THE FAMILY OF ANCILLARY STATISTICS

By D. BASU

Indian Statistical Institute, Calcutta

SUMMARY. Though the marginal distributions of the ancillary statistics are independent of the parameter they are not useless or informationless. A set of ancillaries may sometimes summarise the whole of the information contained in the sample. A classification of the ancillaries in terms of the partial order of their information content is attempted here. In general there are many maximal ancillaries. Among the minimal ancillaries there exists a unique largest one. When there exists a complete sufficient statistic, the problem of tracking down the maximal and minimal ancillaries becomes greatly simplified.

1. Introduction

An ancillary[1] statistic is one whose distribution is the same for all possible values of the unknown parameter. A statistic that is not ancillary may be called 'informative'. The classical example of an ancillary statistic is the following :

Example (a) : Let X and Y be two positive valued random variables with the joint density function

$$f(x, y) = e^{-\theta x - \frac{y}{\theta}}, \; x > 0, y > 0, \theta > 0.$$

Here $F = XY$ is an ancillary statistic. The maximum likelihood estimator $T = \sqrt{Y/X}$ of θ is not a sufficient statistic. However, the pair (F, T) is jointly sufficient.

The above example shows that though an ancillary statistic, by itself, fails to provide any information about the parameter, yet in conjunction with another statistic—which, as we shall presently see, need not be informative—may supply valuable information[2] about the parameter. In the following example we have given a family of ancillary statistics that are jointly equivalent to the whole sample.

Example (b) : Let X and Y be independent normal variables with unknown means θ and unit standard deviations. Here $X-Y$ is an ancillary statistic. It is commonly believed that every ancillary statistic (in this situation) is necessarily a function of $X-Y$. That, however, is not true.

Let

$$F_c = F_c(X, Y) = \begin{cases} X-Y & \text{if } X+Y < c \\ Y-X & \text{if } X+Y \geqslant c \end{cases}$$

where c is a fixed constant.

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[1] The name 'ancillary' is due to Fisher (1925). The name 'distribution-free' is also in use and perhaps would have been more appropriate in the present context.

[2] See Fisher (1956) for a discussion of how the ancillary information may (according to Fisher) be recovered.

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Since $X-Y$ and $Y-X$ are identically distributed and each is independent of $X+Y$ it at once follows that F_c is independent of $X+Y$ and has the same distribution as that of $X-Y$. Thus, F_c is ancillary for each c. Consider now the family $\{F_c\}$, $-\infty < c < \infty$, of ancillary statistics. For fixed X and Y the different values of F_c (for varying c) are either $X-Y$ or $Y-X$. The value c_0 of c where F_c changes sign (F_c does not change sign only if $X-Y = 0$ and that is a null event) is the value of $X+Y$. Thus, given $F_c(X, Y)$ for all c we can find $X+Y$ and $X-Y$. Hence, the family $\{F_c\}$ of ancillary statistics is equivalent to the whole sample (X, Y). The countable family $\{F_c\}$ where c runs through the set of rational numbers is easily seen to be also equivalent to (X, Y).

The author (Basu ; 1955, 1958) has shown that, under very mild restrictions, any statistic independent of a sufficient statistic is ancillary and that the converse proposition is also true, provided the sufficient statistic is complete.

In Example (b) the statistic $T = X+Y$ is a complete sufficient statistic. A statistic F can, therefore, be ancillary if and only if F is independent of T. The following is a general method for constructing statistics independent of T. Start with any ancillary statistic F. In general, there will be many measure-preserving transformations of F (i.e. a mapping φ of the range space of F into itself such that $\varphi(F)$ and F are identically distributed). For each real t, define a measure-preserving transformation φ_t of F. Then, take the statistic $\varphi_T(F)$. Subject to some measurability restrictions, $\varphi_T(F)$ will be independent of T and hence will be ancillary. In Example (b) we took $F = X-Y$ and $\varphi_t(F) = F$ or $-F$ according as $t < c$ or $\geqslant c$.

If a statistic F is ancillary then every (measurable) function of F is also ancillary. The statistic F_2 is said to include (or be more informative than) the statistic F_1 if F_1 can be expressed as a function of F_2. In this case we write $F_2 \supset F_1$ or $F_1 \subset F_2$. Two statistics are said to be equivalent if each can be expressed as a function of the other.

Example (c) : Let $X_1, X_2, \ldots X_n$ be n independent observations on a normal variable with mean θ and s.d. unity. Then each of the $n-1$ statistics

$$F_1 = X_1-X_2,\ F_2 = (X_1-X_2,\ X_1-X_3),\ \ldots\ F_{n-1} = (X_1-X_2, X_1-X_3, \ldots X_1-X_n)$$

is ancillary and

$$F_1 \subset F_2 \subset \cdots \subset F_{n-1}$$

The two ancillary statistics F_{n-1} and $F = (X_2-X_1,\ X_2-X_3, \ldots X_2-X_n)$ are easily seen to be equivalent.

From Example (b) it is obvious that F_{n-1} does not include all ancillary statistics.

An ancillary statistic M is said to be 'maximal' if there exists no non-equivalent ancillary M^* such that $M \subset M^*$. Thus, given any ancillary F, either it is maximal or there exists an ancillary $F^* \supset F$. Given any ancillary F_0, there exists (Theorem 2) a maximal ancillary $M \supset F_0$. In general there exists many non-equivalent maximal ancillaries. A typical property (Cor. to Theorem 4) of a maximal ancillary M is that, for any ancillary F not included in M, the pair (M, F) is informative.

A minimal ancillary is one that is included in every maximal ancillary. Among the class of minimal ancillaries there exists (Theorem 5) a unique largest one G_0. In the absence of a better name we prefer to call G_0 the laminal ancillary. G_0 includes every minimal ancillary and is included in every maximal ancillary. A typical property (Theorem 6) of a minimal ancillary G is that, for any ancillary F, the pair (G, F) is ancillary.

If there exists a complete sufficient statistic G, then, any ancillary statistic F, such that the pair (G, F) is essentially equivalent to the whole sample, is shown (Theorem 7) to be essentially maximal. Under some further restrictions, the laminal ancillary is shown (Theorem 8) to be essentially equivalent to a constant.

In the following sections we elaborate on the above sketch of the family-tree of ancillary statistics. For the sake of elegance and brevity of exposition we use the language of sub σ-fields. Reference may be made to Bahadur (1954, 1955) for excellent expositions of the sub σ-field approach.

2. Definitions

Let $(\mathscr{X}, \mathscr{B})$ be an arbitrary measurable space and let $\{P_\theta\}$, $\theta \in \Omega$ be a family of probability measures on $\mathscr{B}$. Any statistic T induces a sub σ-field $\mathscr{B}_T \subset \mathscr{B}$. Instead of dealing with statistics it is more convenient (in the present context) to deal with the corresponding sub σ-fields.

Definition 1: The event $A \in \mathscr{B}$ is said to be ancillary if $P_\theta(A)$ is the same for all $\theta \in \Omega$. The family of all ancillary events is denoted by $\mathscr{A}$.

It is easy to check that the family $\mathscr{A}$ is closed for complementation and countable disjoint unions. However, in general $\mathscr{A}$ is not closed for intersection (i.e. $\mathscr{A}$ is not a σ-field).

In order to show that the family $\mathscr{A}$ in Example (b) do not constitute a σ-field, we have only to check that

$$P_\theta\,[X-Y > 0 \text{ and } F_c\,(X, Y) > 0] = P_\theta\,(X-Y > 0 \text{ and } X+Y < c)$$

$$= \tfrac{1}{2} \int_{-\infty}^{c} \frac{1}{\sqrt{4\pi}}\, e^{-\frac{1}{4}(x-2\theta)^2}\, dx$$

which varies with θ.

In Example (b) the Borel-extension of $\mathscr{A}$ is $\mathscr{B}$.

Example (d) : Let $\mathscr{X}$ consist of the three points a, b and c and let the corresponding probability measures be $\frac{1}{4}-\theta$, $\frac{1}{2}$, and $\frac{1}{4}+\theta$ respectively, where $0 < \theta < \frac{1}{4}$. Here $\mathscr{A}$ consists of the four sets ϕ, $[b]$, $[a, c]$ and $\mathscr{X}$ and so $\mathscr{A}$ is a sub σ-field of $\mathscr{B}$.

Definition 2 : A σ-field $\mathscr{F}$ is said to be ancillary if $\mathscr{F} \subset \mathscr{A}$. A σ-field that is not ancillary is called informative.

A statistic is ancillary or informative according as the corresponding σ-field is so.

Definition 3: Two ancillary sets A and B are said to conform if AB is also ancillary. If A conforms to B then we write $A \sim B$. Since $P_\theta (AB)+P_\theta (AB') = P_\theta (A)$ it follows that $A \sim B$ if and only if $A \sim B'$.

If A conforms to every one of a sequence of disjoint sets $B_1, B_2 \ldots$ then it is easy to check that $A \sim \bigcup B_i$.

Definition 4 : Let Γ_0 be the family of all ancillary sets B such that $B \sim A$ for all $A \in \mathscr{A}$.

Clearly ϕ and $\mathscr{X}$ belong to Γ_0. From what we have said before it follows that Γ_0 is closed for complementation and countable disjoint unions.

Theorem 1 : *The family* Γ_0 *is a* σ*-field.*

Proof: It is enough to show that Γ_0 is closed for intersection. Let B_1 and B_2 both belong to Γ_0 and let $A \in \mathscr{A}$. From $B_2 \in \Gamma_0$ it follows that $B_2A \in \mathscr{A}$. From $B_1 \in \Gamma_0$ it then follows that $B_1B_2A \in \mathscr{A}$. Since A is an arbitrary ancillary set, it follows that $B_1B_2 \in \Gamma_0$.

We shall later on see that the ancillary σ-field Γ_0 corresponds to the laminal ancillary G_0 that we have referred to in §1.

The family $\mathscr{A}$ of ancillary sets is a σ-field if and only if every pair of ancillary sets conform to one another, i.e. if $\mathscr{A} = \Gamma_0$.

Example (e) : Let $\mathscr{X}$ consist of the five points a, b, c, d and e with the corresponding probabilities $\frac{1}{2}$, θ, θ, $\frac{1}{4}-\theta$ and $\frac{1}{4}-\theta$ respectively, where $0 < \theta < \frac{1}{4}$. In this case Γ_0 consists of the four sets ϕ, [a], $[b, c, d, e]$, and $\mathscr{X}$. The two sets $[b, d]$ and $[b, e]$ are both ancillary but they do not conform. Here $\mathscr{A}$ is wider than Γ_0 and is not a σ-field.

Definition 5 : The ancillary σ-field $\mathscr{F}_2$ is said to include the ancillary σ-field $\mathscr{F}_1$ (in symbols $\mathscr{F}_2 \supset \mathscr{F}_1$ or $\mathscr{F}_1 \subset \mathscr{F}_2$) if every element of $\mathscr{F}_1$ is an element of $\mathscr{F}_2$.

The above partial order on ancillary σ-fields corresponds to the inclusion relationship for ancillary statistics.

Definition 6 : The ancillary σ-field $\mathscr{M}$ is said to be maximal if there exists no other ancillary σ-field $\mathscr{M}^*$ such that $\mathscr{M}^* \supset \mathscr{M}$.

Definition 7 : The intersection of all the maximal ancillary σ-fields is called the laminal ancillary.

The laminal ancillary is the largest ancillary that is included in all maximal ancillaries.

3. Existence and characterizations of maximal and laminal ancillaries

The following theorem is fundamental.

Theorem 2 : *Given any ancillary σ-field $\mathscr{F}_0$ there exists a maximal ancillary σ-field $\mathscr{M} \supset \mathscr{F}_0$.*

Proof : We first prove that given any family $\{\mathscr{F}_j\}$, $j \in J$ of ancillary σ-fields that are linearly ordered (by the inclusion relationship), the Borel-extension $\mathscr{F}$ of $\bigcup \mathscr{F}_j$ is also ancillary.

Clearly, $\bigcup \mathscr{F}_j$ contains ϕ and $\mathscr{X}$ and is closed for complementation. Since $\{\mathscr{F}_j\}$ is linearly ordered it follows that $\bigcup \mathscr{F}_j$ is also closed for finite unions. That is, $\bigcup \mathscr{F}_j$ is a field of sets.

Since each $\mathscr{F}_j$ is ancillary, the restriction of P_θ to $\bigcup \mathscr{F}_j$ is a measure Q that does not depend on θ. From the fundamental Extension Theorem of measures (Kolmogorov, 1933) we know that the extension of Q to $\mathscr{F}$ is unique.

It follows at once that the restriction of P_θ to $\mathscr{F}$ is the same for all θ, i.e. $\mathscr{F}$ is an ancillary σ-field.

Now let $\mathscr{C}$ be the family of all ancillary σ-fields that include $\mathscr{F}_0$. Since corresponding to any linearly ordered sub-family of $\mathscr{C}$ there exists an ancillary σ-field that includes every member of the sub-family it follows from Zorn's Lemma that $\mathscr{C}$ has a maximal element.

Let $\{\mathscr{M}_i\}$, $i \in I$ be the family of all maximal ancillary σ-fields. We at once have the

Theorem 3 : $\mathscr{A} = \bigcup \mathscr{M}_i$

Proof : We have only to note that corresponding to any element A of $\mathscr{A}$ there exists an ancillary σ-field that contains A as an element and then apply Theorem 2.

Corollary : *If $\{\mathscr{M}_i\}$ consists of only one σ-field $\mathscr{M}_0$ then $\mathscr{A} = \mathscr{M}_0 = \Gamma_0$.*

Thus, in any situation where there are non-conforming ancillary sets, the family $\{\mathscr{M}_i\}$ has at least two members.

In Example (d) there is a unique maximal ancillary. In Example (e) there are exactly two maximal ancillaries namely :

$$\mathscr{M}_1 = \text{the } \sigma\text{-field spanned by } [a] \text{ and } [b, d]$$

and

$$\mathscr{M}_2 = \text{the } \sigma\text{-field spanned by } [a] \text{ and } [b, e].$$

Theorem 4 : *If the ancillary set A does not belong to the maximal ancillary $\mathscr{M}$ then A does not conform to at least one element of $\mathscr{M}$.*

Proof : Suppose on the contrary that A cnforms to every element of $\mathcal{M}$. Consider the family $\mathcal{M}^*$ of sets $AX \bigcup A'Y$ where X and Y are arbitrary elements of $\mathcal{M}$. Clearly $\mathcal{M} \subset \mathcal{M}^*$ but not conversely.

Since $$(AX \bigcup A'Y)' = AX' \bigcup A'Y',$$

and $$\bigcup (AX_i \bigcup A'Y_i) = A\,(\bigcup X_i) \bigcup A'(\bigcup Y_i)$$

and $$P_\theta\,(AX \bigcup A'Y) = P_\theta\,(AX) + P_\theta\,(A'Y),$$

it follows that $\mathcal{M}^*$ is also an ancillary σ-field.

This, however, contradicts the maximality of $\mathcal{M}$.

Corollary : *If $\mathcal{M}$ be any maximal ancillary and if the ancillary σ-field $\mathcal{F}$ is not included in $\mathcal{M}$ then the smallest σ-field containing both $\mathcal{M}$ and $\mathcal{F}$ is informative.*

Theorem 5 : $\bigcap \mathcal{M}_i = \Gamma_0$

Proof : Since every element of Γ_0 conforms (by definition) to every ancillary event, if follows from Theorem 4 that $\Gamma_0 \subset \mathcal{M}_i$ for all i, i.e. $\Gamma_0 \subset (\bigcap \mathcal{M}_i)$.

Now let $B \in \bigcap \mathcal{M}_i$ and A be an arbitrary ancillary set. From Theorem 3 it follows that $A \in \mathcal{M}_i$ for some i.

Hence B and A are together as elements of some $\mathcal{M}_i$ and so $B \sim A$.

Since A is arbitrary it follows that $B \in \Gamma_0$.

$\therefore$ $(\bigcap \mathcal{M}_i) \subset \Gamma_0$ and so the equality is proved.

Theorem 6 : *For any ancillary σ-field $\mathcal{F}$ the smallest σ-field containing both $\mathcal{F}$ and Γ_0 is also ancillary.*

Proof : Consider the family of 'rectangular' sets $X \bigcap Y$ where $X \in \mathcal{F}$ and $Y \in \Gamma_0$. From the definition of Γ_0 it follows that all such sets are ancillary and that they conform to one another. The family of sets that may be formed by finite unions of rectangular sets form a field of sets and each of them is ancillary. The rest follows from the Extension Theorem of Measures.

4. When a complete sufficient statistic exists

In general there exist many maximal ancillaries. For instance, in Example (b) there are uncountably many maximal ancillaries. In order to see this, let us consider the family $\{A_c\}$ of ancillary events where $A_c = \{(X, Y) \mid F_c(X, Y) > 0\}$. If $c < d$, then

$$P_\theta\,(A_cA_d) = P_\theta(X-Y > 0 \text{ and } X+Y < c) + P_\theta(Y-X > 0 \text{ and } X+Y \geqslant d)$$

$$\doteq \tfrac{1}{2}\left[\,1 - \int_c^d \frac{1}{2\sqrt{\pi}}\, e^{-\frac{1}{4}(x-2\theta)^2}\, dx\,\right]$$

which varies with θ.

Thus, the members of the family $\{A_c\}$ of ancillary sets are mutually non-conforming. Hence the maximal ancillaries including the different members of the family are all different.

Though there may exist many maximal ancillaries, it is not, in general, easy to prove the maximality of a particular ancillary. However, in the situations where we have a complete sufficient statistic, it is rather easy to demonstrate the maximality of a large class of ancillaries.

The following property of complete sufficient statistics is useful.[1] Here we state and prove the result in terms of σ-fields.

Lemma (Basu, 1955) : *If $\mathcal{G} \subset \mathcal{B}$ be a boundedly complete sufficient σ-field and A any ancillary event, then A is independent of $\mathcal{G}$.*

Proof : Let $\varphi = P(A \mid \mathcal{G})$ be the conditional probability of A given $\mathcal{G}$. That is, φ is a $\mathcal{G}$-measurable function such that

$$P_\theta(AG) = \int_G \varphi dP_\theta \text{ for all } \theta \in \Omega \text{ and } G \in \mathcal{G}.$$

Since $\mathcal{G}$ is sufficient, it follows that φ may be chosen to be independent of θ. Also the set of x's for which $\varphi(x)$ lies outside the interval (0, 1) is of zero-measure for each $\theta \in \Omega$.

Taking $G = \mathcal{X}$ we have

$$P_\theta (A) = \int_{\mathcal{X}} \varphi \, dP_\theta \quad \text{for all } \theta \in \Omega.$$

Since $P_\theta (A)$ is independent of θ and φ is $\mathcal{G}$-measurable, it follows from the bounded completeness of $\mathcal{G}$ that $\varphi = P_\theta (A)$ almost surely for all $\theta \in \Omega$.

$$\therefore \qquad P_\theta (AG) = \int_G \varphi dP_\theta$$
$$= P_\theta (A) P_\theta (G) \text{ for all } \theta \in \Omega \text{ and } G \in \mathcal{G}.$$

That is, A is independent of all $G \in \mathcal{G}$

Before proceeding further we need a slightly wider definition of maximality for an ancillary σ-field.

Definition 8 : The two $\mathcal{B}$-measurable sets A and B are said to be essentially equal if

$$P_\theta (A \Delta B) \equiv P_\theta (AB' \cup A'B)$$
$$= 0 \quad \text{for all } \theta \in \Omega.$$

Definition 9 : Two sub σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be essentially equivalent if corresponding to any set belonging to one of them there exists an essentially equal set belonging to the other.

Definition 10 : Any ancillary σ-field that is essentially equivalent to a maximal ancillary is called essentially maximal.

Theorem 7 : *If $\mathcal{G}$ be a boundedly complete sufficient σ-field then any ancillary $\mathcal{F}$ such that the Borel-extension of $\mathcal{G} \cup \mathcal{F}$ is essentially equivalent to $\mathcal{B}$, is essentially maximal.*

[1] See Basu (1955) and Hogg and Craig (1956) for some other interesting applications.

Proof : Let $\mathcal{M}$ be a maximal ancillary including $\mathcal{F}$ and let M be an arbitrary element of $\mathcal{M}$. For proving the essential maximality of $\mathcal{F}$ we have to establish the existence of an $F_0 \in \mathcal{F}$ such that F_0 is essentially equal to M.

Let $\mathcal{B}^*$ be the Borel extension of $\mathcal{F} \cup \mathcal{G}$. Since $\mathcal{B}^*$ is essentially equivalent to $\mathcal{B}$, there exists an $M^* \in \mathcal{B}^*$ such that M^* is essentially equal to M.

Since $M \in \mathcal{M} \supset \mathcal{F}$ and M^* is essentially equal to M, it follows that M^* is an ancillary set conforming to every $F \in \mathcal{F}$. Clearly, the two measures P and Q on $\mathcal{F}$, defined by the relations $P(F) = P_\theta(F)$ and $Q(F) = P_\theta\,(M^*F)$, are both independent of θ.

Therefore, the conditional probability function

$$\varphi = P_\theta(M^* | \mathcal{F}) = \frac{dQ}{dP}$$

is independent of θ.

Thus, φ is an $\mathcal{F}$-measurable function on $\mathcal{X}$ such that

$$P_\theta\,(M^*F) = \int_F \varphi dP_\theta \quad \text{for all} \quad \theta \in \Omega \quad \text{and} \quad F \in \mathcal{F}.$$

Let F and G be typical elements of $\mathcal{F}$ and $\mathcal{G}$ respectively. Since $\mathcal{F}$ is ancillary and $\mathcal{G}$ is boundedly complete sufficient, it follows (from the Lemma) that $\mathcal{F}$ and $\mathcal{G}$ are independent.

$$\therefore \quad \int_{FG} \varphi dP_\theta = \int_{\mathcal{X}} (\varphi\, \mathcal{X}_F)\, \mathcal{X}_G\, dP_\theta \quad (\mathcal{X}_F \text{ and } \mathcal{X}_G \text{ are characteristic functions of } F \text{ and } G)$$

$$= \int_{\mathcal{X}} (\varphi\, \mathcal{X}_F)\, dP_\theta \int_{\mathcal{X}} \mathcal{X}_G\, dP_\theta \quad (\because \mathcal{F} \text{ and } \mathcal{G} \text{ are independent})$$

$$= P_\theta\,(M^*F)\, P_\theta\,(G) \qquad \ldots \quad (\alpha)$$

Again, since $M^* \sim F$ it follows (from the Lemma) that M^*F is independent of G.

$$\therefore \quad \int_{FG} \mathcal{X}_{M^*}\, dP_\theta = P_\theta\,(M^*FG) = P_\theta\,(M^*F)\, P_\theta\,(G). \qquad \ldots \quad (\beta)$$

From (α) and (β) we have

$$\int_{FG} (\varphi - \mathcal{X}_{M^*})\, dP_\theta = 0 \text{ for all } F \in \mathcal{F} \text{ and } G \in \mathcal{G}.$$

Since $\varphi - \mathcal{X}_{M^*}$ is $\mathcal{B}^*$-measurable it at once follows that

$$\int_B (\varphi - \mathcal{X}_{M^*})\, dP_\theta = 0 \text{ for all } B \in \mathcal{B}^*.$$

Therefore, for each $\theta \in \Omega$, $\varphi(x) - \mathcal{X}_{M^*}\,(x) = 0$ for almost all x.

Let $F_0 = \{x \,|\, \varphi(x) = 1\}$. Clearly $F_0 \in \mathcal{F}$ and is essentially equal to M^*. Since M^* is essentially equal to M the Theorem is proved.

In Example (b), $X+Y$ is a complete sufficient statistic. Also for any fixed c, the pair $(X+Y, F_c)$ is equivalent to the sample (X, Y). Hence it follows that every F_c is an essentially maximal ancillary. In Example (c), the ancillary F_{n-1} together with the complete sufficient statistic $X_1+X_2+\ldots+X_n$ is equivalent to the whole sample and, therefore, is essentially maximal. A large number of similar situations are covered by Theorem 7.

Having partially settled the question of maximal ancillaries let us turn our attention to the laminal ancillary.

The laminal ancillary is the largest ancillary σ-field that is included in all maximal ancillaries. From Theorem 5 we have that the class Γ_0 of ancillary sets C that conform to every ancillary set is the laminal ancillary.

Let Λ be the family of sets that are essentially equal to either the empty set ϕ or the whole space $\mathscr{X}$. That is, Λ is the family of all sets E such that $P_\theta(E)$ is either $\equiv 0$ or $\equiv 1$ for all $\theta \in \Omega$. It is easy to check that Λ is a σ-field and that $\Lambda \subset \Gamma_0$. The following theorem covers a number of important cases.

Theorem 8 : *If the following conditions are satisfied then* $\Gamma_0 = \Lambda$.

i) *$\mathscr{F}$ is an essentially maximal ancillary.*

ii) *There exists an informative set G which is independent of $\mathscr{F}$.*

iii) *For every $F \in \mathscr{F}$ such that $0 < P_\theta(F) < 1$ there exists $F^* \in \mathscr{F}$ such that* $P_\theta(F^*) = P_\theta(F)$ and $P_\theta(FF^*) < P_\theta(F)$.

Proof: Let C be an arbitrary element of Γ_0. We have to prove that $P_\theta(C) = 0$ or 1. If possible let $0 < P_\theta(C) < 1$.

Now, $\mathscr{F}$ is essentially equivalent to a maximal ancillary and C belongs to every maximal ancillary. Hence, there exists $F \epsilon \mathscr{F}$ which is essentially equal to C. Thus, F conforms to every ancillary set and $0 < P_\theta(F) < 1$.

Let G and F^* satisfy conditions (ii) and (iii) respectively and let $A = GF \cup G'F^*$. Since G is independent of $\mathscr{F}$, we have

$$\begin{aligned} P_\theta(A) &= P_\theta(G)P_\theta(F)+P_\theta(G')P_\theta(F^*) \\ &= P_\theta(F)[P_\theta(G)+P_\theta(G')] \\ &= P_\theta(F). \end{aligned}$$

That is, A is an ancillary set.

Now

$$AF = GF \cup G'(FF^*)$$

and, therefore,

$$\begin{aligned} P_\theta(AF) &= P_\theta(G)P_\theta(F)+P_\theta(G')P_\theta(FF^*) \\ &= P_\theta(FF^*)+P_\theta(G)[P_\theta(F)-P_\theta(FF^*)]. \end{aligned}$$

Let us note that $P_\theta(FF^*)$ and $P_\theta(F)-P_\theta(FF^*)$ are both independent of θ and that the latter is not zero. Again since G is informative $P_\theta(G)$ is not independent of θ. Hence AF is informative, which is a contradiction. Therefore, $P_\theta(C) = 0$ or 1, i.e. $C \epsilon \Lambda$, which proves the theorem.

If the conditions of Theorem 7 are satisfied then $\mathscr{F}$ and any informative $G \epsilon \mathscr{G}$ satisfies conditions (i) and (ii) of Theorem 8. We have then only to check whether condition (iii) is satisfied or not. If the restriction of P_θ to $\mathscr{F}$ be non-atomic then it is very easy to see that condition (iii) is also satisfied.

In Examples (b) and (c) the (essentially) maximal ancillaries have continuous (non-atomic) distributions and so Theorem 8 holds. Most of the familiar cases where a complete sufficient statistic exists fall under the above category.

Example (f) : Let X be a single observation on a normal variable with mean zero and standard deviation σ. Here X^2 is a complete sufficient statistic.

Let $$Y = \begin{cases} -1 & \text{of } X < 0 \\ 1 & \text{if } X \geqslant 0 \end{cases}$$

Here the pair (Y, X^2) is equivalent to the whole sample X.

$\therefore$ Y is an essentially maximal ancillary.

The sub σ-field generated by Y consists of the four sets ϕ, $(-\infty, 0)$, $[0, \infty)$ and $\mathscr{F}$. Condition (iii) of Theorem 8 is clearly satisfied. Therefore, the laminal ancillary Γ_0 is the same as Λ.

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