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Delay Dependent Robust Stability of Singular Systems with Additive Time-varying Delays

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Abstract: This paper considers the problem of delay-dependent robust stability for uncertain singular systems with additive time-varying delays. The purpose of the robust stability problem is to give conditions such that the uncertain singular system is regular, impulse free, and stable for all admissible uncertainties. The results are expressed in terms of linear matrix inequalities (LMIs). Finally, two numerical examples are provided to illustrate the effectiveness of the proposed method.

Keywords: Singular systems, additive time-varying delays, linear matrix inequalities (LMIs), robust stability, delay-dependent conditions.

1 Introduction

Singular systems have been extensively studied in the past years due to the fact that singular systems describe physical systems better than regular ones[1]. It is also referred to as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems or semi-state systems[2, 3].

The delay-dependent problem for singular systems is much more complicated than that for regular systems because it requires to consider not only stability, but also regularity and absence of impulses (for continuous singular systems), and causality (for discrete singular systems[4−7]).

In the past few years, much attention has been paid on the stability analysis of singular delay systems[8−18]. The stability and robust stabilization problem of uncertain singular time-delay systems has been studied in [19] by using the Lyapunov-Krasovskii functional and Jensen integral inequality. Gao et al.[14, 20] discussed the design of the guaranteed cost controller for the uncertain singular time-delay systems with norm-bounded parameter uncertainties. On the other hand, the delay-dependent stability and stabilization with H_∞ prescribed performance for singular time-delay systems has been investigated[21, 22]. Saadni et al.[1] studied the robust stability and stabilization for singular systems with multiple time-varying delays by using the slack variables.

It was pointed out in [23−25] that in networked controlled system (NCS), if the signal transmitted from one point to another passes through few segments of networks, then successive delays are induced with different properties due to varying transmission conditions. Thus, it is appropriate to consider different time-delays $h_1(t)$ and $h_2(t)$ in the same state where $h_1(t)$ is the time-delay induced from sensor to controller, and $h_2(t)$ is the delay induced from controller to the actuator. The stability analysis for regular continuous systems with additive time-varying delays is studied[24−26]. Motivated by this idea, we study the problem of robust stability for singular systems with two additive time-varying delays. We develop some delay-dependent sufficient conditions in terms of LMIs, which guarantee the singular time-delay system to be regular, impulse free, and stable. To the best of our knowledge, there is no result in the literature dealing with singular systems with additive time-varying delays.

The paper is organized as follows. In Section 2, the problem is formulated and the required lemmas are given. In Section 3, the asymptotic stability problem is established. In Section 4, we address the robust stability problem. And in Section 5, we present numerical examples to show the effectiveness of the proposed results.

2 System description and preliminaries

Consider an uncertain singular system described by

$$\left\{\begin{array}{l} E\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + \\ \qquad (A_d + \Delta A_d(t))x(t - h_1(t) - h_2(t)), t > 0 \\ \quad x(t) = \phi(t),\ t \in [-\bar{h},\ 0] \end{array}\right. \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $h_1(t)$ and $h_2(t)$ are the time-varying delays satisfying

$$\begin{array}{l} 0 \leqslant h_1(t) \leqslant \bar{h}_1,\ \dot{h}_1(t) \leqslant d_1, \\ 0 \leqslant h_2(t) \leqslant \bar{h}_2,\ \dot{h}_2(t) \leqslant d_2, \\ \bar{h} = \bar{h}_1 + \bar{h}_2, \\ d = d_1 + d_2. \end{array} \tag{2}$$

The matrix $E \in \mathbf{R}^{n\times n}$ may be singular and $\mathrm{rank}(E) = r \leqslant n$ is assumed. $A_0 \in \mathbf{R}^{n\times n}$ and $A_d \in \mathbf{R}^{n\times n}$ are known real constant matrices, $\phi(t)$ is a compatible vector valued continuous initial function. $\Delta A_0(t)$ and $\Delta A_d(t)$ are real matrices representing parameter uncertainties, and are assumed to be norm-bounded as

$$\left[\begin{array}{cc} \Delta A_0(t) & \Delta A_d(t) \end{array}\right] = DF(t)\left[\begin{array}{cc} N_0 & N_d \end{array}\right] \tag{3}$$

where $D \in \mathbf{R}^{n\times p}$, $N_0 \in \mathbf{R}^{p\times n}$ and $N_d \in \mathbf{R}^{p\times n}$ are known real constant matrices, and $F(t)$ is a $p \times p$ unknown real

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and possibly time-varying matrix satisfying

$$F^{\mathrm{T}}(t)F(t) \leqslant I. \tag{4}$$

The following lemmas are very useful for the ideas developed in this paper.

Lemma 1[27]**.** Let $Q = Q^{\mathrm{T}}$, H, E and $F(t)$ satisfying $F^{\mathrm{T}}(t)F(t) \leqslant I$ are appropriately dimensioned matrices, the inequality $Q + HF(t)E + E^{\mathrm{T}}F^{\mathrm{T}}(t)H^{\mathrm{T}} < 0$ is true, if and only if the following inequality holds for any matrix $Y > 0$, $Q + HY^{-1}H^{\mathrm{T}} + E^{\mathrm{T}}YE < 0$.

Lemma 2 (Finsler′s lemma)[28]**.** Consider a vector $\chi \in \mathbf{R}^n$, a symmetric matrix $Q \in \mathbf{R}^{n\times n}$ and a matrix $B \in \mathbf{R}^{m\times n}$, such that $\mathrm{rank}(B) < n$. The following statements are equivalent:

1) $\chi^{\mathrm{T}}Q\chi < 0$, $\forall\chi$ such that $B\chi = 0$, $\chi \neq 0$,
2) $B^{\neg\mathrm{T}}QB^{\neg} < 0$,
3) $\exists\mu \in \mathbf{R}: \ Q - \mu B^{\mathrm{T}}B < 0$,
4) $\exists F \in \mathbf{R}^{n\times m}: \ Q + FB + B^{\mathrm{T}}F^{\mathrm{T}} < 0$,

where $B^{\neg}$ denotes a basis for the null-space of B.

Lemma 3[29]**.** For any constant matrix $M = M^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, $M > 0$, scalar $\gamma \geqslant \eta(t) > 0$, vector function $\omega : [0,\gamma] \to \mathbf{R}^n$ such that the integrations in the following are well defined, then

$$\eta(t)\int_0^{\eta(t)} \omega^{\mathrm{T}}(\beta)M\omega(\beta)\mathrm{d}\beta \geqslant$$

$$\left[\int_0^{\eta(t)} \omega(\beta)\mathrm{d}\beta\right]^{\mathrm{T}} M \left[\int_0^{\eta(t)} \omega(\beta)\mathrm{d}\beta\right].$$

Consider the nominal system of (1) given by

$$E\dot{x}(t) = A_0x(t) + A_dx(t - h_1(t) - h_2(t)). \tag{5}$$

Definition 1[2]**.** 1) The pair $(E,\ A_0)$ is said regular if $\det(sE - A_0)$ is not identically zero. 2) The pair $(E,\ A_0)$ is said to be impulse free if $\deg(\det(sE - A_0)) = \mathrm{rank}(E)$.

Definition 2[15]**.** 1) The singular time-delay system (5) is said to be regular and impulse free if the pair $(E,\ A_0)$ is regular and impulse free. 2) Singular time-delay system (5) is said to be stable if for any $\varepsilon > 0$, there exists a scalar $\delta(\varepsilon) > 0$, such that for any compatible initial conditions $\phi(t)$ satisfying $\sup_{-\bar{h}\leqslant t\leqslant 0}\|\phi(t)\| \leqslant \delta(\varepsilon)$, the solution $x(t)$ of system (5) satisfies $\|x(t)\| \leqslant \varepsilon$ for $t \geqslant 0$. Furthermore, $x(t) \to 0, t \to \infty$. 3) Singular time-delay system (5) is said to be admissible if it is regular, impulse-free and stable.

3 Stability analysis

In this section, we will present the delay-dependent stability conditions for singular systems with two additive time-varying delays. The conditions depend on the upper bound of the delays as given in (2).

Theorem 1. Given $\bar{h}_1 > 0$ and $\bar{h}_2 > 0$, then system (5) is regular, impulse free, and asymptotically stable if there exist $n \times n$ matrices P, F_0, F_1, F_2, and $n \times n$ positive definite matrices Q_1, Q_2, Z_1, and Z_2 such that the following conditions hold.

$$PE = E^{\mathrm{T}}P^{\mathrm{T}} \geqslant 0 \tag{6}$$

$$Q_1 - Q_2 \geqslant 0 \tag{7}$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ * & M_{22} & M_{23} & M_{24} \\ * & * & M_{33} & M_{34} \\ * & * & * & M_{44} \end{bmatrix} < 0 \tag{8}$$

where "*" denotes the symmetric part in a symmetric matrix, and

$$\begin{aligned}
M_{11} &= Q_1 - \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E + F_0A_0 + A_0^{\mathrm{T}}F_0^{\mathrm{T}} \\
M_{12} &= \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E \\
M_{13} &= F_0A_d + A_0^{\mathrm{T}}F_1^{\mathrm{T}} \\
M_{14} &= P - F_0 + A_0^{\mathrm{T}}F_2^{\mathrm{T}} \\
M_{22} &= -(1-d_1)(Q_1 - Q_2) - \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E - \frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E \\
M_{23} &= \frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E \\
M_{24} &= 0 \\
M_{33} &= -\frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E - (1-d)Q_2 + F_1A_d + A_d^{\mathrm{T}}F_1^{\mathrm{T}} \\
M_{34} &= -F_1 + A_d^{\mathrm{T}}F_2^{\mathrm{T}} \\
M_{44} &= \bar{h}_1Z_1 + \bar{h}_2Z_2 - F_2 - F_2^{\mathrm{T}}.
\end{aligned} \tag{9}$$

Proof. From (8), it follows that

$$\begin{bmatrix} M_{11} & M_{14} \\ M_{14}^{\mathrm{T}} & M_{44} \end{bmatrix} < 0. \tag{10}$$

Let $J = \begin{bmatrix} I & A_0^{\mathrm{T}} \end{bmatrix}$. Pre-multiplying and post-multiplying (10) by J and J^{T}, respectively, we get

$$Q_1 - \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E + PA_0 + A_0^{\mathrm{T}}P^{\mathrm{T}} + \bar{h}_1A_0^{\mathrm{T}}Z_1A_0 + \bar{h}_2A_0^{\mathrm{T}}Z_2A_0 < 0. \tag{11}$$

Now, choose two nonsingular matrices M and N such that

$$\begin{aligned}
MEN &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \\
\overline{A}_0 = MA_0N &= \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix} \\
\overline{P} = N^{\mathrm{T}}PM^{-1} &= \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_3 & \bar{P}_4 \end{bmatrix}
\end{aligned} \tag{12}$$

and denote

$$\overline{Q}_1 = N^{\mathrm{T}}Q_1N$$

$$\overline{Z}_1 = M^{-\mathrm{T}}Z_1M^{-1} = \begin{bmatrix} \bar{z}_{11} & \bar{z}_{12} \\ \bar{z}_{13} & \bar{z}_{14} \end{bmatrix}$$

$$\overline{Z}_2 = M^{-\mathrm{T}}Z_2M^{-1} = \begin{bmatrix} \bar{z}_{21} & \bar{z}_{22} \\ \bar{z}_{23} & \bar{z}_{24} \end{bmatrix}.$$

By using (6), it can be shown that $\bar{P}_3 = 0$. Pre-multiplying and post-multiplying (11) by N^{T} and N, respectively, we get

$$\overline{Q}_1 - \frac{1}{\bar{h}_1}\begin{bmatrix} \bar{z}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \overline{P}\ \overline{A}_0 + \overline{A}_0^{\mathrm{T}}\overline{P}^{\mathrm{T}} + \bar{h}_1\overline{A}_0^{\mathrm{T}}\bar{Z}_1\bar{A}_0 + \bar{h}_2\overline{A}_0^{\mathrm{T}}\bar{Z}_2\bar{A}_0 < 0. \tag{13}$$

Since $\bar{Q}_1 > 0$, $\bar{h}_1\overline{A}_0^{\mathrm{T}}\bar{Z}_1\bar{A}_0 \geqslant 0$, $\bar{h}_2\overline{A}_0^{\mathrm{T}}\bar{Z}_2\bar{A}_0 \geqslant 0$, it can be easily observed that $\overline{A}_4^{\mathrm{T}}\overline{P}_4^{\mathrm{T}} + \overline{P}_4\overline{A}_4 < 0$, which implies that $\overline{A}_4$ is nonsingular, consequently, the pair (E, A_0) is regular and impulse free. Therefore, according to Definition 1, system (5) is regular and impulse free.

Now, we prove the stability. Let $x_t = x(t+\theta)$ for $\theta \in [-\bar{h},\ 0]$, and consider the following Lyapunov functional

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{14}$$

where

$$\begin{aligned}
&V_1(x_t) = x^{\mathrm{T}}(t)PEx(t)\\
&V_2(x_t) = \int_{-\bar{h}_1}^{0}\int_{t+\sigma}^{t}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_1E\dot{x}(s)\mathrm{d}s\mathrm{d}\sigma+\\
&\qquad\int_{-\bar{h}_1-\bar{h}_2}^{-\bar{h}_1}\int_{t+\sigma}^{t}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_2E\dot{x}(s)\mathrm{d}s\mathrm{d}\sigma\\
&V_3(x_t) = \int_{t-h_1(t)}^{t}x^{\mathrm{T}}(s)Q_1x(s)\mathrm{d}s+\\
&\qquad\int_{t-h_1(t)-h_2(t)}^{t-h_1(t)}x^{\mathrm{T}}(s)Q_2x(s)\mathrm{d}s.
\end{aligned}$$

Then, the time-derivative of $V(x_t)$ along the solution of system (5) gives

$$\dot{V}_1(x_t) = 2x^{\mathrm{T}}(t)PE\dot{x}(t) \tag{15}$$

$$\begin{aligned}
&\dot{V}_2(x_t) = \int_{-\bar{h}_1}^{0}[\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_1E\dot{x}(t)-\\
&\quad\dot{x}^{\mathrm{T}}(t+\sigma)E^{\mathrm{T}}Z_1E\dot{x}\ (t+\sigma)]\mathrm{d}\sigma+\\
&\quad\int_{-\bar{h}_1-\bar{h}_2}^{-\bar{h}_1}[\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_2E\dot{x}(t)-\\
&\quad\dot{x}^{\mathrm{T}}(t+\sigma)E^{\mathrm{T}}Z_2E\dot{x}(t+\sigma)]\mathrm{d}\sigma\\
&\dot{V}_2(x_t) = \quad \bar{h}_1\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_1E\dot{x}(t)-\\
&\quad\int_{t-\bar{h}_1}^{t}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_1E\dot{x}(s)\mathrm{d}s + \bar{h}_2\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_2E\dot{x}(t)-\\
&\quad\int_{t-\bar{h}}^{t-\bar{h}_1}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_2E\dot{x}(s)\mathrm{d}s.
\end{aligned}$$

For any symmetric positive definite matrices Z_1 and Z_2, the following inequalities always hold[25].

$$-\int_{t-\bar{h}_1}^{t}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_1E\dot{x}(s)\mathrm{d}s \leqslant -\int_{t-h_1(t)}^{t}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_1E\dot{x}(s)\mathrm{d}s$$

$$\begin{aligned}
&-\int_{t-\bar{h}}^{t-\bar{h}_1}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_2E\dot{x}(s)\mathrm{d}s \leqslant -\int_{t-h(t)}^{t-h_1(t)}\dot{x}^{\mathrm{T}}(s)\times\\
&\quad E^{\mathrm{T}}Z_2E\dot{x}(s)\mathrm{d}s
\end{aligned}$$

where $h(t) = h_1(t) + h_2(t)$.

$$\begin{aligned}
&\dot{V}_2(x_t) \leqslant \bar{h}_1\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_1E\dot{x}(t)-\\
&\quad\int_{t-h_1(t)}^{t}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_1E\dot{x}(s)\mathrm{d}s + \bar{h}_2\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_2E\dot{x}(t)-\\
&\quad\int_{t-h(t)}^{t-h_1(t)}\dot{x}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_2E\dot{x}(s)\mathrm{d}s
\end{aligned}$$

By Lemma 3,

$$\begin{aligned}
&\dot{V}_2(x_t) \leqslant \bar{h}_1\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_1E\dot{x}(t) + \bar{h}_2\dot{x}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_2E\dot{x}(t)-\\
&\quad\frac{1}{\bar{h}_1}\left[x(t) - x(t-h_1(t)]^{\mathrm{T}}E^{\mathrm{T}}Z_1E\left[x(t) - x(t-h_1(t)\right]\right.-\\
&\quad\frac{1}{\bar{h}_2}\left[x(t-h_1(t)) - x(t-h(t)]^{\mathrm{T}}E^{\mathrm{T}}Z_2E[x(t-h_1(t))-\right.\\
&x(t-h(t)]
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\dot{V}_3(x_t) = x^{\mathrm{T}}(t)Q_1x(t) - (1-\dot{h}_1(t))x^{\mathrm{T}}(t-h_1(t))Q_1\times\\
&\quad x(t-h_1(t)) + (1-\dot{h}_1(t))x^{\mathrm{T}}(t-h_1(t))Q_2x(t-h_1(t))-\\
&\quad(1-\dot{h}_1(t)-\dot{h}_2(t))x^{\mathrm{T}}(t-h(t))Q_2x(t-h(t))
\end{aligned}$$

$$\begin{aligned}
&\dot{V}_3(x_t) \leqslant x^{\mathrm{T}}(t)Q_1x(t) - (1-d_1)x^{\mathrm{T}}(t-h_1(t))(Q_1-Q_2)\times\\
&x(t-h_1(t)) - (1-d_1-d_2)x^{\mathrm{T}}(t-h(t))Q_2x(t-h(t))
\end{aligned} \tag{17}$$

where $Q_1 - Q_2 \geqslant 0$. Now, let

$$\chi(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-h_1(t)) & x^{\mathrm{T}}(t-h(t)) & (E\dot{x}(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Taking account of (15), (16) and (17), we have

$$\dot{V}(x_t) \leqslant \chi^{\mathrm{T}}(t)\Phi\chi(t) \tag{18}$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & P\\ * & \Phi_{22} & \Phi_{23} & 0\\ * & * & \Phi_{33} & 0\\ * & * & * & \Phi_{44} \end{bmatrix}$$

and

$$\begin{aligned}
&\Phi_{11} = Q_1 - \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E\\
&\Phi_{12} = \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E\\
&\Phi_{22} = -\frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E - \frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E - (1-d_1)(Q_1-Q_2)\\
&\Phi_{23} = \frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E\\
&\Phi_{33} = -\frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E - (1-d)Q_2\\
&\Phi_{44} = \bar{h}_1Z_1 + \bar{h}_2Z_2.
\end{aligned}$$

Now, let

$$B = \begin{bmatrix} A_0 & 0 & A_d & -I \end{bmatrix}$$

$$F = \begin{bmatrix} F_0\\ 0\\ F_1\\ F_2 \end{bmatrix}.$$

Then we can verify that $B\chi = 0$. The matrix M in (8) can be written as

$$M = \Phi + FB + B^{\mathrm{T}}F^{\mathrm{T}} < 0.$$

Applying Lemma 2, we have $\chi^{\mathrm{T}}\Phi\chi < 0$ which implies that $\dot{V}(x_t) < 0$. Thus, system (5) is asymptotically stable. □

Remark 1. When the time-delays $h_1(t)$ and $h_2(t)$ are constant, system (5) is regular, impulse free, and asymptotically stable, if only the conditions (6) and (8) hold with $d_1 = d_2 = 0$.

Remark 2. In the derivation of our stability conditions, it can be clearly observed that neither model transformation is involved, nor bounding techniques for cross terms are required. Our paper presents a new approach based on Finsler′s lemma to establish delay-dependent stability of singular systems with additive time-varying delays.

Remark 3. The condition developed in this theorem are delay-dependent. The result of this theorem can be easily extended to handle the case of systems with multiple additive time-varying delays described by

$$E\dot{x}(t) = A_0x(t) + A_dx(t - \sum_{j=1}^{q} h_j(t))$$

where $0 \leqslant h_j(t) \leqslant \bar{h}_j$, $\dot{h}_j(t) \leqslant d_j, j = 1, \cdots, q, \bar{h} = \sum_{j=1}^{q} \bar{h}_j$, and $d = \sum_{j=1}^{q} d_j$.

And this theorem can be further extended to multiple delay case described by

$$E\dot{x}(t) = A_0x(t) + \sum_{i=1}^{p} A_{di}x(t - \sum_{j=1}^{q} h_{ij}(t))$$

where $0 \leqslant h_{ij}(t) \leqslant \bar{h}_{ij}$, $\dot{h}_{ij}(t) \leqslant d_{ij}$, $i = 1, \cdots, p$, $j = 1, \cdots, q$, $\bar{h}_i = \sum_{j=1}^{q} \bar{h}_{ij}$, and $d_i = \sum_{j=1}^{q} d_{ij}$.

4 Robust stability

In this section, we will extend the results of the above section to uncertain systems (1).

Theorem 2. Given $\bar{h}_1 > 0$ and $\bar{h}_2 > 0$, the uncertain system (1) satisfying conditions (1) is regular, impulse free, and robustly stable, if there exist $n \times n$ matrices P, F_0, F_1, F_2, and $n \times n$ positive definite matrices Q_1, Q_2, Z_1, Z_2 and $Y \in \mathbf{R}^{p\times p}$, such that condition (6) and (7) hold and

$$\begin{bmatrix} M_{11}+f_{11} & M_{12} & M_{13}+f_{12} & M_{14} & F_0D \\ * & M_{22} & M_{23} & M_{24} & 0 \\ * & * & M_{33}+f_{22} & M_{34} & F_1D \\ * & * & * & M_{44} & F_2D \\ * & * & * & * & -Y \end{bmatrix} < 0 \quad (19)$$

where $f_{11} = N_0^{\mathrm{T}}YN_0$, $f_{12} = N_0^{\mathrm{T}}YN_d$, $f_{22} = N_d^{\mathrm{T}}YN_d$, M_{ij} are given by (9).

Proof. Replacing A_0 and A_d respectively by $A_0 + DF(t)N_0$ and $A_d + DF(t)N_d$ in condition (8) gives

$$\begin{bmatrix} \tilde{M}_{11} & M_{12} & \tilde{M}_{13} & \tilde{M}_{14} \\ * & M_{22} & M_{23} & M_{24} \\ * & * & \tilde{M}_{33} & \tilde{M}_{34} \\ * & * & * & M_{44} \end{bmatrix} < 0 \quad (20)$$

where

$$\begin{aligned} \tilde{M}_{11} &= Q_1 - \frac{1}{\bar{h}_1}E^{\mathrm{T}}Z_1E + F_0(A_0 + \\ &\quad DF(t)N_0) + (A_0 + DF(t)N_0)^{\mathrm{T}}F_0^{\mathrm{T}} \\ \tilde{M}_{13} &= F_0(A_d + DF(t)N_d) + (A_0 + DF(t)N_0)^{\mathrm{T}}F_1^{\mathrm{T}} \\ \tilde{M}_{14} &= P - F_0 + (A_0 + DF(t)N_0)^{\mathrm{T}}F_2^{\mathrm{T}} \\ \tilde{M}_{33} &= -\frac{1}{\bar{h}_2}E^{\mathrm{T}}Z_2E - (1 - d_1 - d_2)Q_2 + \\ &\quad F_1(A_d + DF(t)N_d) + (A_d + DF(t)N_d)^{\mathrm{T}}F_1^{\mathrm{T}} \\ \tilde{M}_{34} &= -F_1 + (A_d + DF(t)N_d)^{\mathrm{T}}F_2^{\mathrm{T}} \end{aligned} \quad (21)$$

which can be written as

$$M + \mathrm{sym}\left\{ \begin{bmatrix} F_0 \\ 0 \\ F_1 \\ F_2 \end{bmatrix} DF(t) \begin{bmatrix} N_0 & 0 & N_d & 0 \end{bmatrix} \right\} < 0$$

where $\mathrm{sym}\{A\} = A + A^{\mathrm{T}}$. Applying Lemma 1, there exists $Y > 0$ such as

$$M + \begin{bmatrix} F_0 \\ 0 \\ F_1 \\ F_2 \end{bmatrix} DY^{-1}D^{\mathrm{T}} \begin{bmatrix} F_0 \\ 0 \\ F_1 \\ F_2 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} N_0^{\mathrm{T}} \\ 0 \\ N_d^{\mathrm{T}} \\ 0 \end{bmatrix} Y \begin{bmatrix} N_0^{\mathrm{T}} \\ 0 \\ N_d^{\mathrm{T}} \\ 0 \end{bmatrix}^{\mathrm{T}} < 0. \quad (22)$$

The use of the Schur complement leads to the desired result. □

Remark 4. In the case of constant delays $h_1(t)$ and $h_2(t)$, we require only the conditions (6) and (19) with $d_1 = d_2 = 0$ for checking regularity, impulse free state, and robust stability for the system (1).

5 Numerical examples

In this section, we aim to demonstrate the effectiveness and the less conservatism of the proposed approach.

Example 1[21]**.** Consider the following singular system with constant time delay,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0.6341 & 0.5413 \\ -0.6121 & -1.1210 \end{bmatrix}$$

$$A_d = \begin{bmatrix} -0.4500 & 0 \\ 0 & -0.1210 \end{bmatrix}$$

Letting $d_1 = d_2 = 0$ and applying Theorem 1, our purpose is to find the upper bound $\bar{h}_1$ of delay $h_1(t)$ when $\bar{h}_2$ is given, or to compute the upper bound $\bar{h}_2$ of delay $h_2(t)$ when $\bar{h}_1$ is given. The upper bound $\bar{h}$ is obtained by summing the two delay bounds $\bar{h}_1$ and $\bar{h}_2$. It has been proved in Theorem 1 of [30] that its result is equivalent to those of [31−34]. For comparison, we report the results in Table 1.

It is well known that the results of time-varying delays are more conservative than that of time invariant ones. In the existing literature results[8, 10, 13, 19, 21, 30, 32−37], the time delay considered is constant, and the delay is time-varying in [31]. However, our result is derived for time-varying delay and gives less conservative bounds.

From Table 1, it can be seen that our method gives less conservative results.

Table 1 Comparison of delay-dependent stability conditions of Example 1

Method	Upper bound $\bar{h} = \bar{h}_1 + \bar{h}_2$						
Reference [35]	–						
Reference [8]	2.1328						
Reference [10, 13]	2.1372						
Reference [36, 37]	2.4841						
Reference [21]	2.4865						
Reference [30, 31−34]	2.4865						
Reference [19]	2.4865						
	Upper bound $\bar{h}_2$ for given $\bar{h}_1$				Upper bound $\bar{h}_1$ for given $\bar{h}_2$		
Theorem 1	$\bar{h}_1 = 0.00001$	$\bar{h}_1 = 0.4$	$\bar{h}_1 = 1$	$\bar{h}_1 = 1.5$	$\bar{h}_2 = 1$	$\bar{h}_2 = 1.4$	$\bar{h}_2 = 1.8$
	2.4865	2.1008	1.5153	1.0167	1.5165	1.1167	0.7096

Table 2 Comparison of delay-dependent robust stability conditions of Example 2

α	0.25	0.30	0.35	0.40	0.45	0.50
Reference [36]	0.4209	0.3939	0.3637	0.3279	0.2817	0.2106
Reference [14]	0.8087	0.7942	0.7689	0.7262	0.6521	0.5054
Reference [16]	0.8514	0.8249	0.7924	0.7438	0.6641	0.5110
Reference [33]	0.8962	0.8787	0.8616	0.8448	0.8283	0.8121
Theorem 2 for $\bar{h}_1 = 0.00001$	0.8962	0.8787	0.8616	0.8448	0.8283	0.8121
Theorem 2 for $\bar{h}_1 = 0.4$	0.9298	0.9109	0.8922	0.8737	0.8555	0.8374

Example 2[16]**.** Consider the uncertain singular delay system (1) with

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A_d = \begin{bmatrix} -2.4 & 2 \\ 0 & 1 \end{bmatrix}$$

where $\|\Delta A_0\| \leqslant \alpha$, $\|\Delta A_d\| \leqslant \alpha$ $(\alpha > 0)$, and the delay is constant. The uncertainty is written as $D = \alpha I$, $N_0 = N_d = 0.5I$. In order to apply our results, we set $d_1 = d_2 = 0$, we compute $\bar{h}_2$ when $\bar{h}_1 = 0.4$ and $\bar{h}_1 = 0.00001$, and we deduce the delay bound $\bar{h}$. Table 2 gives the comparison of the maximum allowed delay $\bar{h}$ for various values of the parameter α. It is clear that the conditions in this paper give better results than those in [14, 16, 33, 36], showing the advantage of the stability result in this paper.

6 Conclusions

Delay-dependent conditions are presented in terms of linear matrix inequalities (LMIs) for asymptotic stability and robust stability of singular systems with two additive time-varying delays. The LMIs proposed have been obtained by utilizing a Lyapunov Krasovskii functional. Numerical examples are given to illustrate the effectiveness of the proposed method and to show that our criteria give less conservative results.

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