Ann. Math. Québec (2016) 40:203–220
DOI 10.1007/s40316-015-0041-7

Finite polynomial cohomology for general varieties

Amnon Besser[1] · **David Loeffler**[2] · **Sarah Livia Zerbes**[3]

In honour of Glenn Stevens' 60th birthday

Received: 16 December 2014 / Accepted: 20 January 2015 / Published online: 11 January 2016

Abstract Nekovář and Nizioł (Syntomic cohomology and p-adic regulators for varieties over p-adic fields, 2013) have introduced in a version of syntomic cohomology valid for arbitrary varieties over p-adic fields. This uses a mapping cone construction similar to the rigid syntomic cohomology of (Besser, Israel J Math 120(1):291–334, 2000) in the good-reduction case, but with Hyodo–Kato (log-crystalline) cohomology in place of rigid cohomology. In this short note, we describe a cohomology theory which is a modification of the theory of Nekovář–Nizioł, modified by replacing $1 - \varphi$ with other polynomials in φ. This is the analogue for bad-reduction varieties of the finite-polynomial cohomology of (Besser, Invent Math 142(2):397–434, 2000); and we use this cohomology theory to give formulae for p-adic regulator maps, extending the results of (Besser, Invent Math 142(2):397–434, 2000; Besser, Israel J Math 120(1):335–360, 2000; Besser, Israel J Math 190(1):29–66, 2012) to varieties over p-adic fields, without assuming any good reduction hypotheses.

Keywords Finite-polynomial cohomology · Syntomic cohomology · Regulators

The authors' research was supported by the following Grants: Royal Society University Research Fellowship (Loeffler); EPSRC First Grant EP/J018716/1 (Zerbes).

✉ Sarah Livia Zerbes
s.zerbes@ucl.ac.uk

Amnon Besser
bessera@math.bgu.ac.il

David Loeffler
d.a.loeffler@warwick.ac.uk

[1] Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er-Sheva 84105, Israel

[2] Mathematics Institute, University of Warwick, Zeeman Building, Coventry CV4 7AL, UK

[3] Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

Résumé Nekovar et Niziol (Syntomic cohomology and p-adic regulators for varieties over p-adic fields, 2013) ont introduit une version de la cohomologie syntomique pour les variétés definis sur un corps p-adique. Leur construction généralise la cohomologie rigide syntomique de (Besser, Israel JMath 120(1):291–334, 2000) dans le cas de bonne reduction, remplacant la cohomologie rigide par la cohomologie de Hyodo-Kato. Nouse décrivons une modification de la theorie de Nekovar-Niziol, remplacant l'operateur 1-Φ par d'autres polynomes en Φ. Ceci est l'analogue de la "finite-polynomial cohomology" de (Besser, Invent. Math 142(2):397 434, 2000) pour les variétés de mauvaise réduction. Nous utilisons cette cohomologie pour obtenir des formules pour les régulateurs p-adiques, généralisant les résultats de (Besser, Invent. Math 142(2):397–434, 2000; Besser, 11 Israel J Math 120(1):335–360, 2000; Besser, Israel J Math 190(1):29–66, 2012) dans le cas de bonne réduction.

Mathematics Subject Classification 11G25 (Arithmetic geometry: Varieties over finite and local fields) · 11G55 (Arithmetic geometry: polylogarithms and relations with K-theory) · 14F30 (Algebraic geometry: p-adic cohomology, crystalline cohomology)

1 Preliminaries from p-adic Hodge theory

1.1 Filtered (φ, N)-modules and their cohomology

We recall some standard constructions for (ϕ, N)-modules, following §2.4 of [10]. Let K be a p-adic field (i.e. the field of fractions of a complete DVR V of mixed characteristic $(0, p)$, with perfect residue field k).

Definition 1.1.1 We define a filtered (φ, N, G_K)-module over K to be the data of a finite-dimensional K_0^{nr}-vector space D, where K_0 is the maximal unramified subfield of K and K_0^{nr} its maximal unramified extension, equipped with the following structures:

- a K_0^{nr}-semilinear, bijective Frobenius φ;
- a K_0^{nr}-linear monodromy operator N satisfying $N\varphi = p\varphi N$;
- a K_0^{nr}-semilinear action of G_K;
- and a decreasing, separated, exhaustive filtration of the K-vector space $D_K = (D \otimes_{K_0^{\mathrm{nr}}} \overline{K})^{G_K}$ by K-vector subspaces $\mathrm{Fil}^i D_K$.

These are the objects of a pre-abelian category $\mathrm{MF}_{(\phi,N,G_K)}$. (It is not an abelian category, since morphisms may not be strictly compatible with the filtration.)

Theorem 1.1.2 (Colmez–Fontaine) *If K is a finite extension of $\mathbf{Q}_p$, the subcategory of "weakly admissible" filtered (φ, N, G_K)-modules is equivalent to the category of potentially semistable $\mathbf{Q}_p$-linear representations of G_K, via the functor $V \mapsto \mathbf{D}_{\mathrm{pst}}(V) = \bigcup_M (V \otimes \mathbf{B}_{\mathrm{st}})^{G_M}$, where the direct limit is taken over finite extensions M/K.*

We write $D_{\mathrm{st}} = D^{G_K}$, which is a K_0-vector space. We say that D is *semistable* if the G_K-action on D is unramified, in which case we have

$$D = D_{\mathrm{st}} \otimes_{K_0} K_0^{\mathrm{nr}} \quad \text{and} \quad D_K = D_{\mathrm{st}} \otimes_{K_0} K;$$

and we say D is *crystalline* if it is semistable with $N = 0$. These notions are compatible with the usual ones for weakly admissible D via the Colmez–Fontaine theorem.

Definition 1.1.3 Let D be a filtered (φ, N, G_K)-module and define a complex $C^\bullet_{\mathrm{st}}(D)$ by

$$D_{\mathrm{st}} \oplus \mathrm{Fil}^0 D_K \longrightarrow D_{\mathrm{st}} \oplus D_{\mathrm{st}} \oplus D_K \longrightarrow D_{\mathrm{st}},$$

with morphisms given by $(u, v) \mapsto ((1-\varphi)u, Nu, u-v)$ and $(w, x, y) \mapsto Nw - (1 - p\varphi)x$.

Theorem 1.1.4 (Bloch–Kato, Nekovář) *The complex $C^\bullet_{\mathrm{st}}$ computes Ext groups in the category of filtered (φ, N, G_K)-modules; the subcategory of weakly admissible objects is closed under extensions, and the functor $\mathbf{D}_{\mathrm{pst}}$ induces functorial maps*

$$H^i_{\mathrm{st}}(D) \to H^i(K, V).$$

These maps are isomorphisms for $i = 0$ and injective for $i = 1$.

1.2 Variants

We now construct a variant of the complex $C_{\mathrm{st}}(D)$ in which the semilinear Frobenius is replaced by a "partially linearized" one, and $1-\varphi$ by a more general polynomial.

We choose a finite extension $L/\mathbf{Q}_p$ contained in K, and we write $f = [L_0 : \mathbf{Q}_p]$ and $q = p^f$. We can then define $D_{\mathrm{st},L} = D_{\mathrm{st}} \otimes_{L_0} L$, which we equip with an L-linear operator Φ given by extending scalars from the L_0-linear operator φ^f on D_{st}.

Definition 1.2.1 Let $P \in 1 + TL[T]$ be a polynomial. We write $C^\bullet_{\mathrm{st},L,P}(D)$ for the complex of L-vector spaces

$$D_{\mathrm{st},L} \oplus \mathrm{Fil}^0 D_K \longrightarrow D_{\mathrm{st},L} \oplus D_{\mathrm{st},L} \oplus D_K \longrightarrow D_{\mathrm{st},L},$$

with the maps given by $u, v \mapsto (P(\Phi)u, Nu, u-v)$ and $(w, x, y) \mapsto Nw - P(q\Phi)x$.

Thus $C^\bullet_{\mathrm{st}}(D) = C^\bullet_{\mathrm{st},\mathbf{Q}_p,P_0}(D)$ where $P_0(T) = 1 - T$. We write $H^\bullet_{\mathrm{st},L,P}(D)$ for the cohomology of the complex $C^\bullet_{\mathrm{st},L,P}(D)$.

Remark 1.2.2 In applications of the theory, we are almost always interested in the case when K is finite over $\mathbf{Q}_p$ and $L = K$; but in order to set up the theory we need a base-change compatibility which seems to be easier to prove for varying K but fixed L, which is why we have set up the theory for $L \neq K$.

We have natural compatibilities when we change the polynomial P or the field L:

Definition 1.2.3 If $P, Q \in 1 + TL[T]$, we define natural transformations $C^\bullet_{\mathrm{st},L,P}(-) \to C^\bullet_{\mathrm{st},L,PQ}(-)$ via

$$\begin{tikzcd} D_{\mathrm{st},L} \oplus \mathrm{Fil}^0 D_K \arrow[r] \arrow[d, "{(\mathrm{id},\,\mathrm{id})}"'] & D_{\mathrm{st},L} \oplus D_{\mathrm{st},L} \oplus D_K \arrow[r] \arrow[d, "{(Q(\Phi),\,\mathrm{id},\,\mathrm{id})}"'] & D_{\mathrm{st},L} \arrow[d, "Q(q\Phi)"'] \\ D_{\mathrm{st},L} \oplus \mathrm{Fil}^0 D_K \arrow[r] & D_{\mathrm{st},L} \oplus D_{\mathrm{st},L} \oplus D_K \arrow[r] & D_{\mathrm{st},L} \end{tikzcd}$$

Definition 1.2.4 Suppose $L \subset L'$ are two subfields of K finite over $\mathbf{Q}_p$, let $P' \in 1 + TL[T]$, and define $P(T) = P'(T^d)$ where $d = [L'_0 : L_0] = f'/f$. Then we have natural transformations $C^\bullet_{\mathrm{st},L,P}(-) \to C^\bullet_{\mathrm{st},L',P'}(-)$ arising from the inclusions

$$D_{\mathrm{st}} \otimes_{L_0} L \to D_{\mathrm{st}} \otimes_{L'_0} L'.$$

Note that the map of Definition 1.2.4 is an isomorphism if L'/L is an unramified extension.

Remark 1.2.5 One can define, in the obvious fashion, a pre-abelian category $\mathrm{MF}_{(\Phi,N,G_K,L)}$ of "filtered (Φ, N, G_K)-modules with coefficients in L", whose objects are finite-dimensional LK_0^{nr}-vector spaces D_L with an L-linear q-th power Frobenius Φ, a monodromy operator N, a G_K-action, and a filtration on $D_K = (D_L \otimes_{LK_0^{\mathrm{nr}}} \overline{K})^{G_K}$. There is a natural extension-of-coefficients functor

$$\mathrm{MF}_{(\varphi,N,G_K)} \to \mathrm{MF}_{(\Phi,N,G_K,L)},$$

and the functors $C^\bullet_{\mathrm{st},L,P}(-)$ naturally factor through this.

For each polynomial $P \in 1 + TL[T]$, one can define an object $\mathbf{1}_P$ of $\mathrm{MF}_{(\Phi,N,G_K,L)}$ by taking $\mathbf{1}_P = L[X]/P(X)$, with Φ acting as multiplication by X (and N and G_K acting trivially, and filtration concentrated in degree 0). Then the complex $C^\bullet_{\mathrm{st},L,P}(D)$ computes the Ext groups $\mathrm{Ext}^i(\mathbf{1}_P, D)$ in this category, and the change-of-P map of Definition 1.2.3 is then obtained by contravariant functoriality from the obvious map $\mathbf{1}_{PQ} \to \mathbf{1}_P$. Note, however, that $\mathbf{1}_P$ is not weakly admissible in general.

1.3 Cup products

Definition 1.3.1 (cf. [4, Definition 4.1]) If $P, Q \in 1 + TL[T]$, then we define $P \star Q \in 1 + TL[T]$ as the polynomial with roots $\{\alpha_i \beta_j\}$, where $\{\alpha_i\}$ and $\{\beta_j\}$ are the roots of P and Q respectively.

Proposition 1.3.2 *There are cup-products $C^\bullet_{\mathrm{st},L,P_1}(D_1) \times C^\bullet_{\mathrm{st},L,P_2}(D_2) \to C^\bullet_{\mathrm{st},L,P_1\star P_2}(D_1 \otimes D_2)$, associative and graded-commutative up to homotopy, compatible with the change-of-P maps of Definition 1.2.3 and the change-of-L maps of Definition 1.2.4.*

Proof This is a straightforward exercise in homological algebra. Let $\lambda \in K$ and choose polynomials $a(T_1, T_2)$ and $b(T_1, T_2)$ such that $a(T_1, T_2)P_1(T_1) + b(T_1, T_2)P_2(T_2) = (P_1 \star P_2)(T_1T_2)$. Then the cup-products in the various degrees are given in Table 1. One verifies easily that changing the value of λ, or the polynomials a and b, changes the product by a chain homotopy. □

1.4 Convenient modules

Definition 1.4.1 Let D be a filtered (φ, N, G_K)-module. We say D is *convenient* (for some choice of L and P) if D is crystalline and $P(\Phi)$ and $P(q\Phi)$ are bijective as endomorphisms of $D_{\mathrm{st},L}$.

Table 1 Cup-products on the complexes $C^\bullet_{\mathrm{st},L,P}$

	(u', v')	(w', x', y')	z'
(u, v)	$(u \otimes u', v \otimes v')$	$\begin{pmatrix} b(\Phi_1, \Phi_2)(u \otimes w'), \\ u \otimes x', \\ (\lambda u + (1-\lambda)v) \otimes y' \end{pmatrix}$	$b(\Phi_1, q\Phi_2)(u \otimes z')$
(w, x, y)	$\begin{pmatrix} a(\Phi_1, \Phi_2)(w \otimes u'), \\ x \otimes u', \\ y \otimes ((1-\lambda)u' + \lambda v') \end{pmatrix}$	$-a(\Phi_1, q\Phi_2)(w \otimes x') + b(q\Phi_1, \Phi_2)(x \otimes w')$	0
z	$a(q\Phi_1, \Phi_2)(z \otimes u')$	0	0

Note that D is convenient if and only if $D^*(1)$ is convenient. If D is convenient, then the inclusion $D_K \hookrightarrow C^1_{\mathrm{st},L,P}(D)$ induces an isomorphism

$$\frac{D_K}{\mathrm{Fil}^0 D_K} \cong H^1_{\mathrm{st},L,P}(D). \tag{1}$$

The inverse of this isomorphism is given by

$$(w, x, y) \mapsto y - \iota(P(\Phi)^{-1}w) \mod \mathrm{Fil}^0, \tag{2}$$

where ι is the natural inclusion $D_{\mathrm{st},L} \hookrightarrow D_K$; note that we must have $Nw = P(q\Phi)x$ since (w, x, y) is a cocycle, but by assumption $N = 0$ and $P(q\Phi)$ is bijective, so in fact we have $x = 0$. Note that this map commutes with the change-of-P and change-of-L maps (where defined).

The following special case will be of importance below:

Proposition 1.4.2 *If $P(1) \neq 0$ and $P(q^{-1}) \neq 0$, then there is an isomorphism*

$$H^1_{\mathrm{st},L,P}(\mathbf{Q}_p(1)) \cong K.$$

The key to our description of syntomic regulators is the following. We now take $L = K$.

Proposition 1.4.3 *Suppose that D is convenient (for $L = K$ and some polynomial $P \in 1 + TK[T]$). Then for any $\lambda \in \mathrm{Fil}^0 D^*(1)_K = (D_K/\mathrm{Fil}^0)^*$, there is a polynomial $Q \in 1 + XK[X]$ such that $\lambda \in H^0_{\mathrm{st},K,Q}(D^*(1))$ and $(P \star Q)(1) \neq 0$, $(P \star Q)(q^{-1}) \neq 0$; and if P is such a polynomial, then we have a commutative diagram*

$$\begin{tikzcd} D_K/\mathrm{Fil}^0 \arrow[r, "\cong"'] \arrow[d, "\lambda"'] & H^1_{\mathrm{st},K,P}(D) \arrow[d, "\cup\lambda"] \\ K \arrow[r, equal] & H^1_{\mathrm{st},K,P\star Q}(\mathbf{Q}_p(1)) \end{tikzcd}$$

where the right-hand vertical map is the cup-product of the previous section.

Remark 1.4.4 The utility of this proposition is that if D is convenient, it allows us to completely describe an element of $H^1_{\mathrm{st},L,P}(D)$ via its cup-products with elements of $H^0_{\mathrm{st},L,Q}(D^*(1))$ for suitable polynomials Q. This is the key to our computations of regulator maps below. Note that even if P is some very simple polynomial (such as $1 - T$), we still need to be able to choose Q freely for this to work.

2 Finite polynomial cohomology for general varieties

2.1 Summary of the theory of Nekovář–Niziol

We briefly summarize some of the main results of the paper [10]. Let $\mathrm{Var}(K)$ be the category of varieties over K (i.e. reduced separated K-schemes of finite type).

Theorem 2.1.1 *There exists a functor $X \mapsto \mathbb{R}\Gamma_{\mathrm{syn}}(X_h, *)$ from $\mathrm{Var}(K)$ to the category of graded-commutative differential graded E_∞-algebras over $\mathbf{Q}_p$, equipped with period morphisms*

$$\rho_{\mathrm{syn}} : \mathbb{R}\Gamma_{\mathrm{syn}}(X_h, r) \to \mathbb{R}\Gamma_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(r)).$$

The cohomology theory $X \mapsto \mathbb{R}\Gamma_{\text{syn}}(X_h, r)$ has pushforward maps for projective morphisms, and has a functorial map from Voevodsky's motivic cohomology, compatible with the étale realization map via ρ_{syn}.

For X a variety over K, let $H^j_{\text{HK}}(X_h)$ and $H^j_{\text{dR}}(X_h)$ be the extensions of Hyodo–Kato and de Rham cohomologies defined by Beilinson [1], and $\iota^B_{\text{dR}} : H^j_{\text{HK}}(X_h) \otimes_{K_0} K \to H^j_{\text{dR}}(X_h)$ the comparison morphism relating them (which is an isomorphism if X has a semistable model over O_K).

The main result of [1] shows that we have

$$H^j_{\text{HK}}(X_h) = D^j(X_h)_{\text{st}}, \quad H^j_{\text{dR}}(X_h) = D^j(X_h)_K,$$

where $D^j(X_h)$ is the filtered (φ, N, G_K)-module given by

$$D^j(X_h) = \varinjlim_{K'} H^j_{\text{HK}}(X_{K',h}),$$

where K' varies over finite extensions of K, and that there is a canonical isomorphism of (φ, N, G_K)-modules

$$D^j(X_h) \cong \mathbf{D}_{\text{pst}}\left(H^j(X_{\overline{K},\text{ét}}, \mathbf{Q}_p)\right).$$

Theorem 2.1.2 *There is a "syntomic descent spectral sequence"*

$$H^i_{\text{st}}(D^j(X_h)(r)) \Rightarrow H^{i+j}_{\text{syn}}(X_h, r),$$

and the morphisms

$$H^i_{\text{st}}(D^j(X_h)(r)) \to H^i(K, H^j_{\text{ét}}(X_{\overline{K},\text{ét}}, \mathbf{Q}_p(r))$$

from Theorem 1.1.4 assemble into a morphism from the syntomic descent spectral sequence to the Hochschild–Serre spectral sequence $H^i(K, H^j_{\text{ét}}(X_{\overline{K},\text{ét}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(r))$, compatible with the period morphism ρ_{syn} on the abutment.

2.2 Definition of P-syntomic cohomology

We now develop a theory which very closely imitates that of [10], but modified to use general polynomials in Frobenius in the place of $1 - \varphi$, and replacing the p-power Frobenius φ with a "partially linearized" Frobenius. We choose a finite extension $L/\mathbf{Q}_p$ is a finite extension contained in K, with residue class degree $f = [L_0 : \mathbf{Q}_p]$, and $P \in 1 + TL[T]$.

Let $(U, \overline{U})$ be an "arithmetic pair" (in the sense of *op.cit.*), log-smooth over $V^\times$ (i.e. $\text{Spec}(V)$ with the log structure associated to the closed point).

Let $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$ be the rational crystalline cohomology (defined as

$$\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \text{holim}_n \, \mathbb{R}\Gamma\left(\overline{U}_{\text{ét}}, Ru_{U_n^\times/W_n(k)*}\mathscr{O}_{U_n^\times/W_n(k)}\right),$$

cf. Sect. 3.1 of *op.cit.*). This is a complex of K_0-vector spaces, hence of L_0-vector spaces, and we write $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L = L \otimes_{L_0} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$. There is a K_0-semilinear Frobenius φ on $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$, and we let $\Phi = \varphi^f$, which is L_0-linear and thus extends to an L-linear operator on $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L$. For $r \geq 0$ we write $\Phi_r = q^{-r}\Phi = (p^{-r}\varphi)^f$.

Remark 2.2.1 Curiously, the complex $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$ is already a complex of K-vector spaces, but this K-linear structure interacts very badly with the Frobenius, and passing to Frobenius eigenspaces "kills off" all contributions from $K \backslash K_0$. So we must "add the K-linear structure a second time" in order to obtain a K-linearized theory of syntomic cohomology.

For any $r \geq 0$, the quasi-isomorphism $\gamma_r^{-1} : \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbf{Q}} \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r$ gives rise to a morphism (not, of course, a quasi-isomorphism)

$$\gamma_{r,L}^{-1} : \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L \to \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r .$$

Definition 2.2.2 We define

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(, \overline{U}, r) = \left[\mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L \xrightarrow{P(\Phi_r),\gamma_{r,L}^{-1}} \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r\right]$$

Remark 2.2.3 Note that the use $P(\Phi_r)$, rather than $P(\Phi)$; this choice gives somewhat cleaner formulations of some results (e.g. the pushforward maps and the syntomic descent spectral sequence), but has the disadvantage of introducing a notational discrepancy between $\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r)$ and the finite-polynomial cohomology of [4].

The complexes $\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r)$ are complexes of L-vector spaces, functorial in pairs $(U, \overline{U})$, equipped with L-linear cup-products

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r) \times \mathbb{R}\Gamma_{\mathrm{syn},L,Q}(U, \overline{U}, s) \longrightarrow \mathbb{R}\Gamma_{\mathrm{syn},L,P\star Q}(U, \overline{U}, r+s)$$

which are associative and graded-commutative, up to coherent homotopy (i.e. the direct sum

$$\bigoplus_{r,P} \mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r)$$

is an E_∞-algebra over L). These cup-products are given explicitly by the same recipe as in 1.3.2 above.

The following properties are immediate:

Proposition 2.2.4 (1) (*Compatibility with syntomic cohomology*) *If $L = \mathbf{Q}_p$ and $P(T)$ is the polynomial $1 - T$, then $\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r)$ coincides with $\mathbb{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbf{Q}}$ as defined in* [10].

(2) (*Change of P*) *If P, Q are two polynomials, there is a map*

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r)_{\mathbf{Q}} \to \mathbb{R}\Gamma_{\mathrm{syn},L,PQ}(U, \overline{U}, r)_{\mathbf{Q}}$$

functorial in $(U, \overline{U})$ and compatible with cup-products, which is given by the diagram

$$\begin{array}{ccc}
\mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L & \xrightarrow{P(\Phi_r),\, \gamma_{r,L}^{-1}} & \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle (Q(\Phi_r),\, \mathrm{id})} \\
\mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L & \xrightarrow{PQ(\Phi_r),\, \gamma_{r,L}^{-1}} & \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r
\end{array}$$

(3) (*Change of L*) *Let L'/L be a finite extension, and suppose that we have $P(T) = P'(T^d)$ for some polynomial P', where $d = [L'_0 : L_0] = f'/f$. Then there is a natural map*

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r) \longrightarrow \mathbb{R}\Gamma_{\mathrm{syn},L',P'}(U, \overline{U}, r),$$

functorial in $(U, \overline{U})$ and compatible with cup-products. If L'/L is an unramified extension, this is an isomorphism.

Proof (1) and (2) are obvious by construction. For (3), note that we have $\Phi'_r = (p^{-r}\varphi)^{f'}$ and $\Phi_r = (p^{-r}\varphi)^f$, so $P(\Phi_r) = P'(\Phi'_r)$. Hence we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L & \xrightarrow{P(\Phi_r),\, \gamma_{r,L}^{-1}} & \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_{L'} & \xrightarrow{P'(\Phi'_r),\, \gamma_{r,L'}^{-1}} & \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_{L'} \oplus \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r
\end{array}$$

where the vertical maps are the natural maps induced from the inclusion $L \subset L'$.

If L'/L is unramified, so $L' = LL'_0$, then we have a canonical isomorphism of complexes of L'_0-vector spaces

$$L \otimes_{L_0} \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_{\mathbf{Q}} \cong L' \otimes_{L'_0} \mathbb{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_{\mathbf{Q}},$$

so the vertical maps are isomorphisms.

We will be particularly interested in a special case of this:

Proposition 2.2.5 *If $P(1) = 0$ then there is a morphism of complexes*

$$\mathbb{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbf{Q}} \to \mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r),$$

functorial in $(U, \overline{U})$ and compatible with cup-products.

Proof This is built up by combining all three parts of the above proposition. Firstly, (1) identifies Nekovář–Nizioł's cohomology $\mathbb{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbf{Q}}$ with the special case $P(T) = 1 - T$, $L = \mathbf{Q}_p$ of our construction. If P is any polynomial with $P(1) = 0$, then $P(T^f)$ is divisible by $1 - T$, so (2) gives a morphism

$$\mathbb{R}\Gamma_{\mathrm{syn},\mathbf{Q}_p,1-T}(U, \overline{U}, r) \to \mathbb{R}\Gamma_{\mathrm{syn},\mathbf{Q}_p,P(T^f)}(U, \overline{U}, r)_{\mathbf{Q}_p}.$$

Finally (3) gives a morphism

$$\mathbb{R}\Gamma_{\mathrm{syn},\mathbf{Q}_p,P(T^f)}(U, \overline{U}, r) \to \mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r).$$

All of these are visibly functorial in pairs $(U, \overline{U})$.

We now show a relation analogous to Proposition 3.7 of *op.cit.*. Suppose that $(U, \overline{U})$ is of Cartier type, so Hyodo–Kato cohomology is defined. We use Beilinson's variant of Hyodo–Kato cohomology, which has the advantage of having a comparison map $\iota_{\mathrm{dR}}^B : \mathbb{R}\Gamma_{\mathrm{HK}}^B(U, \overline{U})_K \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)$ which is defined at the level of complexes and does not depend on making a choice of uniformizer of K.

Proposition 2.2.6 *For each uniformizer π of V, there is a quasi-isomorphism*

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r) \xrightarrow[\sim]{\alpha'_{\pi,P,L}} \left[\begin{array}{ccc}
\mathbb{R}\Gamma_{\mathrm{HK}}^B(U, \overline{U})_L & \xrightarrow{(P(\Phi_r),\, \iota_{\mathrm{dR}}^B)} & \mathbb{R}\Gamma_{\mathrm{HK}}^B(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/\operatorname{Fil}^r \\
\downarrow N & & \downarrow (N, 0) \\
\mathbb{R}\Gamma_{\mathrm{HK}}^B(U, \overline{U})_L & \xrightarrow{P(\Phi_{r-1})} & \mathbb{R}\Gamma_{\mathrm{HK}}^B(U, \overline{U})_L
\end{array}\right]$$

where $q = p^f$.

Remark 2.2.7 This map does actually depend on the choice of a uniformizer π, although its source and target are independent of any such choice.

Proof We follow exactly the same argument as the proof given in *op.cit.*. It suffices to check that $P(\Phi)$ is invertible on $I \otimes_{W(k)} H^i_{\mathrm{HK}}(U, \overline{U})_L$, for any $P \in 1 + TL[T]$, where I is the ideal in a divided power series ring over $W(k)$ considered in *op.cit.*. We note that since P is monic the formal Laurent series $1/P(T) = \sum_{n\geq 0} a_n T^n$ has positive radius of convergence, so there is some A such that $\mathrm{ord}_p(a_n) \geq -nA$ for $n \gg 0$. This implies the convergence of the series $\sum a_n \Phi^n$ on $I \otimes_{W(k)} H^i_{\mathrm{HK}}(U, \overline{U})_L$, which gives an inverse of $P(\Phi)$.

Continuing to follow [10], we have

Proposition 2.2.8 *If $(T, \overline{T})$ is the base-change of $(U, \overline{U})$ to a finite Galois extension K'/K with Galois group G, then the natural map $f : (T, \overline{T}) \to (U, \overline{U})$ induces a quasi-isomorphism*

$$f^* : \mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r) \xrightarrow{\sim} \mathbb{R}\Gamma(G, \mathbb{R}\Gamma_{\mathrm{syn},L,P}(T, \overline{T}, r)).$$

Proof This follows by exactly the same proof as in *op.cit.*: the proof proceeds by constructing compatible maps between the various variants of crystalline, de Rham, and Hyodo–Kato cohomology, and these all remain compatible after extending scalars to L.

Remark 2.2.9 We are principally interested in the case $L = K$, but it seems to be easier to prove base-change compatibility in K for a fixed L. It seems eminently natural that if K'/K is totally ramified and both fields are finite over $\mathbf{Q}_p$ then we should get a quasi-isomorphism

$$\mathbb{R}\Gamma_{\mathrm{syn},P}(U, \overline{U}, r)_K \xrightarrow{\sim} \mathbb{R}\Gamma(G, \mathbb{R}\Gamma_{\mathrm{syn},P}(T, \overline{T}, r)_{K'}),$$

but this does not seem to be so easy to prove, and it is not needed for the calculations below.

2.3 h-sheafification

We now sheafify in the h-topology. We write $\mathscr{S}_{L,P}(r)$ for the sheafification of $(U, \overline{U}) \mapsto \mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r)$, and we define

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(X_h, r) = \mathbb{R}\Gamma(X_h, \mathscr{S}_{L,P}(r)).$$

Proposition 2.3.1 *For any arithmetic pair $(U, \overline{U})$ that is fine, log-smooth over $V^\times$ and of Cartier type, the canonical maps*

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P}(U, \overline{U}, r) \to \mathbb{R}\Gamma_{\mathrm{syn},L,P}(U_h, r)$$

are quasi-isomorphisms.

Proof This is exactly the generalization to our setting of Proposition 3.16 of *op.cit.*. The long and highly technical proof fortunately carries over verbatim to general P.

Proposition 2.3.2 *There is a "P-syntomic descent spectral sequence"*

$$E_2^{ij} = H^i_{\mathrm{st},L,P}\left(D^j(X_h)(r)\right) \Rightarrow H^{i+j}_{\mathrm{syn},L,P}(X_h, r),$$

where as above $D^j(X_h) \cong \mathbf{D}_{\mathrm{pst}}\left(H^j(X_{\overline{K},\mathrm{\acute{e}t}}, \mathbf{Q}_p)\right)$ is the (φ, N, G_K)-module defined by the comparison between Hyodo–Kato and de Rham cohomology. This spectral sequence is compatible with extension of L (where defined) and change of P. Moreover, it is compatible with cup-products, where the cup-product on the terms E_2^{ij} is given by the construction of Sect. 1.3.

Proof For the existence of the spectral sequence, see Proposition 3.17 of *op.cit.*. The compatibility with cup-products is immediate from the definition of the cup-product on P-syntomic cohomology (see Sect. 2.4 below for explicit formulae).

Note that the spectral sequence implies that the cohomology groups $H^i_{\mathrm{syn},L,P}(X_h, r)$ are zero if $i > 2\dim X + 2$, and if $K/\mathbf{Q}_p$ is a finite extension, then the cohomology groups are finite-dimensional L-vector spaces (with dimension bounded independently of P). Moreover, the spectral sequence obviously degenerates at E_3, and this gives a 3-step filtration on the groups $H^i_{\mathrm{syn},L,P}(X_h, r)$:

Definition 2.3.3 We write $\mathrm{Fil}^m H^i_{\mathrm{syn},L,P}(X_h, r)$ for the 3-step decreasing filtration on $H^i_{\mathrm{syn},L,P}(X_h, r)$ induced by the P-syntomic spectral sequence. Concretely, we have

$$\mathrm{Fil}^0 / \mathrm{Fil}^1 = \ker\left(H^i_{\mathrm{HK}}(X_h)_L^{P(\Phi_r)=0, N=0} \cap \mathrm{Fil}^r H^i_{\mathrm{dR}}(X_h) \longrightarrow \frac{H^{i-1}_{\mathrm{HK}}(X_h)_L}{\mathrm{im}\, P(\Phi_{r-1}) + \mathrm{im}(N)} \right),$$

$$\mathrm{Fil}^1 / \mathrm{Fil}^2 = \frac{\left\{(x, y, z) \in H^{i-1}_{\mathrm{HK}}(X_h)_L \oplus H^{i-1}_{\mathrm{HK}}(X_h)_L \oplus H^{i-1}_{\mathrm{dR}}(X_h)/\mathrm{Fil}^r : Nx = P(\Phi_{r-1})y\right\}}{\left\{(P(\Phi_r)x, Nx, \iota_{\mathrm{dR}}(x) : x \in H^{i-1}_{\mathrm{HK}}(X_h)_L\right\}},$$

$$\mathrm{Fil}^2 / \mathrm{Fil}^3 = \mathrm{coker}\left(H^{i-1}_{\mathrm{HK}}(X_h)_L^{P(\Phi_r)=0, N=0} \cap \mathrm{Fil}^r H^{i-1}_{\mathrm{dR}}(X_h) \longrightarrow \frac{H^{i-2}_{\mathrm{HK}}(X_h)_L}{\mathrm{im}\, P(\Phi_{r-1}) + \mathrm{im}(N)} \right).$$

Remark 2.3.4 It seems natural to conjecture that the "knight's move" maps

$$H^0_{\mathrm{st},L,P}(D^j(X_h)(r)) \to H^2_{\mathrm{st},L,P}(D^{j-1}(X_h)(r))$$

should be zero for smooth proper X, extending the conjecture for syntomic cohomology formulated in Remark 4.10 of [10]. We do not know if this conjecture holds in general, but the applications below will all concern cases where either the source or the target of this map is zero.

2.4 Explicit cup product formulae

We now use Proposition 2.2.6 to give an explicit formula for the cup product

$$H^i_{\mathrm{syn},L,P}(X_h, r) \times H^j_{\mathrm{syn},L,Q}(X_h, s) \longrightarrow H^{i+j}_{\mathrm{syn},L,P\star Q}(X_h, r+s),$$

which generalizes the description of the cup-product on the complexes $C^\bullet_{\mathrm{st},L,P}(D)$ given in the previous section.

A class $\eta \in H^i_{\mathrm{syn},P}(X_h, r)$ is represented by the following data:

$$\begin{aligned} &u \in \mathbb{R}\Gamma^{B,i}_{\mathrm{HK}}(X_h)_L, && v \in \mathrm{Fil}^r\, \mathbb{R}\Gamma^i_{\mathrm{dR}}(X_h), \\ &w, x \in \mathbb{R}\Gamma^{B,i-1}_{\mathrm{HK}}(X_h)_L, && y \in \mathbb{R}\Gamma^{i-1}_{\mathrm{dR}}(X_h)_L, \\ &z \in \mathbb{R}\Gamma^{B,i-2}_{\mathrm{HK}}(X_h)_L \end{aligned}$$

which satisfy the relations

$$\begin{aligned} &\mathrm{d}u = 0, && \mathrm{d}v = 0, && \\ &dw = P(\Phi)u, && dx = Nu, && \mathrm{d}y = \iota^B_{\mathrm{dR}}(u) - v, \\ &dz = Nw - P(q\Phi)x. && && \end{aligned}$$

Let $\eta' = [u', v'; w', x', y'; z']$ be a corresponding representation of $\eta' \in H^j_{\mathrm{syn},L,Q}(X_h, s)$. We want to find explicit formulae for the class $\eta'' = \eta \cup \eta'$.

As before, fix polynomials $a(t,s)$ and $b(t,s)$ such that

$$(P \star Q)(ts) = a(t,s)P(t) + b(t,s)Q(s),$$

and $\lambda \in K$.

Proposition 2.4.1 *The cup-product is given by* $\eta'' = [u'', v''; w'', x'', y''; z'']$*, where*

$$\begin{aligned}
u'' &= u \cup u',\\
v'' &= v \cup v',\\
w'' &= a(\Phi,\Phi)(w \cup u') + (-1)^i b(\Phi,\Phi)(u \cup w'),\\
x'' &= (x \cup u') + (-1)^i (u \cup x'),\\
y'' &= y \cup \iota^B_{\mathrm{dR}}\Big((1-\lambda)u' + \lambda v'\Big) + (-1)^i \iota^B_{\mathrm{dR}}\Big(\lambda u + (1-\lambda)v\Big) \cup y',\\
w'' &= a(q\Phi,\Phi)(z \cup u') - (-1)^i a(\Phi, q\Phi)(w \cup x') + (-1)^i b(q\Phi,\Phi)(x \cup w')\\
&\quad - b(\Phi, q\Phi)(u \cup z').
\end{aligned}$$

Here, we write e.g. $a(\Phi,\Phi)(w \cup u')$ *for the image of* $a(\Phi_1,\Phi_2)(w \otimes u') \in \mathbb{R}\Gamma^{B,i}_{\mathrm{HK}}(X_h)_L \otimes \mathbb{R}\Gamma^{B,j-1}_{\mathrm{HK}}(X_h)_L$ *under the cup product map into* $\mathbb{R}\Gamma^{B,i+j-1}_{\mathrm{HK}}(X_h)_L$*, where* Φ_1 *and* Φ_2 *are the* q*-power Frobenius maps of the two factors.*

Proof This is simply a translation of the formulae of Sect. 1.3 into the setting of complexes.

2.5 Pushforward maps and compact supports

We now extend to our setup the constructions of Déglise's Appendix B to [10].

Theorem 2.5.1 *Let* $f : X \to Y$ *be a smooth proper morphism of smooth* K*-varieties, of relative dimension* d*. Then there are pushforward maps*

$$f_* : H^i_{\mathrm{syn},L,P}(X_h, r) \to H^{i+2d}_{\mathrm{syn},L,P}(X_h, r+d),$$

which are compatible with change of L *and change of* P *in the obvious sense.*

Proof Each of the sheaves $\mathscr{S}_{L,P}(r)$ is h-local, by definition. They are also $\mathbf{A}^1$-local: this is virtually immediate from the corresponding result for the underlying Hyodo–Kato and de Rham cohomology theories, as in Prop 5.4 of *op.cit.*. The same methods also yield the projective bundle theorem.

Hence we can consider the direct sum of h-sheaves given by

$$\mathscr{S}_{L,P} := \bigoplus_r \mathscr{S}_{L,P}(r).$$

Cup-product gives us maps $\mathscr{S}_L(1) \otimes \mathscr{S}_{L,P}(r) \to \mathscr{S}_{L,P}(r+1)$, where $\mathscr{S}_L(1)$ is defined using the polynomial $1-T$; this gives the direct sum $\mathscr{S}_{L,P}$ the structure of a Tate Ω-spectrum. The same argument as in *op.cit.* now gives pushforward maps (and the projection formula for these holds by construction). It is clear that this argument is compatible with change of L and of P.

The same argument also gives compactly-supported cohomology complexes:

Theorem 2.5.2 *There exist compactly-supported cohomology complexes* $\mathbb{R}\Gamma_{\mathrm{syn},L,P,c}(X_h, r)$*, contravariantly functorial with respect to proper morphisms and covariantly functorial (with a degree shift) with respect to smooth morphisms; and there is a functorial morphism*

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P,c}(X_h, r) \to \mathbb{R}\Gamma_{\mathrm{syn},L,P}(X_h, r)$$

which is an isomorphism for proper X*.*

The compactly-supported cohomology has a descent spectral sequence

$$E^2_{ij} = H^i_{\mathrm{st},L,P}\left(D^j_c(X_h)(r)\right) \Rightarrow H^{i+j}_{\mathrm{syn},L,P,c}(X_h, r)$$

where $D^j_c(X_h) = \mathbf{D}_{\mathrm{pst}}(H^i_{\text{ét},c}(X_{\overline{K},\text{ét}}, \mathbf{Q}_p))$.

Finally, we have a projection formula:

Theorem 2.5.3 *There are cup-products*

$$\mathbb{R}\Gamma_{\mathrm{syn},L,P,c}(X_h, r) \times \mathbb{R}\Gamma_{\mathrm{syn},L,Q}(X_h, s) \to \mathbb{R}\Gamma_{\mathrm{syn},L,P\star Q,c}(X_h, r+s),$$

and for f *a smooth proper morphism, we have the projection formula*

$$f_*(\alpha) \cup \beta = f_*(\alpha \cup f^*\beta).$$

Proof This follows from the construction of the pushforward map, cf. [9].

3 Application to computation of regulators

We'll now apply the constructions above to give formulae describing the syntomic regulator map $H^i_{\mathrm{mot}}(X, j) \to H^i_{\mathrm{syn}}(X_h, j)$, where X is a product of copies of an affine curve Y. More specifically, we will give formulae in the following cases:

- $X = Y$ is a single curve and we are given a class in $H^2_{\mathrm{mot}}(X, 2)$ that is the cup product of two units on Y;
- $X = Y^2$ is the product of two curves, and we are given a class in $H^3_{\mathrm{mot}}(X, 2)$ that is the pushforward of a unit along the diagonal embedding $Y \hookrightarrow Y^2$;
- $X = Y^3$ is the product of three curves and we consider the class in $H^4_{\mathrm{mot}}(X, 2)$ given by the cycle class of the diagonal $Y \hookrightarrow Y^3$.

In the case where Y has a smooth model over $\mathbf{Z}_p$ these regulators have been described using finite-polynomial cohomology (in [4,6,7] respectively); and these have been applied in proving explicit reciprocity laws for Euler systems (in [2,3,8] respectively). We use the generalization of finite-polynomial cohomology to arbitrary varieties described in the preceding section to extend this description to the case of arbitrary smooth curves over K. However, our results are less complete, in that we only obtain a full description of the image of the regulator in a "convenient" quotient of the cohomology of X in the sense of Sect. 1.4. (In other words, we allow X to have bad reduction, but we project to a quotient of its cohomology which looks like the cohomology of a variety with good reduction.)

3.1 Syntomic regulators

Now let us consider the following general setting: X is a smooth connected d-dimensional affine K-variety, and $z \in H^{d+1}_{\mathrm{mot}}(X, j)$ for some j. We have an etale realization map

$$r_{\text{ét}} : H^{d+1}_{\mathrm{mot}}(X, j) \longrightarrow H^{d+1}_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(j)).$$

The Hochschild–Serre descent spectral sequence gives a map

$$H^{d+1}_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(j)) \to H^0(K, H^{d+1}(X_{\overline{K},\text{ét}}, \mathbf{Q}_p(j));$$

but the latter group is zero (because an affine variety of dimension d has étale cohomological dimension d) and thus we obtain a map

$$H^{d+1}(X_{K,\text{ét}}, \mathbf{Q}_p(j)) \to H^1(K, H^d(X_{\overline{K},\text{ét}}, \mathbf{Q}_p(j))).$$

Theorem B of [10] shows that we have $r_{\text{ét}} = \rho_{\mathrm{syn}} \circ r_{\mathrm{syn}}$, where

$$r_{\mathrm{syn}} : H^{d+1}_{\mathrm{mot}}(X, j) \longrightarrow H^{d+1}_{\mathrm{syn}}(X_{K,h}, j)$$

is the syntomic realization map, and

$$\rho_{\mathrm{syn}} : H^{d+1}_{\mathrm{syn}}(X_{K,h}, j) \to H^{d+1}_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(j))$$

is the syntomic period morphism of Theorem 2.1.1 above. Theorem 2.1.2 shows that we have a diagram

$$\begin{tikzcd}
H^{d+1}_{\mathrm{syn}}(X_{K,h}, j) \arrow[r, "\rho_{\mathrm{syn}}"] \arrow[d] & H^{d+1}_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(j)) \arrow[d] \\
H^1_{\mathrm{st}}(D^d(X_h)(j)) \arrow[r, "-\exp_{\mathrm{st}}"] & H^1(K, H^d(X_{\overline{K},\text{ét}}, \mathbf{Q}_p(j)))
\end{tikzcd}$$

where $D^d(X_h) = \mathbf{D}_{\mathrm{pst}}\left(H^d(X_{\overline{K},\text{ét}}, \mathbf{Q}_p)\right)$, the vertical maps are induced by the syntomic and Hochschild–Serre descent spectral sequences, and in the bottom horizontal map, $\exp_{\mathrm{st}}$ denotes the generalized Bloch–Kato exponential for the de Rham G_K-representation $H^d(X_{\overline{K},\text{ét}}, \mathbf{Q}_p(j))$.

Remark 3.1.1 The appearance of $-\exp_{\mathrm{st}}$, rather than $\exp_{\mathrm{st}}$, in the bottom horizontal map is due to the sign $v = (-1)^p v'$ (for $p = 1$) in Step 2 of [10, Theorem 2.17].

Now let D be a quotient of $D^d(X_h)(j)$ (in the category of weakly-admissible (φ, N, G_K)-modules) which is crystalline and convenient, in the sense of Sect. 1.4 above. Then we have isomorphisms

$$H^1_{\mathrm{st}}(D) \cong H^1_{\mathrm{st},K,1-T}(D) \cong D_K/\operatorname{Fil}^0 = (\operatorname{Fil}^0 D^*(1)_K)^*,$$

where $D^*(1)$ is the Tate dual of D, which is a submodule of $D^d_c(X_h)(d+1-j)$. As we saw above, for any class $\eta \in \operatorname{Fil}^0 D^*(1)_K$, we may choose a polynomial P such that $\eta \in H^0_{\mathrm{st},K,P}(D^*(1))$ and $P(1) \neq 0$, $P(q^{-1}) \neq 0$; and the natural perfect pairing $\left(D_K/\operatorname{Fil}^0\right) \times \left(\operatorname{Fil}^0 D^*(1)_K\right) \to K$ coincides with the cup-product

$$H^1_{\mathrm{st},K}(D) \times H^0_{\mathrm{st},K,P}(D^*(1)) \to H^1_{\mathrm{st},K,P}(\mathbf{Q}_p(1)) \cong K.$$

We now relate this to cup-products in P-syntomic cohomology.

Definition 3.1.2 If $P(1) \neq 0$ and $P(q^{-1}) \neq 0$, then we denote by

$$\operatorname{tr}_{X,\mathrm{syn},K,P} : \frac{H^{2d+1}_{\mathrm{syn},X,P,c}(X_h, d+1)}{\mathrm{Fil}^2} \xrightarrow{\cong} H^1_{\mathrm{st},K,P}(\mathbf{Q}_p(1)) \cong K$$

the isomorphism given by the descent spectral sequence and Proposition 1.4.2 above.

Theorem 3.1.3 *Let D be a convenient crystalline quotient of $D^d(X_h)(j)$, and let $\eta \in \mathrm{Fil}^0 D^*(1)_K \subseteq \mathrm{Fil}^{d+1-j} H^d_{\mathrm{dR},c}(X_h)$. If*

$$\mathrm{pr}_D : H^{d+1}_{\mathrm{syn}}(X_h, j) \longrightarrow H^1_{\mathrm{st}}(D^d(X_h)(j)) \longrightarrow H^1_{\mathrm{st}}(D) \cong D_K / \mathrm{Fil}^0$$

is the natural projection, then for any $z \in H^{d+1}_{\mathrm{mot}}(X, j)$, any polynomial P such that $P(\Phi)(\eta) = 0$ and $P(1) \neq 0$, $P(q^{-1}) \neq 0$, and any class

$$\tilde{\eta} \in H^d_{\mathrm{syn},K,P,c}(X_h, d+1-j)$$

lifting η, we have

$$\eta\left(\mathrm{pr}_D(r_{\mathrm{syn}}(z))\right) = \operatorname{tr}_{X,\mathrm{syn},K,P}\left(r_{\mathrm{syn},K}(z) \cup \tilde{\eta}\right)$$

where $r_{\mathrm{syn},K}(z)$ is the image of $r_{\mathrm{syn}}(z)$ under the natural map $H^1_{\mathrm{st}}(D^d(X_h)(j)) \to H^1_{\mathrm{st},K,1-T}(D^d(X_h)(j))$.

Proof This is immediate from Proposition 1.4.3 and the compatibility of the P-syntomic descent spectral sequence with cup-products.

Using the above formula together with the projection formula for cup-products (Theorem 2.5.3), we obtain the following consequences (in which $j = 2$ and $1 \leq d \leq 3$). For $\eta \in H^d_{\mathrm{dR},c}(X/K)$ satisfying the hypotheses of Theorem 4.2, we write λ_η for the map $H^{d+1}_{\mathrm{syn}}(X_h, j) \to K$ given by the composition of pr_D and pairing with η.

Proposition 3.1.4 *If $d = 3$ and $z = \sum_i n_i[Z_i] \in H^4_{\mathrm{mot}}(X, 2)$ is the class of a codimension 2 cycle with smooth components, and η is as in Theorem 3.1.3, then we have*

$$\lambda_\eta(r_{\mathrm{syn}}(z)) = \sum_i n_i \operatorname{tr}_{Z_i,\mathrm{syn},K,P}(\iota_i^*(\tilde{\eta})),$$

where ι_i is the inclusion of Z_i into X.

Proof Cf. [4, Theorem 1.2]. By the compatibility of motivic and syntomic pushforward maps, we have

$$r_{\mathrm{syn},K}(z) = \sum_i n_i \ (\iota_i)_* \left(\mathbf{1}_{Z_i}\right),$$

where $\mathbf{1}_{Z_i} \in H^0_{\mathrm{syn},K}(Z_i, 0)$ is the identity class. By the projection formula, we have

$$r_{\mathrm{syn},K}(z) \cup \tilde{\eta} = \sum_i n_i \ (\iota_i)_* \left(\mathbf{1}_{Z_i} \cup \iota_i^* \tilde{\eta}\right).$$

Since the maps $\operatorname{tr}_{X,\mathrm{syn},K,P}$ are compatible with pushforward (as is clear by comparison with their de Rham analogues), the result is now immediate from Theorem 2.5.3.

Proposition 3.1.5 *If $d = 2$ and $z = \sum_i (Z_i, u_i) \in H^3_{\mathrm{mot}}(X, 2)$, where Z_i are codimension 1 cycles and $u_i \in \mathcal{O}(Z_i)^\times$, and η is as in Theorem 3.1.3, then we have*

$$\lambda_\eta(r_{\mathrm{syn}}(z)) = \sum_i n_i \operatorname{tr}_{Z_i,\mathrm{syn},K,P}(r_{\mathrm{syn},K}(u_i) \cup \iota_i^*(\tilde{\eta})),$$

where ι_i *is the inclusion of* Z_i *into* X*. Here,* r_{syn} *is the composition*

$$\mathcal{O}(Z_i)^\times = H^1_{\mathrm{mot}}(Z_i, 1) \longrightarrow H^1_{\mathrm{syn}}(Z_i, 1).$$

Proof Again, by the compatibility of motivic and syntomic pushforwards, we have

$$r_{\mathrm{syn},K}(z) = \sum_i (\iota_i)_*(r_{\mathrm{syn},K}(u_i)),$$

and the result is immediate from Theorem 2.5.3.

Proposition 3.1.6 *If* $d = 1$ *and* $z = \sum_i \{u_i, v_i\} \in H^2_{\mathrm{mot}}(X, 2)$*, where* $\{u_i, v_i\}$ *is the Steinberg symbol of two units* $u_i, v_i \in \mathcal{O}(X)^\times$*, and* η *is as in Theorem* 3.1.3*, then we have*

$$\lambda_\eta(r_{\mathrm{syn}}(z)) = \sum_i \operatorname{tr}_{X,\mathrm{syn},K,P}(r_{\mathrm{syn},K}(u_i) \cup r_{\mathrm{syn},K}(v_i) \cup \tilde{\eta}).$$

Proof Observe that the image of $\{u_i, v_i\}$ in $H^2_{\mathrm{syn}}(X, 2)$ is equal to $r_{\mathrm{syn},K}(u_i) \cup r_{\mathrm{syn},K}(v_i)$.

3.2 The cohomological triple symbol

We now define a "triple symbol" attached to three de Rham classes on a curve Y, which is closely related to the global triple index of [7]. Let Y be a connected smooth affine curve over K.

Let us choose a class $\eta \in H^1_{\mathrm{dR},c}(Y/K)$; and let $\omega_1, \omega_2 \in \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y/K)$.

Assumption 1 The following conditions are satisfied:

- There is a crystalline (φ, N, G_K)-submodule $D_0 \subseteq D^1_c(X_h)$ such that $\eta \in (D_0)_K$, and similarly $D_1, D_2 \subseteq D^1(X_h)$ such that $\omega_i \in (D_i)_K$ for $i = 1, 2$.
- There are polynomials $P_0, P_1, P_2 \in 1 + TK[T]$ such that $P_0(\Phi)$ annihilates η and $P_i(q^{-1}\Phi)$ annihilates ω_i for $i = 1, 2$, so that
$$\eta \in H^0_{\mathrm{st},K,P_0,c}\left(D^1_c(Y_{K,h})\right),\ \omega_1 \in H^0_{\mathrm{st},K,P_1}\left(D^1(Y_{K,h})(1)\right),\ \omega_2 \in H^0_{\mathrm{st},K,P_2}\left(D^1(Y_{K,h})(1)\right).$$
- The class ω_1 is in the kernel of the "knight's move" map
$$H^0_{\mathrm{st},K,P_1}\left(D^1(Y_{K,h})(1)\right) \to H^2_{\mathrm{st},K,P_1}\left(D^0(Y_{K,h})(1)\right),$$
and similarly for ω_2.
- The polynomial $P_0 \star P_1 \star P_2$ does not vanish at 1 or q^{-1}.
- We have $\eta \cup \omega_1 = \eta \cup \omega_2 = 0$ as elements of $H^2_{\mathrm{dR},c}(Y/K) \cong K$.

Remark 3.2.1 We have

$$H^2_{\mathrm{st},K,P}(D^0(Y_{K,h})(1)) = H^2_{\mathrm{st},K,P}(\mathbf{Q}_p(1)) = \frac{K}{P(1)K},$$

so if $P_1(1) \neq 0$, then ω_1 is automatically in the kernel of the "knight's move" map. In particular, this applies if ω_1 is pure of weight 1. On the other hand, if $\omega_1 = \operatorname{dlog} u$ for a unit u, then we have no choice but to take $P_1(1) = 0$, but the existence of the syntomic regulator r_{syn} and its compatibility with the de Rham regulator forces ω_1 to be in the kernel of this map since we know it is in the image of $H^1_{\mathrm{syn}}(Y_h, 1)$.

If all three de Rham classes are pure of weight 1, then $P_0 \star P_1 \star P_2$ has all its roots of weight -1, so in particular it is non-vanishing at 1 and q^{-1}.

Definition 3.2.2 Under the above assumptions, we define the *triple symbol* $[\eta; \omega_1, \omega_2] \in K$ by the formula

$$[\eta; \omega_1, \omega_2] = \mathrm{tr}_{Y,\mathrm{syn},P_0\star P_1\star P_2}\,(\tilde{\eta} \cup \tilde{\omega}_1 \cup \tilde{\omega}_2)$$

where $\tilde{\eta} \in H^1_{\mathrm{syn},K,P_0,c}(Y_{K,h}, 0)$ and $\tilde{\omega}_i \in H^1_{\mathrm{syn},K,P_i}(Y_{K,h}, 1)$ are liftings of η and the ω_i.

Proposition 3.2.3 *The above quantity is independent of the choice of liftings and of the polynomials P_i.*

Proof Firstly, we note that the natural map $H^1_{\mathrm{syn},K,P_0,c}(Y_{K,h}, 0) \to H^1_{\mathrm{dR},c}(Y/K)$ is an isomorphism, since the degree 0 compactly-supported cohomology is zero. Thus the class $\tilde{\eta}$ is uniquely defined.

If $\tilde{\omega}_1$ is a class in H^1_{syn,K,P_1} lifting ω_1, and $[u, v; w, x, y; z]$ is a representative of $\tilde{\omega}_1$ in $\mathbb{R}\Gamma^1_{\mathrm{syn},K,P_1}$ (with $z = 0$, necessarily, for degree reasons), then any other choice of lifting can be represented by $[u, v; w+\lambda, x+\mu, y; z]$ for some constants $\lambda, \mu \in K$ (with $\mu = 0$ unless $P_1(1) = 0$). From the definition of the cup-product, one sees that varying μ has no effect on $\tilde{\eta} \cup \tilde{\omega}_1 \cup \tilde{\omega}_2$; while varying λ changes the cup product by a multiple of $\eta \cup \omega_2$, which is zero by assumption. Similarly, the assumption that $\eta \cup \omega_1 = 0$ implies that the cup-product is independent of the choice of lifting of ω_2.

So $[\eta; \omega_1, \omega_2]$ is well-defined for a fixed choice of polynomials P_i. However, both the cup-product and the map $\mathrm{tr}_{Y,\mathrm{syn},P_0\star P_1\star P_2}$ are compatible with change of the polynomials P_i, so the symbol $[\eta; \omega_1, \omega_2]$ is also independent of these choices.

Remark 3.2.4 We may also carry out the same construction if Y is projective, rather than affine, if we add the assumption that $P_0(1) \neq 0$ and $P_0(q) \neq 0$; this assumption implies that there is still a unique lifting of η to $H^1_{\mathrm{syn},K,P_0,c}(Y_{K,h}, 0) = H^1_{\mathrm{syn},K,P_0}(Y_{K,h}, 0)$. In particular, this holds if P_0 is pure of weight 1.

3.3 Products of curves

We now use the theory of the previous section to reformulate special cases of Propositions 3.1.4–3.1.6 in terms of the triple symbol. We continue to assume that Y is a connected smooth affine curve over K.

Proposition 3.3.1 *Let $\eta \in H^1_{\mathrm{dR},c}(Y/K)$, and let $\omega_1, \omega_2 \in \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y/K)$ be classes lying in the parabolic subspace (the image of $H^1_{\mathrm{dR},c}$ in H^1_{dR}). Suppose that the triple symbol $[\eta; \omega_1, \omega_2]$ is defined.*

Let $X = Y \times Y \times Y$ and let $z \in H^4_{\mathrm{mot}}(X, 2)$ be the class of the diagonal embedding $Y \hookrightarrow Y \times Y \times Y$. If μ denotes the class $\eta \otimes \omega_{1,c} \otimes \omega_{2,c} \in \mathrm{Fil}^2 H^3_{\mathrm{dR},c}(X/K)$, where $\omega_{i,c}$ are any liftings of ω_i to compactly-supported cohomology, then

$$\lambda_\mu(r_{\mathrm{syn}}(z)) = [\eta; \omega_1, \omega_2].$$

Proposition 3.3.2 *Let $\eta \in H^1_{\mathrm{dR},c}(Y/K)$ and let $\omega \in \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y/K)$ be a class lying in the parabolic subspace. Let $u \in \mathcal{O}(Y)^\times$, and suppose that the triple symbol $[\eta; \omega_1, \mathrm{dlog}\, u]$ is defined.*

Let $X = Y \times Y$ and let $z \in H^3_{\mathrm{mot}}(X, 2)$ be the pushforward of $u \in H^1_{\mathrm{mot}}(Y, 1) \cong \mathcal{O}(Y)^\times$ along the diagonal embedding $Y \hookrightarrow Y \times Y$. If μ denotes the class $\eta \otimes \omega_c \in \mathrm{Fil}^1 H^2_{\mathrm{dR},c}(X/K)$, where ω_c is any lifting of ω to compactly-supported cohomology, then

$$\lambda_\mu(r_{\mathrm{syn}}(z)) = [\eta; \omega, \mathrm{dlog}\, u].$$

Proposition 3.3.3 *Let $\eta \in H^1_{\mathrm{dR},c}(Y/K)$. Let $u, v \in \mathcal{O}(Y)^\times$, and suppose that the triple symbol $[\eta; \mathrm{dlog}\, u, \mathrm{dlog}\, v]$ is defined.*

Let $z \in H^2_{\mathrm{mot}}(X, 2)$ be the cup-product of the classes $u, v \in H^1_{\mathrm{mot}}(Y, 1) \cong \mathcal{O}(Y)^\times$. Then

$$\lambda_\eta(r_{\mathrm{syn}}(z)) = [\eta; \mathrm{dlog}\, u, \mathrm{dlog}\, v].$$

3.4 An alternative description of the triple symbol

We conclude by giving an alternative, equivalent description of the symbol $[\eta; \omega_1, \omega_2]$, which we hope may be useful in relating our results to p-adic modular forms (as in the calculations of [8] in the good-reduction case).

The assertion that $(P_0 \star P_1 \star P_2)(q^{-1}) \neq 0$ implies that $(P_1 \star P_2)(q^{-1}\Phi^{-1})$ acts bijectively on the kernel of $P_0(\Phi)$, so that $\frac{1}{(P_1 \star P_2)(q^{-1}\Phi^{-1})}\eta$ is well-defined.

Proposition 3.4.1 *Suppose that $\tilde{\omega}_1 \cup \tilde{\omega}_2 \mod \mathrm{Fil}^2$ is represented by the class*

$$w, x, y] \in H^1_{\mathrm{st}, K, P_1 \star P_2}(D^1(Y_h)(2)).$$

Then we have

$$[\eta; \omega_1, \omega_2] = \eta \cup y - \left(\frac{1}{(P_1 \star P_2)(q^{-1}\Phi^{-1})}\eta\right) \cup w.$$

Proof From the definition of the cup product, we see that it respects the 3-step filtration on P-syntomic cohomology, and the cup-products induced on the graded pieces coincide with the usual de Rham cup products. Since $\tilde{\omega}_1 \cup \tilde{\omega}_2$ lies in Fil^1, and the trace isomorphism factors through $\mathrm{Fil}^1 / \mathrm{Fil}^2$, we obtain the above compatibility using (2). □

This takes a particularly simple form if η is a Φ-eigenvector, say $\Phi(\eta) = \alpha\eta$; then we have

$$[\eta; \omega_1, \omega_2] = \eta \cup \left(y - \frac{1}{(P_1 \star P_2)(q^{-1}\alpha^{-1})} w\right).$$

Acknowledgments We would like to thank Massimo Bertolini, Henri Darmon, and Victor Rotger for encouraging us to work on this project, and for several useful remarks and conversations while the paper was being written; and Jan Nekovář and Wiesława Nizioł for their patience in explaining their work [10] to us. The second and third authors would also like to thank Ehud de Shalit for his invitation to visit Jerusalem in April 2014, which gave us the opportunity to finish the paper. Finally, we would like to thank the two anonymous referees, for pointing out several mistakes and omissions in the original draft.

References

1. Beilinson, A.: On the crystalline period map. Cambridge J. Math. **1**(1), 1–51 (2013)
2. Bertolini, M., Darmon, H.: Kato's Euler system and rational points on elliptic curves I: a p-adic Beilinson formula. Israel J. Math. **199**, 163–188 (2014)
3. Bertolini, M., Darmon, H., Rotger, V.: Beilinson–Flach elements and Euler systems I: syntomic regulators and p-adic Rankin L-series. J. Algebraic Geom. **24**(2), 355–378 (2015)
4. Besser, A.: A generalization of Coleman's p-adic integration theory. Invent. Math. **142**(2), 397–434 (2000)

5. Besser, A.: Syntomic regulators and p-adic integration. I. Rigid syntomic regulators. Israel J. Math. **120**(1), 291–334 (2000)
6. Besser, A.: Syntomic regulators and p-adic integration II: K_2 of curves. Israel J. Math. **120**(1), 335–360 (2000)
7. Besser, A.: On the syntomic regulator for K_1 of a surface. Israel J. Math. **190**(1), 29–66 (2012)
8. Darmon, H., Rotger, V.: Diagonal cycles and Euler systems I: a p-adic Gross-Zagier formula. Ann. Sci. ÉNS **47**(4), 779–832 (2014)
9. Déglise, F., Mazzari, N.: The Rigid Syntomic Ring Spectrum. J. Inst. Math. Jussieu. **14**(4), 753–799 (2015)
10. Nekovář, J., Nizioł, W.: Syntomic Cohomology and p-Adic Regulators for Varieties over p-Adic Fields (2013). arXiv:1309.7620