
CKN Theory of Singularities of Weak Solutions of the Navier-Stokes Equations

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Notations

The lectures are devoted to a complete exposition of the theory of singularities of the Navier Stokes equations solution studied by Leray, in a simple geometrical setting in which the fluid is enclosed in a container Ω with periodic boundary conditions and side size L . The theory is due to the work of Scheffer, Caffarelli, Kohn, Nirenberg and is called here CKN-theory as it is inspired by the work of the last three authors which considerably improved the earlier estimates of Scheffer.

Although the theory of Leray is well known I recall it here getting at the same time a chance at establishing a few notations, [Ga02].

(1) An underlined letter, *e.g.* $\underline{A}, \underline{A}, \dots$, denotes a 3-dimensional vector (*i.e.* three real or complex numbers) and underlined partial derivative symbol $\underline{\partial}, \underline{\partial}, \dots$ denotes the gradient operator $(\partial_1, \partial_2, \partial_3)$. A vector field \underline{u} is a function on Ω .

(2) Repeated labels convention is used (labels are letters or other) when not ambiguous: hence $\underline{A} \cdot \underline{B}$ or $\underline{A} \cdot \underline{B}$ means sum over i of $A_i B_i$. Therefore $\underline{\partial} \cdot \underline{u}$, if \underline{u} is a vector field, is the divergence of \underline{u} , namely $\sum_i \partial_i u_i$. $\underline{\partial} \cdot \underline{\partial} = \Delta$ is the Laplace operator.

(3) Multiple derivatives are tensors, so that $\underline{\partial} \underline{\partial} f$ is the tensor $\partial_{ij} f$. The $L_2(\Omega)$ is the space of the square integrable functions on Ω : the squared norm of $f \in L_2$ will be $\|f\|_2^2 \stackrel{def}{=} \int_{\Omega} |f(\underline{x})|^2 s\underline{x}$.

(4) The Navier-Stokes equation with regularization parameter λ is

$$\dot{\underline{u}} = \nu \Delta \underline{u} - \langle \underline{u} \rangle_{\lambda} \cdot \underline{\partial} \underline{u} - \underline{\partial} p, \quad \underline{\partial} \cdot \underline{u} = 0, \quad \int_{\Omega} \underline{u} d\underline{x} = \underline{0} \quad (0.1)$$

where the unknowns are $\underline{u}(\underline{x}, t), p(\underline{x}, t)$ with zero average and, [Ga02],

(i) \underline{u} is a divergenceless field, p is a scalar field

(ii) $\langle \underline{u} \rangle_\lambda = \int_\Omega \chi_\lambda(\underline{x} - \underline{y}) \underline{u}(\underline{y}) d\underline{y}$ and χ_λ is defined in terms of a $C^\infty(\Omega)$ function $\underline{x} \rightarrow \chi(\underline{x}) \geq 0$ not vanishing in a small neighborhood of the origin and with integral $\int \chi(\underline{x}) d\underline{x} \equiv 1$: the function $\chi(\underline{x})$ can be regarded as a periodic function on Ω or as a function on R^3 with value 0 outside Ω , as we shall imagine that Ω is centered at the origin, to fix the ideas. For $\lambda \geq 1$ also the function $\chi_\lambda(\underline{x}) \stackrel{def}{=} \lambda^3 \chi(\lambda \underline{x})$ can be regarded as a periodic function on Ω or as a function on R^3 : it is an “approximate Dirac’s δ -function”. Usually the NS equation contains a volume force too: here we set it equal to $\underline{0}$.¹

(iii) The initial datum is a divergenceless velocity field $\underline{u}^0 \in L_2(\Omega)$ with $\underline{0}$ average; no initial datum for p as p is determined from \underline{u}^0 .

(5) A *weak solution* of the NS equations with initial datum $\underline{u}_0 \in L_2(\Omega)$ is a limit on subsequences of $\lambda \rightarrow \infty$ of solutions $\underline{u}^\lambda, p^\lambda$ of (0.1). This means that the Fourier transforms of \underline{u}^λ , and p^λ exist and have components $\underline{u}_k^\lambda(t) \equiv \int_\Omega e^{-ik\underline{x}} \underline{u}^\lambda(\underline{x}) d\underline{x}$ ($\underline{k} = \frac{2\pi}{L} \underline{n}$, $\underline{n} \in Z^3$), and $p_k^\lambda(t)$ which have a limit as $\lambda \rightarrow \infty$ (on subsequences) for each $\underline{k} \neq \underline{0}$ and the limit of the $\underline{u}_k(t)$ is absolutely continuous. This is equivalent to the existence of the limits of the L_2 products $(\underline{u}(t), \underline{\varphi})_{L_2}$ and $(p(t), \psi)_{L_2}$ for all $t \in (0, \infty]$ and for all test functions $\underline{f}(\underline{x}), \psi(\underline{x})$. There might be several such limits (*i.e.* the limit may depend on the subsequence) and what follows applies to any one among them.

The core of the analysis will deal with the regularized equation and the properties of its solutions, which are easily shown to be C^∞ in $\underline{x} \in \Omega$ and in $t \in (0, \infty)$ if the initial datum is $\underline{u} \in L_2(\Omega)$. The limit $\lambda \rightarrow \infty$ will be taken at the end and it is where the theory becomes non constructive because there is need to consider the limit on subsequences.

Of course the point is to obtain bounds which are uniform in $\lambda \rightarrow \infty$ and the limit $\lambda \rightarrow \infty$ only intervenes at the end to formulate the results in a nice form.

The theory of Leray is based on the following *a priori* bounds, see section 3.2 in [Ga02], on solutions of (0.1) with initial datum \underline{u}_0 with L_2 square norm E_0

$$\|\underline{u}^\lambda(t)\|_2^2 \leq E_0, \quad \int_0^t d\tau \|\partial \underline{u}^\lambda(\tau)\|_2^2 \leq \frac{1}{2} E_0 \nu^{-1} \quad (0.2)$$

satisfied by the solution \underline{u}^λ .

The notes are extracted from reference [Ga02] to which the reader is referred for details on the above results and have been made independent from [Ga02] modulo the above results (in fact, essentially, only modulo the statements on the regularized equation (0.1)).

The proof is conceptually quite simple and is based on a few (clever) *a priori* Sobolev inequalities: the estimates are discussed in Sect. 1-3, which form an introduction.

¹ This is a simplicity assumption as the extension of the theory to cases with time independent smooth (*e.g.* C^∞) volume forces would be immediate and just a notational nuisance.

Their application to the analysis of (0.1) is in Sect. 4 where the main theorems are discussed and the CKN main result is reduced via the inequalities to Scheffer's theorem. The method is a kind of multiscale analysis which allows us to obtain regularity *provided* a control quantity, identified here as the “*local Reynolds number*” on various scales, is small enough. Unfortunately it is not (yet) possible to prove that the local Reynolds number is small on all small regions (physically this would mean that in such regions the flow would be laminar, hence smooth on small scale).

However the *a priori* bounds give the information that the local Reynolds number must be small near many points in Ω and, via standard techniques, an estimate of the dimension of the possibly bad points follows. The application to the fractal dimension bound is essentially an “abstract reasoning” consequence of the results of Sect. 4 and is in Sect. 5.

The proofs of the various Sobolev inequalities necessary to obtain the key Scheffer's theorem and of the new ones studied by CKN is described concisely but in full detail in the series of problems at the end of the text: the hints describe quickly the various steps of the proofs (however without skipping any detail, to my knowledge).

1 Leray's Solutions and Energy

The theory of space–time singularities will be partly based upon simple general *kinematic inequalities*, which therefore have little to do with the Navier–Stokes equation, and partly they will be based on the local energy conservation which follows as a consequence of the Navier–Stokes equations *but it is not equivalent to them*.

Energy conservation for the regularized equations (0.1) says that the kinetic energy variation in a given volume element Δ of the fluid, in a time interval $[t_0, t_1]$, plus the energy dissipated therein by friction, equals the sum of the kinetic energy that in the time interval $t \in [t_0, t_1]$ enters in the volume element plus the work performed by the pressure forces (on the boundary element) plus the work of the volume forces (none in our case). The analytic form of this relation is simply obtained by multiplying both sides of the first of the (0.1) by \underline{u} and integrating on the volume element Δ and over the time interval $[t_0, t_1]$.

The relation that one gets can be generalized to the case in which the volume element has a time dependent shape. And an even more general relation can be obtained by multiplying both sides of (0.1) by $\varphi(\underline{x}, t)\underline{u}(\underline{x}, t)$ where φ is a $C^\infty(\Omega \times (0, s])$ function with $\varphi(\underline{x}, t)$ zero for t near 0 (here s is a positive parameter).

Energy conservation in a sharply defined volume Δ and time interval $t \in [t_0, t_1]$ can be obtained as limiting case of choices of φ in the limit in which it becomes the characteristic function of the space–time volume element $\Delta \times [t_0, t_1]$.

Making use of a regular function $\varphi(\underline{x}, t)$ is useful, particularly in the rather “desperate” situation in which we are when using the theory of Leray. The “solutions” \underline{u} (obtained by removing, in (0.1), the regularization, *i.e.* letting $\lambda \rightarrow \infty$) are only “weak solutions”. Therefore, the relations that are obtained in the limit $\lambda\omega\infty$ can be interpreted as valid only after suitable integrations by parts that allow us to avoid introducing derivatives of \underline{u} (whose existence is not guaranteed by the theory) at the “expense” of differentiating the “test function” φ .

Performing analytically the computation of the energy balance, described above in words, in the case of the regularized equation (0.1) and via a few integrations by parts² we get the following relation

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} d\underline{\xi} |\underline{u}(\underline{\xi}, s)|^2 \varphi(\underline{\xi}, s) + \nu \int_0^s dt \int_{\Omega} \varphi(\underline{\xi}, t) |\underline{\partial} \underline{u}(\underline{\xi}, t)|^2 d\underline{x} = \\ & = \int_0^s \int_{\Omega} \left[\frac{1}{2} (\varphi_t + \nu \Delta \varphi) |\underline{u}|^2 + \frac{1}{2} |\underline{u}|^2 \langle \underline{u} \rangle_{\lambda} \cdot \underline{\partial} \varphi + p \underline{u} \cdot \underline{\partial} \varphi \right] dt d\underline{\xi} \end{aligned} \quad (1.1)$$

where $\varphi_t \equiv \partial_t \varphi$ and $\underline{u} = \underline{u}^{\lambda}$ is in fact depending also on the regularization parameter λ ; here p is the pressure $p = -\sum_{ij} \Delta^{-1} \partial_i \partial_j (u_i u_j)$.

Suppose that the solution of (0.1) with fixed initial datum \underline{u}_0 converges (weakly in L_2), for $\lambda \rightarrow \infty$, to a “Leray solution” \underline{u} possibly only over a subsequence $\lambda_n \rightarrow \infty$.

The (1.1) implies that any (in case of non uniqueness) Leray solution \underline{u} verifies the *energy inequality*:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\underline{u}(\underline{\xi}, s)|^2 \varphi(\underline{\xi}, s) d\underline{\xi} + \nu \int_{t \leq s} \int_{\Omega} \varphi(\underline{\xi}, t) |\underline{\partial} \underline{u}(\underline{\xi}, t)|^2 d\underline{\xi} dt \leq \\ & \leq \int_{t \leq s} \int_{\Omega} \left[\frac{1}{2} (\varphi_t + \nu \Delta \varphi) |\underline{u}|^2 + \frac{1}{2} |\underline{u}|^2 \underline{u} \cdot \underline{\partial} \varphi + p \underline{u} \cdot \underline{\partial} \varphi \right] d\underline{\xi} dt \end{aligned} \quad (1.2)$$

where the pressure p is given by $p = -\sum_{ij} \Delta^{-1} \partial_i \partial_j (u_i u_j) \equiv -\Delta^{-1} \underline{\partial} \underline{\partial} (\underline{u} \underline{u}) \equiv -\Delta^{-1} (\underline{\partial} \underline{u})^2$.

Remark 1.1.

(1) It is important to remark that in this relation one might expect the *equality sign*: as we shall see the fact that we cannot do better than just obtaining an inequality means that the limit necessary to reach a Leray solution can introduce a “spurious dissipation” that we are simply unable to understand on the basis of what we know (today) about the Leray solutions.

(2) The above “strange” phenomenon reflects our inability to develop a complete theory of the Navier–Stokes equation, but one can conjecture that no other dissipation can take place and that a (yet to come) complete theory of the equations could show this. Hence we should take the inequality sign in (1.2) as one more manifestation of the inadequacy of the Leray’s solution.

² The solutions of (0.1) are $C^{\infty}(\Omega \times [0, \infty))$ so that there is no need to justify integrating by parts.

The proof of (1.2) and of the other inequalities that we shall quote and use in this section is elementary and based, *c.f.r.* problem [15] below, on a few general “kinematic inequalities” that we now list (all of them will be used in the following but only (S) and (CZ) are needed to check (1.2)).

2 Kinematic Inequalities

A first “kinematic” inequality, *i.e.* the first inequality that we shall need and that holds for any function f , is ³

(P) *Poincaré inequality:*

$$\int_{B_r} d\underline{x} |f - F|^\alpha \leq C_\alpha^P r^{3-2\alpha} \left(\int_{B_r} d\underline{x} |\partial f| \right)^\alpha, \quad 1 \leq \alpha \leq \frac{3}{2} \quad (2.1)$$

where F is the average of f on the ball B_r with radius r and C_α^P is a suitable constant. We shall denote (2.1) by (P).

A second kinematic inequality that we shall use is

(S) *Sobolev inequality:*

$$\begin{aligned} \int_{B_r} |\underline{u}|^q d\underline{x} \leq C_q^S \left[\left(\int_{B_r} (\partial \underline{u})^2 d\underline{x} \right)^a \cdot \left(\int_{B_r} |\underline{u}|^2 d\underline{x} \right)^{q/2-a} + \right. \\ \left. + r^{-2a} \left(\int_{B_r} |\underline{u}|^2 d\underline{x} \right)^{q/2} \right] \quad \text{if } 2 \leq q \leq 6, \quad a = \frac{3}{4}(q-2) \end{aligned} \quad (2.2)$$

where B_r is a ball of radius r and the integrals are performed with respect to $d\underline{x}$. The C_q^S is a suitable constant; the second term of the right hand side can be omitted if \underline{u} has zero average over B_r . We shall denote (2.2) by (S), [So63].

A third necessary kinematic inequality will be

(CZ) *Calderon–Zygmund inequality:*

$$\int_{\Omega} \left| \sum_{i,j} (\Delta^{-1} \partial_i \partial_j)(u_i u_j) \right|^q d\underline{\xi} \leq C_q^L \int_{\Omega} |\underline{u}|^{2q} d\underline{\xi}, \quad 1 < q < \infty \quad (2.3)$$

which we shall denote (CZ): here Ω is the torus of side L and C_q^L is a suitable constant, [St93].

And finally

(H) *Hölder inequality:*

³ The inequalities should be regarded as inequalities for C^∞ functions; they can be extended to the appropriate Sobolev spaces by continuity.

$$\left| \int f_1 f_2 \dots f_n \right| \leq \prod_{i=1}^n \left(\int |f_i|^{p_i} \right)^{\frac{1}{p_i}}, \quad \sum_{i=1}^n \frac{1}{p_i} = 1 \quad (2.4)$$

which we shall denote (H): the integrals are performed over an arbitrary domain with respect to an arbitrary measure (of course the same for all integrals).

Remark 2.1. (H) are a trivial extension of the Schwartz-Hölder inequalities; while (S) and (P) (mainly in the cases, important in what follows, $q = 6$ and $\alpha = \frac{3}{2}$) and (CZ) are less elementary and we refer to the literature, [So63], [St93], [LL01], for their proofs.

An important consequence of the inequalities is

Proposition 2.1. *Let \underline{u} be a Leray solution verifying (therefore) the a priori bounds in (0.2): $\int_{\Omega} |\underline{u}(\underline{x}, t)|^2 d\underline{x} \leq E_0$ and $\int_0^T dt \int_{\Omega} |\underline{\partial} \underline{u}(\underline{x}, t)|^2 d\underline{x} \leq E_0 \nu^{-1}$ then*

$$\int_0^T dt \int_{\Omega} d\underline{x} |\underline{u}|^{10/3} + \int_0^T dt \int_{\Omega} d\underline{x} |p|^{5/3} \leq C \nu^{-1} E_0^{5/3} \quad (2.5)$$

where C can be chosen $C_{\frac{10}{3}}^S (1 + C_{\frac{5}{3}}^L)$.

proof: Apply (S) with $q = \frac{10}{3}$ and $a = 1$:

$$\int_{\Omega} |\underline{u}|^{\frac{10}{3}} d\underline{x} \leq C_{\frac{10}{3}}^S \left(\int_{\Omega} (\underline{\partial} \underline{u})^2 d\underline{x} \right)^1 \cdot \left(\int_{\Omega} \underline{u}^2 d\underline{x} \right)^{\frac{5}{3}-1} \leq C_{\frac{10}{3}}^S E_0^{\frac{2}{3}} \int_{\Omega} |\underline{\partial} \underline{u}|^2 \quad (2.6)$$

hence integrating over t between 0 and T using also the second a priori estimate, we find

$$\int_0^T dt \int_{\Omega} |\underline{u}|^{\frac{10}{3}} d\underline{x} \leq C_{\frac{10}{3}}^S E_0^{\frac{2}{3}} \int_0^T dt \int_{\Omega} d\underline{x} (\underline{\partial} \underline{u})^2 \leq C_{\frac{10}{3}}^S \nu^{-1} E_0^{1+\frac{2}{3}} \quad (2.7)$$

while the (CZ) yields: $\int_{\Omega} d\underline{x} |p|^{\frac{5}{3}} \leq C_{\frac{5}{3}}^L \int_{\Omega} d\underline{x} |\underline{u}|^{\frac{10}{3}}$ which, integrated over t and combined with (2.7), gives the announced result.

3 Pseudo Navier Stokes Velocity – Pressure Pairs. Scaling Operators

As already mentioned the CKN theory will not fully use that \underline{u} verifies the Navier–Stokes equation: in order to better realize this (unpleasant) property it is convenient to define separately the only properties of the Leray solutions that are really needed to develop the theory, *i.e.* to obtain an estimate of the fractal dimension of the space–time singularities set S_0 . This leads to the following notion

Definition 3.1. (*pseudo NS velocity field*): Let $t \rightarrow (\underline{u}(\cdot, t), p(\cdot, t))$ be a function with values in the space of zero average square integrable “velocity” and “pressure” fields on Ω . Suppose that for each $\varphi \in C^\infty(\Omega \times (0, T])$ with $\varphi(\underline{x}, t)$ vanishing for t near zero the following properties hold. For each $T < \infty$ and $s \leq T$:

$$\begin{aligned}
(a) \quad & \int_{\Omega} \underline{u} \, d\underline{x} = \underline{0}, \quad \underline{\partial} \cdot \underline{u} = \underline{0}, \quad p = -\sum_{i,j} \partial_i \partial_j \Delta^{-1}(u_i u_j) \\
(b) \quad & \int_0^T dt \int_{\Omega} d\underline{x} |\underline{u}|^{10/3} + \int_0^T dt \int_{\Omega} d\underline{x} |p|^{5/3} < \infty \\
(c) \quad & \frac{1}{2} \int_{\Omega} d\underline{x} |\underline{u}(\underline{x}, s)|^2 \varphi(\underline{x}, s) + \nu \int_{t \leq s} \int_{\Omega} \varphi(x, t) |\underline{\partial} \underline{u}|^2 d\underline{x} dt \leq \quad (3.1) \\
& \leq \int_{t \leq s} \int_{\Omega} \left[\frac{1}{2} (\varphi_t + \nu \Delta \varphi) |\underline{u}|^2 + \frac{1}{2} |\underline{u}|^2 \underline{u} \cdot \underline{\partial} \varphi + p \underline{u} \cdot \underline{\partial} \varphi \right] d\underline{x} dt
\end{aligned}$$

Then we shall say that the pair (\underline{u}, p) is a pseudo NS velocity and pressure pair. The singularity set in the time interval $[0, T]$ of (\underline{u}, p) will be defined as the set S_0 of the points $(\underline{x}, t) \in \Omega \times [0, T]$ that do not admit a vicinity U where $|\underline{u}|$ is bounded.⁴

The name given to the set S_0 is justified by a general result on the theory of NS equations which shows that is a Leray’s solution of the NS equations is essentially bounded in a neighborhood of a space time point then it is C^∞ near such point.

Proposition 3.1. (*velocity is unbounded near singularities*): Let $\underline{u}(\underline{x}, t)$ be a Leray’s solution of the NS equation in L_2 . Given $t_0 > 0$ suppose that $|\underline{u}(\underline{x}, t)| \leq M$, $(\underline{x}, t) \in U_\rho(\underline{x}_0, t_0) \equiv$ sphere of radius ρ ($\rho < t_0$) around (\underline{x}_0, t_0) , for some $M < \infty$: then $\underline{u} \in C^\infty(U_{\rho/2}(\underline{x}_0, t_0))$.

Remark 3.1.

(1) This means that the only way a singularity can manifest itself, in a Leray solution of the NS equations, is through a divergence of the velocity field itself. For instance it is impossible to have a singular derivative having the velocity itself unbounded. Hence, if $d \geq 3$ velocity discontinuities are impossible (and even less so shock waves), for instance. Naturally if $\underline{u}(\underline{x}, t)$ is modified on a set of points (\underline{x}, t) with zero measure it remains a weak solution (because the Fourier transform, in terms of which the notion of weak solution is defined, does not change), hence the condition $|\underline{u}(\underline{x}, t)| \leq M$ for each $(\underline{x}, t) \in U_\lambda(\underline{x}_0, t_0)$ can be replaced by the condition: for almost all $(\underline{x}, t) \in U_\lambda(\underline{x}_0, t_0)$.

(2) The above result is not strong enough to overcome the difficulties of a local theory of regularity of the Leray weak solutions. Therefore one looks for other results of the same type and it would be desirable to have results concerning

⁴ Here we mean bounded outside a set of zero measure in U or, as one says, *essentially bounded* because it is clear that, being \underline{u}, p in $L_2(\Omega)$, they are defined up to a set of zero measure and it would not make sense to ask that they are bounded everywhere without specifying which realization of the functions we take.

regularity implied by *a priori* informations on the vorticity. We have already seen that bounded total vorticity implies regularity: however it is very difficult to go really beyond; hence it is interesting to note that also other properties of the vorticity may imply regularity. A striking result in this direction, although insufficient for concluding regularity (if true at all) of Leray weak solutions, is in [CF93].

(3) For a proof of the above (Serrin's) theorem see [Ga02], proposition IV in section 3.3.

The remaining part of this section will concern the general properties of the pseudo NS pairs and their regularity at a given point (\underline{x}, t) : it will not have more to do with the velocity and pressure fields that solve the Navier–Stokes equations. It is indeed easy to convince oneself that the (3.1), in spite of the arbitrariness of φ , *are not equivalent*, not even formally, to the Navier–Stokes equations, and they pose far less severe on \underline{u}, p restrictions. We should not be surprised, therefore, if it turned out possible to exhibit pseudo NS pairs that really have singularities on “large sets” of space–time. In a way it is already surprising that the pseudo NS fields verify the regularity properties discussed below.

The analysis of the latter properties (of pseudo NS fields) is based on the mutual relations between certain quantities that we shall call “*dimensionless operators*” relative to the space–time point (\underline{x}_0, t_0)

Definition 3.2. (*dimensionless “operators” for NS*) Let $(\underline{x}_0, t_0) \in \Omega \times (0, \infty)$ and consider the sets ⁵

$$\begin{aligned} \Delta_r(t_0) &= \{t \mid |t - t_0| < r^2 \nu^{-1}\} \\ B_r(\underline{x}_0) &= \{\xi \mid |\xi - \underline{x}_0| < r\} \equiv B_r \\ Q_r(\underline{x}_0, t_0) &= \{(\xi, \vartheta) \mid |\xi - \underline{x}_0| < r, |\vartheta - t_0| < r^2 \nu^{-1}\} \\ Q_r(\underline{x}_0, t_0) &= \Delta_r(t_0) \times B_r(\underline{x}_0) \equiv Q_r \end{aligned} \quad (3.2)$$

define:

(i) “*dimensionless kinetic energy operator*” on scale r :

$$A(r) = \frac{1}{\nu^2 r} \sup_{|t-t_0| \leq \nu^{-1} r^2} \int_{B_r} |\underline{u}(\underline{\xi}, t)|^2 d\underline{\xi} \quad (3.3)$$

and we say that the dimension of A is 1 : this refers to the factor r^{-1} that is used to make the integral dimensionless.

(ii) “*local Reynolds number*” averaged on scale r :

$$\delta(r) = \frac{1}{\nu r} \int_{Q_r} d\vartheta d\underline{\xi} |\partial \underline{u}|^2 \quad (3.4)$$

⁵ If $r \geq L/2$ this is interpreted as $B_r \equiv \Omega$. If $r^2 \nu^{-1} > t_0$ then $\Delta_r(t_0)$ is interpreted as $0 < t < t_0 + r^2 \nu^{-1}$

and we say that the dimension of δ is 1 : this refers in general to the power $-\alpha$ to which r has to be raised so that an expression becomes dimensionless: in this case $\alpha = 1$.

(iii) “dimensionless energy flux” on scale r :

$$G(r) = \frac{1}{\nu^2 r^2} \int_{Q_r} d\vartheta d\underline{\xi} |\underline{u}|^3 \quad (3.5)$$

The dimension of G is 2.

(iv) “dimensionless pressure power” forces on scale r :

$$J(r) = \frac{1}{\nu^2 r^2} \int_{Q_r} d\underline{\xi} d\vartheta |\underline{u}| |p| \quad (3.6)$$

The dimension of J is 2.

(v) “dimensionless non locality” on scale r :

$$K(r) = \frac{r^{-13/4}}{\nu^{3/2}} \int_{\Delta_r} d\vartheta \left(\int_{B_r} |p| d\underline{\xi} \right)^{5/4} \quad (3.7)$$

The dimension of K is $13/4$.

(vi) “dimensionless intensity” on scale r :

$$S(r) = \nu^{-7/3} r^{-5/3} \int_{Q_r} (|\underline{u}|^{10/3} + |p|^{5/3}) d\vartheta d\underline{\xi} \quad (3.8)$$

where the pressure is always defined by the expression

$$p = - \sum_{i,j=1}^3 \partial_i \partial_j \Delta^{-1} (u_i u_j).$$

The dimension of S is $5/3$.

Remark 3.2.

(1) The $A(r), \dots$ are not operators in the common sense of functional analysis. Their name is due to their analogy with the quantities that appear in problems that are studied with the methods of the “renormalization group” (which, also, are not operators in the common sense of the words). Perhaps a more appropriate name could be “dimensionless observables”: but we shall call them operators to stress the analogy of what follows with the methods of the renormalization group.

(2) The $A(r), G(r), J(r), K(r), S(r)$ are in fact estimates of the quantities that their name evokes. We omit the qualifier “estimate” when referring to them for brevity.

(3) The interest of (i) ÷ (iv) becomes manifest if we note that the energy inequality (3.1) can be expressed in terms of such quantities if φ is suitably chosen. Indeed let

$$\varphi = \chi(\underline{x}, t) \frac{\exp\left(-\frac{(\underline{x}-\underline{x}_0)^2}{4(\nu(t-t_0)+2r^2)}\right)}{(4\pi\nu(t-t_0)+8\pi r^2)^{3/2}} \quad (3.9)$$

where $\chi(\underline{x}, t)$ is C^∞ and has value 1 if $(\underline{x}, t) \in Q_{r/2}$ and 0 if $(\underline{x}, t) \notin Q_r$. Then there exists a constant $C > 0$ such that

$$\begin{aligned} |\varphi| &< \frac{C}{r^3}, & |\partial_t \varphi| &< \frac{C}{r^4}, & |\partial_t \varphi + \nu \Delta \varphi| &< \frac{C}{\nu^{-1} r^5}, & \text{everywhere} \\ |\varphi| &> \frac{1}{Cr^3}, & & & & & \text{if } (\underline{x}, t) \in Q_{r/2} \end{aligned} \quad (3.10)$$

Hence (3.1) implies

$$\frac{\nu^2}{Cr^2} (A(\frac{r}{2}) + \delta(\frac{r}{2})) \leq C \left(\frac{1}{\nu^{-1} r^5} \int_{Q_r} |\underline{u}|^2 + \frac{1}{r^4} \int_{Q_r} |\underline{u}|^3 + \frac{1}{r^4} \int_{Q_r} |\underline{u}| |p| \right) \quad (3.11)$$

and, since $\int_{Q_r} |\underline{u}|^2 \leq C (\int_{Q_r} |\underline{u}|^3)^{2/3} (\nu^{-1} r^5)^{1/3}$ with a suitable C , it follows that for some \tilde{C}

$$A(\frac{r}{2}) + \delta(\frac{r}{2}) \leq \tilde{C} (G(r)^{2/3} + G(r) + J(r)) \quad (3.12)$$

(4) Note that the operator $\delta(r)$ is an average of the “local Reynolds’ number” $r \int_{\Delta_r} |\partial_t \underline{u}|^2 d\xi$

(5) The operator (ν) appears if one tries to bound $J(\frac{r}{2})$ in terms of $A(r) + \delta(r)$: such an estimate is indeed possible and it will lead to the *local Scheffer theorem* discussed in the next section.

4 The Theorems of Scheffer and of Caffarelli–Kohn–Nirenberg

We can state the strongest results known (in general and to date) about the regularity of the weak solutions of Navier Stokes equations (which however hold also for the pseudo Navier Stokes velocity–pressure pairs).

Theorem 4.1 (*upper bound on the dimension of the sporadic set of singular times for NS, (“Scheffer’s theorem”)*): *There are two constants $\varepsilon_s, C > 0$ such that if $G(r) + J(r) + K(r) < \varepsilon_s$ for a certain value of r , then \underline{u} is bounded in $Q_{\frac{r}{2}}(\underline{x}_0, t_0)$:*

$$|\underline{u}(\underline{x}, t)| \leq C \frac{\varepsilon_s^{1/3}}{r}, \quad (\underline{x}, t) \in Q_{\frac{r}{2}}(\underline{x}_0, t_0), \quad \text{almost everywhere} \quad (4.1)$$

having set $\nu = 1$.

Remark 4.1.

- (1) *c.f.r.* problems (5)÷(11) for a guide to the proof.
- (2) This theorem can be conveniently combined, for the purpose of checking its

hypotheses, with the inequality: $J(r) + G(r) + K(r) \leq C (S(r)^{9/10} + S(r)^{3/4})$, which follows immediately from inequality (H) and from the definitions of the operators, with a suitable C .

(3) In other words *if the operator $S(r)$ is small enough then (\underline{x}_0, t_0) is a regular point.*

(4) This will imply that the fractal dimension of the space–time singularities set is $\leq 5/3$. In fact, see section 5 below, an *a priori* estimate on the global value of an operator with dimension α implies that the Hausdorff’ measure of the set of points around which the operator is large does not exceed α ; here the operator $S(r)$ has dimension $5/3$ and therefore together with the *a priori* bound (2.5) it yields an estimate $\leq 5/3$ for the Hausdorff dimension of the singularity set. This also justifies the introduction of the operator $S(r)$.

It is easy, in terms of the just defined operators, to illustrate the strategy of the proof of the following theorem which will immediately imply, via a classical argument reproduced in section 5 below, that the fractal dimension of the space time singularities set S_0 for a pseudo NS field is ≤ 1 and that its 1–measure of Hausdorff $\mu_1(S_0)$ vanishes.

Theorem 4.2 (*sufficient condition for local regularity space-time (“CKN theorem”)*): *There is a constant ε_{ckn} such that if (\underline{u}, p) is a pseudo NS pair of velocity and pressure fields and*

$$\limsup_{r \rightarrow 0} \frac{1}{\nu r} \int_{Q_r(\underline{x}_0, t_0)} |\partial \underline{u}(\underline{x}', t')|^2 d\underline{x}' dt' \equiv \limsup_{r \rightarrow 0} \delta(r) < \varepsilon_{ckn} \quad (4.2)$$

then $\underline{u}(\underline{x}', t')$, $p(\underline{x}', t')$ are C^∞ in the vicinity of (\underline{x}_0, t_0) .⁶

For fixed (\underline{x}_0, t_0) , consider the “sequence of length scales”: $r_n \equiv L2^n$, with $n = 0, -1, -2, \dots$. We shall set $\alpha_n \equiv A(r_n)$, $\kappa_n = K_n^{8/5}$, $j_n = J_n$, $g_n = G_n^{2/3}$, $\delta_n = \delta(r_n)$ which is a natural definition as it will shortly appear. And define $\underline{X}_n \equiv (\alpha_n, \kappa_n, j_n, g_n) \in R_+^4$. Then the proof of this theorem is based on a bound that allows us to estimate the size of \underline{X}_n , defined as the sum of its components, in terms of the size of \underline{X}_{n+p} provided the Reynolds number δ_{n+p} on scale $n + p$ is $\leq \delta$.

The inequality will have the form (if $p > 0$ and $0 < \delta < 1$)

$$\underline{X}_n \leq \mathcal{B}_p(\underline{X}_{n+p}; \delta) \quad (4.3)$$

where $\mathcal{B}_p(\cdot; \delta)$ is a map of the whole R_+^4 into itself and the inequality has to be understood “component wise”, *i.e.* in the sense that each component of the l.h.s. is bounded by the corresponding component of the r.h.s. We call $|\underline{X}|$ the sum of the components of $\underline{X} \in R_+^4$.

⁶ This means that near (\underline{x}, t) the functions $\underline{u}(\underline{x}', t')$, $p(\underline{x}', t')$ coincide with C^∞ functions apart from a set of zero measure (recall that the pseudo NS fields are defined as fields in $L_2(\Omega)$)

The map $\mathcal{B}_p(\cdot; \delta)$, which to some readers will appear as strongly related to the “beta function” for the “running couplings” of the “renormalization group approaches”,⁷ will enjoy the following property

Proposition 4.1. *Suppose that p is large enough; given $\rho > 0$ there exists $\delta_p(\rho) > 0$ such that if $\delta < \delta_p(\rho)$ then the iterates of the map $\mathcal{B}_p(\cdot; \delta)$ contract any given ball in R_+^4 , within a finite number of iterations, into the ball of radius ρ : i.e. $|\mathcal{B}_p^k(\underline{X}; \delta)| < \rho$ for all large k 's.*

Assuming the above proposition the main theorem II follows:

proof: Let $\rho = \varepsilon_s$, c.f.r. theorem I, and let p be so large that the above proposition holds. We set $\varepsilon_{ckn} = \delta_p(\varepsilon_s)$ and it will be, by the assumption (4.2), that $\delta_n < \varepsilon_{ckn}$ for all $n \leq n_0$ for a suitable n_0 (recall that the scale labels n are negative).

Therefore it follows that $|\mathcal{B}_p^k(\underline{X}_{n_0}; \varepsilon_{ckn})| < \varepsilon_s$ for some k . Therefore by the theorem I we conclude that (\underline{x}_0, t_0) is a regularity point.

Proposition 5.1 follows immediately from the following general “Sobolev inequalities”

(1) “Kinematic inequalities”: i.e. inequalities depending only on the fact that \underline{u} is a divergence zero, average zero and is in $L_2(\Omega)$ and $p = -\Delta^{-1}(\partial \underline{u})^2$

$$\begin{aligned} J_n &\leq C (2^{-p/5} A_{n+p}^{1/5} G_n^{1/5} K_{n+p}^{4/5} + 2^{2p} A_{n+p}^{1/2} \delta_{n+p}) \\ K_n &\leq C (2^{-p/2} K_{n+p} + 2^{5p/4} A_{n+p}^{5/8} \delta_{n+p}^{5/8}) \\ G_n^{2/3} &\leq C (2^{-2p} A_{n+p} + 2^{2p} A_{n+p}^{1/2} \delta_{n+p}^{1/2}) \end{aligned} \quad (4.4)$$

where C denotes a suitable constant (independent on the particular pseudo NS field). The proof of the inequalities (4.4) is not difficult, assuming the (S,H,CZ,P) inequalities above, and it is illustrated in the problems (1), (2), (3).

(2) “Dynamical inequality”: i.e. an inequality based on the energy inequality (c) in (3.1) which implies, quite easily, the following “dynamic inequality”⁸

$$A_n \leq C (2^p G_{n+p}^{2/3} + 2^p A_{n+p} \delta_{n+p} + 2^p J_{n+p}) \quad (4.5)$$

whose proof is illustrated in problem (4).

⁷ Indeed it relates properties of operators on a scale to those on a different scale. Note, however, that the couplings on scale n , i.e. the components of \underline{X}_n , provide information on those of X_{n+p} rather than on those of \underline{X}_{n-p} as usual in the renormalization group methods, see [BG95].

⁸ We call it “dynamic” because it follows from the energy inequality, i.e. from the equations of motion.

Proof of proposition: Assume the above inequalities (4.4), (4.5) and setting $\alpha_n = A_n, \kappa_n = K_n^{8/5}, j_n = J_n, g_n = G_n^{2/3}, \delta_{n+p} = \delta$ and, as above, $\underline{X}_n = (\alpha_n, \kappa_n, j_n, g_n)$. The r.h.s. of the inequalities defines the map $\mathcal{B}_p(\underline{X}; \delta)$.

If one stares long enough at them one realizes that the contraction property of the proposition is an immediate consequence of

- (1) The exponents to which $\varepsilon = 2^{-p}$ is raised in the various terms are either positive or not; in the latter cases the inverse power of ε is always appearing multiplied by a power of δ_{n+p} which we can take so small to compensate for the size of ε to any negative power, *except in the one case corresponding to the last term in (4.5)* where we see ε^{-1} without any compensating δ_{n+p} .
- (2) Furthermore the sum of the powers of the components of \underline{X}_n in each term of the inequalities is *always* ≤ 1 : this means that the inequalities are “almost linear” and a linear map that “bounds” \mathcal{B}_p exists and it is described by a matrix with small entries *except one off-diagonal element*. The iterates of the matrix therefore contract unless the large matrix element “ill placed” in the matrix: and one easily sees that it is not.

A formal argument can be devised in many ways: we present one in which several choices appear that are quite arbitrary and that the reader can replace with alternatives. In a way one should really try to see why a formal argument is not necessary.

The relation (4.5) can be “iterated” by using the expressions (4.4) for G_{n+p}, J_{n+p} and then the first of (4.4) to express $G_{n+p}^{1/5}$ in terms of A_{n+2p} with n replaced by $n+p$:

$$\begin{aligned} \alpha_n \leq C & (2^{-p}\alpha_{n+2p} + 2^{3p}\delta_{n+2p}^{1/2}\alpha_{n+2p}^{1/2} + \\ & + 2^{p/5}(\alpha_{n+2p}\kappa_{n+2p})^{1/2} + 2^{7p/5}\delta_{n+2p}^{7/20}\alpha_{n+2p}^{1/2}\kappa_{n+2p}^{1/2} + \\ & + 2^{3p}\delta_{n+2p}\alpha_{n+2p}) \end{aligned} \quad (4.6)$$

It is convenient to take advantage of the simple inequalities $(ab)^{\frac{1}{2}} \leq za + z^{-1}b$ and $a^x \leq 1 + a$ for $a, b, z, x > 0, x \leq 1$.

The (4.6) can be turned into a relation between α_n and $\alpha_{n+p}, \kappa_{n+p}$ by replacing p by $\frac{1}{2}p$. Furthermore, in the relation between α_n and $\alpha_{n+p}, \kappa_{n+p}$ obtained after the latter replacement, we choose $z = 2^{-p/5}$ to disentangle $2^{p/10}(\alpha_{n+p}\kappa_{n+p})^{1/2}$ we obtain recurrent (generous) estimates for α_n, κ_n

$$\begin{aligned} \alpha_n & \leq C(2^{-p/10}\alpha_{n+p} + 2^{3p/10}\kappa_{n+p} + \xi_{n+p}^\alpha) \\ \kappa_n & \leq C(2^{-4p/5}\kappa_{n+p} + \xi_{n+p}^\kappa) \\ \xi_{n+p}^\alpha & \stackrel{def}{=} 2^{3p}\delta_{n+p}(\alpha_{n+p} + \kappa_{n+p} + 1) \\ \xi_{n+p}^\kappa & \stackrel{def}{=} 2^{3p}\delta_{n+p}\alpha_{n+p} \end{aligned} \quad (4.7)$$

We fix p once and for all such that $2^{-p/10}C < \frac{1}{3}$.

Then if $C2^{3p}\delta_n$ is small enough, *i.e.* if δ_n is small enough, say for $\delta_n < \bar{\delta}$ for all $|n| \geq \bar{n}$, the matrix $M = C \begin{pmatrix} 2^{-p/10} + 2^{3p}\delta_{n+p} & 2^{3p/10} + 2^{3p}\delta_{n+p} \\ 0 & 2^{-4p/5} + 2^{3p}\delta_{n+p} \end{pmatrix}$ will have the two eigenvalues $< \frac{1}{2}$ and iteration of (4.6) will contract any ball in the plane α, κ to the ball of radius $2\bar{\delta}$.

If α_n, κ_n are bounded by a constant $\bar{\delta}$ for all $|n|$ large enough the (4.4) show that also g_n, j_n are going to be eventually bounded proportionally to $\bar{\delta}$.

Hence by imposing that δ is so small that $|X_n| = \alpha_n + \kappa_n + j_n + g_n < \rho$ we see that proposition 3 holds (hence theorem 2 as a consequence of theorem 1).

5 Fractal Dimension of Singularities of the Navier–Stokes Equation, $d = 3$

Here we ask which could be the structure of the possible set of the singularity points of the solutions of the Navier–Stokes equation in $d = 3$. The answer is an immediate consequence of theorem II and we describe it here for completeness: the technique is a classic method (Almgren) to link *a priori* estimates to fractal dimension estimates.

It has been shown already by Leray that the set of times at which a singularity is possible has zero measure (on the time axis), see §3.4 in [Ga02].

Obviously sets of zero measure can be quite structured and even large in other senses. One can think to the example of the Cantor set which is non denumerable and obtained from an interval I by deleting an open concentric subinterval of length $1/3$ that of I and then repeating recursively this operation on each of the remaining intervals (called n -th generation intervals after n steps); or one can think to the set of rational points which is dense.

5.1 Dimension and Measure of Hausdorff

An interesting geometric characteristic of the size of a set is given by the Hausdorff dimension and by the Hausdorff measure, *c.f.r.* [DS60], p.174.

Definition 5.1. (*Hausdorff α -measure*): The Hausdorff α -measure of a set A contained in a metric space M is defined by considering for each $\delta > 0$ all coverings \mathcal{C}_δ of A by closed sets F with diameter $0 < d(F) \leq \delta$ and setting

$$\mu_\alpha(A) = \lim_{\delta \rightarrow 0} \inf_{\mathcal{C}_\delta} \sum_{F \in \mathcal{C}_\delta} d(F)^\alpha \quad (5.1)$$

Remark 5.1.

(1) The limit over δ exists because the quantity $\inf_{\mathcal{C}_\delta} \dots$ is monotonic nondecreasing.

(2) It is possible to show that the function defined on the sets A of M by

$A \rightarrow \mu_\alpha(A)$ is completely additive on the smallest family of sets containing all closed sets and invariant with respect to the operations of complementation and countable union (which is called the σ -algebra Σ of the Borel sets of M), *c.f.r.* [DS60].

One checks immediately that given $A \in \Sigma$ there is α_c such that

$$\mu_\alpha(A) = \infty \text{ if } \alpha < \alpha_c, \quad \mu_\alpha(A) = 0 \text{ if } \alpha > \alpha_c \quad (5.2)$$

and it is therefore natural to set up the following definition

Definition 5.2. (*Hausdorff measure and Hausdorff dimension*): Given a set $A \subset R^d$ the quantity α_c , (5.2), is called Hausdorff dimension of A , while $\mu_{\alpha_c}(A)$ is called Hausdorff measure of A .

It is not difficult to check that

- (1) Denumerable sets in $[0, 1]$ have zero Hausdorff dimension and measure.
- (2) Hausdorff dimension of n -dimensional regular surfaces in R^d is n and, furthermore, the Hausdorff measure of their Borel subsets defines on the surface a measure μ_{α_c} that is equivalent to the area measure μ : namely there is a $\rho(x)$ such that $\mu_{\alpha_c}(dx) = \rho(x)\mu(dx)$.
- (3) The Cantor set, defined also as the set of all numbers in $[0, 1]$ which in the representation in base 3 do not contain the digit 1, has

$$\alpha_c = \log_3 2 \quad (5.3)$$

as Hausdorff dimension.⁹

5.2 Hausdorff Dimension of Singular Times in the Navier–Stokes Solutions ($d = 3$)

We now attempt to estimate the Hausdorff dimension of the sets of times $t \leq T < \infty$ at which appear singularities of a given weak solution of Leray,

⁹ Indeed with 2^n disjoint segments with size 3^{-n} , uniquely determined (the n -th generation segments), one covers the whole set C ; hence

$$\mu_{\alpha,\delta} \stackrel{\text{def}}{=} \inf_{\mathcal{C}_\delta} \sum_{F \in \mathcal{C}_\delta} d(F)^\alpha \leq 1 \quad \text{if } \alpha = \alpha_0 = \log_3 2$$

and $\mu_{\alpha_0}(C) \leq 1$: *i.e.* $\mu_\alpha(C) = 0$ if $\alpha > \alpha_0$. Furthermore, *c.f.r.* problem (16) below, if $\alpha < \alpha_0$ one checks that the covering \mathcal{C}^0 realizing the smallest value of $\sum_{F \in \mathcal{C}_\delta} d(F)^\alpha$ with $\delta = 3^{-n}$ is precisely the just considered one consisting in the 2^n intervals of length 3^{-n} of the n -th generation and the value of the sum on such covering diverges for $n \rightarrow \infty$. Hence $\mu_\alpha(C) = \infty$ if $\alpha < \alpha_0$ so that $\alpha_0 \equiv \alpha_c$ and $\mu_{\alpha_c}(C) = 1$.

i.e. a solution of the type discussed in (0.1). Here T is an *a priori* arbitrarily prefixed time.

We need a key property of Leray's solutions, namely that if at time t_0 it is $J_1(t_0) = L^{-1} \int (\underline{\partial}u)^2 d\underline{x} < \infty$, *i.e.* if the Reynolds number $R(t_0) = J_1(t_0)^{1/2}/V_c \equiv V_1/V_c$ with $V_c \stackrel{def}{=} \nu L^{-1}$ is $< +\infty$, then the solution stays regular in a time interval $(t_0, t_0 + \tau]$ with (see proposition II in §3.3 of [Ga02], eq. (3.3.34)):

$$\tau = \min F \frac{T_c}{R(t_0)^4}, T_c \quad T_c = \frac{L^2}{\nu} \quad (5.4)$$

From this it will follow, see below, that there are $A > 0, \gamma > 0$ such that if

$$\liminf_{\sigma \rightarrow 0} \left(\frac{\sigma}{T_c} \right)^\gamma \int_{t-\sigma}^t \frac{d\vartheta}{\sigma} R^2(\vartheta) < A \quad (5.5)$$

then $\tau > \sigma$ and the solution is regular in an interval that contains t so that the instant t is an instant at which the solution is regular. Here, as in the following, we could fix $\gamma = 1/2$: but γ is left arbitrary in order to make clearer why the choice $\gamma = 1/2$ is the "best".

We first show that, indeed, from (5.5) we deduce the existence of a sequence $\sigma_i \rightarrow 0$ such that

$$\int_{t-\sigma_i}^t \frac{d\vartheta}{\sigma_i} R^2(\vartheta) < A \left(\frac{\sigma_i}{T_c} \right)^{-\gamma} \quad (5.6)$$

therefore, the l.h.s. being a time average, there must exist $\vartheta_{0i} \in (t - \sigma_i, t)$ such that

$$R^2(\vartheta_{0i}) < A \left(\frac{\sigma_i}{T_c} \right)^{-\gamma} \quad (5.7)$$

and then the solution is regular in the interval $(\vartheta_{0i}, \vartheta_{0i} + \tau_i)$ with length τ_i at least

$$\tau_i = FT_c \frac{(\sigma_i/T_c)^{2\gamma}}{A^2} > \sigma_i \quad (5.8)$$

provided $\gamma \leq 1/2$, and σ_i is small enough and if A is small enough (if $\gamma = \frac{1}{2}$ then this means $2A^2 < F$). Under these conditions the size of the regularity interval is longer than σ_i and *therefore it contains t itself*.

It follows that, if t is in the set S of the times at which a singularity is present, it must be

$$\liminf_{\sigma \rightarrow 0} \left(\frac{\sigma}{T_c} \right)^\gamma \int_{t-\sigma}^t \frac{d\vartheta}{\sigma} R^2(\vartheta) \geq A \quad \text{if } t \in S \quad (5.9)$$

i.e. every singularity point is covered by a family of infinitely many intervals F with diameters σ *arbitrarily small* and satisfying

$$\int_{t-\sigma}^t d\vartheta R^2(\vartheta) \geq \frac{A}{2} \sigma \left(\frac{\sigma}{T_c} \right)^{-\gamma} \quad (5.10)$$

From Vitali's covering theorem (*c.f.r.* problem (19)) it follows that, given $\delta > 0$, one can find a denumerable family of intervals F_1, F_2, \dots , with $F_i = (t_i - \sigma_i, t_i)$, pairwise disjoint and verifying the (5.10) and $\sigma_i < \delta/4$, such that the intervals $5F_i \stackrel{\text{def}}{=} (t_i - 7\sigma_i/2, t_i + 5\sigma_i/2)$ (obtained by dilating the intervals F_i by a factor 5 about their center) cover S

$$S \subset \cup_i 5F_i \quad (5.11)$$

Consider therefore the covering \mathcal{C} generated by the sets $5F_i$ and compute the sum in (5.1) with $\alpha = 1 - \gamma$:

$$\begin{aligned} \sum_i (5\sigma_i) \left(\frac{5\sigma_i}{T_c} \right)^{-\gamma} &= 5^{1-\gamma} \sum_i \sigma_i \left(\frac{\sigma_i}{T_c} \right)^{-\gamma} < \\ &< \frac{2 \cdot 5^{1-\gamma}}{A\sqrt{T_c}} \sum_i \int_{F_i} d\vartheta R^2(\vartheta) \leq \frac{2 \cdot 5^{1-\gamma}}{A\sqrt{T_c}} \int_0^T d\vartheta R^2(\vartheta) < \infty \end{aligned} \quad (5.12)$$

where we have made use of the *a priori* estimates on vorticity (0.2) and we must recall that $\gamma \leq 1/2$ is a necessary condition in order that what has been derived be valid (*c.f.r.* comment to (5.8)).

Hence it is clear that for each $\alpha \geq 1/2$ it is $\mu_\alpha(S) < \infty$ (pick, in fact, $\alpha = 1 - \gamma$, with $\gamma \leq 1/2$) hence the Hausdorff dimension of S is $\alpha_c \leq 1/2$. Obviously the choice that gives the best regularity result (with the informations that we gathered) is precisely $\gamma = 1/2$.

Moreover one can check that $\mu_{1/2}(S) = 0$: indeed we know that S has zero measure, hence there is an open set $G \supset S$ with measure smaller than a prefixed ε . And we can choose the intervals F_i considered above so that they also verify $F_i \subset G$: hence we can replace the integral in the right hand side of (5.12) with the integral over G hence, since the integrand is summable, we shall find that the value of the integral can be supposed as small as wished, so that $\mu_{1/2}(S) = 0$.

5.3 Hausdorff Dimension in Space–Time of the Solutions of NS, ($d = 3$)

The problem of which is the Hausdorff dimension of the points $(\underline{x}, t) \in \Omega \times [0, T]$ which are singularity points for the Leray's solutions is quite different.

Indeed, *a priori*, it could even happen that, at one of the times $t \in S$ where the solution has a singularity as a function of time, *all* points (\underline{x}, t) , with $\underline{x} \in \Omega$, are singularity points and therefore the set S_0 of the singularity points thought of as a set in space–time could have dimension 3 (and perhaps even 3.5 if we take into account the dimension of the singular times discussed in (B) above).

From theorem II we know that if

$$\limsup_{r \rightarrow 0} r^{-1} \int_{t-r^2/2\nu}^{t+r^2/2\nu} \int_{S(\underline{x}, r)} \frac{d\vartheta}{\nu} d\underline{\xi} (\underline{\partial}u)^2 < \varepsilon \quad (5.13)$$

then regularity at the point (\underline{x}, t) follows.

It follows that the set S_0 of the singularity points in space–time can be covered by sets $C_r = S(\underline{x}, r) \times (t - r^2\nu^{-1}, t + \frac{1}{2}r^2\nu^{-1}]$ with r arbitrarily small and such that

$$\frac{1}{r\nu} \int_{t-\frac{r^2}{2\nu}}^{t+\frac{r^2}{2\nu}} d\vartheta \int_{S(\underline{x}, r)} d\underline{x} (\underline{\partial}u)^2 > \varepsilon \quad (5.14)$$

which is the negation of the property in (5.13).

Again by a covering theorem of Vitali (*c.f.r.* problems (16), (17)), we can find a family F_i of sets of the form $F_i = S(\underline{x}_i, r_i) \times (t_i - \frac{r_i^2}{2\nu}, t_i + \frac{r_i^2}{2\nu}]$ pairwise disjoint and such that the sets $6F_i =$ set of points (\underline{x}', t') at distance $\leq 6r_i$ from the points of F_i covers the singularity set S_0 .¹⁰ One can then estimate the sum in (5.1) for such a covering, by using that the sets F_i are pairwise disjoint and that $5F_i$ has diameter, if $\max r_i$ is small enough, not larger than $18r_i$:

$$\sum_i (18r_i) \leq \frac{36}{\nu\varepsilon} \sum_i \int_{F_i} (\underline{\partial}u)^2 d\underline{\xi} dt \leq \frac{36}{\nu\varepsilon} \int_0^T \int_{\Omega} (\underline{\partial}u)^2 d\underline{\xi} dt < \infty \quad (5.15)$$

i.e. the 1-measure of Hausdorff $\mu_1(S_0)$ would be $< \infty$ hence the Hausdorff dimension of S_0 would be ≤ 1 .

Since S_0 has zero measure, being contained in $\Omega \times S$ where S is the set of times at which a singularity occurs somewhere, see (5.9), it follows (still from the covering theorems) that in fact it is possible to choose the sets F_i so that their union U is contained into an open set G which differs from S_0 by a set of measure that exceeds by as little as desired that of S_0 (which is zero); one follows the same method used above in the analysis of the time–singularity. Hence we can replace the last integral in (5.15) with an integral extended to the union U of the F_i 's: the latter integral can be made as small as wished by letting the measure of G to 0. It follows that not only the Hausdorff dimension of S_0 is ≤ 1 , but also the $\mu_1(S_0) = 0$.

Remark 5.2.

(1) In this way we exclude that the set S_0 of the space–time singularities contains a regular curve: singularities, *if existent*, cannot move along trajectories (like flow lines) otherwise the dimension of S would be $1 > 1/2$) nor they can distribute, at fixed time, along lines and, hence, in a sense they must appear isolates and immediately disappear (always assuming their real existence).

(2) A conjecture (much debated and that I favor) that is behind all our discussions is that *if the initial datum \underline{u}^0 is in $C^\infty(\Omega)$ then there exists a solution*

¹⁰ Here the constant 5, as well as the other numerical constants that we meet below like 5, 6, 18 have no importance for our purposes and are just simple constants for which the estimates work.

to the Navier Stokes equation that is of class C^∞ in (\underline{x}, t) ", i.e. $S_0 = \emptyset!$ The problem is, still, open: counterexamples to the conjecture are not known (i.e. singular Leray's solutions with initial data and external force of class C^∞) but the matter is much debated and different alternative conjectures are possible (c.f.r. [PS87]).

(3) In this respect one should keep in mind that if $d \geq 4$ it is possible to show that *not all* smooth initial data evolve into regular solutions: counterexamples to smoothness can indeed be constructed, c.f.r. [Sc77]).

Acknowledgements: The notes are extracted from reference [Ga02].

6 Problems. The Dimensional Bounds of the CKN Theory

In the following problems we shall set $\nu = 1$, with no loss of generality, thus fixing the units so that time is a square length. The symbols (\underline{u}, p) will denote a pseudo NS field, according to definition 1. Moreover, for notational simplicity, we shall set $A_\rho \equiv A(\rho)$, $G_\rho \equiv G(\rho), \dots$, and sometimes we shall write $A_{r_n}, G_{r_n} \dots$ as A_n, \dots with an abuse that should not generate ambiguities. The validity of the (3.1) for Leray's solution is checked in problem (15), at the end of the problems section, to stress that the theorems of Scheffer and CKN concern pseudo NS velocity–pressure fields: however it is independent of the first 14 problems. There will many constants that we generically denote C : they are not the same but one should think that they increase monotonically by a finite amount at each inequality. The integration elements like $d\underline{x}$ and dt are often omitted to simplify the notations and they should be easily understood from the integration domains.

(1) Let $\rho = r_{n+p}$ and $r = r_n$, with $r_n = L2^n$, c.f.r. lines following (4.2), and apply (S),(2.2), with $q = 3$ and $a = \frac{3}{4}$, to the field \underline{u} , at t fixed in Δ_r and using definition 2 deduce

$$\begin{aligned} \int_{B_r} |\underline{u}|^3 d\underline{x} &\leq C_3^S \left[\left(\int_{B_r} |\underline{\partial} \underline{u}|^2 d\underline{x} \right)^{\frac{3}{4}} \left(\int_{B_r} |\underline{u}|^2 d\underline{x} \right)^{\frac{3}{4}} + r^{-3/2} \left(\int_{B_r} |\underline{u}|^2 \right)^{3/2} \right] \leq \\ &\leq C_3^S \left[\rho^{3/4} A_\rho^{3/4} \left(\int_{B_r} |\underline{\partial} \underline{u}|^2 d\underline{x} \right)^{3/4} + r^{-3/2} \left(\int_{B_r} |\underline{u}|^2 \right)^{3/2} \right] \end{aligned}$$

Infer from the above the third of (4.4). (Idea Let $\overline{|\underline{u}|_\rho^2}$ be the average of \underline{u}^2 on the ball B_ρ ; apply the inequality (P), with $\alpha = 1$, to show that there is $C > 0$ such that

$$\begin{aligned} \int_{B_r} d\underline{x} |\underline{u}|^2 &\leq \left(\int_{B_\rho} d\underline{x} \left| |\underline{u}|^2 - \overline{|\underline{u}|_\rho^2} \right| \right) + \overline{|\underline{u}|_\rho^2} \int_{B_r} d\underline{x} \leq \\ &\leq C \rho \int_{B_\rho} d\underline{x} |\underline{u}| |\underline{\partial} \underline{u}| + C \left(\frac{r}{\rho} \right)^3 \int_{B_\rho} d\underline{x} |\underline{u}|^2 \leq C \rho^{3/2} A_\rho^{1/2} \left(\int_{B_\rho} d\underline{x} |\underline{\partial} \underline{u}|^2 \right)^{1/2} + \\ &+ C \left(\frac{r}{\rho} \right)^3 \rho A_\rho \end{aligned}$$

where the dependence from $t \in \Delta_r$ is not explicitly indicated; hence

$$\int_{B_r} d\underline{x} |\underline{u}|^3 \leq C (r\rho^{-1})^3 A_\rho^{3/2} + C (\rho^{3/4} + \rho^{9/4} r^{-3/2}) A_\rho^{3/4} \left(\int_{B_\rho} d\underline{x} |\underline{\partial u}|^2 \right)^{3/4}$$

then integrate both sides with respect to $t \in \Delta_r$ and apply (H) and definition 2.)

(2) Let $\varphi \leq 1$ be a non negative C^∞ function with value 1 if $|\underline{x}| \leq 3\rho/4$ and 0 if $|\underline{x}| > 4\rho/5$; we suppose that it has the “scaling” form $\varphi = \varphi_1(\underline{x}/\rho)$ with $\varphi_1 \geq 0$ a C^∞ function fixed once and for all. Let B_ρ be the ball centered at \underline{x} with radius ρ ; and note that, if $\rho = r_{n+p}$ and $r = r_n$, pressure can be written, at each time (without explicitly exhibiting the time dependence), as $p(\underline{x}) = p'(\underline{x}) + p''(\underline{x})$ with

$$\begin{aligned} p'(\underline{x}) &= \frac{1}{4\pi} \int_{B_\rho} \frac{1}{|\underline{x}-\underline{y}|} p(\underline{y}) \Delta \varphi(\underline{y}) d\underline{y} + \frac{1}{2\pi} \int_{B_\rho} \frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3} \cdot \underline{\partial} \varphi(\underline{y}) p(\underline{y}) d\underline{y} \\ p''(\underline{x}) &= \frac{1}{4\pi} \int_{B_\rho} \frac{1}{|\underline{x}-\underline{y}|} \varphi(\underline{y}) (\underline{\partial u}(\underline{y})) \cdot (\underline{\partial u}(\underline{y})) d\underline{y} \end{aligned}$$

if $|\underline{x}| < 3\rho/4$; and also $|p'(\underline{x})| \leq C\rho^{-3} \int_{B_\rho} d\underline{y} |p(\underline{y})|$ and all functions are evaluated at a fixed $t \in \Delta_r$. Deduce from this remark the first of the (4.4). (*Idea* First note the identity $p = -(4\pi)^{-1} \int_{B_\rho} |\underline{x} - \underline{y}|^{-1} \Delta(\varphi p)$ for $\underline{x} \in B_r$ because if $\underline{x} \in B_{3\rho/4}$ it is $\varphi p \equiv p$. Then note the identity $\Delta(\varphi p) = p \Delta \varphi + 2\underline{\partial} p \cdot \underline{\partial} \varphi + \varphi \Delta p$ and since $\Delta p = -\underline{\partial} \cdot (\underline{u} \cdot \underline{\partial} \underline{u}) = -(\underline{\partial} \underline{u}) \cdot (\underline{\partial} \underline{u})$: the second of the latter relations generates p'' while $p \Delta \varphi$ combines with the contribution from $2\underline{\partial} p \cdot \underline{\partial} \varphi$, after integrating the latter by parts, and generates the two contributions to p' .

From the expression for p'' we see that

$$\begin{aligned} \int_{B_r} d\underline{x} |p''(\underline{x})|^2 &\leq \int_{B_\rho \times B_\rho} d\underline{y} d\underline{y}' |\underline{\partial u}(\underline{y})|^2 |\underline{\partial u}(\underline{y}')|^2 \int_{B_r} d\underline{x} \frac{1}{|\underline{x}-\underline{y}||\underline{x}-\underline{y}'|} \leq \\ &\leq C\rho \left(\int_{B_\rho} d\underline{y} |\underline{\partial u}(\underline{y})|^2 \right)^2 \end{aligned} \quad (!)$$

The part with p' is more interesting: since its expression above contains inside the integral kernels apparently singular at $\underline{x} = \underline{y}$ like $|\underline{x} - \underline{y}|^{-1} \Delta \varphi$ and $|\underline{x} - \underline{y}|^{-1} \underline{\partial} \varphi$ one remarks that, in fact, there is no singularity because the derivatives of φ vanish if $\underline{y} \in B_{3\rho/4}$ (where $\varphi \equiv 1$). Hence $|\underline{x} - \underline{y}|^{-k}$ can be bounded “dimensionally” by ρ^{-k} in the whole region $B_\rho/B_{3\rho/4}$ for all $k \geq 0$ (this remark also motivates why one should think p as sum of p' and p'').

Thus replacing the (apparently) singular kernels with their dimensional bounds we get

$$\int_{B_r} d\underline{x} |\underline{u}| |p'| \leq \frac{C}{\rho^3} \left(\int_{B_r} d\underline{x} |\underline{u}| \right) \cdot \left(\int_{B_\rho} d\underline{x} |p| \right)$$

which can be bounded by using inequality (H) as

$$\begin{aligned}
&\leq \frac{C}{\rho^3} \left(\int_{B_r} d\underline{x} |\underline{u}|^{2/5} \cdot |\underline{u}|^{3/5} \cdot 1 \right) \cdot \left(\int_{B_\rho} d\underline{x} |p| \right) \leq \\
&\leq \frac{C}{\rho^3} \left(\int_{B_r} d\underline{x} |\underline{u}|^2 \right)^{1/5} \cdot \left(\int_{B_r} d\underline{x} |\underline{u}|^3 \right)^{1/5} (r^3)^{3/5} \cdot \int_{B_\rho} d\underline{x} |p| \leq \\
&\leq \frac{C}{\rho^3} (\rho A_\rho)^{1/5} \left(\int_{B_r} d\underline{x} |\underline{u}|^3 \right)^{1/5} \cdot \left(\int_{B_\rho} d\underline{x} |p| \right)
\end{aligned}$$

where all functions depend on \underline{x} (and of course on t) and then, integrating over $t \in \Delta_r$ and dividing by r^2 one finds, for a suitable $C > 0$:

$$\frac{1}{r^2} \int_{Q_r} dt d\underline{x} |\underline{u}| |p'| \leq C \left(\frac{r}{\rho} \right)^{1/5} G_r^{1/5} K_\rho^{4/5} A_\rho^{1/5}$$

that is combined with $\int_{B_r} d\underline{x} |\underline{u}| |p''| \leq (\int_{B_r} d\underline{x} |\underline{u}|^2)^{1/2} (\int_{B_r} d\underline{x} |p''|^2)^{1/2}$ which, integrating over time, dividing by ρ^2 and using inequality (!) for $\int_{B_r} d\underline{x} |p''|^2$ yields: $r^{-2} \int_{Q_r} dt d\underline{x} |\underline{u}| |p''| \leq C(\rho r^{-1})^2 A_\rho^{1/2} \delta_\rho$.

(3) In the context of the hint and notations for p of the preceding problem check that $\int_{B_r} d\underline{x} |p'| \leq C(r\rho^{-1})^3 \int_{B_\rho} d\underline{x} |p|$. Integrate over t the power $5/4$ of this inequality, rendered adimensional by dividing it by $r^{13/4}$; one gets: $r^{-13/4} \int_{\Delta_r} (\int |p'|)^{5/4} \leq C(r\rho^{-1})^{1/2} K_\rho$, which yields the first term of the second inequality in (4.4). Complete the derivation of the second of (4.4). (*Idea* Note that $p''(\underline{x}, t)$ can be written, in the interior of B_r , as $p'' = \tilde{p} + \hat{p}$ with:

$$\tilde{p}(\underline{x}) = -\frac{1}{4\pi} \int_{B_\rho} \frac{\underline{x} - \underline{y}}{|\underline{x} - \underline{y}|^3} \varphi(\underline{y}) \underline{u} \cdot \underline{\partial} \underline{u} d\underline{y}, \quad \hat{p}(\underline{x}) = -\frac{1}{4\pi} \int_{B_\rho} \frac{\underline{\partial} \varphi(\underline{y}) \cdot (\underline{u} \cdot \underline{\partial}) \underline{u}}{|\underline{x} - \underline{y}|} d\underline{y}$$

(always at fixed t and not declaring explicitly the t -dependence). Hence by using $|\underline{x} - \underline{y}| > \rho/4$, for $\underline{x} \in B_r$ and $\underline{y} \in B_\rho/B_{3\rho/4}$, *i.e.* for \underline{y} in the part of B_ρ where $\underline{\partial} \varphi \neq \underline{0}$) we find

$$\begin{aligned}
\int_{B_r} |\tilde{p}| d\underline{x} &\leq C \int_{B_\rho} d\underline{y} \left(\int_{B_r} \frac{d\underline{x}}{|\underline{x} - \underline{y}|^2} |\underline{u}(\underline{y})| |\underline{\partial} \underline{u}(\underline{y})| \right) \leq \\
&\leq C r \left(\int_{B_\rho} |\underline{u}|^2 \right)^{1/2} \left(\int_{B_\rho} |\underline{\partial} \underline{u}|^2 \right)^{1/2} \leq C r \rho^{1/2} A_\rho^{1/2} \left(\int_{B_\rho} |\underline{\partial} \underline{u}|^2 \right)^{1/2} \\
\int_{B_r} |\hat{p}| d\underline{x} &\leq C \frac{r^3}{\rho^2} \int_{B_\rho} |\underline{u}| |\underline{\partial} \underline{u}| \leq C r \rho^{1/2} A_\rho^{1/2} \left(\int_{B_\rho} |\underline{\partial} \underline{u}|^2 \right)^{1/2}
\end{aligned}$$

and $\left(\int_{B_r} |p''| \right)^{5/4}$ is bounded by raising the right hand sides of the last inequalities to the power $5/4$ and integrating over t , and finally applying inequality (H) to generate the integral $\left(\int_{Q_\rho} |\underline{\partial} \underline{u}|^2 \right)^{5/8}$.

(4) Deduce that (4.5) holds for a pseudo-NS field (\underline{u}, p) , *c.f.r.* definition 2.1. (*Idea* Let $\varphi(\underline{x}, t)$ be a C^∞ function which is 1 on $Q_{\rho/2}$ and 0 outside Q_ρ ; it is: $0 \leq \varphi(\underline{x}, t) \leq 1$, $|\underline{\partial} \varphi| \leq \frac{C}{\rho}$, $|\Delta \varphi + \partial_t \varphi| \leq \frac{C}{\rho^2}$, if we suppose that φ has the

form $\varphi(\underline{x}, t) = \varphi_2(\frac{\underline{x}}{\rho}, \frac{t}{\rho^2}) \geq 0$ for some φ_2 suitably fixed and smooth. Then, by applying the third of (3.1) and using the notations of the preceding problems, if $\bar{t} \in \Delta_{\rho/2}(t_0)$, it is

$$\begin{aligned} \int_{B_r \times \{\bar{t}\}} |\underline{u}(\underline{x}, t)|^2 d\underline{x} &\leq \frac{C}{\rho^2} \int_{Q_\rho} dt d\underline{x} |\underline{u}|^2 + \int_{Q_\rho} dt d\underline{x} (|\underline{u}|^2 + 2p) \underline{u} \cdot \underline{\partial} \varphi \leq \\ &\leq \frac{C}{\rho^2} \int_{Q_\rho} dt d\underline{x} |\underline{u}|^2 + \left| \int_{Q_\rho} dt d\underline{x} (|\underline{u}|^2 - \overline{|\underline{u}|_\rho^2}) \underline{u} \cdot \underline{\partial} \varphi \right| + 2 \int_{Q_\rho} dt d\underline{x} p \underline{u} \cdot \underline{\partial} \varphi \leq \\ &\leq \frac{C}{\rho^{1/3}} \left(\int_{Q_\rho} dt d\underline{x} |\underline{u}|^3 \right)^{2/3} + \left| \int_{Q_\rho} dt d\underline{x} (|\underline{u}|^2 - \overline{|\underline{u}|_\rho^2}) \underline{u} \cdot \underline{\partial} \varphi \right| + \frac{2C}{\rho} \int_{B_\rho} dt d\underline{x} |p| |\underline{u}| \leq \\ &\leq C \rho G_\rho^{2/3} + C \rho J_\rho + \rho \left| \frac{1}{\rho} \int_{Q_\rho} dt d\underline{x} (|\underline{u}|^2 - \overline{|\underline{u}|_\rho^2}) \underline{u} \cdot \underline{\partial} \varphi \right| \quad (*) \end{aligned}$$

We now use the following inequality, at t fixed and with the integrals over $d\underline{x}$

$$\begin{aligned} \frac{1}{\rho} \left| \int_{B_\rho} d\underline{x} (|\underline{u}|^2 - \overline{|\underline{u}|_\rho^2}) \underline{u} \cdot \underline{\partial} \varphi \right| &\leq \frac{C}{\rho^2} \int_{B_\rho} d\underline{x} |\underline{u}| \left| |\underline{u}|^2 - \overline{|\underline{u}|_\rho^2} \right| \leq \\ &\leq \frac{C}{\rho^2} \left(\int_{B_\rho} d\underline{x} |\underline{u}|^3 \right)^{1/3} \left(\int_{B_\rho} |\underline{u}|^2 - \overline{|\underline{u}|_\rho^2} \right)^{2/3} \end{aligned}$$

and we also take into account inequality (P) with $f = \underline{u}^2$ and $\alpha = 3/2$ which yields (always at t fixed and with integrals over $d\underline{x}$):

$$\left(\int_{B_\rho} \left| \underline{u}^2 - \overline{|\underline{u}|_\rho^2} \right|^{3/2} \right)^{2/3} \leq C \left(\int_{B_\rho} |\underline{u}| |\underline{\partial} \underline{u}| \right)$$

then we see that

$$\begin{aligned} \int_{B_\rho} \left| |\underline{u}|^2 - \overline{|\underline{u}|_\rho^2} \right| |\underline{u}| |\underline{\partial} \varphi| &\leq \frac{C}{\rho} \left(\int_{B_\rho} |\underline{u}|^3 \right)^{1/3} \left(\int_{B_\rho} |\underline{u}| |\underline{\partial} \underline{u}| \right) \leq \\ &\leq \frac{C}{\rho} \left(\int_{B_\rho} |\underline{u}|^3 \right)^{1/3} \left(\int_{B_\rho} |\underline{u}|^2 \right)^{1/2} \left(\int_{B_\rho} |\underline{\partial} \underline{u}|^2 \right)^{1/2} \leq \\ &\leq \frac{C}{\rho} \rho^{1/2} A_\rho^{1/2} \left(\int_{B_\rho} |\underline{u}|^3 \right)^{1/3} \cdot \left(\int_{B_\rho} |\underline{\partial} \underline{u}|^2 \right)^{1/2} \cdot 1 \end{aligned}$$

Integrating over t and applying (H) with exponents 3, 2, 6, respectively, on the last three factors of the right hand side we get

$$\frac{1}{\rho^2} \int_{Q_\rho} |\underline{u}| \left| |\underline{u}|^2 - \overline{|\underline{u}|_\rho^2} \right| \leq C A_\rho^{1/2} G_\rho^{1/3} \delta_\rho^{1/2} \leq C (G_\rho^{2/3} + A_\rho \delta_\rho)$$

and placing this in the first of the preceding inequalities (*) we obtain the desired result).

The following problems provide a guide to the proof of theorem II. Below we replace, unless explicitly stated the sets B_r, Q_r, Δ_r introduced in definition 2, in (C) above, and employed in the previous problems with B_r^0, Δ_r^0, Q_r^0 with $B_r^0 = \{\underline{x} \mid |\underline{x} - \underline{x}_0| < r\}$, $\Delta_r^0 = \{t \mid t_0 > t > t_0 - r^2\}$, $Q_r^0 = \{(\underline{x}, t) \mid |\underline{x} - \underline{x}_0| < r, t_0 > t > t_0 - r^2\} = B_r^0 \times \Delta_r^0$. Likewise we shall set $B_{r_n}^0 = B_{r_n}^0, \Delta_{r_n}^0 = \Delta_{r_n}^0, Q_{r_n}^0 = Q_{r_n}^0$ and we shall define new operators A, δ, G, J, K, S by the same

expressions in (3.2)–(3.8) in (C) above but with the just defined new meaning of the integration domains. However, to avoid confusion, we shall call them A^0, δ^0, \dots with a superscript 0 added.

(5) With the above conventions check the following inequalities

$$A_n^0 \leq CA_{n+1}^0, \quad G_n^0 \leq CG_{n+1}^0, \quad G_n^0 \leq C(A_n^{0\ 3/2} + A_n^{0\ 3/4} \delta_n^{0\ 3/4})$$

(Idea The first two are trivial consequences of the fact that the integration domains of the right hand sides are larger than those of the left hand sides, and the radii of the balls differ only by a factor 2 so that C can be chosen 2 in the first inequality and 4 in the second. The third inequality follows from (S) with $a = \frac{3}{4}$, $q = 3$:

$$\begin{aligned} \int_{B_r^0} |\underline{u}|^3 &\leq C \left[\left(\int_{B_r^0} |\partial \underline{u}|^2 \right)^{3/4} \left(\int_{B_r^0} |\underline{u}|^2 \right)^{3/4} + r^{-3/2} \left(\int_{B_r^0} |\underline{u}|^2 \right)^{3/2} \right] \leq \\ &\leq C \left[r^{3/4} A_r^{0\ 3/4} \left(\int_{B_r^0} |\partial \underline{u}|^2 \right)^{3/4} + A_r^{0\ 3/2} \right] \end{aligned}$$

where the integrals are over $d\underline{x}$ at t fixed; and integrating over t we estimate G_r^0 by applying (H) to the last integral over t .)

(6) Let $n_0 = n+p$ and $Q_n^0 = \{(x, t) \mid |x - x_0| < r_n, t_0 > t > t - r_n^2\} \stackrel{def}{=} B_n^0 \times \Delta_n^0$ consider the function:

$$\varphi_n(\underline{x}, t) = \frac{\exp(-(\underline{x} - \underline{x}_0)^2/4(r_n^2 + t_0 - t))}{(4\pi(r_n^2 + t_0 - t))^{3/2}}, \quad (\underline{x}, t) \in Q_{n_0}^0$$

and a function $\chi_{n_0}(\underline{x}, t) = 1$ on $Q_{n_0-1}^0$ and 0 outside $Q_{n_0}^0$, for instance choosing, a function which has the form $\chi_{n_0}(\underline{x}, t) = \tilde{\varphi}(r_{n_0}^{-1}\underline{x}, r_{n_0}^{-1/2}t) \geq 0$, with $\tilde{\varphi}$ a C^∞ function fixed once and for all. Then write (3.1) using $\varphi = \varphi_n \chi_{n_0}$ and deduce the inequality

$$\frac{A_n^0 + \delta_n^0}{r_n^2} \leq C \left[r_{n+p}^{-2} G_{n+p}^{0\ 2/3} + \sum_{k=n+1}^{n+p} r_k^{-2} G_k^0 + r_{n+p}^{-2} J_{n+p}^0 + \sum_{k=n+1}^{n+p-1} r_k^{-2} L_k \right] \quad (\textcircled{A})$$

where $L_k = r_k^{-2} \int_{Q_k^0} d\underline{x} dt |\underline{u}| |p - \overline{p^k}|$ with $\overline{p^k}$ equals the average of p on the ball B_k^0 ; for each $p > 0$. (Idea Consider the function φ and note that $\varphi \geq (Cr_n^3)^{-1}$ in Q_n^0 , which allows us to estimate from below the left hand side term in (3.1), with $(Cr_n^2)^{-1}(A_n^0 + \delta_n^0)$. Moreover one checks that

$$\begin{aligned} |\varphi| &\leq \frac{C}{r_m^3}, & |\partial \varphi| &\leq \frac{C}{r_m^4}, & n \leq m \leq n+p \equiv n_0, & \text{in } Q_{m+1}^0/Q_m^0 \\ |\partial_t \varphi + \Delta \varphi| &\leq \frac{C}{r_{n_0}^5} & & & & \text{in } Q_{n_0}^0 \end{aligned}$$

and the second relation follows from $\partial_t \varphi + \Delta \varphi \equiv 0$ in the “dangerous places”, i.e. $\chi = 1$, because φ is a solution of the heat equation (backward in time).

Hence the first term in the right hand side of (3.1) can be bounded *from above* by

$$\int_{Q_{n_0}^0} |\underline{u}|^2 |\partial_t \varphi_n + \Delta \varphi_n| \leq \frac{C}{r_{n_0}^5} \int_{Q_{n_0}^0} |\underline{u}|^2 \leq \frac{C}{r_{n_0}^5} \left(\int_{Q_{n_0}^0} |\underline{u}|^3 \right)^{2/3} r_{n_0}^{5/3} \leq \frac{C}{r_{n_0}^2} G_{n_0}^{02/3}$$

getting the first term in the r.h.s. of (@).

Using here the scaling properties of the function φ the second term is bounded by

$$\begin{aligned} \int_{Q_{n_0}^0} |\underline{u}|^3 |\partial \varphi_n| &\leq \frac{C}{r_n^4} \int_{Q_{n+1}^0} |\underline{u}|^3 + \sum_{k=n+2}^{n_0} \frac{C}{r_k^4} \int_{Q_k^0/Q_{k-1}^0} |\underline{u}|^3 \leq \\ &\leq \sum_{k=n+1}^{n_0} \frac{C}{r_k^4} \int_{Q_k^0} |\underline{u}|^3 \leq C \sum_{k=n+1}^{n_0} \frac{G_k^0}{r_k^2} \end{aligned}$$

Calling the third term (*c.f.r.* (1.1)) Z we see that it is bounded by

$$\begin{aligned} Z &\leq \left| \int_{Q_{n_0}^0} p \underline{u} \cdot \partial \chi_{n_0} \varphi_n \right| \leq \left| \int_{Q_{n+1}^0} p \underline{u} \cdot \partial \chi_{n+1} \varphi_n \right| + \\ &+ \sum_{k=n+2}^{n_0} \left| \int_{Q_k^0} p \underline{u} \cdot \partial (\chi_k - \chi_{k-1}) \varphi_n \right| \leq \left| \int_{Q_{n+1}^0} (p - \overline{p^{n+1}}) \underline{u} \cdot \partial \chi_{n+1} \varphi_n \right| + \\ &+ \sum_{k=n+2}^{n_0-1} \left| \int_{Q_k^0} (p - \overline{p^k}) \underline{u} \cdot \partial (\chi_k - \chi_{k-1}) \varphi_n \right| + \int_{Q_{n_0}^0} |\underline{u}| |p| |\partial (\chi_{n_0} - \chi_{n_0-1}) \varphi_n \end{aligned}$$

where $\overline{p^m}$ denotes the average of p over B_m^0 (which only depends on t): the possibility of replacing p by $p - \overline{p}$ in the integrals is simply due to the fact that the 0 divergence of \underline{u} allows us to add to p *any* constant because, by integration by parts, it will contribute 0 to the value of the integral.

From the last inequality it follows

$$Z \leq \sum_{k=n+1}^{n_0-1} \frac{C}{r_k^4} \int_{Q_k^0} |p - \overline{p^k}| |\underline{u}| + J_{n_0}^0 r_{n_0}^{-2} = \sum_{k=n+1}^{n_0-1} \frac{C}{r_k^2} L_k + J_{n_0}^0 r_{n_0}^{-2}$$

then sum the above estimates.)

(7) If \underline{x}_0 is the center of Ω the function $\chi_{n_0} p$ can be regarded, if $n_0 < -1$, as defined on the whole R^3 and zero outside the torus Ω . Then if Δ is the Laplace operator *on the whole R^3* note that the expression of p in terms of \underline{u} (*c.f.r.* (a) of (3.1)) implies that in $Q_{n_0}^0$:

$$\chi_{n_0} p = \Delta^{-1} \Delta \chi_{n_0} p \equiv \Delta^{-1} \left(p \Delta \chi_{n_0} + 2(\partial \chi_{n_0}) \cdot (\partial p) - \chi_{n_0} \partial \partial \cdot (\underline{u} \underline{u}) \right)$$

Show that this expression can be rewritten, for $n < n_0$, as

$$\begin{aligned} \chi_{n_0} p &= -\Delta^{-1} (\chi_{n_0} \partial \partial (\underline{u} \underline{u})) + [\Delta^{-1} (p \Delta \chi_{n_0}) + 2(\partial \Delta^{-1})((\partial \chi_{n_0}) p)] = \\ &= [-\partial \partial \Delta^{-1} (\chi_{n_0} \underline{u} \underline{u})] + [2(\partial \Delta^{-1}) (\partial \chi_{n_0} \underline{u} \underline{u}) - \Delta^{-1} ((\partial \partial \chi_{n_0}) \underline{u} \underline{u})] + \\ &+ [\Delta^{-1} (p \Delta \chi_{n_0}) + 2(\partial \Delta^{-1}) ((\partial \chi_{n_0}) p)] \stackrel{def}{=} p_1 + p_2 + p_3 + p_4 \end{aligned}$$

with $p_1 = -(\partial\partial\Delta^{-1})(\chi_{n_0}\vartheta_{n+1}\underline{u}\underline{u})$ and $p_2 = -(\partial\partial\Delta^{-1})(\chi_{n_0}(1 - \vartheta_{n+1})\underline{u}\underline{u})$ where ϑ_k is the characteristic function of B_k^0 and p_3, p_4 are the last two terms in square brackets. (*Idea Use*, for $\underline{x}, t \in Q_{n_0}^0$, Poisson formula

$$\begin{aligned}\chi_{n_0}(\underline{x}, t)p(\underline{x}, t) &= \frac{-1}{4\pi} \int_{B_{n_0}^0} \frac{\Delta((\chi_{n_0}p)(\underline{y}, t))}{|\underline{x}-\underline{y}|} d\underline{y} = \\ &= \frac{-1}{4\pi} \int_{B_{n_0}^0} \frac{p\Delta\chi_{n_0} + 2\partial\chi_{n_0} \cdot \partial p - \chi_{n_0} \partial\partial \cdot (\underline{u}\underline{u})}{|\underline{x}-\underline{y}|} d\underline{y}\end{aligned}$$

and suitably integrate by parts).

(8) In the context of the previous problem check that the formulae derived there can be written more explicitly as

$$\begin{aligned}p_1 &= -(\partial\partial\Delta^{-1}) \cdot (\chi_{n_0}\vartheta_{n+1}\underline{u}\underline{u}), & p_2 &= -\frac{1}{4\pi} \int_{B_{n_0}^0/B_{n+1}^0} \left(\partial\partial \frac{1}{|\underline{x}-\underline{y}|}\right) \cdot \chi_{n_0}\underline{u}\underline{u} \\ p_3 &= \frac{1}{2\pi} \int_{B_{n_0}^0} \frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3} (\partial\chi_{n_0})\underline{u}\underline{u} + \frac{1}{4\pi} \int_{B_{n_0}^0} \frac{1}{|\underline{x}-\underline{y}|} (\partial\partial\chi_{n_0})\underline{u}\underline{u} \\ p_4 &= -\frac{1}{4\pi} \int \frac{1}{|\underline{x}-\underline{y}|} p(\underline{y})\Delta\chi_{n_0} + \frac{2}{4\pi} \int p(\underline{y}) \frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^3} \cdot \partial\chi_{n_0}\end{aligned}$$

where $n < n_0$ and the integrals are over \underline{y} at t fixed, and the functions in the left hand side are evaluated in \underline{x}, t .

(9) Consider the quantity L_n , introduced in (6),

$$L_n \stackrel{def}{=} r_n^{-2} \int_{Q_n} |\underline{u}| |p - \bar{p}_n(\vartheta)| d\xi d\vartheta$$

and show that, setting $n_0 = n + p$, $p > 0$, it is

$$\begin{aligned}L_n \leq C \left[\left(\frac{r_{n+1}}{r_{n_0}}\right)^{7/5} A_{n+1}^{0^{1/5}} G_{n+1}^{0^{1/5}} K_{n_0}^{0^{4/5}} + \left(\frac{r_{n+1}}{r_{n_0}}\right)^{5/3} G_{n+1}^{0^{1/3}} G_{n_0}^{0^{2/3}} + \right. \\ \left. + G_{n+1}^0 + r_{n+1}^3 G_{n+1}^{0^{1/3}} \sum_{k=n+1}^{n_0} r_k^{-3} A_k^0 \right]\end{aligned}$$

(*Idea Refer* to (8) to bound L_n by: $\sum_{i=1}^4 r_n^{-2} \int_{Q_n^0} |\underline{u}| |p_i - \bar{p}_i^n|$ where \bar{p}_i^n is the average of p_i over B_n^0 ; and estimate separately the four terms. For the first it is not necessary to subtract the average and the difference $|p_1 - \bar{p}_1^n|$ can be divided into the sum of the absolute values each of which contributes equally to the final estimate which is obtained via the (CZ), and the (H)

$$\int_{B_{n+1}^0} |p_1 - \bar{p}_1^n| |\underline{u}| \leq 2 \left(\int_{B_{n+1}^0} |p_1|^{3/2} \right)^{2/3} \left(\int_{B_{n+1}^0} |\underline{u}|^3 \right)^{1/3} \leq C \int_{B_{n+1}^0} |\underline{u}|^3$$

and the contribution of p_1 at L_n is bounded, therefore, by CG_{n+1}^0 : note that this would not be true with p instead of p_1 because in the right hand side there would be $\int_{\Omega} |p|^{3/2}$ rather than $\int_{B_{n+1}^0} |p|^{3/2}$, because the (CZ) is a “nonlocal” inequality. The term with p_2 is bounded as

$$\begin{aligned}
& \int_{\Delta_n^0} \int_{B_n^0} |p_2 - \overline{p_2}| |\underline{u}| \leq \int_{\Delta_n^0} \int_{B_n^0} |\underline{u}| r_n \max |\partial p_2| \leq \\
& \leq r_n \left(\int_{Q_n^0} \frac{|\underline{u}|^3}{r_n^2} \right)^{1/3} r_n^{2/3} r_n^{10/3} \max_{Q_n^0} |\partial p_2| \leq \\
& \leq r_n^5 G_n^{0^{1/3}} \sum_{m=n+1}^{n_0-1} \max_{t \in \Delta_n^0} \int_{B_{m+1}^0/B_m^0} \frac{|\underline{u}|^2}{r_m^4} = r_n^5 G_n^{0^{1/3}} \sum_{m=n+1}^{n_0-1} \frac{A_m^0}{r_m^3}
\end{aligned}$$

Analogously the term with p_3 is bounded by using $|\partial p_3| \leq C r_{n_0}^{-4} \int_{B_{n_0}^0} |\underline{u}|^2$ which is majorized by $C r_{n_0}^{-3} (\int_{B_{n_0}^0} |\underline{u}|^3)^{2/3}$ obtaining

$$\begin{aligned}
& \frac{1}{r_n^2} \int_{Q_n^0} |\underline{u}| |p_3 - \overline{p_3}| \leq \frac{C}{r_n^2} r_{n_0}^{-3} \int_{\Delta_n^0} [(\int_{B_{n_0}^0} |\underline{u}|^3)^{2/3} r_n \int_{B_n^0} |\underline{u}|] \leq \\
& \leq \frac{C}{r_n^2} r_n^3 r_{n_0}^{-3} \int_{\Delta_n^0} (\int_{B_n^0} |\underline{u}|^3)^{2/3} (\int_{B_n^0} |\underline{u}|^3)^{1/3} \leq \\
& \leq \frac{C}{r_n^2} \left(\frac{r_n}{r_{n_0}} \right)^3 r_{n_0}^{4/3} r_n^{2/3} G_{n_0}^{0^{2/3}} G_n^{0^{1/3}} = C \left(\frac{r_n}{r_{n_0}} \right)^{5/3} G_{n_0}^{0^{2/3}} G_n^{0^{1/3}}
\end{aligned}$$

Finally the term with p_4 is bounded (taking into account that the derivatives $\Delta \chi_n, \partial \chi_n$ vanish where the kernels become bigger than what suggested by their dimension) by noting that

$$\int_{B_n^0} |p_4 - \overline{p_4}| |\underline{u}| \leq C r_n \int_{B_n^0} |\underline{u}| \max_{B_n^0} |\partial p_4| \leq C r_n \left(\int_{B_n^0} |\underline{u}| \right) \left(\int_{B_n^0} \frac{|p|}{r_{n_0}^4} \right)$$

Denoting with $\tilde{K}_{n_0}^0$ the operator $K_{n_0}^0$ without the factor $r_{n_0}^{-13/4}$ which makes it dimensionless, and introducing, similarly, $\tilde{A}_n^0, \tilde{G}_n^0$ we obtain the following chain of inequalities, using repeatedly (H)

$$\begin{aligned}
& \frac{1}{r_n^2} \int_{Q_n^0} |p_4 - \overline{p_4}| |\underline{u}| \leq \frac{C}{r_n^2} r_n \left(\int_{\Delta_n^0} \left(\int_{B_n^0} \frac{|p|}{r_{n_0}^4} \right)^{5/4} \right)^{4/5} \left(\int_{\Delta_n^0} \left(\int_{B_n^0} |\underline{u}| \right)^5 \right)^{1/5} \leq \\
& \leq \frac{C}{r_n^2} \frac{r_n}{r_{n_0}^4} \tilde{K}_{n_0}^{0^{4/5}} \left(\int_{\Delta_n^0} \left(\int_{B_n^0} |\underline{u}|^{2/5} |\underline{u}|^{3/5} \cdot 1 \right)^5 \right)^{1/5} \leq \\
& \leq \frac{C}{r_n^2} \frac{r_n}{r_{n_0}^4} \tilde{K}_{n_0}^{0^{4/5}} \left(\int_{B_n^0} |\underline{u}|^2 \right)^{1/5} \left(\int_{Q_n^0} |\underline{u}|^3 \right)^{1/5} r_n^{9/5} \leq \\
& \leq \frac{C}{r_n^2} \frac{r_n}{r_{n_0}^4} r_n^{12/5} \tilde{K}_{n_0}^{0^{4/5}} \tilde{A}_n^{0^{1/5}} \tilde{G}_n^{0^{1/5}} \leq C \left(\frac{r_n}{r_{n_0}} \right)^{7/5} A_n^{0^{1/5}} G_n^{0^{1/5}} K_{n_0}^{0^{4/5}}
\end{aligned}$$

Finally use the inequalities of (5) and combine the estimates above on the terms $p_j, j = 1, \dots, 4$.)

(10) Let $T_n = (A_n^0 + \delta_n^0)$; combine inequalities of (6) and (9), and (5) to deduce

$$\begin{aligned}
T_n & \leq 2^{2n} \left(2^{-2n_0} \varepsilon + \sum_{k=n+1}^{n_0-1} 2^{-2k} T_k^{3/2} + 2^{-2n_0} \varepsilon + \right. \\
& \quad \left. + 2^{-7n_0/5} \varepsilon \sum_{k=n+1}^{n_0-1} 2^{-3k/5} T_k^{1/2} + \varepsilon 2^{-5n_0/3} \sum_{k=n_0+2}^{n_0-1} 2^{-k/3} T_k^{1/2} + \right. \\
& \quad \left. \sum_{k=n+1}^{n_0-1} 2^{-2k} T_k^{3/2} + \sum_{k=n+1}^{n_0-1} 2^k T_k^{1/2} \sum_{p=k}^{n_0} 2^{-3p} T_p \right) \\
\varepsilon & \equiv C \max(G_{n_0}^{0^{2/3}}, K_{n_0}^{0^{4/5}}, J_{n_0}^0)
\end{aligned}$$

and show that, by induction, if ε is small enough then $r_n^{-2} T_n \leq \varepsilon^{2/3} r_{n_0}^{-2}$.

(11) If $G(r_0) + J(r_0) + K(r_0) < \varepsilon_s$ with ε_s small enough, then given $(\underline{x}', t') \in Q_{r_0/4}(\underline{x}_0, t_0)$, show that if one calls $G_r^0, J_r^0, K_r^0, A_r^0, \delta_r^0$ the operators associated with $Q_r^0(\underline{x}', t')$ then

$$\limsup_{n \rightarrow \infty} \frac{1}{r_n^2} A_n^0 \leq C \frac{\varepsilon_s^{2/3}}{r_0^2}$$

for a suitable constant C . (*Idea* Note that $Q_{r_0/4}^0(\underline{x}', t') \subset Q_{r_0}(\underline{x}_0, t_0)$ hence $G_{r_0/4}^0, J_{r_0/4}^0, \dots$ are bounded by a constant, ($\leq 4^2$), times $G(r_0), J(r_0), \dots$ respectively. Then apply the result of (10)).

(12) Check that the result of (11) implies theorem II. (*Idea* Indeed

$$\frac{1}{r_n^2} A_n^0 \geq \frac{1}{r_n^3} \int_{B_n^0} |\underline{u}(\underline{x}, t')|^2 d\underline{x} \xrightarrow{n \rightarrow \infty} \frac{4\pi}{3} |\underline{u}(\underline{x}', t')|^2$$

where B_n^0 is the ball centered at \underline{x}' , for almost all the points $(\underline{x}', t') \in Q_{r_0/4}^0$; hence $|\underline{u}(\underline{x}', t')|$ is bounded in $Q_{r_0/4}^0$ and one can apply proposition 2).

(13) Let f be a function with zero average over B_r^0 . Since $f(\underline{x}) = f(\underline{y}) + \int_0^1 ds \partial f(\underline{y} + (\underline{x} - \underline{y})s) \cdot (\underline{x} - \underline{y})$ for each $\underline{y} \in B_r^0$, averaging this identity over \underline{y} one gets

$$f(\underline{x}) = \int_{B_r^0} \frac{d\underline{y}}{|B_r^0|} \int_0^1 ds \partial f(\underline{y} + (\underline{x} - \underline{y})s) \cdot (\underline{x} - \underline{y})$$

Assuming $\alpha = 1$ prove (P). (*Idea* Change variables as $\underline{y} \rightarrow \underline{z} = \underline{y} + (\underline{x} - \underline{y})s$ so that for α integer

$$\int_{B_r^0} |f(\underline{x})|^\alpha \frac{d\underline{x}}{|B_r^0|} \equiv \int_{B_r^0} \frac{d\underline{x}}{|B_r^0|} \left| \int_0^1 \int_{B_r^0} \frac{d\underline{z}}{|B_r^0|} \frac{ds}{(1-s)^3} \partial f(\underline{z}) \cdot (\underline{z} - \underline{x}) \right|^\alpha$$

where the integration domain of \underline{z} depends from \underline{x} and s , and it is contained in the ball with radius $2(1-s)r$ around \underline{x} . The integral can then be bounded by

$$\int \frac{d\underline{z}_1}{|B_r^0|} \frac{ds_1}{1-s_1} \dots \frac{d\underline{z}_\alpha}{|B_r^0|} \frac{ds_\alpha}{1-s_\alpha} (2r)^\alpha |\partial f(\underline{z}_1)| \dots |\partial f(\underline{z}_\alpha)| \int \frac{d\underline{x}}{|B_r^0|}$$

where \underline{x} varies in a domain with $|\underline{x} - \underline{z}_i| \leq 2(1-s_i)r$ for each i . Hence the integral over $\frac{d\underline{x}}{|B_r^0|}$ is bounded by $8(1-s_i)^3$ for each i . Performing a geometric average of such bounds (over α terms)

$$\begin{aligned} \int_{B_r^0} |f(\underline{x})|^\alpha \frac{d\underline{x}}{|B_r^0|} &\leq 2^{\alpha+3} r^\alpha \prod_{i=1}^\alpha \int \frac{d\underline{z}_i ds_i}{|B_r^0| (1-s_i)} |\partial f(\underline{z}_i)| (1-s_i)^{3/\alpha} \leq \\ &\leq 2^{\alpha+3} r^\alpha \left(\int_{B_r^0} |\partial f(\underline{z})| \frac{d\underline{z}}{|B_r^0|} \right)^\alpha \cdot \left(\int_0^1 \frac{ds}{(1-s)^{3-3/\alpha}} \right)^\alpha \end{aligned}$$

getting (P) and an explicit estimate of the constant C_α^P only for $\alpha = 1$ and a hint that (P) should hold for $\alpha < \frac{3}{2}$ at least.)

(14) Differentiate twice with respect to α^{-1} and check the convexity of $\alpha^{-1} \rightarrow \|f\|_\alpha \equiv (\int |f(\underline{x})|^\alpha d\underline{x}/|B_r^0|)^{1/\alpha}$. Use this to get (P) for each $1 \leq \alpha < \alpha_0$ if it holds for $\alpha = \alpha_0$. (*Idea* Since (P) can be written $\|f\|_\alpha \leq C_\alpha (\int |\partial f| d\underline{x}/r^2)$ then if $\alpha^{-1} = \vartheta\alpha_0^{-1} + (1 - \vartheta)$ it follows that C_α can be taken $C_\alpha = \vartheta C_{\alpha_0} + (1 - \vartheta) C_1$).

(15) Consider a sequence \underline{u}^λ of solutions of the Leray regularized equations which converges *weakly* (*i.e.* for each Fourier component) to a Leray solution. By construction the $\underline{u}^\lambda, \underline{u}$ verify the *a priori* bounds in (0.2) and (hence) (2.5). Deduce that \underline{u} verifies the (3.1). (*Idea* Only (c) has to be proved. Note that if $\underline{u}^\lambda \rightarrow \underline{u}^0$ weakly, then the left hand side of (1.1) is semi continuous hence the value computed with \underline{u}^0 is not larger than the limit of the right hand side in (1.1). On the other hand the right hand side of (1.1) is *continuous* in the limit $\lambda \rightarrow \infty$. Indeed given $N > 0$ weak convergence implies

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_0^{T_0} dt \int_\Omega |\underline{u}^\lambda - \underline{u}^0|^2 d\underline{x} &\equiv \lim_{\lambda \rightarrow \infty} \int_0^{T_0} dt \sum_{0 < |\underline{k}| < N} |\underline{\gamma}_{\underline{k}}^\lambda(t) - \underline{\gamma}_{\underline{k}}^0(t)|^2 \leq \\ &\leq \lim_{\lambda \rightarrow \infty} \left(\sum_{0 < |\underline{k}| < N} \int_0^{T_0} dt |\underline{\gamma}_{\underline{k}}^\lambda(t) - \underline{\gamma}_{\underline{k}}^0(t)|^2 + \sum_{|\underline{k}| \geq N} \int_0^{T_0} dt \frac{|\underline{k}|^2}{N^2} |\underline{\gamma}_{\underline{k}}^\lambda(t) - \underline{\gamma}_{\underline{k}}^0(t)|^2 \right) \leq \\ &\leq \lim_{\lambda \rightarrow \infty} \left(\sum_{0 < |\underline{k}| < N} \int_0^{T_0} dt |\underline{\gamma}_{\underline{k}}^\lambda(t) - \underline{\gamma}_{\underline{k}}^0(t)|^2 + \frac{1}{N^2} \int_0^{T_0} dt \int_\Omega |\partial(\underline{u}^\lambda - \underline{u}^0)|^2 \right) = \\ &= \lim_{\lambda \rightarrow \infty} \frac{1}{N^2} \int_0^{T_0} dt \int_\Omega |\partial(\underline{u}^\lambda - \underline{u}^0)|^2 \leq \frac{2E_0\nu^{-1}}{N^2} \end{aligned}$$

using the *a priori* bound in (0.2) (with zero force) and componentwise convergence of the Fourier transform $\underline{\gamma}_{\underline{k}}(t)$ of $\underline{u}(t)$ to the Fourier transform $\underline{\gamma}_{\underline{k}}^0(t)$ of \underline{u}^0 . Hence $\int_0^{T_0} \int_\Omega |\underline{u}^\lambda - \underline{u}^0|^2 \rightarrow 0$ showing the convergence of the first two terms of the right hand side of (1.1) to the corresponding terms of (c) in (3.1).

Apply, next, the inequality (S), (2.2), with $q = 3$, $a = \frac{3}{4}$, $\frac{q}{2} - a = \frac{3}{4}$, and again by the *a priori* bounds in (0.2) we get

$$\begin{aligned} \int_0^{T_0} dt \int_\Omega |\underline{u}^\lambda - \underline{u}^0|^3 d\underline{x} &\leq C \int_0^{T_0} dt \|\partial(\underline{u}^\lambda - \underline{u}^0)\|_2^{3/2} \|\underline{u}^\lambda - \underline{u}^0\|_2^{3/2} \leq \\ &\leq C \left(\int_0^{T_0} dt \|\partial(\underline{u}^\lambda - \underline{u}^0)\|_2^2 \right)^{3/4} \left(\int_0^{T_0} dt \|\underline{u}^\lambda - \underline{u}^0\|_2^6 \right)^{1/4} \leq \\ &\leq C(2E_0\nu^{-1})^{3/4} (2\sqrt{E_0}) \int_0^{T_0} dt \|\underline{u}^\lambda - \underline{u}^0\|_2^2 \xrightarrow{\lambda \rightarrow \infty} 0 \end{aligned}$$

showing continuity of the third term in the second member of (3.1). Finally, and analogously, if we recall that $p^\lambda = -\Delta^{-1} \sum_{ij} \partial_i \partial_j (u_i^\lambda u_j^\lambda)$ and if we apply the inequalities (CZ) and (H), we get

$$\begin{aligned} \int_0^{T_0} dt \int_\Omega d\underline{x} |p^\lambda \underline{u}^\lambda - p^0 \underline{u}^0| &\leq \int \int |p^\lambda - p^0| |\underline{u}^\lambda| + \int \int |p^0| |\underline{u}^\lambda - \underline{u}^0| \leq \\ &\left(\int \int |p^\lambda - p^0|^{3/2} \right)^{2/3} \left(\int \int |\underline{u}^\lambda|^3 \right)^{1/3} + \left(\int \int |p^0|^{3/2} \right)^{2/3} \left(\int \int |\underline{u}^\lambda - \underline{u}^0|^3 \right)^{1/3} \end{aligned}$$

where the last integral tends to zero by the previous relation while the first, via (CZ), will be such that $\int_0^{T_0} \int_\Omega |p^\lambda - p^0|^{3/2} \leq \left(\int \int |\underline{u}^\lambda - \underline{u}^0|^3 \right)^{2/3} \xrightarrow{\lambda \rightarrow \infty} 0$ proving the continuity of the fourth term in the right hand side of (c) in (3.1). Hence the right hand side is continuous in the considered limit).

(16) (*covering theorem, (Vitali)*) Let S be an arbitrary set inside a sphere of R^n . Consider a *covering* of S with little open balls with the *Vitali property*: i.e. such that every point of S is contained in a family of open balls of the covering whose radii have a zero greatest lower bound. Given $\eta > 0$ show that if $\lambda > 1$ is large enough it is possible to find a denumerable family F_1, F_2, \dots of pairwise disjoint balls of the covering with diameter $< \eta$ such that $\cup_i \lambda F_i \supset S$ where λF_i denotes the ball with the same center of F_i and radius λ times longer. Furthermore λ can be chosen independent of S , see also problem (17). (*Idea* Let \mathcal{F} be the covering and let $a = \max_{\mathcal{F}} \text{diam}(F)$. Define $a_k = a2^{-k}$ and let \mathcal{F}_1 be a *maximal* family of *pairwise disjoint* ball of \mathcal{F} with radii $\geq a2^{-1}$ and $< a$. Likewise let \mathcal{F}_2 be a maximal set of balls of \mathcal{F} with radii between $a2^{-2}$ and $a2^{-1}$ pairwise disjoint between themselves and with the ones of the family \mathcal{F}_1 . Inductively we define $\mathcal{F}_1, \dots, \mathcal{F}_k, \dots$. It is now important to note that if $x \notin \cup_k \mathcal{F}_k$ it must be: $\text{distance}(x, \mathcal{F}_k) < \lambda a 2^{-k}$ for some k , if λ is large enough. If indeed δ is the radius of a ball S_δ containing x and if $a2^{-k_0} \leq \delta < a2^{-k_0+1}$ then the point of S_δ farthest away from x is at most at distance $\leq 2\delta < 4a2^{-k_0}$; and if, therefore, it was $d(x, \mathcal{F}_{k_0}) \geq 4a2^{-k_0}$ we would find that the set \mathcal{F}_{k_0} could be made larger by adding to it S_δ , against the maximality supposed for \mathcal{F}_{k_0} . Note that $\lambda = 5$ is a possible choice.)

(17) Show that if the balls in problem (16) are replaced by the *parabolic cylinders* which are Cartesian products of a radius r ball in the first k coordinates and one of radius r^α , with $\alpha \geq 1$ in the $n - k$ remaining ones, then the result still holds if one replaces $5F_i$ with λF_i where λ is a suitable homothety factor (with respect to the center of F_i). Show that if $\alpha = 1, 2$ then $\lambda = 5$ is enough (and, in general, $\lambda = (4^2 + 2^{2(1+\alpha)/\alpha})^{1/2}$ is enough).

(18) Check that the Hausdorff dimension of the Cantor set C is $\log_3 2$, c.f.r. (5.3). (*Idea* It remains to see, given the equation in footnote⁹, that if $\alpha < \alpha_0$ then $\mu_\alpha(C) = \infty$. If $\delta = 3^{-n}$ the covering \mathcal{C}_n of C with the n -th generation intervals is “the best” among those with sets of diameter $\leq 3^{-n}$ because another covering could be refined by deleting from each of its intervals the points that are out of the n -th generation intervals. Furthermore the inequality $1 < 23^{-\alpha}$ for $\alpha < \log_3 2$ shows that it will not be convenient to further subdivide the intervals of \mathcal{C}_n for the purpose of diminishing the sum $\sum |F_i|^\alpha$. Hence for $\delta = 3^{-n}$ the minimum value of the sum is $2^n 3^{-n\alpha} \xrightarrow{n \rightarrow \infty} \infty$.)

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