Finite Volume Methods on Non-Matching Grids with Arbitrary Interface Conditions and Highly Heterogeneous Media

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Summary. We are interested in a robust and accurate domain decomposition method with arbitrary interface conditions on non-matching grids using a finite volume discretization. We introduce transmission operators to take into account the non-matching grids. Under compatibility assumptions, we have the well-posedness of the global problem and of the local subproblems with a new discretization of the arbitrary interface conditions. Then, we give two error estimates in the discrete H^1 norm: the first one is in $O(h^{1/2})$ with L^2 orthogonal projections onto piecewise functions along the interface and the second one in $O(h)$ with transmission conditions based on a linear rebuilding along the interface. Finally, numerical results confirm the theory. Particular attention is paid to the situation with non matching grids and highly heterogeneous coefficients both across and inside subdomains. The addition of a third very thin subdomain between geological blocks is necessary to ensure a good accuracy.

1 Introduction

The aim of basin modelling is to simulate maturation of source rocks and migration of oil in sedimentary basins in order to provide quantitative prediction about phenomena leading to oil accumulations. A sedimentary basin is divided by faults in several blocks, which are themselves composed of several layers of different lithology. In order to account for these heterogeneities, the mesh used in each block follows the stratigraphic layers. Blocks displacement along faults results in sliding and therefore leads to non matching grids between two adjacent blocks (eventually between two adjacent layers). Our objective is to develop numerical methods based on finite volume discretization (as it is well adapted to multiphase flow modelling), and to handle efficiently non-matching grids. We work in the context of domain decomposition techniques which offer a general framework to handle non matching grids.

As a first simplified model, we consider the following problem in Ω, bounded polygonal subset of $\mathbb{R}^d$ $(d = 2, 3)$:

$$(\eta - \Delta)(p) = f \text{ in } \Omega \text{ and } p = 0 \text{ on } \partial\Omega \tag{1}$$

where $\eta > 0$. For the sake of simplicity, we assume that the domain Ω is divided in two non overlapping subdomains Ω_i $(i = 1, 2)$, with grids that do not match on the interface.
Previous works have shown that Robin or more general interface conditions in domain decomposition methods ensure robustness and efficiency of the iterative domain decomposition Faille et al. [2000], Achdou et al. [1999]. A continuous domain decomposition formulation of (1) reads:

$$\begin{array}{l} (\eta - \Delta)(p_i^{n+1}) = f \text{ in } \Omega_i \text{ and } p_i^{n+1} = 0 \text{ on } \partial\Omega \cap \partial\Omega_i \\ \frac{\partial p_i^{n+1}}{\partial n_i} + \alpha_j p_i^{n+1} = -\frac{\partial p_j^n}{\partial n_j} + \alpha_j p_j^n \text{ on } \partial\Omega_j \cap \partial\Omega_i,\ i, j = 1, 2 \text{ and } i \neq j \end{array} \tag{2}$$

where $\alpha_j > 0$. Our aim is to combine this domain decomposition algorithm with a cell centered finite volume discretization, while satisfying the following properties. First, the method should be robust enough (at least existence and uniqueness of the discrete solution). Then it should allow a wide range of values for Robin coefficients or even more general interface conditions and it should be accurate enough as our ultimate goal is to consider grids that do not match between layers. Finally, as sliding blocks are considered, the discretization in one block should not depend on the grid of the adjacent block. In the framework of finite volume or mixed finite element method, several discretization methods for non-matching grids have been developed Arbogast et al. [1996], Ewing et al. [1991], Achdou et al. [2002],Cautrés et al. [2000], Aavatsmark et al. [2001] but these methods do not use Robin conditions or loose finite volume accuracy.

The rest of the paper is organized as follows. In the next section, we describe the finite volume discretization inside a subdomain. In Section 3, we introduce the transmission operators used to match the unknowns. In Section 5, error estimates are given. In Section 6, numerical results are shown. In Section 7, discontinuous coefficients are taken into account.

2 Finite volume discretization

We consider a finite volume admissible mesh $\mathcal{T}_i$ associated with each subdomain Ω_i Eymard et al. [2000] which is a set of closed polygonal subsets of Ω_i such that $\Omega_i = \cup_{K \in \mathcal{T}_i} K$ and $\mathcal{E}_{\Omega_i}$ is the set of faces of $\mathcal{T}_i$. We shall use the following notations: Let ϵ_i be a face of $\mathcal{E}_{\Omega_i}$ located on the boundary of Ω_i, $K(\epsilon_i)$ denotes the control cell $K \in \mathcal{T}_i$ such that $\epsilon_i \in K$, $\mathcal{E}_i$ is the set of faces of domain Ω_i located on the interface, $\mathcal{E}(K)$ is the set of faces of $K \in \mathcal{T}_i$, $\mathcal{E}_i(K)$ is the set of faces of $K \in \mathcal{T}_i$ which are on the interface,

$\mathcal{N}_i(K) = \{K' \in \mathcal{T}_i : K \cap K' \in \mathcal{E}_{\Omega_i}\}$ is the set of the control cells adjacent to K and $[K, K']$ denotes the face $K \cap K'$.
We introduce p^i_K an approximation of $p(x_K)$ (where x_K is a point inside the control cell K), p^i_ϵ an approximation of $p(y_\epsilon)$ (where y_ϵ is the center of the face $\epsilon \in \mathcal{E}_i$) and u^i_ϵ an approximation of the flux $\frac{\partial p_i}{\partial n_i}(y_\epsilon)$ outward Ω_i through $\mathcal{E}_i$. Then, not taking into account the Dirichlet boundary condition, a finite volume scheme for (1) can be defined by the set of equations Eymard et al. [2000].

$$\eta p^i_K m(K) - \sum_{K' \in \mathcal{N}_i(K)} \frac{p^i_{K'} - p^i_K}{d(x_{K'}, x_K)} m([K, K']) - \sum_{\epsilon \in \mathcal{E}_i(K)} u^i_\epsilon m(\epsilon) = F^i_K \quad (3)$$

$$\text{with } u^i_\epsilon = \frac{p^i_\epsilon - p^i_K}{d(y_\epsilon, x_K)} \text{ for } \epsilon \in \mathcal{E}_i \quad (4)$$

for all control cells K of $\mathcal{T}_i$ and where $m(A)$ is the measure of $A \subset \Omega$. Discretized Robin interface conditions or more general interface conditions on $\mathcal{E}_i$ are introduced in the next section.

3 Transmission operators

We introduce the operators $Q_i : P^0(\mathcal{E}_j) \mapsto P^0(\mathcal{E}_i)$ $(i, j = 1, 2\ i \neq j)$ where $P^0(\mathcal{E}_i)$ is the space of piecewise constant functions on $\mathcal{E}_i$.
Assumption 1 Operators Q_1 and Q_2 are transposed of each other for the standard L^2 scalar product.

Method Constant The first type of transmission operators that we consider are the restrictions on $P^0(\mathcal{E}_j)$ of P^c_i the L^2 orthogonal projection onto $P^0(\mathcal{E}_i)$. They satisfy Assumption 1.

Method Linear The second type of transmission operators uses a linear rebuilding to ensure a more accurate transmission than P^c_i. We introduce for $i = 1, 2$

- the interface grid: $\mathcal{E}^2_i$ coarsening by a factor 2 of $\mathcal{E}_i$
- $P^1_d(\mathcal{E}^2_i)$ discontinuous piecewise linear functions on $\mathcal{E}^2_i$.
- interpolation operator $I_i : P^0(\mathcal{E}_i) \longmapsto P^1_d(\mathcal{E}^2_i)$ and its transpose I^t_i (w.r.t. the scalar product $L^2(\Gamma)$, $\forall u \in P^0(\epsilon_i)$ and $\forall v \in P^1(\epsilon^2_i)$ $< I_i(u), v >_{L^2(\Gamma)} = < u, I^t_i(v) >_{L^2(\Gamma)}$).
- P^L_i L^2 orthogonal projection on $P^1_d(\mathcal{E}^2_i)$

The definitions of the transmission operators are inspired by previous works Arbogast et al. [1996] in mixed finite element method:

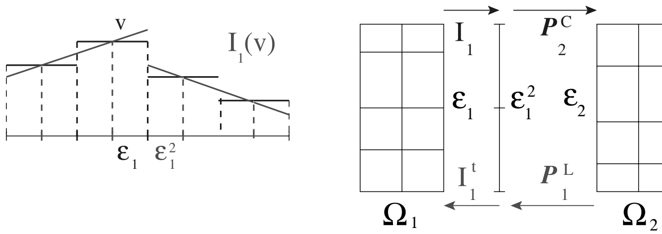

Fig. 1. Linear rebuilding and transmission operators

$$Q_1 = I_1^t P_1^L \tag{5}$$
$$Q_2 = P_2^C I_1$$

They satisfy Assumption 1 but are not projections.

4 Interface Conditions

In analogy with Bernardi et al. [1994], transmission operators are used to write discrete matching conditions ensuring continuity of the solution and of its normal derivative on the interface:

$$p_2 = Q_2(p_1) \text{ on } \mathcal{E}_2 \text{ and } u_1 = Q_1(-u_2) \text{ on } \mathcal{E}_1 \tag{6}$$

where $p_i \in P^0(\mathcal{E}_i)$ is the approximate pressure on $\mathcal{E}_i$ and $u_i \in P^0(\mathcal{E}_i)$ is the approximate flux outward Ω_i on $\mathcal{E}_i$ ($p_i = (p_i^\epsilon)_{\epsilon\in\mathcal{E}_i}$ and $u_i = (u_i^\epsilon)_{\epsilon\in\mathcal{E}_i}$). In mortar terminology Bernardi et al. [1994], domain Ω_1 is called the master because it imposes the pressure and Ω_2 is called the slave.
These matching conditions are made compatible with arbitrary interface conditions defined via operators $S_i : P^0(\mathcal{E}_i) \longmapsto P^0(\mathcal{E}_i)$ which satisfy
Assumption 2 S_i is positive definite
The corresponding interface conditions read:

$$Q_1(S_2(Q_2(p_1))) + u_1 = Q_1(S_2(p_2) - u_2) \tag{7}$$
$$p_2 + Q_2(S_1^{-1}(Q_1(u_2))) = Q_2(p_1 - S_1^{-1}(u_1)) \tag{8}$$

Examples of interface conditions are:

- Discrete Steklov-Poincaré operator ($S_i = (DtN_i)_h$)
- Robin interface conditions de $S_i = diag(\alpha_\epsilon^i)$, $S_i = diag(\alpha_{opt}^i)$
- optimized of order 1 or 2 (S_i tridiagonal)

Lemma 1. *Under Assumptions 1 and 2, mortar matching conditions (6) and arbitrary interface conditions (7)-(8) are equivalent.*

5 Error Estimates

It is proved in Saas et al. [2002] that under Assumptions 1 and 2,

- the global problem defined by the set of equations (3)-(4)-(6) is well-posed and stable.
- the local problem defined in Ω_1 by the set of equations (3)-(4)-(7) and the local problem defined in Ω_2 by the set of equations (3)-(4)-(8) are both well-posed and stable.

Under Assumptions 1 and 2 and additional assumptions on the mesh:
Assumption 3 $\exists C > 0$ such that $\forall \epsilon \in \mathcal{E}_i$, $diam(\epsilon) \leq Cd(x_{K(\epsilon)}, y_\epsilon)^{1/2}$
error estimates can be derived

Theorem 1. *The H^1 discrete norm of the error is in $O(h^{1/2})$ when the piecewise constant projections are used ($Q_i = P_i^c$).*
The H^1 discrete norm of the error is in $O(h)$ when the linear rebuilding (5) is used.

6 Numerical Results in the homogeneous case

Numerical tests have been done with the equation in four subdomains:

$$p - \Delta p = x^3y^2 - 6x^2y^2 - 2x^3 + (1 + x^2 + y^2)sin(xy) \text{ in } \Omega$$
$$p = p_0 \text{ on } \partial\Omega$$

This results have been compared to the analytical solution which is $p(x, y) = x^3y^2 + sin(xy)$. The domain decomposition method is reformulated with a substructuring method and is solved with a GMRES algorithm. For asymptotic study, we use an initial non conforming mesh which we refine successively by a factor 2. We compare different methods TPFA (Two point flux approximation, see Cautrés et al. [2000]), Ceres (like TPFA but a linear interpolation is performed in order to have a consistent flux approximation on the interface, see Faille et al. [1994]), New Cement (Achdou et al. [2002]), Constant and Linear (Section 3). For all these methods, we take different values for $S_i = diag(\alpha)$ with $\alpha = 1$ or $\alpha = 1/h$ or $\alpha = \alpha_{opt} = 1/h^{1/2}$. The numerical solution depends on the choice of S_i only for the New Cement method. Accuracy is given in figure 2 and iteration counts of the GMRES algorithm in figure 3.

7 Numerical Results in the heterogeneous case

We consider now the problem with discontinuous coefficients

$$\eta p - div(\kappa \nabla p) = f \text{ in } \Omega \text{ and } p = 0 \text{ on } \partial\Omega \tag{9}$$

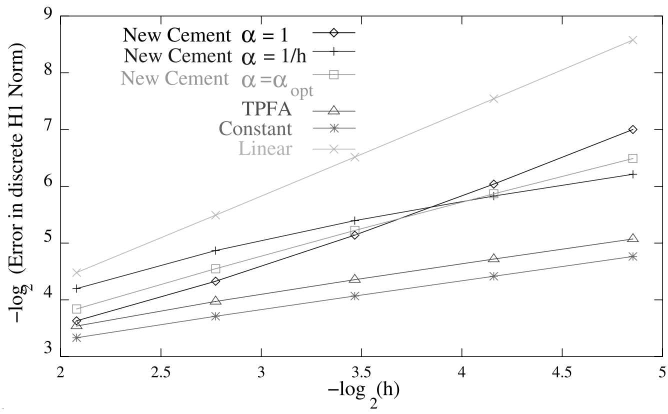

Fig. 2. Slopes: Linear $\simeq 1.3$; New Cement: ($S_i = cte$: $\simeq 1.3$), ($S_i = 1/h^{1/2}$: $\simeq 0.9$), ($S_i = 1/h$: $\simeq 0.6$), Constant: $\simeq 0.5$, TPFA: $\simeq 0.5$

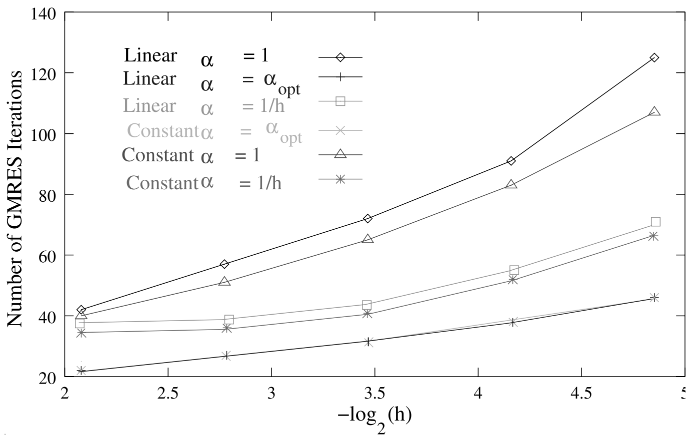

Fig. 3. Iteration counts for the GMRES algorithm

where $\eta > 0$ and κ are highly discontinuous, typically two or three orders of magnitude, see figure 4. For Test 2 for instance, with a very coarse grid methods Constant and Linear (see Section 3) work very poorly especially compared to TPFA and Ceres methods. Typically we have the following relative errors: Linear: 60%, Constant 10%, TPFA 3.2% and Ceres 2%. The errors are computed thanks to a computation on a very fine mesh since we don't have analytic solutions in these cases. Poor results for methods Linear and Constant are due to the fact that the flux on the interface is a very discontinuous function whose jumps are located on the jumps of the coefficients on both blocks. In Ceres and TPFA, a subgrid containing all locations of the jumps of the coefficients on the interface is involved which is not the case for methods Constant and Linear. In order to remedy this situation, a very thin

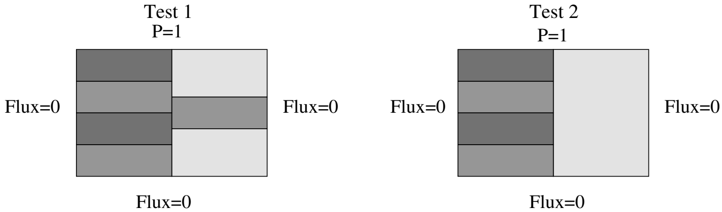

Fig. 4. Heterogeneous media

third subdomain is introduced between the blocks. The mesh of this additional subdomain along the interface is the intersection of the grid interfaces of the neighboring blocks, see figure 5. Methods Constant and Linear are then applied to this three subdomains case. The improvement is dramatic. The relative errors are then: Linear: 1.6%, Constant 1.6% (compared to respectively 60% and 10% in the two-subdomain case).

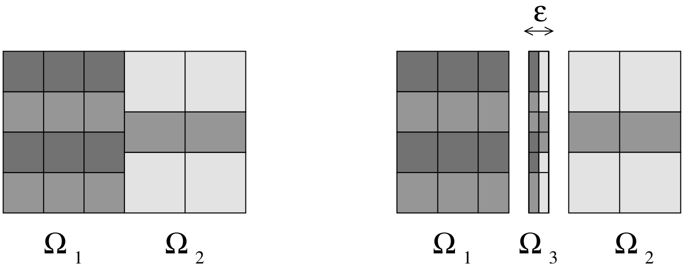

Fig. 5. Addition of a third subdomain

8 Conclusion

We have introduced matching operators to take into account the non-matching grids. Under compatibility assumptions, we have the well-posedness of the global problem and of the local subproblems with a new discretization of the arbitrary interface conditions. We give two error estimates in the discrete H^1 norm: the first one is in $O(h^{1/2})$ with L^2 orthogonal projections onto piecewise functions along the interface and the second one in $O(h)$ with transmission conditions based on a linear rebuilding along the interface. The error estimates depend only on the transmission operators, see Section 3. But, the numerical solutions are independent of the interface conditions whose discretizations are given by (7)-(8). Particular attention was paid to the situation with non matching grids and highly heterogeneous coefficients both across and inside subdomains. The addition of a third very thin subdomain between geological blocks is necessary to ensure a good accuracy. Extension to a finite element discretization would be interesting.

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