Population Planning; a Distributed Time Optimal Control Problem

G.J. Olsder and R.C.W. Strijbos
Department of Applied Mathematics
Twente University of Technology
P.O. Box 217, Enschede
The Netherlands

Abstract

The time evolution of the age profile of a group of people, for instance the population of a certain country, can be described by a first-order partial differential equation. A time optimal control problem arises when the population must be brought from a given age profile to another desired one as quickly as possible. The birth rate, i.e. the number of births per unit of time, is the control variable and it serves as a boundary condition for the partial differential equation. To prevent the age distribution to become undesirable from an economical point of view during the transient to the final situation we require the working population to exceed a given fraction of the total population at each instant of time. This introduces a state constraint to the problem.

For the cases considered the following facts turn out. a) If age and time are discretized properly, a linear programming problem results, the solution of which equals an optimal solution of the continuous version of the problem. b) The time optimal control is not necessarily unique. A complete characterization of the class of all optimal controls can be given. c) Under certain conditions the class of all optimal controls contains a unique non-increasing control.
Two examples are solved analytically.

1. Introduction.

The evolution of a certain group of people, say for instance the population of a country, can be described by a partial differential equation in which the independent variables are time and age. If the initial age profile is given as well as the birth rate and the mortality function, then the evolution is completely determined. The mortality function depends on time and age and is assumed to be known. Immigration and emigration are not considered in the model, though this could easily be built in.

Given a certain initial age profile the population must be "steered" as quickly as possible to another, prescribed, final age profile by means of a suitable chosen birth rate. In this way a time optimal control problem has been formulated. The problem is stated in terms of an overpopulation which should be reduced. The other way around can be dealt with equally well.

The optimal birth rate may unbalance the age distribution during the time interval concerned, which could give rise to economic and social problems. Therefore it is assumed that the working population, which must support the non-working population, must exceed a given fraction of the total population at each instant of time. This becomes a state constraint in the mathematical formulation. It will turn out that the addition of this state constraint makes the control problem nontrivial. Another constraint which is considered is that the birth rate, obviously nonnegative, must be a nonincreasing function of time in order to avoid possible peaks.

This paper is not concerned with the social and political problems involved in establishing the best mechanism for a program of population management. Instead of this, it focusses upon the mathematical solution of the time optimal control problem.

The mathematical problem has not quite been solved in its generality; the mortality function and the definition of the working population should satisfy certain restrictions.

Related problems have been treated by for instance Langhaar [1] and Falkenburg [2]. Instead of time optimality Falkenburg considers a quadratic criterion and no state constraints are included. In [3] a similar control problem is considered using the Leslie model, with demografic data of the Netherlands.

2. The model describing the population dynamics.

The quantity p will stand for population density and it depends on the independent variables time t and age r. The number of people of ages in the age interval $(r, r+dr]$ at a certain time t is given by $p(t,r)dr$. Suppose t increases with dt and hence the age of an individual increases with $dr = dt$ as well. Now

$$p(t+dt,\ r+dt)dr = p(t,r)dr - p(t,r)\mu(t,r)dr\ dt, \tag{1}$$

where $\mu(t,r)$ is the mortality function, i.e. $\mu(t,r)dt$ is the fraction of people of the age class $(r,r+dr]$ who die in the time interval $[t,t+dt)$. If $dr = dt \to 0$, eq. (1) yields

$$\frac{\partial p(t,r)}{\partial t} = - \frac{\partial p(t,r)}{\partial r} - \mu(t,r)p(t,r) \qquad (2)$$

which is a linear partial differential equation of the hyperbolic type. It is assumed that μ and the boundary conditions are sufficiently smooth in order for $\partial p/\partial r$ and $\partial p/\partial t$ to exist almost everywhere. The following boundary and initial conditions will be used:

$$p(0,r) = p_0(r) \quad , \quad 0 < r \leqslant 1, \qquad (3)$$

$$p(t,0) = u(t) \quad , \quad 0 \leqslant t \leqslant T, \qquad (4)$$

where $p_0(r)$ is the given initial age distribution; $u(t)$ is the birth rate and T is the final time. It has been assumed that the age r has been scaled in such a way that nobody reaches an age of $r > 1$. The time t will be measured with respect to the same scale.

From now on it will be assumed that $\mu(t,r)$ is independent of t and by abuse of notation we will now write $\mu(r)$, which is assumed to be known. The solution of eqs. (2)-(4) is

$$p(t,r) = p_0(r-t)\exp(-\int_{r-t}^{r} \mu(s)ds) \quad , \quad 0 \leqslant t < r, \qquad (5)$$

$$p(t,r) = u(t-r)\exp(-\int_{0}^{r} \mu(s)ds) \quad , \quad t \geqslant r. \qquad (6)$$

Because nobody reaches $r = 1^+$, the mortality function should satisfy

$$\exp(-\int_0^{1^+} \mu(s)ds) = 0$$

However, this is not the case for the functions $\mu(r)$ considered in this paper. For these imperfect $\mu(r)$-functions it will be assumed that those people who reach an age of $r > 1$, simply leave the model and are not considered any longer.

The age distribution is called stationary at $t = t_s$ if

$$p(t_s,r) = c.\ \exp(-\int_0^r \mu(s)ds), \quad 0 \leqslant r \leqslant 1 \qquad (7)$$

where c is a positive constant. A stationary age distribution corresponds to a constant birth rate $u(t) = c$ for $t_s - 1 \leqslant t < t_s$.

In the remainder of the text the following abbreviations will be used

$$e(r) = \exp(-\int_0^r \mu(s)ds) \,,$$

$$ei(a;b) = \int_a^b e(r)dr \,.$$

3. Statement of the problem.

We will consider the following problem. Bring the population from a given initial age profile to another desired one as quickly as possible by properly choosing the birth rate. More precisely, given eqs. (2) and (3), the boundary condition (4) should be chosen in such a way that at time $t = T$ the age distribution $p(T,r)$ equals a prescribed function $p_T(r)$ for all $r \in (0,1]$. Moreover the final time T must have its minimal value. It is easily shown that $p(T,r)$, $0 \leqslant r \leqslant 1$ is completely determined by $u(t)$ with $t \in [T-1,T]$ because the people who were born between $t = 0$ and $t = T-1$ have all died at time T and hence do not show up in $p(T,r)$. It is tacitly assumed here that $T \geqslant 1$.

From now on it will be assumed that both $p_0(r)$ and $p_T(r)$ are stationary. Apart from a multiplicative constant p_0 and p_T are equal; we assume that $p_0(r) > p_T(r)$, $0 \leqslant r \leqslant 1$.

The final time T must be minimized and it is clear from above that the minimal T, to be denoted by T^*, equals 1; simply choose $u(t) = u_T$ on $0 \leqslant t \leqslant 1$, where the constant u_T corresponds to the stationary age distribution $p_T(r)$.

The optimal control problem defined above is trivial. However, the solution may unbalance the age distribution and therefore the following state constraint will be added:

$$\int_a^b p(t,r)dr \geqslant \alpha \int_0^1 p(t,r)dr, \qquad 0 \leqslant t \leqslant T,$$

which can be rewritten as

$$h(t) \triangleq \int_0^1 \gamma(r)p(t,r)dr \geqslant 0, \qquad 0 \leqslant t \leqslant T. \tag{8}$$

This ineq. can be thought of as an economic constraint; the working population, defined as all people aged between a and b, should be at least a given percentage of the total population.

Quantities a and b are constants with $0 \leq a < b \leq 1$; $\gamma(r)$ is a stepfunction; $\gamma(r) = -\alpha$ for $0 \leq r < a$ and $b \leq r < 1$ and $\gamma(r) = 1-\alpha$ for $a \leq r < b$. The parameter α is a constant within the bounds

$$0 < \alpha < \bar{\alpha} \triangleq \frac{ei(a;b)}{ei(0;1)}, \tag{9}$$

where the second inequality has been obtained from the fact that ineq. (8) should be valid for a stationary age distribution.
We will also assume that

$$u(t) \geq 0, \qquad 0 < t \leq T$$

If we use eq. (6), the problem can be completely restated in terms of the birth rate $u(t)$;

minimize T subject to

$$\begin{aligned}
u(t) &= u_0 \geq 0, \ -1 \leq t < 0, \\
u(t) &= u_T \geq 0, \ T-1 \leq t \leq T, \\
u(t) &\geq 0 \quad , \ 0 \leq t < T-1, \\
h(t) &= \int_0^1 \gamma(r)u(t-r)e(r)dr \geq 0, \ 0 \leq t \leq T.
\end{aligned}$$

Here u_0 and u_T are the constant birth rates, corresponding to $p_0(r)$ and $p_T(r)$ respectively. Without loss of generality we will take $u_0 = 1$. In a later stage sometimes the condition that the optimal birth rate must be nonincreasing will be added.

The problem will be approached in a slightly different manner. The roles played by the quantities T and u_T are interchanged, i.e. in obtaining a solution we assume that T is fixed and that u_T must be minimized. This trick is justified by the fact that the mapping $T \to u_T$ is nonincreasing and continuous which can be proved for the problems treated. Our problem now reads

minimize u_T subject to

$$u(t) = 1 \quad , \ -1 \leq t < 0, \tag{10}$$

$$u(t) = u_T \quad , \ T-1 \leq t \leq T, \tag{11}$$

$$u(t) \geq 0 \quad , \ 0 \leq t < T-1, \tag{12}$$

$$h(t) = \int_0^1 \gamma(r)u(t-r)e(r)dr \geq 0 \quad , \ 0 \leq t \leq T. \tag{13}$$

4. Elucidation of the constructive solution scheme.

Because the function to be minimized and the constraints (10)-(13) are all linear in u_T and $u(t)$, the problem just stated can considered to be a Linear Programming problem in an abstract space. So a saddlepoint theorem can be applied which gives necessary and sufficient conditions for the optimal solutions u_T^*, $u^*(t)$. A start in this direction has already been made [4].

In this paper however we want to follow a more direct approach, although the class of problems to which it can be applied is not as wide as the class for which the attack in [4] is valid.

We want to make clear certain features of the proposed method by treating two examples. The first example, though not very realistic, serves well to illustrate the method.

Example I.

In this example we take $\mu(r) = 0$ for $0 \leq r < 1$, $a = 0$, $b = \frac{1}{2}$, $\alpha = \frac{1}{3}$, $T = 2$. The problem now is to minimize u_T subject to

$$u(t) = 1 \quad , \qquad -1 \leq t < 0, \tag{14}$$

$$u(t) = u_T \quad , \qquad 1 \leq t < 2, \tag{15}$$

$$u(t) \geq 0 \quad , \qquad 0 \leq t < 1, \tag{16}$$

$$h(t) = \int_0^{\frac{1}{2}} u(t-r)dr - \frac{1}{3} \int_0^1 u(t-r)dr \geq 0, \; 0 \leq t \leq 2 \tag{17}$$

Consider ineq. (17) only for $t = \frac{1}{2}$, 1 and $\frac{2}{3}$ respectively;

$$\int_0^{\frac{1}{2}} u(t)dt \geq \frac{1}{4}, \tag{18}$$

$$\int_{\frac{1}{2}}^1 u(t)dt \geq \frac{1}{2} \int_0^{\frac{1}{2}} u(t)dt, \tag{19}$$

$$\frac{1}{2} u_T \geq \frac{1}{2} \int_{\frac{1}{2}}^1 u(t)dt, \tag{20}$$

from which it follows that $u_T^* \geq \frac{1}{8}$. However, $u_T^* = \frac{1}{8}$, because the piecewise constant control deduced from ineqs. (18)-(20) by imposing the equality-sign

$$u(t) = \frac{1}{2} \quad , \quad 0 \leq t < \frac{1}{2}$$
$$u(t) = \frac{1}{4} \quad , \quad \frac{1}{2} \leq t < 1$$
$$u(t) = u_T = \frac{1}{8} \quad , \quad 1 \leq t \leq 2$$

satisfies all the conditions (14)-(17) and hence is optimal (see figure 1).

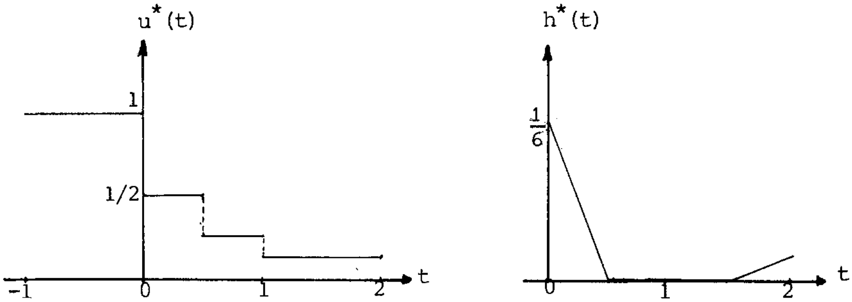

Figure 1. An optimal solution for the first example.

However, the optimal solution is not unique. Another possible solution is for instance

$$u^*(t) = \frac{5}{12} \quad , \quad 0 \leq t < \frac{1}{4} ,$$
$$u^*(t) = \frac{7}{12} \quad , \quad \frac{1}{4} \leq t < \frac{1}{2} ,$$
$$u^*(t) = \frac{1}{6} \quad , \quad \frac{1}{2} \leq t < \frac{3}{4} ,$$
$$u^*(t) = \frac{1}{3} \quad , \quad \frac{3}{4} \leq t < 1 ,$$
$$u^*(t) = u_T^* = \frac{1}{8} , \quad 1 \leq t \leq 2.$$

This optimal solution is sketched in figure 2.

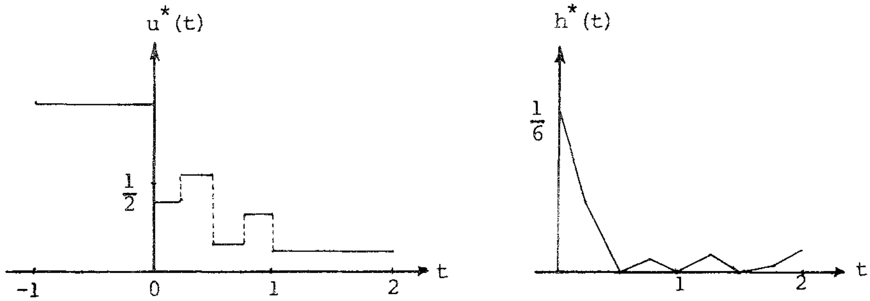

Figure 2. Another optimal solution for the first example.

In order to give a complete characterization of the class of all optimal solutions we define

$$f(t) = \int_0^t u(t)dt, \qquad t \geq 0$$

and

$$x_i(\tau) = f(\tau+(i-1)\tfrac{1}{2}), \qquad i = 1,2, \quad 0 \leq \tau \leq \tfrac{1}{2}\,.$$

It follows from (18)-(20) with equality-signs that $x_2(\frac{1}{2}) = f(1) = \frac{3}{8}$. On the intervals $\frac{1}{2}(i-1) \leq \tau \leq \frac{1}{2}i$, $i = 1,2,3,4$ ineq. (17) becomes:

$$\left.\begin{aligned} x_1(\tau) &\geq \tau - \tfrac{1}{4} \\ 2x_2(\tau) - 3x_1(\tau) &\geq \tfrac{1}{2} - \tau \\ -3x_2(\tau) + x_1(\tau) &\geq -\tfrac{1}{4}\tau - \tfrac{3}{4} \\ x_2(\tau) &\geq \tfrac{1}{8}\tau + \tfrac{1}{4} \end{aligned}\right\} \quad 0 \leq \tau \leq \tfrac{1}{2}\,, \tag{21}$$

where $u_T^* = \frac{1}{8}$ and $x_2(\frac{1}{2}) = \frac{3}{8}$ have been substituted.

In the three dimensional space, spanned by x_1-, x_2- and τ-axes, the points (x_1, x_2, τ) satisfying ineqs. (21) form a bounded set; this set is a tetrahedron and has been drawn in figure 3. For $\tau = 0$ only one point (x_1, x_2) satisfies ineqs. (21), viz. $x_1 = 0$, $x_2 = \frac{1}{4}$. For $\tau = \frac{1}{2}$ again only one point satisfies ineqs. (21), viz. $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{8}$. For intermediate τ, however, infinitely many points (x_1, x_2) satisfy ineq. (21). For each τ, $0 \leqslant \tau \leqslant \frac{1}{2}$, we choose a point (x_1, x_2) subject to the ineqs. (21) in such a way that $x_i(\tau_1) \leqslant x_i(\tau)$ for all $\tau_1 \leqslant \tau$ and $i = 1,2$ in order for ineq. (16) to be satisfied; then the points constitute an optimal solution. Because the optimal $x_i(\tau)$ may have jumps, even optimal birth rates are possible which possess delta-functions.

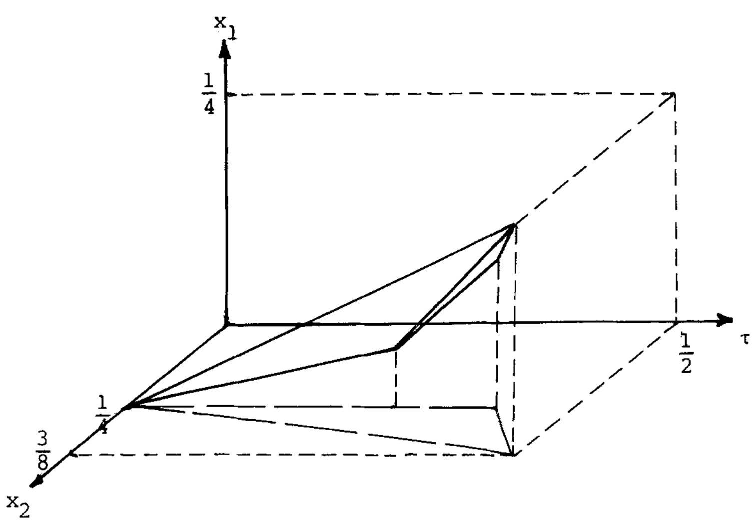

Figure 3. Set of all optimal solutions.

It is easily seen that, if only non-increasing $u(t)$ functions are allowed, the solution is unique and equals the one of fig.1. In fact, if $u(t)$ is nonincreasing, then $x_i(\tau)$ is concave (a connecting bar lies below or on the curve) and the only possibility then is the straight line connecting ($\tau=0$, $x_1=0$, $x_2=\frac{1}{4}$) and ($\tau=\frac{1}{2}$, $x_1=\frac{1}{4}$, $x_2=\frac{3}{8}$); see also figure 3.

We conclude this example by summarizing the facts proved.

(i) An optimal solution (piecewise constant) can be found by considering the ineq. $h(t) \geqslant 0$ only at a finite number of characteristic points ($t = \frac{1}{2}$, 1, $\frac{3}{2}$).

(ii) The class of all optimal solutions can be completely characterized with the help of the $x_i(\tau)$-functions.

(iii) This class contains a unique non-increasing control.

The above analysis can be extended to more realistic situations. For example the mortality function may be taken constant and the final time T arbitrary. Instead of going through this problem in quite its generality, only a rough sketch of the solution method and the results for a specific example will be given. For a more detailed discussion and other generalizations one is referred to [4].

Example II.

In this example we take $\mu(r) = \bar{\mu}$ = constant for $0 \leq r < 1$, $a = \frac{1}{3}$ and $b = \frac{2}{3}$. The parameter α is a constant satisfying the ineq. (9). The final time T is taken arbitrarily. In this case the problem reads

minimize u_T subject to

$$u(t) = 1 \quad , -1 \leq t < 0 , \tag{22}$$

$$u(t) = u_T \quad , \quad T-1 \leq t < T, \tag{23}$$

$$u(t) \geq 0 \quad , \quad 0 \leq t < T-1 \tag{24}$$

$$h(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} u(t-r)e(r)dr - \alpha \int_0^1 u(t-r)e(r)dr \geq 0 \quad , 0 \leq t \leq T. \tag{25}$$

It turns out that the critical points on the time axis, which play a crucial role in the analysis, are the points given by

$$t_k = k.\nu \quad , \quad k = 1,\ldots, N+1 ,$$

and $$\tilde{t}_k = (k-1).\nu + \sigma, \quad k = 1,\ldots, N+1,$$

with $\nu = \frac{1}{3}$, $N = [T/\nu]$ and $\sigma = T-N.\nu$, where $[T/\nu]$ is the largest natural number less then or equal to T/ν.

Now define

$$f(t) = \int_0^t u(t-r)\exp[-\bar{\mu}r]dr \quad , \quad 0 \leq t \leq T , \tag{26}$$

$$x_i(\tau) = f[t_{i-1} + \tau(\tilde{t}_i - t_{i-1})], \; i = 1,\ldots, N-2; \quad 0 \leq \tau \leq 1, \tag{27}$$

$$\tilde{x}_i(\tau) = f[\tilde{t}_i + \tau(t_i - \tilde{t}_i) \qquad i = 1,\ldots, N-3; \quad 0 \leq \tau \leq 1, \tag{28}$$

$$\bar{x} = x_{N-2}(1) \quad , \quad t_0 = 0 \tag{29}$$

The ineq. (25) can be transformed into restrictions on the functions $x_i(\tau)$ and $\tilde{x}_i(\tau)$. In a compact way the restrictions are given by the following matrix inequality

$$A(\tau)\ x\ (\tau)\ \geq b(\tau) \qquad , \quad 0 \leq \tau \leq 1 \tag{30}$$

with

$$x(\tau) = (x_1(\tau), \tilde{x}_1(\tau), x_2(\tau), \ldots\ldots, \tilde{x}_{N-3}(\tau),\ x_{N-2}(\tau),\ \bar{x},\ u_T)' \tag{31}$$

Matrix $A(\tau)$ has size $(2N+1) \times (2N-3)$ and its elements, as well as the components of $b(\tau)$, are given in the appendix.
An optimal solution can be obtained in the following way

1) Minimize u_T subject to the economic constraint (30) at the characteristic points only, i.e. subject to the constraint

$$A(0)x(0) \geq b(0). \tag{32}$$

Note that $x_1(0) = 0$. This is a finite-dimensional linear programming problem which can be solved by standard techniques. Call the solution of this LP-problem

$$x^*(0) = (x_1^*(0),\ \tilde{x}_1{}^*(0), \ldots\ldots,\ x_{N-2}^*(0),\ \bar{x}^*,\ u_T^*)' \tag{33}$$

2) Now choose $u(t)$, $0 \leq t < T-1$, to be piecewise constant, i.e.,

$$\begin{aligned} u(t) &= u_i \qquad , \quad t_{i-1} \leq t < \tilde{t}_i \qquad , \quad i = 1,2,\ldots,N-2 \\ u(t) &= \tilde{u}_i \qquad , \quad \tilde{t}_i \leq t < t_i \qquad , \quad i = 1,2,\ldots,N-3 \end{aligned}$$

The quantities u_i and $\tilde{u}_i$ are uniquely determined by eq. (33) and the formulas (27)-(28). For $\mu = 0$ the calculations have been carried out analytically and the result is

$$u_T = \frac{\nu(1-\xi^N)(1+\xi)\xi^{N-2} + \sigma(1+\xi^{N-1})(\xi^2-1)\xi^{N-2}}{\nu(1-\xi^N)(1+\xi^{2N-3}) + \sigma(1+\xi^{N-1})(\xi^2-1)\xi^{N-2}} \tag{34}$$

$$u_i = \frac{(\nu-\sigma)\xi^{N-2}(1-\xi^2)(1+\xi^{N-1}) + \nu(\xi^i+\xi^{2N-1-i})(1-\xi^{N-2})}{(\nu-\sigma)(1-\xi^N)(1+\xi^{2N-3}) + \sigma(1+\xi^{2N-1})(1-\xi^{N-2})} \quad , \tag{35}$$

$$i = 1,2,\ldots,N-2$$

$$\tilde{u}_i = \frac{\nu(1-\xi^N)(\xi^i+\xi^{2N-3-i}) + \sigma(1+\xi^{N-1})(\xi^2-1)\xi^{N-2}}{\nu(1-\xi^N)(1+\xi^{2N-3}) + \sigma(1+\xi^{N-1})(\xi^2-1)\xi^{N-2}}, \tag{36}$$

$$i = 1,2,\ldots,N-3$$

where ξ is defined as the largest root of

$$-\alpha\xi^2 + (1-\alpha)\xi - \alpha = 0$$

3) It can be easily shown that

$$u_1 > \tilde{u}_1 > u_2 > \ldots\ldots > \tilde{u}_{N-3} > u_{N-2} > u_T > 0$$

So the constraint (24) is satisfied. Moreover it can be proved that the solution (34)-(36) satisfies the economic constraint (30) for all $\tau \in [0,1]$, and hence the solution (34)-(36) is an optimal solution.
Some remarks will be made on the uniqueness for the piecewise constant solution $u^*(t)$ found. Because $\bar{x}$ and u_T are not time dependent and are known from (34)-(36) they will be substituted in (30) with as result:

$$\hat{A}\,\hat{x}(\tau) \geq \hat{b}(\tau) \quad , \quad 0 \leq \tau \leq 1, \tag{37}$$

where $\hat{x}(\tau) = (x_1(\tau), \tilde{x}_1(\tau), \ldots, x_{N\div 2}(\tau))'$ and the size of the constant matrix $\hat{A}$ is $(2N+1) \times (2N-5)$; $\hat{b}(s)$ is reconstructed from b, $\bar{x}$ and u_T.

In the 2(N-2)-dimensional space spanned by the components of $\hat{x}$ and the parameter τ an admissible region for $\hat{x}$ and τ exists with $0 \leq \tau \leq 1$; one can imagine a figure similar to figure 3. Such a region of admissible $\hat{x},\tau$ points, i.e. those $\hat{x}$ and τ which satisfy (37), will now be bounded by curved hypersurfaces because in general $\mu \neq 0$. So the admissible region will be banana-shaped. For each $\tau \in [0,1]$ all the admissible $\hat{x}$ of course constitute a convex set.
As was shown in the first example, the optimal control is unique if only non-increasing solutions are allowed. Is this also true in this example? The answer is affirmative. One has to investigate a matrix D, which can be constructed from the matrix A, on inverse-monotonicity [4].

We conclude this example by sketching the function $u_T(\tau)$ for different values of $\bar{\mu}$ and α. For $\bar{\mu} \neq 0$ the function values has been obtain numerically.

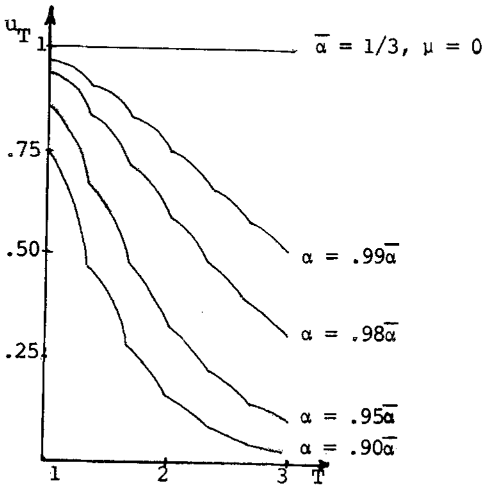

Figure 4, The values of $u_T(T)$ for $\bar{\mu} = 0$ and several values of α,

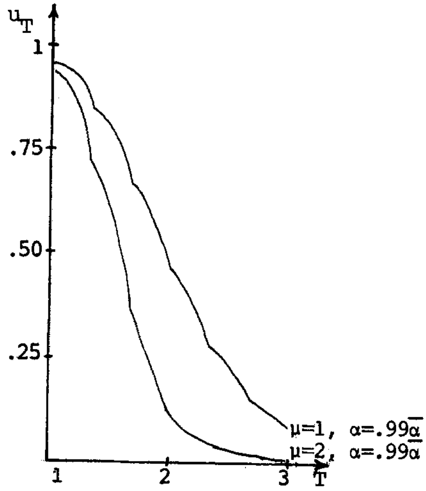

Figure 5. The values of $u_T(T)$ for $\bar{\mu} = 1$ and 2 respectively and $\alpha = .99\bar{\alpha}$.

Note that for $\alpha = \bar{\alpha}$ the working population can just support the non-working population. There is no freedom left to reduce u_T.

5. Conclusion.

In this paper some mathematical features of a population planning problem have been investigated. An open loop control has been found which decreases (or increases) the number of people to a desired level and distribution as quickly as possible subject to the condition that the working population must be at least a given percentage of the total population at each instant of time. Remarkably, the optimal solution to this dynamic problem can be obtained by linear programming provided the working population and mortality function satisfy suitable prerequisities.

A constraint, which has not been considered in this research, is a minimum level of fertility (or maternity functions), i.e. u(t) should satisfy

$$u(t) \geq \int_0^1 \delta(t,r)p(t,r)dr,\ 0 \leq t \leq T, \text{ for some function } \delta(t,r).$$

Only constant mortality functions have been considered. Some mortality functions somewhat closer to reality may be $\mu(r) = \frac{\pi}{2} \operatorname{tg} \frac{\pi r}{2}$, with corresponding stationary population $p(r) = c. \cos \frac{\pi r}{2}$, or $\mu = \frac{1}{1-r}$ with corresponding $p(r) = c(1-r)$.

The first mortality-function may be considered as a crude approximation of a mortality-function of a developed country, whereas the second one may approximate the situation for a developing country. No analytic solutions are known for these cases at this time.

6. Appendix.

Matrix $A(\tau)$ and vector $b(\tau)$, as defined in ineq. (30) will be given here. Matrix $A(\tau)$ has size $(2N+1) \times (2N-3)$ and $b(\tau)$ has $(2N+1)$ components. All elements a_{ij} of $A(\tau)$ are zero except for

$$a_{ii} = -\alpha,\ a_{i+2,i} = e(\nu),\ a_{i+4,i} = -e(\nu)^2,\ a_{i+6,i} = \alpha e(\nu)^3;$$

$$i = 1,2,\ldots,2N-5;$$

$$a_{i,2N-4} = -\alpha.e(\sigma(\tau-1) + \nu(i-N+3));\ a_{i,2N-3} = -\alpha.ei(0;\sigma(\tau-1) + \nu(i-N+3));$$

$$i = 2N-3;2N+1;$$

$$a_{i,2N-4} = (1-\alpha).e(\sigma(\tau-1) + \nu(i-N+3));\ a_{i,2N-3} = (1-\alpha).ei(0;\sigma(\tau-1) + \nu(i-N+3));\quad i = 2N-1;$$

$$a_{i,2N-4} = -\alpha.e(\tau(\nu-\sigma) + \nu(i-N+2));\ a_{i,2N-3} = -\alpha.ei(0;\tau(\nu-\sigma) + \nu(i-N+3));$$

$$i = 2N-4;\ 2N;$$

$$a_{i,2N-4} = (1-\alpha).e(\tau(\nu-\sigma) + \nu(i-N+2));\ a_{i,2N-3} = (1-\alpha).ei(0;\tau(\nu-\sigma) + \nu(i-N+2));\quad i = 2N-2;$$

The components b_i of the vector b are zero except for

$$b_1 = \alpha.ei(\tau\sigma;\nu) - (1-\alpha).\ ei(\nu;2\nu) + \alpha.ei(2\nu;1)\ ;$$

$$b_2 = \alpha.ei(\tau(\nu-\sigma) + \sigma;\nu) - (1-\alpha).ei(\nu;\ 2\nu) + \alpha\ ei(2\nu;1)\ ;$$

$$b_3 = -(1-\alpha).ei(\tau\sigma+\nu;2\nu) + \alpha.ei(2\nu;1) \ ;$$

$$b_4 = -(1-\alpha).ei(\tau(\nu-\sigma) + \sigma + \nu;2\nu) + \alpha.ei(2\nu;1) \ ;$$

$$b_5 = \alpha.ei(\tau\sigma + 2\nu;1) \ ;$$

$$b_6 = \alpha.ei(\tau(\nu-\sigma) + \sigma + 2\nu;1).$$

7. References.

[1] Langhaar, H.L.; General Population Theory in the Age-Time Continuum, J. of the Franklin Inst., vol. 293, no. 3, March 1973, pp. 199 - 214.

[2] Falkenburg, D.R.; Optimal Control in Age Dependent Populations, Proceedings 1973 J.A.C.C., Columbus, Ohio, pp. 112 - 117.

[3] Kwakernaak, H.; Population policy as an optimal control problem, Memorandum nr. 94, Twente University of Technology, The Netherlands, 1975 (in Dutch).

[4] Olsder, G.J. and Strijbos, R.C.W.,; Population Planning; a distributed time optimal control problem, Memorandum nr. 64, Twente University of Technology, The Netherlands, 1975.