HIRIART-URRUTY Jean-Baptiste
Département de Mathématiques Appliquées
Université de Clermont
Boîte Postale n° 45
63170 AUBIERE

ABOUT PROPERTIES OF THE MEAN VALUE FUNCTIONAL AND OF THE CONTINUOUS INFIMAL CONVOLUTION IN STOCHASTIC CONVEX ANALYSIS

Abstract : In stochastic convex programming numerous examples are to be found where the cost functional to be minimized is of the form of a mean value functional $Ef(x) = \int_\Omega f(x, \omega)\, dP(\omega)$ where $x \in \mathbb{R}^n$ and ω is an uncertain quantity-element of a probability space. The problem of minimizing Ef is a deterministic problem related to the stochastic convex program. To be able to apply the methods of convex optimization and the theorems of convex analysis, it is important to know the properties of Ef, both topological and algebraic. The aim of this paper is to determine the main properties and characteristics of the mean value functional Ef resulting from these corresponding to the functions $f(., \omega)$. By the conjugacy operation, the mean value functional is closely related to the continuous infimal convolution of which we shall also give some properties. Finally the different results obtained are applied to stochastic optimization problems.

I - Notations and terminology

We shall adopt throughout the notations and terminology of P.J. Laurent ([3]) for convex analysis and those of R.T. Rockafellar ([7]) for measurable multivalued mappings.Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, a.s. meaning "almost surely". $\mathcal{F}(\mathbb{R}^n \times \Omega, \overline{\mathbb{R}})$ will denote the set of functions defined on $\mathbb{R}^n \times \Omega$ and taking their values in $\overline{\mathbb{R}}$. In what follows C will usually designate a measurable multivalued mapping defined on Ω. We recall that if $C(\omega)$ is a nonempty closed convex set containing no whole lines, the measurability of C is equivalent to the mesurability of the support functions $\chi^*_{C(.)}(x')$ for every x' ([7] Corollary 3.2). We shall say that C is a CK-valued mapping if $C(\omega)$ is a.s. a nonempty compact convex set of $\mathbb{R}^n$.

As for the integrability of a measurable multivalued mapping, we shall recall the following fundamental definition and properties.

I.1. Definition : Let C be a measurable multivalued mapping defined on Ω. By *mathematical expectation* of C, we shall mean the set (it may be empty) denoted E (C) and defined by :

$$E(C) = \{E(X) \,/\, X \text{ integrable selector of } C\}$$

where an integrable selector of C is an integrable function $X : \Omega \to \mathbb{R}^n$ such that $X(\omega) \in C(\omega)$ a.s.

I.2. Definition : Let C be a CK-valued mapping ; it is said that C is P-integrable if and only if C is measurable and the random variable $||C||$ is integrable, where $||C||$ is defined by :

$$\forall\, \omega \in \Omega \qquad ||C||\,(\omega) = ||C(\omega)|| = \text{Sup}\,\{||x|| \,/\, x \in C(\omega)\}$$

I.3. Fundamental property ([11])

If C is a CK-valued mapping, P-integrable, then E (C) is a nonempty compact convex set characterized by the support functions as follows :

$$(1) \qquad \forall\, x \qquad \chi^*_{E(C)}(x) = E\,\{\chi^*_{C(\omega)}(x)\}$$

In this case, the integrability of C is equivalent to the integrability of the functions $\chi^*_{C(.)}(x)$ for every x.

Remarks : I.3.1. If C is a measurable multivalued mapping such that C (ω) is a.s. a closed convex set , then C is said quasi P-integrable if there exists an integrable selector of C. In this case, $\chi^*_{C(.)}(x)$ is a quasi-integrable function and the relation (1) remains true ([12]).

I.3.2. In a wider sense, if C is such that : $\exists\, A,\ P(A) > 0$ $\omega \in A \Longrightarrow C(\omega) = \emptyset$, then E (C) is empty.

I.3.3. If C is a CK-valued mapping, P-integrable, we have the inequality between norms :

$$||E(C)|| \leqslant E(||C||)$$

Our purpose in this paper is to examine the properties of the mean value functional and of the continuous infimal convolution. Let us recall here their definitions as well as those of integrands.

I.4. Definitions ([7]). By integrand we mean a function $f : \mathbb{R}^n \times \Omega \to \mathbb{R} \cup \{+\infty\}$. If $f(., \omega)$ is proper, l.s.c. for every ω and f is $\mathcal{B}_n \otimes \mathcal{A}$-measurable where $\mathcal{B}_n$ denotes the σ-algebra of borelian subsets of $\mathbb{R}^n$, f is said to be a normal integrand. In a wider sense, by normal integrand, we shall also mean f such that $f(., \omega) \in \Gamma_o(\mathbb{R}^n)$ almost surely. If, moreover, $f(., \omega)$ is convex for every ω f is called convex.

Normality ensures in particular that for every random variable X, the function $\omega \to f(X(\omega), \omega)$ is measurable.

I.5. Mean value functional : Let $f \in \mathcal{F}(\mathbb{R}^n \times \Omega, \overline{\mathbb{R}})$ be such that, for every x, $f(x, .)$ is measurable. The mean value functional Ef of the collection $\{f(., \omega)\}_{\omega \in \Omega}$ is defined by :

$$\forall x \qquad Ef(x) = \int_\Omega f(x, \omega)^+ \, dP(\omega) \dot{+} \int_\Omega f(x, \omega)^- dP(\omega)$$

Of course, for any measurable function $g : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ such that, if we denote $\Omega_\infty(f) = \{\omega \mid f(\omega) = +\infty\}$, we have $P(\Omega_\infty(f)) > 0$, then $Ef = +\infty$.

I.6. Continuous infimal convolution ([2], [12]). Let f be a normal integrand on $\mathbb{R}^n \times \Omega$. By continuous infimal convolution of the family $\{f(., \omega)\}_{\omega \in \Omega}$ relating to the probability measure P , we mean the functional denoted by $F = \oint_\Omega f(., \omega) \, dP(\omega)$ and defined by :

$$\forall x \qquad F(x) = \operatorname*{Inf}_{X \in \mathcal{L}^1(x)} \left\{ \int_\Omega f(X(\omega), \omega)^+ \, dP(\omega) \dot{+} \int_\Omega f(X(\omega), \omega)^- \, dP(\omega) \right\}$$

where $\mathcal{L}^1(x) = \{X \in \mathcal{L}^1 \,/\, E(X) = x\}$.

II - Properties of the mean value functional Ef.

A certain class of stochastic optimization problems are characterized by a cost functional of the form Ef (x). It is therefore interesting to know how the properties of the functions f (., ω) are transmitted to the function Ef. We shall examine such topological properties as l.s.c. continuity and determine such convex characteristics as recession function, θ-subdifferential ...

II.1. L.s.c. continuity of Ef.

II.1. Theorem : *Let* $f \in \mathcal{F}(\mathbb{R}^n \times \Omega, \overline{\mathbb{R}})$ *be such that :* $f(x, .)$ *is measurable for every* x *and a.s.* $f(., \ )$ *is l.s.c. If* $f^*(0, .)^+$ *is integrable, then* Ef *is a l.s.c. function taking its values in* $\mathbb{R} \cup \{+\infty\}$.

Proof : The inequality $f(x, .) \geqslant - f^*(0, .)$ implies that for every x $f(x, .)$ is quasi-integrable and $\forall x$ $Ef(x) > -\infty$. Suppose that Ef is a proper function (if Ef is identically $+\infty$, Ef is l.s.c).

Let $\{x_n\}_{n \in \mathbb{N}}$ a sequence converging to x ; a.s. f (., ω) is l.s.c., then :

$$\text{a.s.} \qquad f(x, \omega) \leqslant \varliminf_{n\to\infty} f(x_n, \omega)$$

According to Fatou's lemma ([5]), $E(\varliminf_{n\to\infty} f(x_n, \omega)) \leqslant \varliminf_{n\to\infty} E f(x_n, \omega)$

$$\Longrightarrow \quad EF(x) \leqslant \varliminf_{n\to\infty} Ef(x_n, \omega).$$ Hence the l.s.c. of Ef

II.2. Recession function of Ef.

II.2.1. Theorem : *Let* $f \in \mathcal{F}(\mathbb{R}^n \times \Omega, \overline{\mathbb{R}})$ *be such that :* $f(x, .)$ *is measurable for every* x *and a.s.* $f(., \omega) \in \Gamma_0(\mathbb{R}^n)$*. If moreover* Ef *is l.s.c. and proper, then the recession function* $(Ef)_\infty$ *of* Ef *is given by :*

$$(Ef)_\infty = E(f_\infty).$$

Proof : Ef being proper, there exists x_0 such that $f(x_0, .)$ is integrable : thus

a.s. $x_0 \in \text{dom } f(., \omega)$

a.s. $f(., \omega) \in \Gamma_0(\mathbb{R}^n)$ and the recession function $f_\infty(., \omega)$ is given by the formula : ([3] Proposition 6.8.3).

$$\forall x \qquad f_\infty(x,\omega) = \sup_{\lambda>0} \frac{f(x_o+\lambda x,\omega) - f(x_o,\omega)}{\lambda} = \lim_{\lambda\to\infty} \frac{f(x_o+\lambda x,\omega) - f(x_o,\omega)}{\lambda}$$

This formula is only valid for functions of $\Gamma_o(\mathbb{R}^n)$. Ef is supposed proper and l.s.c., it is also convex. Like previously

$$\forall x \qquad (Ef)_\infty(x) = \lim_{\lambda\to\infty} \frac{Ef(x_o+\lambda x) - Ef(x_o)}{\lambda}$$

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ an increasing sequence of real numbers such that : $\lambda_n > 0$ and $\lim_{n\to\infty} \lambda_n = +\infty$. We take :

$$f_n(x,\omega) = \frac{f(x_o+\lambda_n x,\omega) - f(x_o,\omega)}{\lambda_n}$$

$\{f_n(x,.)\}_{n\in\mathbb{N}}$ is an increasing sequence of quasi-integrable functions ($f_n(x,.)^-$ being integrable) and $\lim_{n\to\infty} f_n(x,.) = f_\infty(x,.)$

It follows that $f_\infty(x,.)$ is quasi-integrable and according to the Beppo-Levi monotone convergence theorem, $Ef_n \to E(f_\infty)$.

II.2.2. Remarks

a) Theorem II.1 gives conditions for Ef being l.s.c.

b) The l.s.c. of Ef was proved in the convex case in [14]. Moreover, R.J.B. Wets ([15]) has proved the inf-compactness of Ef when f (., ω) is inf-compact. Using the result of theorem II.2.1 and making a proof similar to that of Wets, we prove more generally that : *if $f(.,\omega)$ is a.s. inf-compact for a slope $X_o(\omega)$ where X_o is an integrable random variable, then Ef is inf-compact for the slope $E(X_o)$.*

II.3. Conjugate of Ef

The computation of the conjugate of Ef will show us the connection between the mean value functional and the continuous infimal convolution. This will allow us to deduce the properties of one through the properties of the other.

II.3.1. Theorem : *Let f be a convex integrand such that $f(x,.)$ is integrable for every x. Then*

$$(Ef)^* = \oint_\Omega f^*(.,\omega)\, dP(\omega)$$

Proof : We shall use mainly the theorem given by Valadier ([12] Theorem 7) determing the conjugate of the continuous infimal convolution.
"Let f be a normal convex integrand such that, for every x' , $f^*(x', .)$ is integrable. Then the continuous infimal convolution $F = \oint_\Omega f(., \omega)\, dP(\omega)$ is exact (i.e. $\forall x,\ \exists X \in \mathcal{L}^1$ such that $E(X) = x$ and $F(x) = \int_\Omega f(X(\omega), \omega)\, dP(\omega)$).
Moreover $F \in \Gamma_0(\mathbb{R}^n)$ and the conjugate F^* is given by : $F^*(x') = \int_\Omega f^*(x',\omega)\, dP(\omega)$."

We apply this theorem to the normal convex integrand f^* (it will be shown in the proof of theorem II.4.2. that f^* is a normal convex integrand). We have $f(., \omega) = f^{**}(., \omega)$ and $(Ef)^{**} = Ef$. The conjugate of $\oint_\Omega f^*(., \omega)\, dP(\omega)$ is Ef ; hence the result.

II.3.2. Remarks : Let f be anormal convex integrand such that $f^*(x', .)$ is integrable for every x'. Then the continuous infimal convolution F of the family $\{f(., \omega)\}_{\omega \in \Omega}$ is a *co-finite* convex function ([8] p. 116) and $F_\infty = \chi_{\{0\}}$.

Moreover, it is deduced an interesting result of continuity of F : *let X_0 be an integrable random variable, if $f(., \omega)$ is finite and continuous at $X_0(\omega)$ a.s., then F is finite and continuous at $E(X_0)$.* This arises from the relation between continuity of f and inf-compactness of f^* ([3] theorem 6.3.9) and from Remark II.2.2. If f takes particular forms, specially with random matrix, the equality of theorem II.3.1 implies interesting results.

II.3.3. Proposition : *Let f be a normal convex integrand. We suppose that $Ef \in \Gamma_0(\mathbb{R}^n)$ and that $f_\infty(x, .)$ is integrable for every x. Then, the multivalued mapping $\omega \to \overline{dom}\, f^*(., \omega)$ is a CK-valued mapping, P-integrable and :*

$$E\{\overline{dom}\, f^*(., \omega)\} = \overline{dom}\ \oint_\Omega f^*(., \omega)\, dP(\omega)$$

Proof : Let $x_0 \in$ dom Ef ; $x_0 \in$ dom $f(., \omega)$ a.s. We have :

$$\forall x,\quad \forall z \qquad f(z, \omega) \leq f(x, \omega) + f_\infty(z-x, \omega) \quad ([8] \text{ Corollary } 8.5.1)$$

This implies that $f(x, .)$ is integrable for every x. Moreover, $f(., \omega) \in \Gamma_0(\mathbb{R}^n)$ and $f_\infty(., \omega)$ is a function of $\Gamma_0(\mathbb{R}^n)$ such that :

$$f_\infty (x, \omega) = \chi^*_{\overline{\mathrm{dom}}\, f^*(.,\omega)} (x) \qquad ([3] \text{ theorem 6.8.5})$$

Likewise : $(Ef)_\infty(x) = \chi^*_{\overline{\mathrm{dom}}(Ef)^*} (x)$. f^* is a normal convex integrand ([6] lemma 5) and the multivalued mapping $\omega \to$ epi f^* (., ω) is measurable ([7] theorem 4) ; hence the mesurability of the mapping $\omega \to$ dom $f^*(.,\omega) = \mathrm{proj}_{\mathbb{R}^n}$ (epi f^* (., ω)). The multivalued mapping $\omega \to \overline{\mathrm{dom}}\, f^*(., \omega)$ is a CK-valued mapping, P-integrable because the support function $f_\infty(x, .)$ is integrable for every x. Then,

$$\chi^*_{E\{\overline{\mathrm{dom}}\, f^*(.,\omega)\}} = E\,(\chi^*_{\overline{\mathrm{dom}}\, f^*(.,\omega)})$$

It arises from the equality $(Ef)_\infty = E(f_\infty)$ that $\overline{\mathrm{dom}}\,(Ef)^* = E\{\overline{\mathrm{dom}}\, f^*(., \omega)\}$ and the result is deduced from the expression $(Ef)^* = ⨏_\Omega f^*(., \omega)\, dP(\omega)$.

II.3.4 <u>Remark</u> : Under the assumptions of the previous theorem, f (., ω) is Lipchitzian with coefficient α (ω) and α is an integrable random variable. Likewise, Ef is Lipchitzian with coefficient A and we have : $A \leq E(\alpha)$.

Indeed α (ω) is given by $\|\overline{\mathrm{dom}}\, f^*(., \omega)\| = \mathrm{Sup}\, \{\|x^*\| \,/\, x^* \in \mathrm{dom}\, f^*(., \omega)\}$ ([8] Corollary 13.3.3). Similarly $A = \|\overline{\mathrm{dom}}\,(Ef)^*\| = \|E\{\overline{\mathrm{dom}}\, f^*(., \omega)\}\|$
We have seen that $\|E\{\mathrm{dom}\, f^*(., \omega)\}\| \leq E(\|\overline{\mathrm{dom}}\, f^*(., \omega)\|)$ (Remark I.3.3) ; hence the result.

II.4. Determination of the θ-subdifferential of Ef

Convex analysis ([8]) and algorithms for minimizing convex functions ([4]) have brought out the importance of the θ-subdifferential of a convex function ; in this paragraph, we intend to determine the θ-subdifferential of the mean value functional Ef.

II.4.1. <u>Definition</u> : Let $f \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$ be a functional finite at x_0. A vector x^* is called an θ-subgradient of f et x_0 (where $\theta \geq 0$) if

$$\forall x \qquad f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle - \theta$$

The set of all θ-subgradients is denoted by $\partial_\theta f(x_0)$ and is called θ-subdifferential of f at x_0.

II.4.2. Theorem : *If f is a convex integrand on $\mathbb{R}^n \times \Omega$ such that $f(x, .)$ is measurable for every x and such that*

$$(H) \qquad \forall x \quad \int_\Omega |f(x, \omega)|\, dP(\omega) < +\infty$$

Then $$\forall \theta \geqslant 0 \qquad \partial_\theta Ef(x) = \bigcup_{\substack{\theta \in \mathcal{L}^1(\theta) \\ \theta(\omega) \geqslant 0}} E \{\partial_{\theta(\omega)} f(x, \omega)\}$$

$$\text{where } \mathcal{L}^1(\theta) = \{\theta \in \mathcal{L}^1 \,/\, E(\theta) = \theta\}$$

Proof : Let us take a sequence $\{z_p\}$ satisfying the following condition : for every x, x is the barycenter of a finite number of z_p (take $z_p \in \mathbb{Z}^n$ for example) ; $f(., \omega)$ being convex functions, we can deduce that : $\exists\, N \in \mathcal{A}$, $P(N) = 0$ such that : $\forall \omega \notin N$, $f(x, \omega) \in \mathbb{R}$ $\forall x$ ([12] proof of the lemma 6). Thus there are no major drawbacks in supposing f real-valued. Let $\theta \in \mathcal{L}^1(\theta)$, $\theta(\omega) \geqslant 0$. We have :

$$\partial_{\theta(\omega)} f(x, \omega) = \{x^* \,/\, f^*(x^*, \omega) - \langle x, x^* \rangle \leqslant \theta(\omega) - f(x, \omega)\}$$

f is a normal convex integrand ([8]), f^* is also one ([8] lemma 5), as is g^* defined by : $g^*(x^*, \omega) = f^*(x^*, \omega) - \langle x, x^* \rangle$. Moreover, the function $\omega \to \theta(\omega) - f(x, \omega)$ is measurable. Consequently, for every x, the multivalued mapping $\omega \to \partial_{\theta(\omega)} f(x, \omega)$ is measurable ([7] Corollary 4.3).

Let X^* be a measurable selection of this multivalued mapping. By definition, we have :

$$\forall y \qquad f(x+y, \omega) \geqslant f(x, \omega) + \langle X^*(\omega), y \rangle - \theta(\omega)$$

$$f(x-y, \omega) \geqslant f(x, \omega) - \langle X^*(\omega), y \rangle - \theta(\omega)$$

$$\Longrightarrow \forall y \qquad -f(x-y, \omega) + f(x, \omega) - \theta(\omega) \leqslant \langle X^*(\omega), y \rangle$$

$$\leqslant f(x+y, \omega) - f(x, \omega) + \theta(\omega)$$

Then, for every y, $\langle X^*(.), y \rangle$ is an integrable function. Every selection X^* of the multivalued mapping being integrable, the multivalued mapping $\omega \to \partial_{\theta(\omega)} f(x, \omega)$ is P-integrable.

By definition of the θ-subdifferential, we have :

$$\forall y \qquad f(y, \omega) \geqslant f(x, \omega) + \langle X^*(\omega), y-x \rangle - \theta(\omega)$$

$$\Longrightarrow \forall y \qquad Ef(y) \geqslant Ef(x) + \langle E(X^*), y-x \rangle - \theta$$

$$\Longrightarrow \qquad E(X^*) \in \partial_\theta Ef(x)$$

Therefore, according to the definition of E $\{\partial_{\theta(\omega)} f(x, \omega)\}$, we deduce that :

$$E \{\partial_{\theta(\omega)} f(x, \omega)\} \subset \partial_{\theta} Ef(x)$$

$$\Longrightarrow \bigcup_{\substack{\theta \in \mathcal{L}^1(\theta) \\ \theta(\omega) \geq 0}} E\{ \partial_{\theta(\omega)} f(x, \omega)\} \subset \partial_{\theta} Ef(x)$$

Conversely, let $x^* \in \partial_{\theta} Ef(x)$. We can also write :

$$\partial_{\theta} Ef(x) = \{x^* / (Ef)^* (x^*) + Ef(x) - < x, x^* > \leq \theta\}$$

From the theorem II.3.1, there exists $X^* \in \mathcal{L}^1$ such that :

$$E(X^*) = x^* \text{ and } (Ef)^* (x^*) = \int_{\Omega} f^* (X^* (\omega), \omega) \, dP(\omega)$$

Let $\theta(\omega) = f^* (X^* (\omega), \omega) + f(x, \omega) - < x, X^* (\omega) >$. Obviously we have :

$$\theta(\omega) \geq 0, \quad \theta \in \mathcal{L}^1 , E(\theta) \leq \theta \quad \text{ and } X^* (\omega) \in \partial_{\theta(\omega)} f(x, \omega)$$

By definition of the expectation, $x^* \in E \{ \partial_{\theta(\omega)} f(x, \omega)\}$; hence the result.

II.4.3. <u>Remark</u> : in the previous theorem, making $\theta = 0$, we obtain the formula for $\partial_0 Ef = \partial Ef$.

Then the formula of theorem II.4.2 becomes :

$$\partial(Ef) = E(\partial f) \qquad \text{(denoted symbolically)}$$

Thus we again find the known formula of the subdifferential of the mean value functional ([1] ; [9] p. 62)

II.4.4. <u>Application</u> : <u>extremums of the expectation E (C)</u>

For a convex compact K, let K' be the extremum in the direction x^*, that is to say:

$$K'_{x^*} = \{x \in K / < x, x^* > = \text{Sup} (< z, x^* > / z \in K)\} \qquad ([13] \text{ p. } 10)$$

Let us consider a CK-valued mapping, P integrable. For each x^* , we denote by C'_{x^*} the multivalued mapping defined as following : $\forall \omega \in \Omega$

$$C'_{x^*} (\omega) = [C(\omega)]'_{x^*} .$$ C'_{x^*} is also a CK-valued mapping.

II.4.4. <u>Proposition</u> : For each x^*, the multivalued mapping C'_{x^*} is P-integrable and we have : $E(C'_{x^*}) = [E(C)]'_{x^*}$; that is to say : *the extremum of the expectation in the direction x^* is the expectation of the extremum in the same direction.*

Proof : We have the following equivalence ([8] Corollary 23.5.3) : for a non-empty closed convex set K, $K'_{x^*} = \partial\, \chi^*_K \;(x^*)$

Then, $C'_{x^*}(\omega) = \partial \chi^*_{C(\omega)}(x^*)$ and $[E\,(C)]'_{x^*} = \partial \chi^*_{E(C)}(x^*)$

From the fundamental property I.3 and the formula II.4.3., it is deduced that :

$$E\,(C'_{x^*}) = [E\,(C)]'_{x^*}$$

II.4.5. Remark : for a non-empty closed convex set C, it is said that $x \in C$ is "exposed in the direction x^*" if χ^*_C is differentiable at x^* and if $\partial\, \chi^*_C(x^*) = \{x\}$. It follows from the previous proposition that :

x is an exposed point of E (C) in the direction x if and only if x = E (X) where X (ω) is a.s. an exposed point of C (ω) in the direction x*.*

II.5. θ-directional derivative of Ef

In minimization methods, when using certain methods of descent, we replace the directional derivative $f'(x\,;\,d)$ by an approximation $f'_\theta\,(x\,;\,d)$ which is the θ-directional derivative. We consider here the functional Ef and an explicit characterization is given for the θ-directional derivative of Ef.

II.5.1. Definition : Let $f \in \Gamma_o(\mathbb{R}^n)$, finite at x and $\theta \geq 0$. The θ-directional derivative of f at x with respect to a vector d is defined by :

$$f'_\theta\,(x\,;\,d) = \text{Sup}\,\{< c,\, d > /\; c \in \partial_\theta\, f(x)\}$$

II.5.2. Theorem : *Under the assumption of theorem II.4.2., we have :*

$$\forall \theta \geq 0 \qquad (Ef)'_\theta\,(x\,;\,d) = \sup_{\substack{\theta \in \mathcal{L}^1(\theta) \\ \theta(\omega) \geq 0}} E\,\{f'_{\theta(\omega)}\,(x\,;\,d,\,\omega)\}$$

Proof : According to the previous definition, for any $\theta \in \mathcal{L}^1(\theta)$, we write :

$$(Ef)'_\theta\,(x\,;\,d) = \chi^*_{\partial_\theta Ef(x)} \quad \text{and} \quad f'_{\theta(\omega)}(x\,;\,d,\,\omega) = \chi^*_{\partial_{\theta(\omega)} f(x,\omega)}\,(d)$$

From the fundamental property I.3 ;

$$\chi^*_{E\{\partial_{\theta(\omega)} f(x,\omega)\}} = E\{\chi^*_{\partial_{\theta(\omega)}} f(x,\omega)\}$$

The functional Ef is finite and continuous at x ; ∂_θ Ef (x) is a nonempty compact convex set characterized by the formula of Theorem II.4.2. Then, according to the lemma 16.5.1 of [8], it is deduced that :

$$\chi^*_{\partial_\theta} Ef(x) = \sup_{\substack{\theta \in \mathcal{L}^1(\theta) \\ \theta(\omega) \geq 0}} E\{\chi^*_{\partial_{\theta(\omega)}} f(x, \omega)\}$$

II.5.3. Remark : Like in the case of the previous theorem ; if we take $\theta = 0$, the directional derivative of Ef at x with respect to the vector d can be expressed as following :

$$(Ef)'(x ; d) = E\{f'(x ; d, \omega)\}$$

In the present case, f (., ω) being a.s. finite and continuous at x, there is no discrepancy between f' (x ; d) as usually defined and f'_θ (x ; d) for $\theta = 0$. Likewise Ef is finite and continuous at x ; so, $(Ef)'(x ; d) = \chi^*_{\partial Ef(x)}$.

II.5.4. Necessary and sufficient optimality condition for a class of stochastic optimization problems

Let us consider stochastic programming models consisting of two-stage formulations ([14]). A first stage problem is that in which an optimization problem is performed without having the prior knowledge of the random outcomes.

After the random outcomes have been observed, the inaccuracies occured are compensated in another optimization problem : second stage program. Generally, the functional to be minimized is of the form Ef, x may be subject to certain constraints :

$$\begin{cases} x \in C \\ f_i(x) \leq 0 \quad i = 1, \ldots, m \end{cases}$$ where C is a convex set and f_i are convex

This deterministic program can be written as :

($\mathcal{P}$) Find $\bar{x}$ such that : $\bar{x} \in Q$ and $Ef(\bar{x}) = \operatorname{Inf}\{Ef(x) / x \in Q\}$.

II.5.4. Proposition : Let f a convex integrand a $\mathbb{R}^n \times \Omega$ such that f (x, .) is integrable for every x, Q a convex set. Then a necessary and sufficient condition for $\bar{x}$ being a solution of $(\mathcal{P})$ is that :

$$\forall x \in Q \qquad E\{f'(\bar{x}\ ;\ x-\bar{x},\ \omega)\} \geq 0$$

Proof : This arises from the usual optimality conditions in convex programming and from the remark II.5.3. concerning the directional derivative of Ef.

III - Properties of the continuous infimal convolution

The continuous infimal convolution F is closely related to the average functional Ef by the conjugacy operation ; for the operations concerning the measurable multivalued mappings, we remark that ; on the one hand : $\chi^*_{E(C)} = E\,(\chi^*_{C(\omega)})$ under certain assumptions ; on the other hand, more generally : $\chi_{E(C)} = \oint_\Omega \chi_{C(\omega)}\ dP\ (\omega)$. This last equality brings out the fact that $F = \oint_\Omega f\ (.,\ \omega)\ dP\ (\omega)$ may not be l.s.c. for a normal convex integrand f. Let f_1 and $f_2 \in \mathcal{F}(\mathbb{R}^n, \bar{\mathbb{R}})$ (not identically equal to $+\infty$) ; if the infimal convolution $f_1 \nabla f_2$ is exact in $x = x_1 + x_2$, we have : $\partial\,(f_1 \nabla f_2)\ (x) = \partial f_1\ (x_1) \cap \partial f_2(x_2)$ ([3] Proposition 6.6.4). Our purpose is to obtain a similar formula for the subdifferential of the continuous infimal convolution. For that end, we shall define the continuous intersection of a family of sets indexed by Ω.

III.1. Definition : Let A be a multivalued mapping defined on Ω ; we shall call *continuous intersection* of the family $\{A\ (\omega)\}_{\omega \in \Omega}$, the set denoted by A^* and defined by :

$$A^* = \bigcup_{N \in \mathcal{N}} \left(\bigcap_{\omega \in \Omega \setminus N} A\ (\omega) \right)$$

where $\mathcal{N}$ indicates the family of P-null sets of $\mathcal{A}$.

Remark : It is indeed a generalization to the continuous case of the intersecting operation. For example, we know that if we consider two functions f_1 et f_2 we obtain epi $[\sup\,(f_1, f_2)]$ = epi $f_1 \cap$ epi f_2. Similarly, let $f \in \mathcal{F}(\mathbb{R}^n \times \Omega, \bar{\mathbb{R}})$ such that for each x, f (x, .) is measurable. Let Φ defined by :

$$\Phi(x) = \text{ess. sup. } f(x, \omega) = \text{Inf } \{\alpha \ / \ f(x, \omega) \leqslant \alpha \quad \text{a.s.}\}$$

Then, it is easy to see that : epi $\Phi = [\text{epi } f(., \omega)]^*$

III.2. Properties

a) If we modify A on a P-null set, we do not alter A^*.

b) If A is a.s. convex-valued (resp. closed valued, compact valued) mapping, then A^* is convex (resp. closed, compact)

c) The indicator function of A^* is : $\chi_{A^*} = E(\chi_{A(\omega)})$

Proofs :

a) Let A and B two multivalued mappings such that if we denote by N_o the set $\{\omega \in \Omega \ / \ A(\omega) \neq B(\omega)\}$ we have $P(N_o) = 0$

Let $x \in A^*$; $\exists N \in \mathcal{N}$ such that : $\forall \omega \in \Omega \setminus N \quad x \in A(\omega)$; this implies that : $\forall \omega \in \Omega \setminus N \cup N_o \quad x \in B(\omega)$; thus $x \in B^*$. Conversely, a similar proof shows us that $B^* \subset A^*$.

b) According to the property a) we may suppose the required hypothesis assumed for every $\omega \in \Omega$.

Convexity of A^* : let $x, y \in A^*$; $\exists N_x, N_y \in \mathcal{N}$ such that :

$x \in A(\omega) \ \forall \omega \in \Omega \setminus N_x$; $x \in B(\omega) \ \forall \omega \in \Omega \setminus Ny$. Taking $\lambda \in [0, 1]$, $A(\omega)$ being convex, we have : $\lambda x + (1-\lambda) y \in A(\omega) \forall \omega \in \Omega \setminus N_x \cup N_y$. So, $\lambda x + (1-\lambda) y \in A^*$.

Closedness of A^* : let $\{x_n\}_{n \in \mathbb{N}}$ a convergent sequence of A^*. For each $n \in \mathbb{N}$, there exists $N_n \in \mathcal{N}$ such that : $x_n \in A(\omega) \quad \forall \omega \in \Omega \setminus N_n$. But :

$$\bigcup_{n \in \mathbb{N}} \left(\bigcap_{\omega \in \Omega \setminus N_n} A(\omega) \right) \subset \bigcap_{\omega \in \Omega \setminus \bigcup_{n \in \mathbb{N}} N_n} A(\omega)$$

$$\Longrightarrow \quad \{x_n\}_{n \in \mathbb{N}} \quad \bigcap_{\omega \in \Omega \setminus \bigcup_{n \in \mathbb{N}} N_n} A(\omega) \text{ which is closed and included}$$

in A^*. Thus, $\lim_{n\to\infty} x_n \in A^*$ and A^* is a closed set.

Compactness of A^* : the same inclusion as previously shows us that A^* is compact when $A(\omega)$ is compact.

c) Indicator function of A^* : $E(\chi_{A(\omega)}(x)) = 0 \iff \exists N \in \mathcal{N}\ \forall \omega \in \Omega \setminus N \quad \chi_{A(\omega)}(x) = 0$ that is to say : $E(\chi_{A(\omega)}(x)) = 0 \iff x \in A^*$.

III.3. Subdifferential of the continuous infimal convolution F

III.3.1. Theorem : *Let f be a normal convex integrand such that $f^*(x^*, .)$ is integrable for every x^*. Let $x_0 \in \operatorname{dom} F$ and X_0 a random variable whose expectation is x_0 and giving the exactness of the continuous infimal convolution at x_0. If we denote by D_{X_0} the multivalued mapping : $\omega \to \partial f(X_0(\omega), \omega)$, then*

$$\partial F(x_0) = (D_{X_0})^*$$

Proof : Let $x^* \in \partial F(x_0)$; $x^* \in \partial F(x_0) \iff F(x_0) + F^*(x^*) = \langle x_0, x^* \rangle$. Moreover x_0 giving the exactness of the continuous infimal convolution F at x_0, we have :

$$E(X_0) = x_0 \quad \text{and} \quad F(x_0) = \int_\Omega f(X_0(\omega), \omega)\, dP(\omega)$$

By Theorem 7 of [12] , $F^* = Ef^*$ and

$$\int_\Omega f(X_0(\omega), \omega)\, dP(\omega) + \int_\Omega f^*(x^*, \omega)\, dP(\omega) = \int_\Omega \langle X_0(\omega), x^* \rangle\, dP(\omega)$$

We always have : $f(X_0(\omega), \omega) + f^*(x^*, \omega) \geq \langle X_0(\omega), x^* \rangle$. The equality between integrals implies then : a.s. $f(X_0(\omega), \omega) + f^*(x^*, \omega) = \langle X_0(\omega), x^* \rangle$, that is to say : a.s. $x^* \in \partial f(X_0(\omega), \omega)$. Consequently $\partial F(x_0) \subset (D_{X_0})^*$.

Conversely, let x^* belonging to $\partial f(X_0(\omega), \omega)$ a.s. :

$$f(X_0(\omega), \omega) + f^*(x^*, \omega) = \langle X_0(\omega), x^* \rangle \text{ a.s.}$$

$$\Longrightarrow F(x_0) + F^*(x^*) = \langle x_0, x^* \rangle \text{ that is to say : } x^* \in \partial F(x_0).$$

Thus $\partial F(x_0) = \bigcup_{N \in \mathcal{N}} \left(\bigcap_{\omega \in \Omega \setminus N} \partial f(X_0(\omega), \omega) \right)$ Q.E.D.

III.3.2. Application : normal cone to E(C)

We shall apply the previous result to determining the normal cone to E (C). For a convex set K, the normal cone to K at x_0 is denoted by $N_K(x_0)$ and is defined by :

$$N_K(x_0) = \{x^* / \forall x \in K \quad \langle x^*, x - x_0 \rangle \leq 0\}$$

III.3.2. Proposition : Let C be a CK-valued mapping, P-integrable ; x_o belonging to E(C). If we denote by X_o a random variable such that $E(X_o) = x_o$, $X_o(\omega) \in C(\omega)$ a.s. and by N_{X_o} the multivalued mapping : $\omega \to N_{C(\omega)}(X_o(\omega))$, then we have :

$$N_{E(C)}(x_o) = (N_{X_o})^*$$

Proof : Let f be defined by $f(x, \omega) = \chi_{C(\omega)}(x)$; the properties of C imply that f is a normal convex integrand ([7]). The normal cone is related to the indicator function by the following equality

$$N_{C(\omega)}(x) = \partial\chi_{C(\omega)}(x) \qquad ([8] \text{ Page } 215)$$

It is enough then to apply Theorem III.3.1. whose assumptions are satisfied, bearing in mind that $\chi_{E(C)} = \oint_\Omega \chi_{C(\omega)}\, dP(\omega)$.

III.4. Directional derivative of the continuous infimal convolution

III.4.1. Lemma : Let A be a measurable multivalued mapping defined on Ω such that $A(\omega)$ is a.s. a nonempty closed convex set. Then the support function of A^* is given by :

$$\chi^*_{A^*} = \overline{\oint_\Omega \chi^*_{A(\omega)}\, dP(\omega)}$$

Proof : Let $\Phi = \oint_\Omega \chi^*_{A(\omega)}\, dP(\omega)$. According to the definition of the continuous infimal convolution, Φ is convex and positively homogeneous. The l.s.c. regularization $\overline{\Phi}$ of Φ is the support function of a certain closed convex set C, namely :

$$C = \{x^* \;/\; \forall x \quad < x, x^* > \leqslant \Phi(x)\} \qquad ([8] \text{ Corollary } 13.2.1)$$

Let $X \in \mathcal{L}^1(x)$; $\forall y \in A^* \quad < X(\omega), y > \;\leqslant \chi^*_{A(\omega)}(X(\omega))$ a.s.

$$\Longrightarrow \quad \forall y \in A^* \quad < x, y > \;\leqslant \Phi(x)$$

$$\Longrightarrow \quad \chi^*_{A^*} \leqslant \overline{\Phi} \quad \text{, thus } A^* \subset C$$

Conversely, let $x^* \in C$. For every $A \in \mathcal{A}$ such that $P(A) > 0$, we define the random variable X_A by :

$$X_A(\omega) = \begin{cases} \dfrac{x}{P(A)} & \text{if} \quad \omega \in A \\ 0 & \text{elsewhere} \end{cases}$$

$E(X_A) = x$ and according to the definitions of Φ and C, we deduce that :

$$\text{(R)} \quad \forall x,\ \forall A \in \mathcal{A} \quad P(A) > 0 \quad \langle x, x^* \rangle \leq \frac{1}{P(A)} \int_A \chi^*_{A(\omega)}(x)\, dP(\omega)$$

This inequality implies that :

$$\text{a.s.} \quad \forall x \qquad \langle x, x^* \rangle \leq \chi^*_{A(\omega)}(x)$$

Suppose that this last inequality is not satisfied, there is $A \in \mathcal{A}$, $P(A) > 0$ and $\varepsilon > 0$ such that :

$$\forall \omega \in A \qquad \langle x, x^* \rangle \geq \chi^*_{A(\omega)}(x) + \varepsilon$$

$$\Longrightarrow \qquad \langle x, x^* \rangle \geq \frac{1}{P(A)} \int_A \chi^*_{A(\omega)}(x) + \varepsilon$$

and this is in contradiction with the inequality (R).

III.4.2. Remark : The result of the previous lemma is a generalization to the continuous case of the following formula : if $A_1, \ldots, A_m$ are non-empty closed convex sets of $\mathbb{R}^n$,

$$\chi^*_{\bigcap_{i=1}^m A_i} = \overline{\mathop{\nabla}_{i=1}^{m} \chi^*_{A_i}}$$

III.4.3. Theorem : *Let f be a normal convex integrand such that $f^*(x^*, .)$ is integrable for every x^*. Let x_0 such that $\partial F(x_0) \neq \emptyset$, X_0 a random variable whose expectation is x_0 and giving the exactness of the continuous infimal convolution at x_0. Then, the l.s.c. regularization of the directional derivative F' is given by :*

$$\overline{F'}(x_0, .) = \overline{\oint_\Omega \overline{f'}(X_0(\omega), .)\, dP(\omega)}$$

Proof : We suppose that $\partial F(x_0) \neq \emptyset$. According to Theorem III.3.1., $\partial f(X_0(\omega), \omega)$ is a.s. a non empty closed convex set. The multivalued mapping

$D_{X_0} : \omega \to \partial f(X_0(\omega), \omega)$ is measurable : consequently, the support function of D_{X_0}, i.e. the function g defined by : $g(d, \omega) = \overline{f'}(X_0(\omega), d)$ is a normal convex integrand. We apply then the previous lemma and conclude with the equality :

$\chi^*_{\partial F(x_0)} = \overline{F'}(x_0, .)$ ([3] Theorem 6.4.8).

Remark : the formula of Theorem III.4.3. generalizes the corresponding formula for two convex functions g_1 et g_2 : let g_1, g_2 proper convex functions and $g = g_1 \nabla g_2$. If the infimal convolution g is exact in $x_o = x_o^1 + x_o^2$ and if $\partial g(x_o) \neq 0$, then :

$$\overline{g'}(x_o, .) = \overline{\overline{g'}(x_o^1, .) \nabla \overline{g'}(x_o^2, .)}$$

IV. Applications

A - We first apply the obtained results to determining some convex characteristics of the mean value functional Ef and continuous infimal convolution such as : level sets, l.s.c. conical hull, gauge functional of the polar of the expectation E(C)...

1 - Level sets of the continuous infimal convolution F

For a convex function g, it is interesting to be able to determine the λ-level set of g, i.e. $g^{\leqslant}(\lambda) = \{x / g(x) \leqslant \lambda\}$. For the continuous infimal convolution $F = \oint_\Omega f(., \omega)\, dP(\omega)$, our purpose is to determine $F^{\leqslant}(\lambda)$ by expressing it with the level sets of $f(., \omega)$.

Let $T_f(\omega)$ the projection of epi $f(., \omega)$ on $\mathbb{R}$, that is to say : $\{\lambda / \exists x\ f(x, \omega) \leqslant \lambda\}$. Likewise, $\tau_F = \operatorname{proj}_{\mathbb{R}}(\text{epi}F)$

1.1. Theorem : *Let f be a normal convex integrand such that $f^*(x^*, .)$ is integrable for every x^*. Then :*

$$\forall \lambda \in \tau_F \qquad F^{\leqslant}(\lambda) = \bigcup_{\substack{\Lambda \in \mathcal{L}^1(\lambda) \\ \Lambda(\omega) \in T_f(\omega)\ a.s.}} E\,\{f(., \omega)^{\leqslant}(\Lambda(\omega))\}$$

Proof : We have $T_f(\omega) = [m(\omega), +\infty[$ and $\tau_F = [M, +\infty[$ where we denote by $m(\omega) = \inf\{f(x, \omega) / x \in \mathbb{R}^n\}$ and $M = \inf\{F(x) / x \in \mathbb{R}^n\}$. According to the equality $F^* = E(f^*)$ ([12] Theorem 7), it is easy to remark that $M = E(m)$. Moreover, it is clear that T_f is a measurable multivalued mapping such that, a.s. $T_f(\omega)$ is a non-empty closed set. Let $\lambda \in \tau_F$; $\exists \Lambda \in \mathcal{L}^1(\lambda)$ such that : a.s. $\Lambda(\omega) \in T_f(\omega)$. Let us consider the multivalued mapping : $\omega \to f(., \omega)^{\leqslant}(\Lambda(\omega))$. The conjugate $f^*(., \omega)$ is a.s. finite and continuous at 0 (proof of theorem II.4.2) and $\partial_\theta f^*(0, \omega)$ is a.s. a nonempty compact convex set characterized by :

$$\partial_{\theta} f^{*}(0,\omega)=\{x \,/\, f(x,\omega)\leqslant \theta - f^{*}(0,\omega)\}$$

Let $\theta(\omega)=\Lambda(\omega)+f^{*}(0,\omega)$; $\theta(\omega)\geqslant 0$ a.s. because $\Lambda(\omega)\in T_f(\omega)$ a.s. The random variable $\Lambda + f^{*}(0,.)$ is integrable and we can deduce like in the proof of Theorem II.4.2 that the multivalued mapping $\omega \to \partial_{\theta(\omega)} f^{*}(0,\omega)$ is P-integrable.

On the other hand ; we have : $\forall \lambda \in \tau_F \quad F^{\leqslant}(\lambda)=\partial_{\lambda+E(f^{*})(o)} E(f^{*})(o)$

Then it is enough to apply theorem II.4.2 to obtain that :

$$F^{\leqslant}(\lambda)=\bigcup_{\substack{\Lambda\in\mathcal{L}^1(\lambda)\\ \Lambda(\omega)+f^{*}(0,\omega)\geqslant 0}} E\{f(.,\omega)^{\leqslant}(\Lambda(\omega))\}$$

a.s. Hence the result.

2 - Level sets of the mean value functional Ef.

2.1. Theorem : *Let $f \in \mathcal{F}(\mathbb{R}^n \times \Omega, \overline{\mathbb{R}})$ be such that $f(x,.)$ is measurable for every x and Ef the mean value functional. If, for each random variable Λ, we denote by S_Λ the multivalued mapping : $\omega \to f(.,\omega)^{\leqslant}(\Lambda(\omega))$, then :*

$$(Ef)^{\leqslant}(\lambda)=\bigcup_{\Lambda\in\mathcal{L}^1(\lambda)} (S_\Lambda)^{*} \qquad ((S_\Lambda)^{*} \textit{is defined in III.1})$$

Proof : It is trivial to see that : $\forall \Lambda \in \mathcal{L}^1(\lambda)\ (S_\Lambda)^{*}\subset Ef^{\leqslant}(\lambda)$. Conversely, let $x\in Ef^{\leqslant}(\lambda)$ and define Λ_o by

$\Lambda_o(\omega)=f(x,\omega)-Ef(x)+\lambda$ a.s. Then $\Lambda_o \in \mathcal{L}^1(\lambda)$ and $x\in(S_{\Lambda_o})^{*}$.

This theorem is proved to show the analogy between the expressions of the level sets of the mean value functional and continuous infimal convolution.

3 - L.s.c. conical hull of Ef

For $g\in\mathcal{F}(\mathbb{R}^n,\overline{\mathbb{R}})$, we denote by g^c the l.s.c. conical hull of g ([3] definition 6.8.6).

3.1. Theorem : *Let f be a convex integrand such that, for every x, $f(x,.)$ is integrable. We suppose that $Ef(0)\geqslant 0$. Then :*

$$(Ef)^{c}\neq-\infty \ \textit{and}\ (Ef)^{c}=\sup_{\substack{\Lambda\in\mathcal{L}^1(0)\\ f(0,\omega)+\Lambda(\omega)\geqslant 0\ a.s.}}\left\{\int_\Omega [f(.,\omega)+\Lambda(\omega)]^{c}\,dP(\omega)\right\}$$

Proof : Ef is real-valued ; $\partial Ef(0) \neq \emptyset$. Let $x^* \in \partial Ef(0)$, we have :

$\forall x \quad Ef(x) \geqslant Ef(o) + \langle x^*, x \rangle$; this implies that $(Ef)^*(x^*) \leqslant 0$. Moreover $(Ef)^c$ is the support function of C, where $C = \{x^* / (Ef)^*(x^*) \leqslant 0\}$. ([3] Theorem 6.8.7). Therefore $C \neq \emptyset$ and $(Ef)^c \not\equiv -\infty$. $(Ef)^* = \oint_\Omega f^*(., \omega)\, dP(\omega)$, then, according to Theorem 1.1.

$$C = \bigcup_{\substack{\Lambda \in \mathcal{L}^1(0) \\ f(0,\omega)+\Lambda(\omega) \geqslant 0 \text{ a.s.}}} E\{f^*(.,\omega)^{\leqslant}(\Lambda(\omega))\}$$

The conjugate of $f + \Lambda$ being $f^* - \Lambda$, $[f(., \omega) + \Lambda(\omega)]^c$ is the support function of the set $f^*(., \omega)^{\leqslant}(\Lambda(\omega))$.

$f(0, .) + \Lambda$ is integrable and the multivalued mapping

$\omega \to f^*(., \omega)^{\leqslant}(\Lambda(\omega)) = \partial_{f(0,\omega)+\Lambda(\omega)} f(0, \omega)$ is P-integrable. Hence :

$$\chi^*_C = \operatorname*{Sup}_{\Lambda} \chi^*_{E\{f^*(.,\omega)^{\leqslant}(\Lambda(\omega))\}} = \operatorname*{Sup}_{\Lambda} \int_\Omega [f(., \omega) + \Lambda(\omega)]^c\, dP(\omega)$$

4 - L.s.c. conical hull of F

4.1. Theorem : *Let f a normal convex integrand such that $f^*(x^*, .)$ is integrable for every x^* and $F = \oint_\Omega f(., \omega)\, dP(\omega)$. The l.s.c. conical hull of the continuous infimal convolution F is given by :*

$$F^c = \operatorname*{Sup}_{\substack{\Lambda \in \mathcal{L}^1(0) \\ f(0,\omega)+\Lambda(\omega)>0 \text{ a.s.}}} \overline{\oint [f(.,\omega) + \Lambda(\omega)]^c\, dP(\omega)}$$

Proof : For a function $g \in \Gamma(\mathbb{R}^n)$, it is easy to state the following equivalence :

$$g_c \not\equiv -\infty \iff g(0) > 0$$

Let $\Lambda \in \mathcal{L}^1(0)$ such that $f(0, \omega) + \Lambda(\omega) > 0$ a.s. . $[f(., \omega) + \Lambda(\omega)]^c$ which is the support function of the multivalued mapping $S_\Lambda : \omega \to f^*(.,\omega)^{\leqslant}(\Lambda(\omega))$ is a normal convex integrand. If $D = \{x / F^*(x) \leqslant 0\}$, according to Theorem 2.1, we have :

$$\chi^*_D = \operatorname*{Sup}_{\Lambda \in \mathcal{L}^1(0)} \chi^*_{(S_\Lambda)^*}$$

According to lemma III.4.1. $\chi^*_{(S_\Lambda)^*} = \overline{\oint \chi^*_{S_{\Lambda(\omega)}} dP(\omega)}$. Hence the formulation of F^c.

5 - Polar of E (C). Gauge function of $[E\ (C)]^\circ$

5.1. Theorem : *Let C be a measurable multivalued mapping such that, for every* ω, *C* (ω) *is a closed convex set containing 0. Then the gauge function* $\gamma_{[E(C)]^\circ}$ *and the polar* $[E\ (C)]^\circ$ *are given by :*

a) $\gamma_{[E(C)]^\circ} = E\ (\gamma_{C^\circ})$

b) $[E(C)]^\circ = \bigcup_{\Lambda \in \mathcal{L}^1(1)} [\Lambda\ C^\circ]^*$, ΛC° *indicating the multivalued mapping* $\omega \rightarrow \Lambda\ (\omega)\ C^\circ\ (\omega)$.

Proof : C being measurable, the multivalued mapping C° : $\omega \rightarrow [C\ (\omega)]^\circ$ is also measurable ([7] Corollary 3.5).

a) E(C) is a convex set containing 0 and : $\gamma_{[E(C)]^\circ} = \chi^*_{E(C)}$ ([8] Theorem 14.5)
Moreover, $0 \in C\ (\omega)$ and C is a quasi P-integrable multivalued mapping. Therefore, according to Remark I.3.1.

$$\chi^*_{E(C)} = \int_\Omega \chi^*_{C(\omega)}\ dP\ (\omega) = \int_\Omega \gamma_{C^\circ(\omega)}\ dP\ (\omega)$$

b) $\gamma_{[E(C)]^\circ}$ is a l.s.c. and positively homogeneous function such that :

$$\forall \lambda > 0 \quad \gamma_{[E(C)]^\circ}^{\leqslant}\ (\lambda) = \lambda\ [E\ (C)]^\circ \quad ([8]\ \text{Corollary } 9.7.1)$$

Thus $[E(C)]^\circ = \gamma_{[E(C)]^\circ}^{\leqslant}\ (1) = \bigcup_{\Lambda \in \mathcal{L}^1(1)} (S_\Lambda)^*$ with $S_\Lambda\ (\omega) = \gamma_{C^\circ(\omega)}^{\leqslant}(\Lambda(\omega)) = \Lambda(\omega) C^\circ(\omega)$

IV.1.6. Remark : If, furthermore C (ω) is a cone, we have : $\chi_{C(\omega)} = \gamma_{C(\omega)}$ and $\chi_{[E(C)]^\circ} = E\ (\chi_{C^\circ})$. So, $[E\ (C)]^\circ = (C^\circ)^*$.

B - Optimization problems

In this second part of applications, we consider different optimization problems where the function to be minimized and the constraints may depend on the random outcome ω.

6 - A first minimization problem

Let f be a real-valued convex function and C a CK-valued mapping which is supposed P-integrable. Let us consider the following problems :

$(\mathcal{P}_\omega)$ $\qquad \varphi(x, \omega) = Inf \{f(x-y) \ / \ y \in C(\omega)\}$

$(\mathcal{P})$ $\qquad \Phi(x) = Inf \{f(x-y) \ / \ y \in E(C)\}$

The relation between φ and Φ is given by the proposition below :

6.1. Proposition : $\Phi = \oint_\Omega \varphi(., \omega) \, dP(\omega)$

Proof :

a) f being convex, the definition of φ shows us that φ is a real-valued convex function. C is a measurable multivalued mapping such that C (ω) is nonempty and closed. Then, there exists a countable collection of measurable functions $\{f_i\}_{i \in D}$ such that :

$$C(\omega) = \overline{\{f_i(\omega) \ / \ i \in D\}} \qquad ([7] \text{ Theorem } 1)$$

Then $\varphi(x, \omega) = \inf_{i \in D} f(x - f_i(\omega))$ is measurable ; φ is then a normal convex integrand. Let $S(x, \omega) = \{(y \in C(\omega) \ / \ f(x-y) = \varphi(x, \omega)\}$. For every x, S (x, .) is a CK-valued measurable multivalued mapping. According to the theorem of Kuratowski - Ryll-Nardzewski ([7] Corollary 1.1), there exists a measurable selector of S (x, .) i.e. a measurable function Y such that : $Y(\omega) \in S(x, \omega)$ a.s. Then, a.s. $f(x-Y(\omega)) = \varphi(x, \omega)$ and for every random variable X such that $E(X) = x$, we have :

$$\int_\Omega f(X(\omega) - Y(\omega)) \, dP(\omega) \geqslant f(x - E(Y)) \geqslant \inf \{f(x-y) \ / \ y \in E(C)\}$$

$$\Longrightarrow \quad \oint_\Omega \varphi(., \omega) \, dP(\omega) \geqslant \Phi$$

b) Conversely, let $\overline{Y} \in E(C)$ such that $f(x-\overline{Y}) = \Phi(x)$. Denote by Y a measurable selector of C such that $E(Y) = \overline{Y}$. Let $X(\omega) = x + Y(\omega) - \overline{Y}$; we deduce that :

$$E(X) = x \text{ and } \varphi(X(\omega), \omega) \leqslant f(X(\omega) - Y(\omega)) = f(x - \overline{Y})$$

$$\Longrightarrow \int_\Omega \varphi(X(\omega), \omega) \, dP(\omega) \leqslant f(x-\overline{Y})$$

Consequently : $\Phi \geqslant \oint_\Omega \varphi(., \omega) \, dP(\omega)$. We also conclude that the continuous infimal convolution is exact.

Application

Let $f(x) = ||x||$; then $\varphi(x, \omega) = d(x, C(\omega))$. According to the previous result ; $d(., E(C)) = \oint_\Omega d(., C(\omega))\, dP(\omega)$.

7 - Minimization of the continuous infimal convolution

Let f a normal integrand and F the continuous infimal convolution of the family $\{f(., \omega)\}_{\omega \in \Omega}$ relating to the probability measure P. We consider the following optimization problems :

$(\mathcal{P}_\omega)$ *Find* $\bar{x}$ *such that :* $f(\bar{x}, \omega) = \mathrm{Inf}\left\{f(x, \omega) / x \in \mathbb{R}^n\right\}$

$(\mathcal{P})$ *Find* $\tilde{x}$ *such that :* $F(\tilde{x}) = \mathrm{Inf}\left\{F(x) / x \in \mathbb{R}^n\right\}$

We call $S(\omega)$ and $\sum$ the solution sets of respectively $(\mathcal{P}_\omega)$ and $(\mathcal{P})$. The following theorem allow us to compare $S(\omega)$ and $\sum$ as well as the optimal values.

7.1. Theorem :*Let f be a normal integrand on* $\mathbb{R}^n \times \Omega$. *We denote by* $m(\omega) = \mathrm{Inf}\{f(x, \omega) / x \in \mathbb{R}^n\}$ *and* $M = \inf\{F(x) / x \in \mathbb{R}^n\}$. *Then*

a) If there exists a measurable selector for the multivalued mapping S, we have :

$$M = E(m).$$

b) Moreover, if f is a normal convex integrand such that f (x', .) is integrable for every x' , S is a CK-valued mapping, P-integrable and $E(S) = \sum$. *In other words :*

$\tilde{x}$ *solution of* $(\mathcal{P}) \iff \tilde{x} = E(X)$ *where* $X(\omega)$ *is a.s. solution of* $(\mathcal{P}_\omega)$.

Proof :

a) Let X_0 a measurable selector of S and $E(X_0) = x_0$. From the definition of F, $F(x_0) \leq \int_\Omega f(X_0(\omega), \omega)\, dP(\omega) = \int_\Omega m(\omega)\, dP(\omega)$; thus $M \leq E(m)$. Conversely, $\forall x, \forall X \in \mathcal{L}^1(x) \int_\Omega f(X(\omega), \omega)\, dP(\omega) \geq \int_\Omega m(\omega)\, dP(\omega)$ because a.s. $f(X(\omega), \omega) \geq m(\omega)$. Therefore : $\forall x \quad F(x) \geq E(m)$ and $M \geq E(m)$

b) We have already seen that F and a.s. $f(., \omega)$ are inf-compact functions. $f(., \omega) \in \Gamma_0(\mathbb{R}^n)$; the solution set of $(\mathcal{P}_\omega)$ is $\partial f^*(0, \omega)$ ([8] Theorem 27.1) Likewise, the solution set $\sum$ of $(\mathcal{P})$ is $\partial F^*(0) = \partial E(f^*)(0)$. According to Theorem II.4.1., we have $\partial E(f^*)(0) = E(\partial f^*(0, \omega))$. Q.E.D.

7.2. Remarks :

a) More generally, for a normal integrand f, we have only the following inequality : $E(m) \leqslant M$ in $\overline{R}$. The equality $M = E(m)$ appears in a different form and in the convex case in [10] (Proposition 1).

b) Under the assumptions b) of Theorem 7.1., concerning the uniqueness of solutions of $(\mathcal{P})$ and $(\mathcal{P}_\omega)$, we may assert that :

$$\sum = \{\overline{x}\} \iff S(\omega) = \{\overline{X}(\omega)\} \text{ a.s. with } E(\overline{X}) = \overline{x}$$

The previous theorem may be extended when considering approximated optimization problems. Let $(\mathcal{P}^\alpha_\omega)$ and $(\mathcal{P}^\theta)$ the following approximated minimization problems.

$(\mathcal{P}^\alpha_\omega)$ Find $\overline{x}$ such that : $m(\omega) \leqslant f(\overline{x}, \omega) \leqslant m(\omega) + \alpha$

$(\mathcal{P}^\theta)$ Find $\tilde{x}$ such that : $M \leqslant F(\tilde{x}) \leqslant M + \theta$.

7.3. Theorem : *Let f be a normal convex integrand such that $f^*(x', .)$ is integrable for every x'. Then, for each* $\theta \geqslant 0$,

$$\tilde{x} \text{ solution of } (\mathcal{P}^\theta) \iff \begin{cases} \exists \overline{X},\ E(\overline{X}) = \tilde{x} \\ \exists \theta,\ \theta(\omega) \geqslant 0 \qquad E(\theta) = \theta \\ \text{such that : a.s. } \overline{X}(\omega) \text{ is solution of } \mathcal{P}^{\theta(\omega)}_\omega \end{cases}$$

Proof : According to the definition of the θ-subdifferential, we have :

$\overline{X}(\omega)$ solution of $(\mathcal{P}^\alpha_\omega) \iff 0 \in \partial_\alpha f(\overline{X}(\omega), \omega) \iff \overline{X}(\omega) \in \partial_\alpha f^*(0,\omega)$

Likewise : $\tilde{x}$ solution of $(\mathcal{P}^\theta) \iff \tilde{x} \in \partial_\theta E(f^*)(0)$ because $F^* = E(f^*)$. It is enough to apply Theorem II.4.1. to determine $\partial_\theta E(f^*)(0)$

8 - Minimization of a normal convex integrand on a random closed convex set.

Let f be a normal convex integrand and C a measurable multivalued mapping such That C (ω) is a nonempty closed convex set (in a stochastic linear program, C (ω) may be $\pi(\omega) = \{x \,/\, A(\omega)\, x \leq b(\omega)\}$ where A and b are measurable). Let :

$$\tilde{f}(x, \omega) = f(x, \omega) + \chi_{C(\omega)}(x)$$

8.1. Theorem : *Let f be a normal convex integrand such that for every x' $f^*(x', .)$ is integrable, C a nonempty closed convex valued measurable mapping. Moreover, assume that $\tilde{F} = \oint_\Omega \tilde{f}(., \omega)\, dP(\omega)$ is not identically equal to + ∞. Then $\tilde{f}$ is a normal convex integrand and if we denote by S^1_C the set of integrable selectors of C, we have :*

$$\tilde{M} = Inf\left\{\tilde{F}(x)/x \in \mathbb{R}^n\right\} = \underset{X \in S^1_C}{Inf} \int_\Omega f(X(\omega),\omega)\,dP(\omega) = \int_\Omega Inf\left\{f(x,\omega)/x \in C(\omega)\right\} dP(\omega)$$

Proof : The assumption $\tilde{F} \not\equiv +\infty$ implies that : $\exists x_0$ such that $\tilde{F}(x_0) < +\infty$. Then, there exists $X_0 \in \mathcal{L}^1$ such that $\tilde{f}(X_0(.), .)^+$ is integrable.
Necessarily : a.s. $X_0(\omega) \in C(\omega)$. So, S^1_C is a nonempty set.
Let $\Omega_0 = \{\omega \in \Omega / \exists\, x \in \mathbb{R}^n \;\; f(x,\omega) + \chi_{C(\omega)}(x) < +\infty\}$. We have shown that $P(\Omega_0) = 1$.
Thus, $\tilde{f}$ a normal convex integrand ([7] Corollary 4.2).
Moreover :

$$f(., \omega) \leq \tilde{f}(.,\omega) \Longrightarrow \tilde{f}^*(., \omega) \leq f^*(., \omega)$$

$$\forall x' \quad f^*(x', \omega) \geq \langle X_0(\omega), x'\rangle - f(X_0(\omega), \omega)$$

$$\Longrightarrow X_0(\omega), x'\rangle - f(X_0(\omega), \omega) \leq \tilde{f}^*(x', \omega) \leq f^*(x', \omega)$$

These inequalities imply that for every x', $\tilde{f}^*(x', .)$ is integrable. According to results of Theorem 7.1., we obtain :

$$M = Inf\left\{\tilde{F}(x) \,/\, x \in \mathbb{R}^n\right\} = \int_\Omega Inf \;\; f(x, \omega) \,/\, x \in C(\omega)\} \, dP(\omega)$$

On the other hand, it arises from the definition of $\tilde{F}$ that :

$$\tilde{M} = \underset{X \in S^1_C}{Inf} \int_\Omega f(X(\omega), \omega)\, dP(\omega)$$

8.2. Remark : The required assumptions for f and C imply that $\tilde{F} \in \Gamma_0(\mathbb{R}^n)$. Moreover, it is obvious that dom $\tilde{F} \subset E(C)$. The problem we have dealt with is quite different from the minimization of $F = \oint_\Omega f(., \omega)\, dP(\omega)$ on E (C).

9 - Minimization of the continuous infimal convolution on compact convex set Q

9.1. Theorem : *Let f be a normal convex integrand such that* $f^*(x', .)$ *is integrable for every x' and C a CK-valued mapping P-integrable. We set :*

$$F = \oint_\Omega f(., \omega)\, dP(\omega) \qquad \bar{\alpha} = Min\{F(x) / x \in E(C)\}$$

For every $u \in \mathcal{L}^1(O)$, $\alpha_u(\omega) = Inf\{f(x+u(\omega), \omega) / x \in C(\omega)\}$.

Then :

$$\bar{\alpha} = Min\left\{ \int_\Omega \alpha_u(\omega)\, dP(\omega) / u \in \mathcal{L}^1(O)\right\}$$

Proof : Denote $f(., \omega) \nabla \chi_{-C(\omega)}$ by $g(., \omega)$. g is obviously a normal convex integrand. For each $u \in \mathcal{L}^1(O)$, $\alpha_u(\omega)$ may be written : $\alpha_u(\omega) = g(u(\omega), \omega)$. Likewise :

$\bar{\alpha} = [F \nabla \chi_{-E(C)}](O)$. The functions $f^*(x', .)$ and $\chi^*_{C(.)}(x')$ are integrable for every x', thus $F \nabla \chi_{-E(C)} = \oint_\Omega g(., \omega)\, dP(\omega)$ and the continuous infimal convolution is exact. That is to say :

$$F \nabla \chi_{-E(C)}(O) = \underset{u \in \mathcal{L}^1(O)}{Min} \int_\Omega g(u(\omega), \omega)\, dP(\omega)$$

So, these exists $\bar{u} \in \mathcal{L}^1(O)$ such that : $\bar{\alpha} = \int_\Omega \alpha_{\bar{u}}(\omega)\, dP(\omega)$. Hence, the result :

If previously we take $C(\omega) = Q$ where Q is a nonempty convex set, the previous theorem takes the following form :

9.2. Corollary : Let g be a normal convex integrand such that for every x' $f^*(x', .)$ is integrable. Q is a nonempty compact convex set. Let :

$$(\mathcal{P}) \qquad \bar{\alpha} = Inf\{F(x) / x \in Q\}$$

$(\mathcal{P}_\varepsilon)$ the perturbed problems :

$$(\mathcal{P}_\varepsilon) \qquad \alpha_\varepsilon(\omega) = Inf\{f(x, \omega) / x \in Q + \varepsilon(\omega)\}$$

Then : $\exists\, \varepsilon \in \mathcal{L}^1(O)$ such that $\bar{\alpha} = \int_\Omega \alpha_\varepsilon(\omega)\, dP(\omega)$

Proof : We take in the previous theorem $\varepsilon(\omega) = -\bar{u}(\omega)$.

REFERENCES

[1] D.P. BERTSEKAS, Stochastic optimization problems with non differentiable cost functionals. Journal of Optimization Theory and Applications. 1973, Vol. 12, n° 2, p. 218-231.

[2] A.D. IOFFE - V.M. TIHOMIROV, Dualité des fonctions convexes et problèmes d'extrêmum. Uspehi Mat. N. 23-6 (1968) p. 51-116.

[3] P.J. LAURENT, Approximation et optimisation, Hermann, Paris 1972.

[4] C. LEMARECHAL, Sur les méthodes de gradient conjugué appliquées aux critères non différentiables.C.R. Acad. Sc. Paris, t. 278 (1974).

[5] J. NEVEU, Bases mathématiques du Calcul des Probabilités. Masson, Paris 1970.

[6] R.T. ROCKAFELLAR, Integrals which are convex functionals. Pacific Journal of Mathematics 1968. Vol. 24, n° 3, p. 525-539.

[7] R.T. ROCKAFELLAR, Measurable dependance of convex sets and functions on parameters. Journal of Mathematical Analysis and applications, 1969, 28, p. 4-25.

[8] R.T. ROCKAFELLAR, Convex analysis. Princeton 1970.

[9] R.T. ROCKAFELLAR, Conjugate duality and optimization, Lecture Notes, Série n° 16, SIAM Publications, 1974.

[10] R.T. ROCKAFELLAR, R.J.B. WETS, Stochastic convex programming : Basic Duality. Pacific Journal of Mathematics (to appear)

[11] M. VALADIER, Multiapplications mesurables à valeurs convexes compactes, Journal de Mathématiques Pures et Appliquées 50, 1971, p. 265-297.

[12] M. VALADIER, Intégration d'ensembles convexes fermés, notamment d'épigraphes. Rev. Franç. Inf. Rech. Op. 4 (1970) p. 57-73.

[13] B. VAN CUTSEM, Eléments aléatoires à valeurs convexes compactes. Thèse Université de Grenoble, 1971.

[14] D.W. WALKUP - R.J.B. WETS, Stochastic programs with recourse II : on the continuity of the objective. Siam J. Appl. Math. 1969, Vol. 17, n° 1, p. 98-103.

[15] R.J.B. WETS, On inf-compact mathematical programs, Proceedings 5th I.F.I.P. Conference on optimization, Springer Verlag, (1974).