

Wodzicki residue and minimal operators on a noncommutative 4-dimensional torus

Andrzej Sitarz

Received: 9 July 2013 / Revised: 6 April 2014 / Accepted: 16 June 2014 /
Published online: 24 July 2014
© The Author(s) 2014. This article is published with open access at Springerlink.com

Abstract We compute the Wodzicki residue of the inverse of a conformally rescaled Laplace operator over a 4-dimensional noncommutative torus. We show that the straightforward generalisation of the Laplace-Beltrami operator to the noncommutative case is not the minimal operator.

1 Introduction

Noncommutative geometry as proposed in [2] aims to use geometric methods to study noncommutative algebras in a similar way that differential geometry is used to study spaces. One of the appealing potential applications is its use in physics to describe the structure of space-time and fundamental interactions at high energies. Although the construction of basic data in noncommutative geometry is equivalent in the classical case to the standard data of Riemannian geometry [1] in the genuine noncommutative examples this aspect has not been sufficiently explored until recently.

In a series of papers first [3,4] and [7–12] a conformally rescaled metric has been proposed and studied for the noncommutative two and four-tori. This led to the expressions of Gauss-Bonnet theorem and formulae for the noncommutative counterpart of scalar curvature. Independently, another class of Dirac operators and metrics with the geometric interpretation as arising from the $U(1)$ connections on noncommutative

Partially supported by NCN grant 2011/01/B/ST1/06474.

A. Sitarz (✉)
Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland
e-mail: andrzej.sitarz@uj.edu.pl

A. Sitarz
Institute of Mathematics of the Polish Academy of Sciences,
Śniadeckich 8, 00-950 Warszawa, Poland

circle bundles has been proposed by the author and Dabrowski in [5]. Following this lead, a more general type of metric on the two torus has been suggested and studied perturbatively for the two-torus [6].

However, in all the mentioned approaches one major assumption was made. The second term of the heat-kernel asymptotics of the rescaled Laplace operator (or square of the Dirac operator) was identified as a linear functional of the scalar curvature. In Riemannian geometry this is certainly true, provided that the Laplace operator (or Dirac operator) comes from the Levi-Civita connection. In the presence of the nontrivial torsion the term is modified and includes the integral of the square of the torsion, as has been observed already in the early computations of Seeley-Gilkey-de Witt coefficients. As in the noncommutative geometry there is no implicit notion of torsion, one may wonder whether the rescaled Laplace operators are those, which minimise the second term of the heat-kernel expansion for a fixed metric. Additionally, in the classical case the computation of second heat kernel coefficient is closely related to the computations of the Wodzicki residue of a certain power of the Dirac operator.

As it has been shown [12] and more generally in [15] Wodzicki residue exists also in the case of the pseudodifferential calculus over noncommutative tori.

In this note we shall address the question of the minimal operators (using the Wodzicki residue to check minimality) and compute the Wodzicki residue for a class of operators on the noncommutative torus.

2 Noncommutative tori and their pseudodifferential calculus

We use the usual presentation of the algebra of d -dimensional noncommutative torus as generated by d unitary elements $U_i, i = 1, \dots, d$, with the relations

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j,$$

where $0 < \theta_{jk} < 1$ is real. The smooth algebra $\mathcal{A}(\mathbb{T}_\theta^d)$ is then taken as an algebra of elements

$$a = \sum_{\beta \in \mathbb{Z}^d} a_\beta U^\beta,$$

where a_β is a rapidly decreasing sequence and

$$U^\beta = U_1^{\beta_1} \dots U_d^{\beta_d}.$$

The natural action of $U(1)^d$ by automorphisms, gives, in its infinitesimal form, two linearly independent derivations on the algebra: given on the generators as:

$$\delta_k(U_j) = \delta_{jk} U_j, \quad \forall j, k = 1, \dots, d. \quad (2.1)$$

where δ_{jk} denotes the Kronecker delta.

The canonical trace on $\mathcal{A}(\mathbb{T}_\theta^d)$ is

$$\mathfrak{t}(a) = \alpha_0,$$

where $\mathbf{0} = \{0, 0, \dots, 0\} \in \mathbb{Z}^d$. The trace is invariant with respect to the action of $U(1)^d$, hence

$$\mathfrak{t}(\delta_j(a)) = 0, \quad \forall j = 1, \dots, d, \forall a \in \mathcal{A}(\mathbb{T}_\theta^d).$$

By \mathcal{H} we denote the Hilbert space of the GNS construction with respect to the trace \mathfrak{t} on the C^* completion of $\mathcal{A}(\mathbb{T}_\theta^d)$ and π the associated faithful representation. The elements of the smooth algebra $\mathcal{A}(\mathbb{T}_\theta^d)$ act on \mathcal{H} as bounded operators by left multiplication, whereas the derivations δ_i extend to densely defined selfadjoint operators on \mathcal{H} with the smooth elements of the Hilbert space, $\mathcal{A}(\mathbb{T}_\theta^d)$, in their common domain.

2.1 Pseudodifferential operators on \mathbb{T}_θ^d

The symbol calculus defined in [4] and developed further in [3] (see also [15]) is easily generalised to the d -dimensional case and to the operators defined above. We shall briefly review the basic definitions and methods used further in the note. Let us recall that a differential operator of order at most n is of the form

$$P = \sum_{0 \leq k \leq n} \sum_{|\beta_k|=k} a_{\beta_k} \delta^{\beta_k},$$

where a_{β_k} are assumed to be in the algebra $\mathcal{A}(\mathbb{T}_\theta^d)$, $\beta_k \in \mathbb{Z}^d$ and:

$$|\beta_k| = \beta_1 + \dots + \beta_d, \quad \delta^\beta = \delta_1^{\beta_1} \dots \delta_d^{\beta_d}.$$

Its symbol is:

$$\rho(P) = \sum_{0 \leq k \leq n} \sum_{|\beta_k|=k} a_{\beta_k} \xi^{\beta_k},$$

where

$$\xi^\beta = \xi_1^{\beta_1} \dots \xi_d^{\beta_d}.$$

On the other hand, let ρ be a symbol of order n , which is assumed to be a C^∞ function from \mathbb{R}^d to $\mathcal{A}(\mathbb{T}_\theta^d)$, which is homogeneous of order n , satisfying certain bounds (see [4] for details). With every such symbol ρ there is associated an operator P_ρ on a dense subset of \mathcal{H} spanned by elements $a \in \mathcal{A}(\mathbb{T}_\theta^d)$:

$$P_\rho(a) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-i\sigma \cdot \xi} \rho(\xi) \alpha_\sigma(a) d\sigma d\xi,$$

where

$$\alpha_\sigma(U^\alpha) = e^{i\sigma \cdot \alpha} U^\alpha, \quad \sigma \in \mathbb{R}^d, \alpha \in \mathbb{Z}^d.$$

For two operators P, Q with symbols:

$$\rho(P) = \sum p_\alpha \xi^\alpha, \quad \rho(Q) = \sum q_\beta \xi^\beta,$$

we use the formula, which follows directly from the same computations as in the case of classical calculus of pseudodifferential operators:

$$\rho(PQ) = \sum_{\gamma \in \mathbb{N}^d} \frac{1}{\gamma!} \partial_\xi^\gamma (\rho(P)) \delta^\gamma (\rho(Q)), \tag{2.2}$$

where $\gamma! = \gamma_1! \cdots \gamma_d!$.

2.2 Wodzicki residue

In this part we shall provide an elementary proof that there exists a trace on the above defined algebra of symbols on the d -dimensional noncommutative torus:

Proposition 2.1 *Let $\rho = \sum_{j \leq k} \rho_j(\xi)$ be a symbol over the noncommutative torus $\mathcal{A}(\mathbb{T}_\theta^d)$. Then the functional:*

$$\rho \mapsto \int_{S^{d-1}} \mathfrak{t}(\rho_{-d}(\xi)) d\xi,$$

is a trace over the algebra of symbols.

Although the above proposition as well as the proof are not new, we provide it to make the paper self-contained.

Let us start with a simple lemma about homogeneous functions.

Lemma 2.2 *Let f be a smooth function on $\mathbb{R}^d \setminus \{0\}$, homogeneous of degree ρ . Then*

$$\int_{S^{d-1}} \partial_{\xi_i} f(\xi) d\xi = 0,$$

holds for every $1 \leq i \leq d$ if and only if $\rho = 1 - d$.

Proof The \Rightarrow part is trivial, as it is sufficient to take $f_j(\xi) = \xi^j (\xi^2)^{\frac{\rho-1}{2}}$. Then:

$$\partial_i (\xi^j (\xi^2)^{\frac{\rho-1}{2}}) = \delta^{ij} (\xi^2)^{\frac{\rho-1}{2}} + 2 \frac{\rho-1}{2} \xi^j \xi^i (\xi^2)^{\frac{\rho-1}{2}-1},$$

which, when restricted to the sphere $\xi^2 = 1$ and integrated, gives:

$$V(d) \left(1 + (\rho - 1) \frac{1}{d} \right),$$

where $V(d)$ is the volume of $d-1$ dimensional sphere. This vanishes only if $\rho = 1-d$.

Assume now that we have a homogeneous function on \mathbb{R}^d of degree ρ , denoted f . Observe that using the freedom of the choice of coordinates we can safely assume that $i = 1$. Using the spherical coordinates $r, \phi_1, \dots, \phi_{d-1}$:

$$\xi^1 = r \sin \phi_1, \xi^2 = r \cos \phi_1 \sin \phi_2, \dots, \xi^d = r \cos \phi_1 \cdots \cos \phi_{d-2} \cos \phi_{d-1},$$

we know the volume form on the sphere:

$$\omega = (\sin \phi_1)^{d-2} (\sin \phi_2)^{d-3} \cdots \sin \phi_{d-2} d\phi_1 \cdots d\phi_{d-1},$$

and we can express the partial derivative $\frac{\partial}{\partial x^1}$ as:

$$\frac{\partial}{\partial \xi^1} = \cos \phi_1 \frac{\partial}{\partial r} - \frac{\sin \phi_1}{r} \frac{\partial}{\partial \phi_1}.$$

Since the function f is homogeneous in r of order α we have:

$$\frac{\partial f}{\partial \xi^1} = \cos \phi_1 \frac{\alpha}{r} f - \frac{\sin \phi_1}{r} \frac{\partial f}{\partial \phi_1}.$$

Consider now the following function in the coordinates $\phi_1, \dots, \phi_{d-1}$:

$$\begin{aligned} \omega \left(\frac{\partial f}{\partial \xi^1} \right) \Big|_{r=1} &= \left(\alpha (\sin \phi_1)^{d-2} (\cos \phi_1) f|_{r=1} - (\sin \phi_1)^{d-1} \frac{\partial f|_{r=1}}{\partial \phi_1} \right) \\ &\quad \times (\sin \phi_2)^{d-3} \cdots \sin \phi_{d-2} = \dots \end{aligned}$$

If $\alpha = (1-d)$ it could be written as:

$$\dots = \frac{\partial}{\partial \phi_1} \left(-(\sin \phi_1)^{d-1} f|_{r=1} \right) (\sin \phi_2)^{d-3} \cdots (\sin \phi_{d-2}).$$

Since the integral of a function f over the sphere S^{d-1} in the spherical coordinates is:

$$\int_{S^{d-1}} F = \int_0^{2\pi} d\phi_1 \int_0^\pi d\phi_2 \cdots \int_0^\pi d\phi_{d-1} (\sin \phi_1)^{d-2} (\sin \phi_2)^{d-3} \cdots \sin \phi_{d-2} f(\phi_1, \dots, \phi_{d-1}),$$

we have that:

$$\int_{S^{d-1}} \left(\frac{\partial f}{\partial \xi^1} \right) \Big|_{r=1} = \int_0^\pi d\phi_{d-1} \int_0^\pi d\phi_{d-2} \sin \phi_{d-2} \cdots \int_0^\pi d\phi_2 (\sin \phi_2)^{d-3} \int_0^{2\pi} d\phi_1 \left(\frac{\partial}{\partial \phi_1} \left(-(\sin \phi_1)^{d-1} f|_{r=1} \right) \right) = 0.$$

□

Proof of Proposition 2.1 Let ρ and σ be two symbols, that is C^∞ maps from \mathbb{R}^d to $\mathcal{A}(\mathbb{T}_\theta^d)$, which are decomposed into the sum of homogeneous symbols:

$$\sigma = \sum_{j \leq S} \sigma_j, \quad \rho = \sum_{j \leq R} \rho_j.$$

We shall prove that the the Wodzicki residue is a trace, that is:

$$\text{Wres}(\rho\sigma) = \text{Wres}(\sigma\rho).$$

Using the formula for the product (2.2) we have:

$$(\rho\sigma)_{-d} = \sum_{\substack{\gamma, j, k \\ |\gamma|+j+k=-d}} \frac{1}{\gamma!} \partial_\xi^\gamma (\rho_j) \delta^\gamma (\sigma_k),$$

and since derivation in ξ decreases the degree of homogeneity by 1, the sum is necessarily finite.

First we shall prove the trace property for $|\gamma| = 0$ and $|\gamma| = 1$. If $|\gamma| = 0$ we have:

$$\mathfrak{t}(\rho_j \sigma_k) = \mathfrak{t}(\sigma_k \rho_j),$$

since \mathfrak{t} is a trace on the algebra $\mathcal{A}(\mathbb{T}_\theta^d)$.

Next, if $|\gamma| = 1$, we have:

$$\begin{aligned} \int_{S^{d-1}} d\xi \mathfrak{t}(\partial_{\xi^i}(\rho_j) \delta_i(\sigma_k)) &= \int_{S^{d-1}} d\xi \mathfrak{t}(-\delta_i(\partial_{\xi^i}(\rho_j))\sigma_k) \\ &= \int_{S^{d-1}} d\xi \mathfrak{t}(-\partial_{\xi^i}(\delta_i(\rho_j))\sigma_k) \\ &= \int_{S^{d-1}} d\xi \mathfrak{t}(\delta_i(\rho_j) \partial_{\xi^i}(\sigma_k)) \\ &= \int_{S^{d-1}} d\xi \mathfrak{t}(\partial_{\xi^i}(\sigma_k) \delta_i(\rho_j)), \end{aligned}$$

where we have used that t is invariant with respect to $U(1)$ symmetry generated by δ_i , the fact that δ_i and ∂_{ξ_i} commute, then Lemma 2.2 (which we can use because the product $\sigma_j \rho_j$ is homogeneous of degree $1 - d$), and finally the trace property of t .

For any $|\gamma| > 1$ we repeat the above argument sufficient number of times. \square

3 Laplace-type operator of a conformally rescaled metric

In this section we shall fix our attention on a family of Laplace-type operators, which originate from a conformally rescaled fixed metric on manifold. We begin with the classical situation. Let us take a closed Riemannian manifold M of dimension d with a fixed metric tensor g . If h is a positive smooth function on M then we take the conformally rescaled metric to be given by:

$$g_{ab} \rightarrow h^2 g_{ab} = \tilde{g}_{ab},$$

where g_{ab} is the original metric tensor.

Lemma 3.1 *Let Δ be the usual Laplace operator on M with the metric given by the metric tensor g_{ab} and \mathcal{H} be the Hilbert space of $L^2(M, g)$ (where the measure is taken with respect to the metric g_{ab}). Let $\tilde{\Delta}$ be the Laplace operator on M with the conformally rescaled metric acting on the Hilbert space $\tilde{\mathcal{H}} = L^2(M, \tilde{g})$.*

Then $\tilde{\Delta}$ is unitarily equivalent to $\Delta_h = h^{\frac{d}{2}} \tilde{\Delta} h^{-\frac{d}{2}}$ acting on \mathcal{H} . Moreover, the operator Δ_h written in local coordinates is:

$$\Delta_h = h^{-2} \Delta - 2h^{-3} g^{ab} (\partial_a h) \partial_b + h^{\frac{d}{2}-2} (\Delta h^{-\frac{d}{2}}).$$

Proof Let $U : \Psi \rightarrow h^{-\frac{d}{2}} \Psi$ be a map between Hilbert spaces $U : \mathcal{H} \rightarrow \mathcal{H}_h$. Since:

$$\|\Psi\|_{\mathcal{H}} = \int_M \sqrt{g} |\Psi|^2 = \int_M \sqrt{gh^{2d}} |h^{-\frac{d}{2}} \psi|^2 = \|U\Psi\|_{\tilde{\mathcal{H}}}.$$

the map U is unitary. Then $\Delta_h = U^{-1} \tilde{\Delta} U$ is an operator on \mathcal{H} unitary equivalent to $\tilde{\Delta}$. The formula in local coordinates follows by explicit computations. \square

Next we shall restrict ourselves now to the case when M is a d -dimensional torus, \mathbb{T}^d , and the metric we begin with is a constant, flat metric.

3.1 The case of d -dimensional torus

Consider a flat d -dimensional torus, $\mathbb{T}^d = (S^1)^d$, with a constant diagonal metric $g_{ab} = \delta_{ab}$. We take the usual system of coordinates on the torus (each circle parametrised by an angle) and from now on we assume that δ_a are the associated derivations. Take as \mathcal{H}_0 the Hilbert space of square summable functions with respect to the flat metric measure. An immediate consequence of Lemma 3.1 is:

Lemma 3.2 *Let h be a positive smooth function on the torus \mathbb{T}^n . The following operator on \mathcal{H}_0 :*

$$\Delta_h = \sum_{a=1}^n h^{-\frac{d}{2}} \partial_a (h^{d-2} \partial_a) h^{-\frac{d}{2}},$$

is unitarily equivalent to the Laplace operator of the conformally rescaled metric $h^2 \delta_{ab}$.

Proof This Laplace operator on the torus with the metric $g_{ab} = h^2 \delta_{ab}$ is:

$$\tilde{\Delta} = \sum_{a=1}^n h^{-d} \delta_a (h^{d-2} \delta_a).$$

Applying Lemma 3.1 we get the above explicit formula for Δ_h . □

This formula has been generalised to the noncommutative case in dimension 4 [10] in order to compute the curvature of the conformally rescaled Laplace operator on the noncommutative four-torus. However, even though the noncommutative generalisation of the above prescription for the conformally rescaled Laplace operator makes sense, it does not exclude the possibility that the quantity computed is not exactly the scalar curvature. The reason for this is the existence of torsion and the possibility that the above operator might not be torsion-free in the noncommutative generalisation.

3.2 Laplace-type operators on noncommutative tori

We shall investigate the family of operators, which are noncommutative generalisations of the above Laplace operator and which differ from them only by terms of lower order. This guarantees that their principal symbol is unchanged and hence using the natural (albeit naive) notion of noncommutative metric we could say both operators determine the same metric. We begin with the following definition:

Definition 3.3 Let us take $h \in \mathcal{A}(\mathbb{T}_\theta^d)$ to be a positive element with a bounded inverse and take the following densely defined operator on \mathcal{H} :

$$\Delta_h = \sum_{a=1}^n h^{-\frac{d}{2}} \delta_a (h^{d-2} \delta_a) h^{-\frac{d}{2}}, \tag{3.1}$$

to be the Laplace operator on d -dimensional noncommutative torus.

In both cases of $d = 2$ and $d = 4$ this has been studied as the Laplace operator of the conformally rescaled noncommutative torus.

The family which we intend to investigate now is,

Definition 3.4 A generalised family of Laplace operator for the conformally rescaled metric over the torus has a form:

$$\Delta = \Delta_h + \sum_{a=1}^n \left(T^a \delta_a + \frac{1}{2} \delta_a (T^a) \right) + X, \tag{3.2}$$

where T^a and X are some selfadjoint elements of $\mathcal{A}(\mathbb{T}_\theta^d)$. This form could be rewritten as:

$$\Delta = h^{-2} \left(\sum_a \delta_a^2 \right) + \sum_a Y^a \delta_a + \Phi, \tag{3.3}$$

where

$$Y^a = h^{\frac{d}{2}-2} \delta_a(h^{-\frac{d}{2}}) + h^{-\frac{d}{2}} \delta_a(h^{\frac{d}{2}-2}) + T^a,$$

and

$$\Phi = h^{\frac{d}{2}-2} \left(\sum_a \delta_a^2(h^{-\frac{d}{2}}) \right) + h^{-\frac{d}{2}} \left(\sum_a \delta_a(h^{d-2}) \delta_a(h^{-\frac{d}{2}}) \right) + \frac{1}{2} \sum_a \delta_a T^a + X.$$

The above Laplace-type operator (3.3) has the following symbol:

$$\rho(\Delta) = h^{-2} \xi^2 + \sum_a Y^a \xi_a + \Phi, \tag{3.4}$$

From now on, we shall work only with the symbol $\rho(\Delta)$ (3.4).

4 Wodzicki residue in dimension 4

Our aim will be to compute the Wodzicki residue of $\Delta^{-\frac{d}{2}+k}$ in the case of $d = 4$ for $k = 0, 1$. As the only difficulty in considering the general case is purely computational, we postpone it for future work, concentrating instead on the example case of $d = 4$ (which is most relevant for physics). The significance of these computations lies in the classical relation between Wodzicki residue of the inverse of the Laplace operator (in the sense of pseudodifferential calculus) and the scalar of curvature in the classical case [11]. We shall see, whether this extends to the noncommutative case. For simplicity we keep $X = 0$, focusing on the parameters h and T_a .

4.1 The symbol of Δ^{-2} and Δ^{-1}

We fix here $d = 4$. To simplify the notation above and in the remaining part of the note we use the convenient Einstein notation (implicit summation over repeated indices).

Proposition 4.1 *The Wodzicki residue of Δ^{-2} depends only on h :*

$$Wres(\Delta^{-2}) = 2\pi^2 \text{t}(h^4).$$

whereas for Δ^{-1} :

$$Wres(\Delta^{-1}) = \frac{\pi^2}{2} \left(\text{t}(h^2 T_a h^2 T_a h^2) + \text{t}(h^2 [T_a, \delta_a(h^2)]) - \text{t}(\delta_a(h^2) h^{-2} \delta_a(h^2)) \right).$$

Proof Let us compute the relevant symbols of both operators. Observe, that for a differential operator of degree 2 its symbol (split into part of homogeneous degrees) reads:

$$\rho(\Delta) = a_2 + a_1 + a_0.$$

Here we have:

$$\begin{aligned} a_2 &= h^{-2}\xi^2, \\ a_1 &= \left(\delta_a(h^{-2}) + T_a\right)\xi^a, \\ a_0 &= \delta_a\delta_a(h^{-2}) + h^{-2}\left(\delta_a(h^2)\delta_a(h^{-2})\right) + \frac{1}{2}\delta_a T^a. \end{aligned} \tag{4.1}$$

To see the first result, it is sufficient to observe that the symbol $\rho(\Delta^{-2})$ starts with a homogeneous symbol of order -4 , which is exactly $h^4|\xi|^{-4}$. Hence, computing the Wodzicki residue as in the proposition (2.1) gives the above result.

To compute the Wodzicki residue of Δ^{-1} we need to calculate further terms of its symbol using the parametrix in the pseudodifferential calculus. For a pseudodifferential operator of order -2 we have:

$$b = b_0 + b_1 + b_2 + \dots,$$

where each b_k is homogeneous of degree $-2 - k$ and could be iteratively computed from the following sequence of identities, which arise from comparing homogeneous terms of the product Δ^{-1} and Δ using the following formulae, which follow directly from the product rules of pseudodifferential operators:

$$\begin{aligned} b_0a_2 &= 1, \\ b_1a_2 + b_0a_1 + \partial_k(b_0)\delta_k(a_2) &= 0, \\ b_2a_2 + b_1a_1 + b_0a_0 + \partial_k(b_0)\delta_k(a_1) + \partial_k(b_1)\delta_k(a_2) \\ &\quad + \frac{1}{2}\partial_k\partial_j(b_0)\delta_k\delta_j(a_2) = 0, \end{aligned} \tag{4.2}$$

where δ_k ($k = 1, \dots, 4$) are the standard derivations on the noncommutative torus and ∂_j ($j = 1, \dots, 4$) are partial derivatives with respect to ξ_j .

The relations could be solved explicitly (compare [4]), giving:

$$\begin{aligned} b_0 &= (a_2)^{-1}, \\ b_1 &= -(b_0a_1b_0 + \partial_k(b_0)\delta_k(a_2)b_0), \\ b_2 &= -\left(b_0a_0b_0 + b_1a_1b_0 + \partial_j(b_0)\delta_j(a_1)b_0 + \partial_j(b_1)\delta_j(a_2)b_0 \right. \\ &\quad \left. + \frac{1}{2}\partial_{jk}(b_0)\delta_j\delta_k(a_2)b_0\right), \end{aligned} \tag{4.3}$$

For the pseudodifferential operator (4.1) we obtain first:

$$\begin{aligned} b_1(T, h) &= \left(-h^2(\delta_a(h^{-2}) + T_a)h^2 + 2h^2\delta_a(h^{-2})h^2\right) |\xi|^{-4}\xi^a \\ &= \left(h^2(\delta_a(h^{-2}) - T_a)h^2\right) |\xi|^{-4}\xi^a, \end{aligned}$$

and then:

$$\begin{aligned} b_2(T, h)(\xi) &= |\xi|^{-6}\xi^a\xi^b \left(h^2T_a h^2 T_b h^2 + h^2T_a \delta_b(h^2) + 2h^2\delta_a(T^1)h^2\right. \\ &\quad \left.+ 3\delta_a(h^2)T^a h^2 - \delta_a(h^2)h^{-2}\delta_b(h^2) + 2\delta_a\delta_b(h^2)\right) \\ &\quad + |\xi|^{-4} \left(-\frac{1}{2}h^2\delta_a(T_a)h^2 - \delta_a(h^2)T^a h^2\right) \end{aligned}$$

Finally, taking trace, using the Leibniz rule, the fact that the trace is closed and integrating over 3-dimensional sphere we obtain the result (4.1). \square

Before we pass to the interpretation of the above result, let us consider the classical limit $\theta = 0$.

4.2 The commutative case

We assume here that $\theta = 0$, so h and T_a are smooth functions on a torus, which commute with each other (and with their derivations).

Lemma 4.2 *For the commutative torus the Wodzicki residue of Δ^{-1} is:*

$$Wres(\Delta^{-1}) = 2\pi^2 \int_{\mathbb{T}^4} \left(h^6(T_a T_a) - \delta_a(h)\delta_a(h)\right) dV,$$

and for a fixed h the term reaches an absolute extremum if and only if $T_a = 0$, which has the interpretation of torsion-free Laplace operator.

Observe that the classical results of Kastler and Kalau-Walze [13, 14] give (for Laplace-Beltrami operator):

$$Wres(\Delta^{-1}) = 2\pi^2 \int_M \sqrt{g} \left(\frac{1}{6}R\right),$$

where R is the scalar curvature.

In the conformally rescaled metric the volume form and the curvature (in dimension $d = 4$) are:

$$\sqrt{g} = h^4, \quad R = 6h^{-3}\delta_{aa}(h),$$

so we obtain the same result.

4.3 Nonminimal operators and curvature

In the classical (commutative) situation the additional first-order term in the Laplace-type operator Δ contributes to the Wodzicki residue of Δ^{-1} with a term proportional to $T_a T_a$. As already noted, for *minimal operators* (like Laplace-Beltrami) such term vanishes and, equivalently, one can say that *minimal operators* are such Laplace-type operators, which, at the fixed metric minimise the Wodzicki residue.

In the noncommutative case we have two terms, one which is quadratic in T_a and the second one, which is linear in T_a and involves a commutator with $\delta_a(h^2)$. Therefore one can clearly state the following corollary,

Lemma 4.3 *A naive generalisation of Laplace-Beltrami operator for the conformally rescaled metric to the noncommutative case (in dimension $d = 4$) as proposed in (3.1) is not a minimal operator in the above sense (does not minimise the Wodzicki residue of Δ^{-1} with h fixed).*

Proof Assume that $T_a = 0$ minimises for a fixed h the functional (4.1). If we consider a small perturbation of T_a , $T_a = \varepsilon t_a$, for some fixed t_a we obtain a quadratic function of ε , which has a nonvanishing linear term provided that the commutator $[T_a, \delta_a(h^2)]$ does not vanish. Such function does not have minimum at $\varepsilon = 0$, hence $T_a = 0$ cannot be a minimum of the functional. \square

A significant consequence of this fact is that no simple identification of the scalar of curvature is possible: indeed, if have no means of identifying *minimal* (or unperturbed, or torsion-free) Laplace-type operators we cannot possibly recover the geometric invariants associated with the metric alone.

5 Conclusions and open problems

The computations aimed to show that the Wodzicki residue of the noncommutative generalisation of the Laplace-type operator on the 4-dimensional noncommutative torus with a conformally recalled metric is a nontrivial functional on the parameters h and T_a . There are several interesting problems, which arise that are linked to our result.

First of all, computations of the Wodzicki residue are closely linked to heat-kernel coefficients. As the principal symbol fails to be scalar, this is not happening for the noncommutative tori. An interesting point is then to link the Wodzicki residue in this case to the respective heat-kernel coefficients, which then, in turn, appear in the spectral action.

To study the geometric notions like curvature and geometric constructions like scalar curvature (at least as a noncommutative analogue of the classical one) it is important to identify the class of minimal Laplace-type operators. Otherwise, the functional would not only depend on the metric but also on some additional data. Our result shows that the naive generalisation of minimal operator fails to be minimal in the noncommutative case (at least in the proposed sense).

A natural question, which arises in this context, is about a proper definition of Dirac and Laplace operators on noncommutative tori. Even though the flat situation appears

to be quite well understood, even a slight deviation, like the conformal rescaling, discussed in this note, changes completely the picture. Unlike classical case we still cannot identify the components of such operators, which are of intrinsic geometric origin and distinguish them from the additional degrees of freedom (like torsion).

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Connes, A.: On the spectral characterization of manifolds. *J. Nonc. Gom.* **7**(1), 1–82 (2013)
2. Connes, A.: *Noncommutative Geometry*. Academic Press, London, New York, San Francisco (1994)
3. Connes, A., Moscovici, H.: Modular curvature for noncommutative two-tori. [arXiv:1110.3500](https://arxiv.org/abs/1110.3500)
4. Connes, A., Tretkoff, P.: The Gauss-Bonnet theorem for the noncommutative two torus. In: *Noncommutative Geometry, Arithmetic, and Related Topics*, pp. 141–158. Johns Hopkins University Press (2011)
5. Dąbrowski, L., Sitarz, A.: Noncommutative circle bundles and new Dirac operators. *Comm. Math. Phys.* **318**(1), 111–130 (2013)
6. Dąbrowski, L., Sitarz, A.: Curved noncommutative torus and Gauss-Bonnet. *J. Math. Phys.* **54**, 013518 (2013)
7. Fathizadeh, F., Khalkhali, M.: The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. *J. Noncommut. Geom.* **6**(3), 457–480 (2012)
8. Fathizadeh, F., Khalkhali, M.: Scalar curvature for the noncommutative two torus. [arXiv:1110.3511](https://arxiv.org/abs/1110.3511)
9. Fathizadeh, F., Khalkhali, M.: Weyl's law and Connes' trace theorem for noncommutative two tori. *Lett. Math. Phys.* **103**(1), 1–18 (2013)
10. Fathizadeh, F., Khalkhali, M.: Scalar curvature for noncommutative four-tori. [arXiv:1301.6135](https://arxiv.org/abs/1301.6135) [math.QA]
11. Gilkey, P.: *Invariance Theory, The Heat Equation And The Atiyah-Singer Index Theorem*, vol. 2. CRC Press, Boca Raton (1995)
12. Fathizadeh, F., Wong, M.W.: Noncommutative residues for pseudo-differential operators on the noncommutative two-torus. *J. Pseudo-Differ. Oper. Appl.* **2**(3), 289–302 (2011)
13. Kastler, D.: The Dirac operator and gravitation. *Commun. Math. Phys.* **166**, 633–643 (1995)
14. Kalau, W., Walze, M.: Gravity, non-commutative geometry, and the Wodzicki residue. *J. Geom. Phys.* **16**, 327–344 (1995)
15. Levy, C., Jiménez, C., Paycha S.: The canonical trace and the noncommutative residue on the noncommutative torus. [arXiv:1303.0241](https://arxiv.org/abs/1303.0241)