aligned}$$

Noting that $9 - 2 = 7$, we compute $f_7 = 24$ and $g_7 = 43$. Indeed,

$$67 \times 43 - 120 \times 24 = 1$$

so that

$$\frac{67}{120} = \frac{1}{5160} + \frac{24}{43}$$

To solve $24x' - 43y' = 1$ we again use Euclid's algorithm, starting with $X_1 = 24/43$ and ending with $f_5 = 5$ and $g_5 = 9$. This gives

$$\frac{24}{43} = \frac{1}{387} + \frac{5}{9}$$

Now $5x'' - 9y'' = 1$ has solution $(2, 1)$ so that

$$\frac{5}{9} = \frac{1}{18} + \frac{1}{2}$$

Take that Pharaoh!
7. We have $X_1 = \sqrt{3}$, $X_{2n} = (1 + \sqrt{3})/2$, and $X_{2n+1} = 1 + \sqrt{3}$. This gives $[X_{2n}] = 1$ and $[X_{2n+1}] = 2$ and

$$\begin{aligned}
g_1 &= 1\\
g_2 &= 1\\
g_3 &= 3\\
g_4 &= 4\\
g_5 &= 11\\
g_6 &= 15\\
g_7 &= 41\\
g_8 &= 56\\
g_9 &= 153\\
g_{10} &= 209\\
g_{11} &= 571\\
g_{12} &= 780\\
g_{13} &= 2131\\
g_{14} &= 2911\\
g_{15} &= 7953\\
g_{16} &= 10864\\
g_{17} &= 29681\\
g_{18} &= 40545\\
g_{19} &= 110771
\end{aligned}$$

We stop at this point since, for $10^{-10}$ accuracy, it suffices to have $g^2 > 10^{10}$. A similar computation gives $f_{19} = 191861$, and the approximation to $\sqrt{3}$ is

$$\frac{191861}{110771}$$

10. The smallest possible number of maids is 292.

# Exercises 11

1 (e) Granted Aristotle's view that PaS implies PiS, this is valid. In modern logic, however, it is not.
2. (c) and (d) are contingent.

# Exercises 12

2. Hint: Write $d : c :: b : a$ in terms of Eudoxus's definition, but switch the $p$ and $q$ (which will not affect the meaning of the definition).
5. $y = (4/9)x^2$.

# Exercises 16

1. Suppose the circle representing the earth has equation

$$x^2 + y^2 + Dx + Ey = F$$

Since $(0, 0)$ is on it, $F = 0$. By symmetry of the other two points on it, $E = 0$. Moreover,

$$(-0.1364)^2 + (0.9907)^2 + D(-0.1364) = 0$$

or $1 = 0.1364D$. Hence $D = 3.66$ so that the answer, also, is 3.66.
2. Move a quarter until it just covers the full moon. Then the distance to the moon is to the distance between your eye and the quarter as the diameter of the moon is to the diameter of the quarter.
3. $x = 10.97 - 23028.41V$.
4. $AC^2 = AB^2 + BC^2 + 2AB.BC = AB^2 + BC^2 + 2BW^2$, and Archimedes' area theorem follows from the fact that the area of a semicircle is proportionate to the square on the diameter. (Archimedes gave this proof in Proposition 4 in his *Book of Lemmas.*) As angles in semicircles, $\angle AUB$ and $\angle WUB$ are right and hence $WUA$ is straight. So is $WVC$. Hence, as an angle in the semicircle on $AC$, $\angle UWV$ is right. Thus $UV$ is a diameter of the circle with diameter $BW$, and, of course, the two diameters are equal and bisect each other. To show that $UV$ is a tangent to the semicircle on $AB$, it suffices to show that $\angle VUB = \angle UAB$. Now

$$\angle VUB = \angle VWB = 90^\circ - \angle AWB = \angle WAB$$

5. Hint: Use mathematical induction.

## Exercises 17

3. The $n$th row ends with $2(n(n+1)/2)-1$. The sum of the first $x$ odds is $x^2$, so the sum of the entries in the $n$th row is

$$(n(n+1)/2)^2 - ((n-1)n/2)^2 = n^3$$

Corollary: the sum of the first $n$ cubes is the sum of the first $n(n+1)/2$ odd numbers, namely,

$$\left(\frac{n(n+1)}{2}\right)^2$$

## Exercises 18

1. We first solve $3x+2 = 5y+3$ to get $x = 5z+2$ and hence $3x+2 = 15z+8$. We then solve $15z+8 = 7w+2$ or $7w - 15z = 6$. Noting that $w = 3v$, we have $7v - 5z = 2$, yielding $z = 7k+1$, so that $15z+8 = 105k+23$, which is the general solution.
4. There are 201 steps in the pyramid.
11. $x = 649$ and $y = 180$ solve the equation.

## Exercises 19

1. $x^2 + (706.02/x)^2 = (36.9 + x)^2$ so that

$$738000x^3 + 13616100x^2 = 4984642404$$

Each of the coefficients is divisible by $41 \times 36$ giving

$$500x^3 + 41 \times 225x^2 = 7^2 41^3$$

Let $x = (41 \times 7)/5u$. Then $u^3 = 9u + 28$ and $u = 4$ gives a solution. The other two solutions are not real. If $u = 4$ then $x = 14.35$. Hence $y = 706.02/14.35 = 49.2$, and $z = 36.9 + 14.35 = 51.25$.

## Exercises 20

1. By similar triangles, $AB/BB' = BC/AB$ and $AC/CC' = BC/AC$.
5. From the given formula, you can find a diagonal of the quadrilateral, and thus a triangle inscribed in the circle.

# Exercises 21

4. If $x + y = 10$ and $x^2 + y^2 + y/x = 72$ then

$$2x^3 - 20x^2 + 27x + 10 = 0$$

By inspection, one solution is $x = 2$.

# Exercises 22

10. Oresme's sum can be written

$$\begin{gathered}1/2 + 1/4 + 1/8 + 1/16 + \cdots \\ +1/4 + 1/8 + 1/16 + \cdots \\ +1/8 + 1/16 + \cdots\end{gathered}$$

and so on. The first row sums to 1 (think of Zeno!), the second to 1/2, the third to 1/4, and so on. The grand total is thus

$$1 + 1/2 + 1/4 + \cdots = 2$$

# Exercises 25

1.Let $s$ be the real number $(704 + 12\sqrt{3930})^{1/3}$. Then the solutions are

$$-\frac{2}{3} - \frac{26 \times 2^{1/3}}{3s} + \frac{s}{3 \times 2^{1/3}}$$

$$-\frac{2}{3} + \frac{13 \times 2^{1/3}(1 \pm \sqrt{-3})}{3s} - \frac{(1 \mp \sqrt{-3})s}{6 \times 2^{1/3}}$$

with the signs corresponding.

2. Let $s = (27000 + 24\sqrt{1265613})^{1/3}$. The real solution is

$$-2 + \frac{4 \times 2^{1/3}}{s} + \frac{s}{3 \times 2^{1/3}}$$

3. Let $s = (10260 + 162\sqrt{3973})^{1/3}$. The only solution is

$$-4 + \frac{21 \times 2^{1/3}}{s} + \frac{s}{3 \times 2^{1/3}}$$

5. One pound of saffron cost

$$\frac{1569 - 17\sqrt{4785}}{399}$$

6. Let $s = (\pi - \arctan 2\sqrt{2})/3$. Then $x = \sqrt{3}\sin s - \cos s$. This is the distance from the centre at which one must slice a sphere of radius 1 so that one piece has twice the volume of the other.
7. Let $s = ((-51392 + 1584\sqrt{1351})/2)^{1/3}$. Let

$$t = \sqrt{\frac{23}{3} - \frac{572}{3s} + \frac{s}{3}}$$

Then one of the two solutions has

$$x = \frac{3}{2} - \frac{t}{2} - \frac{1}{2}\sqrt{\frac{46}{3} + \frac{572}{3s} - \frac{s}{3} + \frac{6}{t}}$$

8. Let $x$ be the height of the top of the 20 foot ladder and $y$ the height of the top of the 30 foot ladder. Then

$$\frac{8}{x} + \frac{8}{y} = 1$$

Also $y^2 - x^2 = 500$. Let $u = 1/x$ and $v = 1/y$. Then $u + v = 1/8$ and $(1/8)(u - v) = u^2 - v^2 = 500u^2v^2$. Since

$$(u - v)^2 + 4uv = (u + v)^2$$

we have

$$(4000u^2v^2)^2 + 4uv = 1/64$$

Let $k = 400uv$. Then

$$k^4 + 16k = 25$$

Solving this we get an answer approximately equal to 16.2121 feet. More precisely, let

$$s = \sqrt[3]{\frac{6912 + 48\sqrt{67611}}{2}}$$

Let

$$t = \sqrt{\frac{s}{3} - \frac{100}{s}}$$

Then

$$k = \frac{\sqrt{\frac{100}{s} - \frac{s}{3} + \frac{32}{t}}}{2} - \frac{t}{2}$$

Hence

$$x = \frac{80}{5 + \sqrt{25 - 16k}}$$

and the answer is

$$\sqrt{400 - x^2}$$

# Exercises 26

1. Changing to base $e$, we have

$$\frac{\ln 3 + 2\ln x}{\ln 2} = \frac{\ln 4 + 2\ln x}{\ln 3}$$

and we can now solve for $\ln x$ and then for $x$.

# Exercises 26

6. The equations $Y = X^2/4p$ and $Y - x^2/4p = m(X - x)$ meet at points whose $X$-coordinate is a solution of

$$X^2 - x^2 - 4pmX + 4pmx = 0$$

or

$$(X - x)(X - (4pm - x)) = 0$$

The straight line is a tangent just in case there is only one such solution, that is, just in case $x = 4pm - x$, and hence $m = x/2p$.
7. $(x - a)/10 + 10^m a = 4x$ together with Fermat's little theorem leads to 102564 as the solution.

# Exercises 29

2. The probability of not getting double sixes in $n$ throws of two dice is $(35/36)^n$. This is less than 1/2 when

$$n > \frac{\log(1/2)}{\log(35/36)} = 24.6$$

The answer is thus 25.
4. If you do not switch, your expectation is one-third the value of the heir (since one-third is the probability that you have chosen the ace). If you do switch, your expectation is minus 300 plus two-thirds the value of the heir (since two-thirds is the probability that you have chosen a jack and will therefore choose the ace when you switch). So it is better to switch.
5. The wheel has radius 1. It rolls up the right side of the $y$-axis. The point on the rim starts at the origin and ends up at $(2, \pi)$.

## Exercises 30

4. $1.6 \times 10^{-11}$ newtons.

## Exercises 32

3.

$$1 - \frac{32000}{40000} \times \frac{31999}{39999} \times \frac{31998}{39998}$$

## Exercises 33

2. $(2x + 7y - 1)(7x + 2y - 3)$.
6. $125 + 27 + 27 + 27 + 8 + 8 + 8 + 8 + 1$.

## Exercises 37

10. For $m = 2$, 16, and 98.

## Exercises 38

5. There is a one-to-one correspondence that maps

$$(.a_1a_2a_3\ldots,\quad .b_1b_2b_3\ldots)$$

to

$$.a_1b_1a_2b_2a_3b_3\ldots$$

## Exercises 40

2. Let $x = 2$ and $y = 3$.

# Index

## Undergraduate Texts in Mathematics

**Anglin:** Mathematics: A Concise History and Philosophy.
*Readings in Mathematics.*

**Anglin/Lambek:** The Heritage of Thales.
*Readings in Mathematics.*

**Apostol:** Introduction to Analytic Number Theory. Second edition.

**Armstrong:** Basic Topology.

**Armstrong:** Groups and Symmetry.

**Axler:** Linear Algebra Done Right. Second edition.

**Beardon:** Limits: A New Approach to Real Analysis.

**Bak/Newman:** Complex Analysis. Second edition.

**Banchoff/Wermer:** Linear Algebra Through Geometry. Second edition.

**Berberian:** A First Course in Real Analysis.

**Bix:** Conics and Cubics: A Concrete Introduction to Algebraic Curves.

**Brémaud:** An Introduction to Probabilistic Modeling.

**Bressoud:** Factorization and Primality Testing.

**Bressoud:** Second Year Calculus.
*Readings in Mathematics.*

**Brickman:** Mathematical Introduction to Linear Programming and Game Theory.

**Browder:** Mathematical Analysis: An Introduction.

**Buskes/van Rooij:** Topological Spaces: From Distance to Neighborhood.

**Cederberg:** A Course in Modern Geometries.

**Childs:** A Concrete Introduction to Higher Algebra. Second edition.

**Chung:** Elementary Probability Theory with Stochastic Processes. Third edition.

**Cox/Little/O'Shea:** Ideals, Varieties, and Algorithms. Second edition.

**Croom:** Basic Concepts of Algebraic Topology.

**Curtis:** Linear Algebra: An Introductory Approach. Fourth edition.

**Devlin:** The Joy of Sets: Fundamentals of Contemporary Set Theory. Second edition.

**Dixmier:** General Topology.

**Driver:** Why Math?

**Ebbinghaus/Flum/Thomas:** Mathematical Logic. Second edition.

**Edgar:** Measure, Topology, and Fractal Geometry.

**Elaydi:** Introduction to Difference Equations.

**Exner:** An Accompaniment to Higher Mathematics.

**Fine/Rosenberger:** The Fundamental Theory of Algebra.

**Fischer:** Intermediate Real Analysis.

**Flanigan/Kazdan:** Calculus Two: Linear and Nonlinear Functions. Second edition.

**Fleming:** Functions of Several Variables. Second edition.

**Foulds:** Combinatorial Optimization for Undergraduates.

**Foulds:** Optimization Techniques: An Introduction.

**Franklin:** Methods of Mathematical Economics.

**Gordon:** Discrete Probability.

**Hairer/Wanner:** Analysis by Its History.
*Readings in Mathematics.*

**Halmos:** Finite-Dimensional Vector Spaces. Second edition.

**Halmos:** Naive Set Theory.

**Hämmerlin/Hoffmann:** Numerical Mathematics.
*Readings in Mathematics.*

**Hijab:** Introduction to Calculus and Classical Analysis.

**Hilton/Holton/Pedersen:** Mathematical Reflections: In a Room with Many Mirrors.

**Iooss/Joseph:** Elementary Stability and Bifurcation Theory. Second edition.

**Isaac:** The Pleasures of Probability.
*Readings in Mathematics.*

## Undergraduate Texts in Mathematics

**Troutman:** Variational Calculus and Optimal Control. Second edition.

**Valenza:** Linear Algebra: An Introduction to Abstract Mathematics.

**Whyburn/Duda:** Dynamic Topology.

**Wilson:** Much Ado About Calculus.