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From intersection local time to the Rosenblatt process

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Abstract The Rosenblatt process was obtained by Taqqu (Z. Wahr. Verw. Geb. 31:287–302, 1975) from convergence in distribution of partial sums of strongly dependent random variables. In this paper, we give a particle picture approach to the Rosenblatt process with the help of intersection local time and white noise analysis, and discuss measuring its long-range dependence by means of a number called dependence exponent.

Keywords Rosenblatt process · Long-range dependence · Intersection local time · Particle system

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1 Introduction

The Rosenblatt process was introduced and studied by Taqqu [33], motivated by a counterexample of Rosenblatt regarding a strong mixing condition [26]. See Taqqu [36] for the history and an overview of the process, and related work. The Rosenblatt

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process arises from convergence in distribution of normalized partial sums of strongly dependent random variables. It lives in the second Wiener chaos, in contrast to Gaussian processes (which belong to the first chaos) that are obtained from sums of independent or weakly dependent random variables. The Rosenblatt process possesses some of the main properties of the (Gaussian) fractional Brownian motion with Hurst parameter greater than $1/2$: mean zero, Hölder continuity, non-differentiability, self-similarity and stationarity of increments (hence, it has the same form of covariance as fractional Brownian motion), infinite divisibility, long-range dependence in the sense that the sum over k of covariances of increments on the intervals $[0, 1]$ and $[k, k + 1]$ diverges, and it is not a semimartingale. The Rosenblatt process is the simplest non-Gaussian Hermite process (see Taqqu [34]). It is a counterpart of the fractional Brownian motion, which is the most prominent long-range dependent Gaussian process. Significant attention has been attracted by the Rosenblatt process due to its mathematical interest, and possible applications where the Gaussian property may not be assumed. Recent papers on the subject include [39], which develops a related stochastic calculus and mentions areas of application (see also references therein), [23] and [40], where new properties of the process have been found, and [17], which provides a strong approximation for the process. Relevant information for the present paper on the Rosenblatt process and fractional Brownian motion is given in the next section.

Our main objective in this paper was to show a different way of obtaining the Rosenblatt process. The method of Taqqu, which was developed for Hermite processes generally [34], is based on limits of sums of strongly dependent random variables. The Rosenblatt process can also be defined as a double stochastic integral [36,39]. Our approach consists in deriving the Rosenblatt process from a specific random particle system, which hopefully provides an intuitive physical interpretation of this process. A useful tool here is the theory of random variables in the space of tempered distributions $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^d)$ ($d = 1$ in our case), since it permits to employ some nice properties of this space, and the Rosenblatt process can be expressed with the help of an $\mathcal{S}'$-random variable. This was noted by Dobrushin [13]. Relations between random particle systems and random elements of $\mathcal{S}'$ have been studied by many authors, beginning with Martin-Löf [24]. Our approach is in the spirit of Adler and coauthors, e.g., [1], where a general scheme (but still not so general to cover our case) was developed for representing a given random element of $\mathcal{S}'$ as the limit of appropriate functionals of some particle system (with high density in [1]). That approach was later applied in [2,11,32] to give particle picture interpretations of the self-intersection local times of density processes in $\mathcal{S}'$. We stress that our principal aim is to construct the Rosenblatt process by means of a particle system, and to this end, an important step is to study a suitable random element of $\mathcal{S}'$. Particle picture approaches have been used to obtain fractional Brownian motion and subfractional Brownian motion with Hurst parameter H, in different ways for $H < 1/2$ and $H > 1/2$ [8,12].

Our results can be summarized as follows.

The Rosenblatt process with parameter H, defined for $H \in (1/2, 1)$, is represented in the form $\xi_t = \langle Y, \mathbb{1}_{[0,t]} \rangle$, $t \geq 0$, where Y is an $\mathcal{S}'$-random variable which is obtained from the Wick product : $X \otimes X$:, where X is a centered Gaussian $\mathcal{S}'$-random variable

(see (3.1)). X is in a sense a distributional derivative of a suitable fractional Brownian motion, and the relation between Y and X corresponds to the informal formula (43) in [36]. Note that : $X \otimes X$: is a counterpart of the Hermite polynomial of order 2, $H_2(x) = x^2 - 1$, used in [33,36]. The possibility to take "test functions" of the form $\varphi = \mathbf{1}_{[0,t]}$ to bring in a time parameter has been noted for example in [34]. This formulation is in the spirit of white noise analysis.

We define a particle system on $\mathbb{R}$ with initial distribution given by a Poisson random field with Lebesgue intensity measure, and particles evolving independently according to the standard symmetric α-stable Lévy process. The particles are independently assigned charges $+1$ and -1 with probabilities $1/2$. A crucial element of the construction is the intersection local time of two independent α-stable processes, which is known to exist for $\alpha > 1/2$ (Proposition 5.1 of [11]). Consider the process ξ^T defined by

$$\xi_t^T = \frac{1}{T} \sum_{\substack{j,k \\ j \neq k}} \sigma_j \sigma_k \langle \Lambda(x_j + \rho^j, x_k + \rho^k; T), \mathbf{1}_{[0,t]} \rangle, \ t \geq 0, \tag{1.1}$$

where the x_j are the points of the initial Poisson configuration, the ρ^j are the α-stable processes corresponding to those points ($\rho^j(0) = 0$), the σ_j are the respective charges, and $\Lambda(x_j + \rho^j, x_k + \rho^k; T)$ is the intersection local time of the processes $x_j + \rho^j$ and $x_k + \rho^k$ on the interval $[0, T]$. This local time is defined as a process in $\mathcal{S}'$ (Definition 2.2 and Proposition 2.3). ξ^T is given a precise meaning and shown to have a continuous modification (Lemma 3.4). The result is that for $\alpha \in (1/2, 1)$, ξ^T converges as $T \to \infty$, in the sense of weak functional convergence, to the Rosenblatt process with parameter $H = \alpha$, up to a multiplicative constant (Theorem 3.5).

Our second result concerns the analysis of long-range dependence of the Rosenblatt process. We give a precise measure of long-range dependence by means of a number called *dependence exponent* (Theorem 4.1). We show that the increments are asymptotically independent (not just uncorrelated) as the distance between the intervals tends to infinity (Corollary 4.2).

In Sect. 2, we give background on the Rosenblatt process, on the $\mathcal{S}'$-random variable X from which fractional Brownian motion is derived, on how the abovementioned particle system produces X, and on intersection local time. In Sect. 3, we construct the Rosenblatt process by means of the $\mathcal{S}'$-random variable Y obtained from : $X \otimes X$:, we show that the process ξ^T is well defined, and we prove convergence to the Rosenblatt process. In Sect. 4, we discuss the long-range dependence of the process. Section 5 contains some comments of related interest. The proofs are given in Sect. 6.

We use the following notation:

$\mathcal{S}$: the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}$,
$\mathcal{S}'$: the space of tempered distributions (dual of $\mathcal{S}$),
$\overline{}$: complex conjugate,
$\widehat{\varphi}(z) = \int_{\mathbb{R}} e^{ixz} \varphi(x) \mathrm{d}x$: Fourier transform of a function φ,
$\Rightarrow$: convergence in law in an appropriate space,
$\Rightarrow_f$: convergence of finite-dimensional distributions,
$*$: convolution,

$C([0, \tau])$: the space of real continuous functions on $[0, \tau]$,

C, C_i: generic positive constants with possible dependencies in parentheses.

2 Background

2.1 The Rosenblatt process

We recall some facts on the Rosenblatt process $\xi = (\xi_t)_{t \geq 0}$ with parameter $H \in (1/2, 1)$, which are found in [33,36]. The characteristic function of the finite-dimensional distributions of the process in a small neighborhood of 0 has the form

$$E \exp \left\{ i \sum_{j=1}^{p} \theta_j \xi_{t_j} \right\} = \exp \left\{ \frac{1}{2} \sum_{k=2}^{\infty} \frac{(2i\sigma)^k}{k} R_{H,k}(\theta_1, \ldots, \theta_p; t_1, \ldots, t_p) \right\} \quad (2.1)$$

where

$$R_{H,k}(\theta_1, \ldots, \theta_p; t_1, \ldots, t_p) = \int_{\mathbb{R}^k} \psi(x_1)\psi(x_2)\ldots\psi(x_k)$$

$$\cdot |x_1 - x_2|^{H-1}|x_2 - x_3|^{H-1}\ldots|x_{k-1} - x_k|^{H-1}|x_k - x_1|^{H-1}dx_1\ldots dx_k, \quad (2.2)$$

$$\psi(x) = \sum_{j=1}^{p} \theta_j \mathbf{1}_{[0,t_j]}(x), \quad (2.3)$$

and

$$\sigma = \left[\frac{1}{2}H(2H - 1) \right]^{1/2}. \quad (2.4)$$

The value of σ is chosen so that $E\xi_1^2 = 1$. The series in the exponent converges for $\theta_1, \ldots, \theta_p$ in a (small) neighborhood of 0 depending on $t_1, \ldots, t_p$, and (2.1) defined in this neighborhood determines the distribution. The process is also characterized by the cumulants of the random variable $\sum_{j=1}^{p} \theta_j \xi_{t_j}$, which are $\kappa_1 = 0$,

$$\kappa_k = 2^{k-1}(k-1)!\sigma^k R_{H,k}(\theta_1, \ldots, \theta_p; t_1, \ldots, t_p), \ k \geq 2. \quad (2.5)$$

The process ξ arises from a (Donsker-type) limit in distribution as $n \to \infty$ of the processes

$$\xi_n(t) = \frac{\sigma}{n^H} \sum_{j=1}^{\lfloor nt \rfloor} X_j, \ t \in [0, T],$$

($\lfloor \cdot \rfloor$ denotes integer part), where the random variables X_j are defined by $X_j = Y_j^2 - 1$ (i.e., $X_j = H_2(Y_j)$, where H_2 is the Hermite polynomial of order 2), and $(Y_j)_j$ is a Gaussian stationary sequence of random variables with mean 0, variance 1, and covariances

$$r_j = EY_0Y_j = (1 + j^2)^{(H-1)/2} \sim j^{H-1} \text{ as } j \to \infty.$$

The spectral representation of the process is

$$\xi_t \overset{d}{=} A(H) \int_{\mathbb{R}^2}'' \frac{e^{i(\lambda_1+\lambda_2)t} - 1}{i(\lambda_1 + \lambda_2)} \frac{1}{|\lambda_1|^{H/2}|\lambda_2|^{H/2}} d\widetilde{B}(\lambda_1)d\widetilde{B}(\lambda_2) \qquad (2.6)$$

($\overset{d}{=}$ means equality in distribution), where

$$A(H) = \frac{[H(2H - 1)/2]^{1/2}}{2\Gamma(1 - H)\sin(H\pi/2)}, \qquad (2.7)$$

$\widetilde{B}$ is a complex Gaussian measure on $\mathbb{R}$ such that $\widetilde{B} = B^{(1)} + iB^{(2)}$, $B^{(1)}(A) = B^{(1)}(-A)$, $B^{(2)}(A) = -B^{(2)}(-A)$, A is a Borel set of $\mathbb{R}$ with finite Lebesgue measure $|A|$, $B^{(1)}$ and $B^{(2)}$ are independent, and $E(B^{(1)}(A))^2 = E(B^{(2)}(A))^2 = \frac{1}{2}|A|$. ($\widetilde{B}$ can be viewed as a complex-valued Fourier transform of white noise). The double prime on the integral means that the diagonals $\lambda_1 = \pm\lambda_2$ are excluded in the integration. The process also has a time representation as a double integral on $\mathbb{R}^2$ with respect to Brownian motion, and a finite interval integral representation obtained in [39].

We have mentioned in the Introduction some of the main properties of the Rosenblatt process. We recall the self-similarity with parameter H: for any $c > 0$,

$$(\xi_{ct})_{t\geq 0} \overset{d}{=} c^H(\xi_t)_{t\geq 0} \qquad (2.8)$$

This and stationarity of increments imply

$$E(\xi_t - \xi_s)^2 = \sigma^2(t - s)^{2H}, \qquad (2.9)$$

and hence, the covariance function of ξ has the same form as that of the fractional Brownian motion, i.e.,

$$E\xi_s\xi_t = \frac{\sigma^2}{2}(s^{2H} + t^{2H} - |t - s|^{2H}). \qquad (2.10)$$

In particular, the increments are positively correlated (since $H > 1/2$), and

$$\sum_{k=1}^{\infty} E\xi_1(\xi_k - \xi_{k-1}) = \infty. \qquad (2.11)$$

We have not found a published proof of the non-semimartingale property of ξ, but that is easy to show. By (2.9) with $H > 1/2$, it is obvious that the quadratic variation is 0. A deeper result is that self-similarity and increment stationarity imply that the paths have infinite variation [41]. The non-semimartingale property of fractional Brownian motion ($H \neq 1/2$) follows for example from a general criterion for Gaussian processes [7] (Lemma 2.1, Corollary 2.1).

Infinite divisibility was recently proved in [23,40].

There does not seem to be information in the literature on whether or not the Rosenblatt process has the Markov property (it seems plausible that it does not, by analogy with fractional Brownian motion).

The fractional Brownian motion is the only Gaussian process that has the properties (2.8) and (2.9) (with $\sigma = 1$). On the other hand, there are many processes with stationary increments satisfying (2.8) which belong to the second chaos [22]. The Rosenblatt process is the simplest one of them.

2.2 An $\mathcal{S}'$-random variable related to fractional Brownian motion

Recall that fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $(B_t^H)_{t \geq 0}$ with covariance given by the right-hand side of (2.10) with $\sigma = 1$ (see [29] for background on fBm). This process can be represented with the help of the centered Gaussian $\mathcal{S}'$-valued random variable X with covariance functional

$$E\langle X, \varphi_1 \rangle \langle X, \varphi_2 \rangle = \frac{1}{\pi} \int_{\mathbb{R}} \widehat{\varphi_1}(x) \overline{\widehat{\varphi_2}(x)} |x|^{-\alpha} dx, \quad \varphi_1, \varphi_2 \in \mathcal{S}, \qquad (2.12)$$

where $-1 < \alpha < 1$. Namely,

$$\left(\langle X, \mathbb{1}_{[0,t]} \rangle \right)_{t \geq 0} = (K B_t^H)_{t \geq 0}, \qquad (2.13)$$

with $H = \frac{1+\alpha}{2}$ and K is some positive constant. ($\langle X, \mathbb{1}_{[0,t]} \rangle$ is defined by an L^2-extension). For $0 < \alpha < 1$ ($\frac{1}{2} < H < 1$), this random variable X can be obtained from the particle system described in the Introduction, i.e., the system of independent standard α-stable processes (particle motions) starting from a Poisson random field with Lebesgue intensity. Each particle has a charge ± 1 with equal probabilities, and the charges are mutually independent and independent of the initial configuration and of the particle motions. The motions have the form $x_j + \rho^j$, where the x_j's are the points of the initial configuration, and the ρ^j are independent standard α-stable processes independent of $\{x_j\}_j$, $\rho_0^j = 0$. The charges are denoted by σ_j.

The normalized total charge occupation on the interval $[0, T]$ is defined by

$$\langle X_T, \varphi \rangle = \frac{1}{\sqrt{T}} \sum_j \sigma_j \int_0^T \varphi(x_j + \rho_s^j) ds, \quad \varphi \in \mathcal{S}. \qquad (2.14)$$

We have the following proposition.

Proposition 2.1 *If* $0 < \alpha < 1$ *and* $T \to \infty$, *then*

(a) $X_T \Rightarrow X$ *in* $\mathcal{S}'$, *where* X *is as in* (2.12).
(b) $(\langle X_T, \mathbb{1}_{[0,t]} \rangle)_{t \geq 0} \Rightarrow_f (K B_t^H)_{t \geq 0}$ *with* $H = \frac{1+\alpha}{2}$.

This fact is an easy consequence of Theorem 2.1(a) in [9], where the occupation time fluctuations around the mean for the system without charges were considered. It suffices to take two independent copies of such systems and to write the difference of their occupation time fluctuations.

A similar procedure with a different functional of a particle system without charges permits also to obtain fBm with $H < \frac{1}{2}$, as well as the corresponding random variable X (see Theorem 2.9 in [12]).

2.3 Intersection local time

There are several ways to define the intersection local time (ILT) of two processes (see, e.g., [1,14,25]. We will take the definition from [11], which is close to that of [1]. Intuitively, ILT of real cadlag processes $(\eta_t^1)_{t\geq 0}$, $(\eta_t^2)_{t\geq 0}$ up to time T is given by

$$\langle \Lambda(\eta^1, \eta^2; T), \varphi \rangle = \int_0^T \int_0^T \delta(\eta_v^2 - \eta_u^1)\varphi(\eta_u^1) du\, dv, \quad \varphi \in \mathcal{S},$$

where δ is the Dirac distribution. We want to regard Λ as a process in $\mathcal{S}'$.

To make this definition rigorous, one has to apply a limiting procedure.

Let $\mathcal{F}$ denote the class of non-negative symmetric infinitely differentiable functions f on $\mathbb{R}$ with compact support and such that $\int_\mathbb{R} f(x)dx = 1$. For $f \in \mathcal{F}, \varepsilon > 0$, let

$$f_\varepsilon(x) = \varepsilon^{-1} f\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}. \tag{2.15}$$

We will frequently use

$$|\widehat{f_\varepsilon}(x)| \leq 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \widehat{f_\varepsilon}(x) = \lim_{\varepsilon \to 0} \widehat{f}(\varepsilon x) = 1. \tag{2.16}$$

We define

$$\langle \Lambda_\varepsilon^f(\eta^1, \eta^2; T), \varphi \rangle = \int_0^T \int_0^T f_\varepsilon(\eta_v^2 - \eta_u^1)\varphi(\eta_u^1) du\, dv, \quad T \geq 0, \varphi \in \mathcal{S}. \tag{2.17}$$

Definition 2.2 If there exists an $\mathcal{S}'$-process $\Lambda(\eta^1, \eta^2) = (\Lambda(\eta^1, \eta^2; T))_{T\geq 0}$ such that for each $T \geq 0, \varphi \in \mathcal{S}$ and any $f \in \mathcal{F}$, $\langle \Lambda(\eta^1, \eta^2; T), \varphi \rangle$ is the mean square limit of $\langle \Lambda_\varepsilon^f(\eta^1, \eta^2; T), \varphi \rangle$ as $\varepsilon \to 0$, then the process $\Lambda(\eta^1, \eta^2)$ is called the intersection local time (ILT) of the processes η^1, η^2.

In [11], the following result was proved (Theorem 4.2 and Proposition 5.1 therein).

Proposition 2.3 *Let η^1, η^2 be independent standard α-stable processes in $\mathbb{R}$. If $\alpha > \frac{1}{2}$, then for any $x, y \in \mathbb{R}$ the ILT $\Lambda(x+\eta^1, y+\eta^2)$ exists. Moreover, for all $T \geq 0, f \in \mathcal{F}$,*

$\varphi \in \mathcal{S}$, $\langle \Lambda_\varepsilon^f(\cdot + \eta^1, \cdot + \eta^2; T), \varphi \rangle$ converges in $L^2(\mathbb{R}^2 \times \Omega, \lambda \otimes \lambda \otimes P)$, where λ is the Lebesgue measure on $\mathbb{R}$, and P is the probability measure on the underlying sample space Ω.

3 Particle picture for the Rosenblatt process

We begin with another representation of the Rosenblatt process, which is more suitable for our purpose. From [13], it can be deduced that this construction was known to Dobrushin, but we have not been able to find it written explicitly in the literature. Therefore, we will describe it in detail, but the proof will be only sketched.

Let X be the centered Gaussian $\mathcal{S}'$-random variable with covariance (2.12). Recall that the Wick product $: X \otimes X :$ is defined as a random variable in $\mathcal{S}'(\mathbb{R}^2)$ such that

$$\langle : X \otimes X :, \varphi_1 \otimes \varphi_2 \rangle = \langle X, \varphi_1 \rangle \langle X, \varphi_2 \rangle - E \langle X, \varphi_1 \rangle \langle X, \varphi_2 \rangle, \quad \varphi_1, \varphi_2 \in \mathcal{S}. \quad (3.1)$$

(see, e.g., Chapter 6 of [18] or [3,5]). The Wick square of X is an $\mathcal{S}'$-random variable Y that can be written informally as $\langle Y, \varphi \rangle = \langle : X \otimes X :, \varphi(x)\delta_{y-x} \rangle$. To make this rigorous, we use approximation. Fix $f \in \mathcal{F}$ and let f_ε be as in (2.15). For $\varphi \in \mathcal{S}$ we denote

$$\Phi_{\varepsilon,\varphi}^f(x, y) = \varphi(x) f_\varepsilon(y - x), \quad (3.2)$$

and we define an $\mathcal{S}'$-random variable Y_ε^f by

$$\langle Y_\varepsilon^f, \varphi \rangle = \langle : X \otimes X :, \Phi_{\varepsilon,\varphi}^f \rangle, \quad \varphi \in \mathcal{S}. \quad (3.3)$$

The following lemma is an easy consequence of the regularization theorem [20], and the fact that, by Gaussianity,

$$E \langle : X \otimes X :, \Phi \rangle \langle : X \otimes X :, \Psi \rangle$$
$$= \frac{1}{\pi^2} \int_{\mathbb{R}} \widehat{\Phi}(x, y)(\overline{\widehat{\Psi}(x, y)} + \overline{\widehat{\Psi}(y, x)}) |x|^{-\alpha} |y|^{-\alpha} dx\, dy, \quad \Phi, \Psi \in \mathcal{S}(\mathbb{R}^2). \quad (3.4)$$

Lemma 3.1 *If $\frac{1}{2} < \alpha < 1$, then there exists an $\mathcal{S}'$-random variable Y such that for any $f \in \mathcal{F}$,*

$$\langle Y, \varphi \rangle = L^2 - \lim_{\varepsilon \to 0} \langle Y_\varepsilon^f, \varphi \rangle, \quad \varphi \in \mathcal{S}.$$

$\langle Y, \varphi \rangle$ *can be further extended in $L^2(\Omega)$ to test functions of the form $\mathbf{1}_{[0,t]}$.*

The next theorem is an analogue of (2.13) for the Rosenblatt process.

Theorem 3.2 *Let Y be as in Lemma 3.1. Then the real process*

$$(\langle Y, \mathbf{1}_{[0,t]} \rangle)_{t \geq 0} \quad (3.5)$$

is, up to a constant, the Rosenblatt process with parameter $H = \alpha$.

We remark that this theorem gives a rigorous meaning to the informal expression (43) in [36] relating the Rosenblatt process ξ with parameter H and a fBm with parameter $H_1 = \frac{H+1}{2} \in (\frac{3}{4}, 1)$, which is given by

$$\xi_t = C(H_1) \int_0^t \left((B_s^{H_1})' \right)^2 ds.$$

X corresponds to $(B^{H_1})'$ and $: X \otimes X :$ corresponds to $((B^{H_1})')^2$. The relationship between the parameters follows from Proposition 2.1(b) and Theorem 3.2.

In [13], $\mathcal{S}'$-random variables such as Y are represented in terms of complex multiple stochastic integrals related to (2.6).

After the first version of this paper had been submitted, the referee drew our attention to preprint [4] which appeared in the meantime. That paper uses Hida–Kuo type calculus [21] to construct the stochastic integral with respect to the Rosenblatt process, but the representation (3.5) does not seem to be present there.

Representation (3.5) and Proposition 2.1 suggest a way to construct the Rosenblatt process by means of a particle system. We consider the particle system as before with $\frac{1}{2} < \alpha < 1$. By Proposition 2.3, for each pair $\rho^j, \rho^k, j \neq k$, the intersection local time $\Lambda(x_j + \rho^j, x_k + \rho^k; T)$ exists; moreover, it extends in a natural way to test function $\mathbb{1}_{[0,t]}$. Namely, we have the following lemma.

Lemma 3.3 *Let*

$$\psi = \sum_{j=1}^m a_j \mathbb{1}_{I_j}, \quad a_j \in \mathbb{R}, \; I_j \text{ is a bounded interval}, \tag{3.6}$$

Fix ρ^1, ρ^2, independent standard α-stable processes and $x, y \in \mathbb{R}, T > 0, f \in \mathcal{F}$.

(a) *There exists an L^2-limit of $\langle \Lambda_\varepsilon^f(x + \rho^1, y + \rho^2; T), \psi \rangle$ as $\varepsilon \to 0$, where Λ_ε^f is given by (2.17), and this limit does not depend on f. We denote it by $\langle \Lambda(x + \rho^1, y + \rho^2; T), \psi \rangle$.*

(b) *$\langle \Lambda(x + \rho^1, y + \rho^2; T), \psi \rangle = L^2 - \lim_{\varepsilon \to 0} \langle \Lambda(x + \rho^1, y + \rho^2; T), \psi * f_\varepsilon \rangle$.*

(c) *Moreover, $\langle \Lambda_\varepsilon^f(\cdot + \rho^1, \cdot + \rho^2; T), \psi \rangle$ converges in $L^2(\mathbb{R}^2 \times \Omega, \lambda \otimes \lambda \otimes P)$.*

For the convenience of the reader, let us recall (1.1),

$$\xi_t^T = \frac{1}{T} \sum_{\substack{j,k \\ j \neq k}} \sigma_j \sigma_k \langle \Lambda(x_j + \rho^j, x_k + \rho^k; T), \mathbb{1}_{[0,t]} \rangle, \; t \geq 0.$$

Lemma 3.4 *The process ξ^T is well defined (the series converges in L^2), and it has a continuous modification.*

The main result of the paper is stated in the next theorem, which is a counterpart of Proposition 2.1.

Theorem 3.5 *Let $\frac{1}{2} < \alpha < 1$. Then $\xi^T \Rightarrow K\xi$ in $C([0, \tau])$ as $T \to \infty, \tau > 0$, where ξ is the Rosenblatt process with $H = \alpha$ and K is a positive constant.*

4 Dependence exponent

Long-range dependence is a general notion that has not been clearly defined and can be viewed in different ways [19,28,35]. For a Gaussian process η, long-range dependence is usually described as slow (power) decay of the covariance of increments on intervals $[u, v]$, $[s + \tau, t + \tau]$ as $\tau \to \infty$, i.e.,

$$\mathrm{Cov}(\eta_v - \eta_u, \eta_{t+\tau} - \eta_{s+\tau}) \sim C_{u,v,s,t}\tau^{-K},$$

where K is a positive constant, and convergence and divergence of the series

$$\sum_{k=1}^{\infty} \mathrm{Cov}(\eta_1 - \eta_0, \eta_{k+1} - \eta_k)$$

are sometimes referred to as "short-range" dependence and "long-range" dependence, respectively. This criterion is also applied to non-Gaussian processes with finite second moments, such as the Rosenblatt process [36]. The underlying idea is that the increments become uncorrelated (but not necessarily independent) at some rate as the distance τ between the intervals tends to infinity. However, it can happen that $K \le 0$, which should also be regarded as long-range dependence ([16] contains examples).

In order to characterize long-range dependence in some more precise way for infinitely divisible processes (not necessarily Gaussian), the codifference (see [27,29]) can be useful. In [10], we defined the *dependence exponent* of a (real) infinitely divisible process η as the number

$$\kappa = \inf_{z_1, z_2 \in \mathbb{R}} \inf_{0 \le u < v < s < t} \sup\{\gamma > 0 : D_\tau^\eta(z_1, z_2; u, v, s, t) = o(\tau^{-\gamma}) \text{ as } \tau \to \infty\},$$

(4.1)

where

$$D_\tau^\eta(z_1, z_2; u, v, s, t) = |\log E\mathrm{e}^{i(z_1(\eta_v - \eta_u) + z_2(\eta_{t+\tau} - \eta_{s+\tau}))}$$
$$- \log E\mathrm{e}^{iz_1(\eta_v - \eta_u)} - \log E\mathrm{e}^{iz_2(\eta_{t+\tau} - \eta_{s+\tau})}|$$

(4.2)

is the absolute value of the codifference of the random variables $z_1(\eta_v - \eta_u)$ and $-z_2(\eta_{t+\tau} - \eta_{s+\tau})$. Note that if η is Gaussian, then

$$D_\tau^\eta(z_1, z_2; u, v, s, t) = |z_1 z_2 \mathrm{Cov}(\eta_v - \eta_u, \eta_{t+\tau} - \eta_{s+\tau})|.$$

For fractional Brownian motion, $\kappa = K = 2 - 2H$, and for sub-fractional Brownian motion, $\kappa = K = 3 - 2H$ [8].

It turns out that the same idea can be used to measure long-range dependence for the Rosenblatt process, and this can be done without recourse to infinite divisibility. As recalled in Sect. 2.1, the characteristic functions of the finite-dimensional distributions of the process are given by an explicit formula only for small values of the parameters, which are z_1 and z_2 in our case (see (4.2)). We show next that it is enough to take z_1

and z_2 in an appropriate neighborhood of 0 to measure long-range dependence and prove asymptotic independence of increments.

For simplicity we take $u = s$, $v = t$.

Theorem 4.1 *Let ξ be the Rosenblatt process with parameter H. For any $0 \leq s < t$ there exists a neighborhood $U(s, t)$ of 0 in $\mathbb{R}^2$ such that*

$$D_\tau^\xi(z_1, z_2, s, t) := D_\tau^\xi(z_1, z_2, s, t, s, t) \qquad (4.3)$$

is well defined for $(z_1, z_2) \in U(s, t)$ and all $\tau > 0$, and if we modify (4.1) putting

$$\kappa = \inf_{0 \leq s < t} \inf_{(z_1, z_2) \in U(s,t)} \sup\{\gamma > 0 : D_\tau^\xi(z_1, z_2, s, t) = o(\tau^{-\gamma}) \text{ as } \tau \to \infty\}, \quad (4.4)$$

then $\kappa = 2 - 2H$.

So we see that dependence exponent of the Rosenblatt process with parameter H is the same as that for fBm B^H.

From this theorem, by a standard tightness argument, stationarity of increments of ξ, and the fact that the law of ξ_t is determined by its characteristic function in an arbitrarily small neighborhood of 0, we obtain the following corollary.

Corollary 4.2 *For any $0 < s < t$, the increments of the Rosenblatt process $\xi_t - \xi_s$ and $\xi_{t+\tau} - \xi_{s+\tau}$ are asymptotically independent as $\tau \to \infty$, i.e., if $\mu_{(s,t)}$ is the law of $\xi_t - \xi_s$ and $\mu_{(s,t),(s+\tau,t+\tau)}$ is the law of $(\xi_t - \xi_s, \xi_{t+\tau} - \xi_{s+\tau})$, then*

$$\mu_{(s,t),(s+\tau,t+\tau)} \Rightarrow \mu_{(s,t)} \otimes \mu_{(s,t)} \left(= \mu_{(s,t)} \otimes \mu_{(s+\tau,t+\tau)}\right).$$

5 Additional comments

5.1 Sub-Rosenblatt process

It is known that if in the formula (2.13) we put $\mathbb{1}_{[0,t]} - \mathbb{1}_{[-t,0]}$ instead of $\mathbb{1}_{[0,t]}$, we obtain a sub-fractional Brownian motion (sub-fBm), i.e., a centered Gaussian process with covariance

$$t^{2H} + s^{2H} - \frac{1}{2}\left(|t - s|^{2H} + (t + s)^{2H}\right),$$

again with $H = \frac{1+\alpha}{2}$. This process has been studied by several authors, e.g., [8,15, 38,42] and others. In particular, in [12], an analogue of Proposition 2.1(b) was proved for sub-fBm.

We can now extend formula (3.5) and define a new process $(\langle Y, \mathbb{1}_{[0,t]} - \mathbb{1}_{[-t,0]}\rangle)_{t \geq 0}$. It is natural to call it sub-Rosenblatt process, as it has the same covariance as sub-fBm.

Analogues of Theorems 3.5 and 4.1 also hold.

5.2 Rosenblatt process with two parameters

Maejima and Tudor in [22] define a class of self-similar processes with stationary increments that live in the second Wiener chaos. These processes depend on two parameters H_1, H_2, and the Rosenblatt process corresponds to the case $H_1 = H_2$. One can ask about the possibility of extending our construction to those two-parameter processes.

5.3 General Hermite processes

Taqqu [34] studies extensions of the Rosenblatt process living in Wiener chaos of order k, $k \geq 2$, which he calls Hermite processes. One can attempt to find a particle picture interpretation for those processes. It seems that one should employ k-th Wick powers and work with intersection local times of k-tuples of stable processes.

6 Proofs

Proof of Theorem 3.2 (outline). Let ξ be the Rosenblatt process with parameter H. It is known that its distributions are determined by its moments, and therefore, it is enough to prove that for all n, $p \in \mathbb{N}$, $t_1, t_2, \ldots, t_p \geq 0$,

$$E \langle Y, \psi \rangle^n = C^n E \left(\sum_{j=1}^{p} \theta_j \xi_{t_j} \right)^n, \tag{6.1}$$

where ψ has the form (2.3). It is known (see, e.g., [30], Thm II.12.6) that

$$E \left(\sum_{j=1}^{p} \theta_j \xi_{t_j} \right)^n = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} \kappa_{\#B}, \tag{6.2}$$

where κ's are the corresponding cumulants given by (2.5) and $\mathcal{P}(n)$ is the set of all partitions of $\{1, \ldots, n\}$, and # denotes cardinality of a set.

To compute $E \langle Y, \psi \rangle^n$, we will need the formulas for moments of the Wick product $: X \otimes X :$. These moments are expressed with the help of Feynman graphs (see, e.g., [3, p. 1,085] or [31, p. 422]). For fixed n, we consider graphs as follows. Suppose that we have n numbered vertices. Each vertex has two legs numbered 1 and 2. Legs are paired, forming links between vertices, in such a way that each link connects two different vertices, and there are no unpaired legs left. The graph is a set of links. Each link is described by an unordered pair $\{(i, j), (l, m)\}$, $i, l \in \{1, \ldots n\}$, $j, m \in \{1, 2\}$, which means that leg j, growing from vertex i is paired with leg m growing from vertex l. $i \neq l$ since each link connects different vertices, and any (i, j), $i = 1, \ldots, n$, $j = 1, 2$, is a part of one and only one link. The set of all distinct graphs of the above form will be denoted by $\mathcal{G}_n^2$. Let $\tilde{\mathcal{G}}_n^2$ denote the set of all connected graphs in $\mathcal{G}_n^2$.

By formulas (2.7) and (6.10) in [31], we have

$$E\langle : X \otimes X :, \Phi \rangle^n = \sum_{G \in \mathcal{G}_n^2} I^G(\Phi), \qquad \Phi \in \mathcal{S}(\mathbb{R}^{2d}), \tag{6.3}$$

where

$$I^G(\Phi) = \int_{\mathbb{R}^{2n}} \hat{\Phi}(x_{1,1}, x_{1,2}) \dots \hat{\Phi}(x_{n,1}, x_{n,2}) \prod_{\{(l,m),(p,q)\} \in G} \delta_{-x_{p,q}}(\mathrm{d}x_{l,m}) |x_{p,q}|^{-H} \, dx_{p,q}. \tag{6.4}$$

Using Lemma 3.1 and similar arguments as in the proof of Lemma 3.3 below, it is not difficult to see that

$$E\langle Y, \psi \rangle^n = \lim_{\varepsilon \to 0} \sum_{G \in \mathcal{G}_n^2} I^G(\Phi_{\varepsilon,\psi}^f), \tag{6.5}$$

where $\Phi_{\varepsilon,\psi}^f$ is given by (3.2).

For $G \in \tilde{\mathcal{G}}_n^2$ we have

$$I^G(\Phi_{\varepsilon,\psi}^f) = \int_{\mathbb{R}^n} \hat{\psi}(x_1 - x_2) \hat{\psi}(x_2 - x_3) \dots \hat{\psi}(x_n - x_1)$$

$$\times F_\varepsilon^G(x_1, \dots, x_n) |x_1|^{-H} \dots |x_n|^{-H} \, dx_1 \dots dx_n,$$

where F_ε^G is a product of functions of the form $\hat{f}_\varepsilon(x_i)$ or $\overline{\hat{f}_\varepsilon(x_i)}$. By (2.16) and the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} I^G(\Phi_{\varepsilon,\psi}^f) = J_n$$

$$:= \int_{\mathbb{R}^n} \hat{\psi}(x_1 - x_2) \hat{\psi}(x_2 - x_3) \dots \hat{\psi}(x_n - x_1) |x_1|^{-H} \dots |x_n|^{-H} \, dx_1 \dots dx_n. \tag{6.6}$$

In the proof of the integrability of the function under the integral in (6.6), we use $|\hat{\psi}(x_1 - x_2)\hat{\psi}(x_n - x_1)| \le |\hat{\psi}(x_1 - x_2)|^2 + |\hat{\psi}(x_n - x_1)|^2$,

$$|\widehat{\psi}(x)| \le \frac{C}{1 + |x|}, \tag{6.7}$$

and

$$\int_{\mathbb{R}} \frac{1}{1 + |x - y|} |y|^{-H} \, \mathrm{d}y \le C_1.$$

In the general case, if $G \in \mathcal{G}_n^2$, then it has a decomposition of the form $G = G_1 \cup \dots \cup G_k$, where G_j are connected components of G. Then by (6.4) and (6.6), we have

$$\lim_{\varepsilon \to 0} I^G(\Phi_{\varepsilon,\psi}^f) = J_{\#G_1} \dots J_{\#G_k}, \tag{6.8}$$

where $\#G_j$ is the number of vertices of G_j.

Note that each $G \in \mathcal{G}_n^2$ determines a partition $\pi_G \in \mathcal{P}(n)$, whose elements are the sets of vertices of connected components of G. Hence, by (6.5) and (6.8),

$$E \langle Y, \psi \rangle^n = \sum_{G \in \mathcal{G}_n^2} \prod_{B \in \pi_G} J_{\#B}.$$

It is not difficult to see that if $\pi \in \mathcal{P}(n)$ is of the form $\pi = \{B_1, \ldots, B_k\}$, $B_j \geq 2$, $j = 1, \ldots, k$, then the number of different $G \in \mathcal{G}_n^2$ such that $\pi_G = \pi$ is equal to

$$\prod_{j=1}^k 2^{\#B_j - 1} (\#B_j - 1)!$$

Therefore, setting $J_1 = 0$, we obtain

$$E \langle Y, \psi \rangle^n = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} 2^{\#B - 1} (\#B - 1)! J_{\#B}. \tag{6.9}$$

J_k given by (6.6) can be also written as

$$J_k = C^k \int_{\mathbb{R}^k} \psi(x_1) \ldots \psi(x_k) |x_1 - x_2|^{H-1} |x_2 - x_3|^{H-1} \ldots |x_k - x_1|^{H-1} dx_1 \ldots dx_k,$$

hence combining (6.9) with (2.2), (2.5), and (6.2), we obtain (6.1).   □

Proof of lemma 3.3 To prove part (a), it suffices to show that for any $f, g \in \mathcal{F}$,

$$E \langle \Lambda_\varepsilon^f(x + \rho^1, y + \rho^2; T), \psi \rangle \langle \Lambda_\delta^g(x + \rho^1, y + \rho^2; T), \psi \rangle \tag{6.10}$$

has a finite limit as $\varepsilon, \delta \to 0$. Analogously as in (7.1)-(7.4) of [11], (6.10) is equal to

$$\frac{1}{(2\pi)^4} \int_{[0,T]^4} \int_{\mathbb{R}^4} e^{-ix(z+z')} e^{-iy(w+w')} \widehat{\psi}(z + w) \widehat{\psi}(z' + w') \widehat{f}(\varepsilon w) \widehat{g}(\delta w')$$

$$\times \overline{\widehat{\nu_{s,u}}(z, z') \widehat{\nu_{r,v}}(w, w')} dz dz' dw dw' ds du dr dv,$$

where $\nu_{s,u}$ is the law of (ρ_s^1, ρ_u^1). To complete the proof of part (a), it suffices to show that

$$I := \int_{\mathbb{R}^4} |\widehat{\psi}(z + w)| |\widehat{\psi}(z' + w')| \int_{[0,T]^2} |\widehat{\nu_{s,u}}(z, z')| ds \, du$$

$$\times \int_{[0,T]^2} |\widehat{\nu_{r,v}}(w, w')| dr dv dz dz' dw dw' < \infty.$$

$$\tag{6.11}$$

To derive this, we cannot repeat the argument of [11] because $\widehat{\psi} \notin L^1$ for ψ of the form (3.6), we only have (6.7).

Fix $\gamma > 0$ such that $\frac{1}{2} + 4\gamma < \alpha$. It is easy to see that

$$
\int_{[0,T]^2} |\widehat{v}_{s,u}(z, z')| ds du \leq \frac{C(T)}{1 + |z + z'|^\alpha} \left(\frac{1}{1 + |z|^\alpha} + \frac{1}{1 + |z'|^\alpha} \right)
$$

$$
\leq C_1(T) h_\gamma(z, z') \frac{1}{1 + |z|^\gamma} \frac{1}{1 + |z'|^\gamma}, \tag{6.12}
$$

where

$$
h_\gamma(z, z') = \frac{1}{1 + |z + z'|^{\frac{1}{2}+\gamma}} \left(\frac{1}{1 + |z|^{\frac{1}{2}+\gamma}} + \frac{1}{1 + |z'|^{\frac{1}{2}+\gamma}} \right).
$$

We have used

$$
\frac{1}{1 + |z + z'|^\gamma} \leq C \frac{1 + |z|^\gamma}{1 + |z'|^\gamma}. \tag{6.13}
$$

Using $h_\gamma(z, z') h_\gamma(w, w') \leq h_\gamma^2(z, z') + h_\gamma^2(w, w')$, (6.12) and symmetry, we obtain

$$
I \leq C_2(T) \int_{\mathbb{R}^4} \frac{|\widehat{\psi}(z + w)| \, |\widehat{\psi}(z' + w')|}{(1 + |z|^\gamma)(1 + |w|^\gamma)(1 + |z'|^\gamma)(1 + |w'|^\gamma)} h_\gamma^2(z, z') dz dz' dw dw',
$$

and (6.13) permits to replace the denominator by $(1 + |z + w|^\gamma)(1 + |z' + w'|^\gamma)$, hence (6.11) follows by (6.7).

Note that we have also shown that

$$
E \langle \Lambda(x + \rho^1, y + \rho^2; T), \psi \rangle^2 = \frac{1}{(2\pi)^4} \int_{[0,T]^4} \int_{\mathbb{R}^4} e^{-ix(z+z')} e^{-iy(w+w')}
$$

$$
\times \widehat{\psi}(z + w) \widehat{\psi}(z' + w') \overline{\widehat{v}_{s,u}(z, z') \widehat{v}_{r,v}(w, w')} dz dz' dw dw' ds du dr dv. \tag{6.14}
$$

To prove part (b), we observe that the argument above can be carried out for linear combinations of functions of the form (3.6) and from $\mathcal{S}$ instead of ψ. Hence, we see that (6.14) holds for $\psi - \psi * f_\varepsilon$, since

$$
(\widehat{\psi - \psi * f_\varepsilon})(x) = \widehat{\psi}(x)(1 - \widehat{f}(\varepsilon x)) \tag{6.15}
$$

Then (b) follows from (6.11).

The proof of part (c) is the same as that of Proposition 4.4 in [11]. Only the fact that $\psi \in L^2$ is needed here. □

Remark 6.1 From the proof of part (c), it follows that

$$E \int_{\mathbb{R}^2} (\langle \Lambda_\varepsilon^f (x + \rho^1, y + \rho^2; T), \psi \rangle - \langle \Lambda(x + \rho^1, y + \rho^2; T), \psi \rangle)^2 dxdy$$

$$= \frac{1}{(2\pi)^2} \int_{[0,T]^4} \int_{\mathbb{R}^2} |\widehat{\psi}(x + y)|^2 |\widehat{f}(\varepsilon y) - 1|^2 e^{-|s-u||x|^\alpha} e^{-|r-v||y|^\alpha} dxdydsdrdudv,$$

(6.16)

and

$$E \int_{\mathbb{R}^2} (\langle \Lambda(x + \rho^1, y + \rho^2; T), \psi \rangle^2 dxdy$$

$$= \frac{1}{(2\pi)^2} \int_{[0,T]^4} \int_{\mathbb{R}^2} |\widehat{\psi}(x + y)|^2 e^{-|s-u||x|^\alpha} e^{-|r-v||y|^\alpha} dxdydsdrdudv. \qquad (6.17)$$

We need the following lemma which can be proved by repeating the argument of the proof of Lemma 4.1 in [11]

Lemma 6.2 *For any* $F(\cdot + \rho^1, \cdot + \rho^2) \in L^2(\mathbb{R}^2 \times \Omega, \lambda \otimes \lambda \otimes P)$ *the series*

$$\sum_{\substack{j,k \\ j \neq k}} \sigma_j \sigma_k F(x_j + \rho^j, x_k + \rho^k)$$

converges in $L^2(\Omega)$, *and*

$$E \left(\sum_{\substack{j,k \\ j \notin k}} \sigma_j \sigma_k F(x_j + \rho^j, x_k + \rho^k) \right)^2 = \int_{\mathbb{R}^2} E(F^2(x + \rho^1, y + \rho^2)$$

$$+ F(x + \rho^1, y + \rho^2) F(y + \rho^1, x + \rho^2))dxdy. \qquad (6.18)$$

Proof of lemma 3.4 From Lemma 3.3, it follows that $\langle \Lambda(x_j + \rho^j, x_k + \rho^k; T), \mathbf{1}_{[0,t]} \rangle$ are well defined, and $\langle \Lambda(\cdot + \rho^j, \cdot + \rho^k; T), \mathbf{1}_{[0,t]} \rangle$ belongs to $L^2(\mathbb{R}^2 \times \Omega, \lambda \otimes \lambda \otimes P)$, hence by Lemma 6.2 the process ξ^T is well defined and the series in (1.1) converges in $L^2(\Omega)$. Moreover, using the fact that $\Lambda(x + \rho^j, y + \rho^k, T) = \Lambda(y + \rho^k, x + \rho^j; T)$ (see Corollary 3.4 in [11]), we have

$$E\left(\sum_{\substack{j,k \\ j\neq k}}\sigma_j\sigma_k\langle\Lambda(x_j+\rho^j,x_k+\rho^k;T),\psi\rangle\right)^2 = 2\int_{\mathbb{R}^2}E\langle\Lambda(x+\rho^1,y+\rho^2;T),\psi\rangle^2 dxdy,$$

(6.19)

for ψ either of the form (3.6), or $\psi = \psi_1 + \varphi$, ψ_1 of the form (3.6), $\varphi \in \mathcal{S}$. Hence, for $t_2 > t_1 \geq 0$, by (1.1), (6.19) and (6.17),

$$E\left(\xi_{t_2}^T - \xi_{t_1}^T\right)^2$$
$$= \frac{1}{2\pi^2 T^2}\int_{[0,T]^4}\int_{\mathbb{R}^2}\left|\widehat{\mathbb{1}_{(t_1,t_2]}}(x+y)\right|^2 e^{-|s-u||x|^\alpha}e^{-|r-v||y|^\alpha}dxdydsdrdudv.$$

Using $1 > \alpha > \frac{1}{2}$ and

$$\frac{1}{T}\int_0^T\int_0^T e^{-|s-r||x|^\alpha}dsdr \leq \frac{2}{|x|^\alpha}$$

(6.20)

we obtain

$$E(\xi_{t_2}^T - \xi_{t_1}^T)^2 \leq C\int_{\mathbb{R}^2}\frac{|e^{i(t_2-t_1)(x+y)}-1|^2}{|x+y|^2}|x|^{-\alpha}|y|^{-\alpha}dxdy$$
$$\leq C_1|t_2-t_1|^{2\alpha}.$$

(6.21)

Hence, ξ^T has a continuous modification.   □

Before we pass to the proof of Theorem 3.5, we observe that for any $\mathcal{S}'$-random variable Z (not necessarily Gaussian) such that $\mathcal{S} \ni \varphi \mapsto E\langle Z,\varphi\rangle^2$ is continuous, the Wick product $: Z \otimes Z :$ is well defined by an extension of (3.1). Moreover, we have the following lemma.

Lemma 6.3 *Let $(Z_T)_{T\geq 1}$ be a family of $\mathcal{S}'$-random variables such that*

$$\sup_{T\geq 1} E\langle Z_T,\varphi\rangle^2 \leq p^2(\varphi), \quad \varphi \in S,$$

form some continuous Hilbertian seminorm p on $\mathcal{S}$. Assume that $Z_T \Rightarrow Z$ and $E\langle Z_T,\varphi\rangle^2 \to E\langle Z,\varphi\rangle^2, \varphi \in \mathcal{S}$, as $T \to \infty$. Then $: Z_T \otimes Z_T :, : Z \otimes Z :$ are well defined and $: Z_T \otimes Z_T :\Rightarrow: Z \otimes Z :$ in $\mathcal{S}'(\mathbb{R}^2)$ as $T \to \infty$.

This lemma follows by a standard argument using properties of $\mathcal{S}$ [37], so we skip the proof.

Lemma 6.3 together with Proposition 2.1(a) implies the following corollary.

Corollary 6.4 *Let X_T, X be as in Proposition 2.1. Then*

$$: X_T \otimes X_T :\Rightarrow: X \otimes X : \text{ in } \mathcal{S}'(\mathbb{R}^2) \text{ as } T \to \infty.$$

(6.22)

Indeed, it suffices to observe that by (2.14) and the Poisson initial condition, we have

$$E\langle X_T, \varphi\rangle^2 = \frac{1}{T}\int_{\mathbb{R}}\int_0^T\int_0^T E\varphi(x+\rho_s)\varphi(x+\rho_u)\,duds\,dx$$

$$= \frac{1}{2\pi}\frac{1}{T}\int_{\mathbb{R}}\int_0^T\int_0^T |\widehat{\varphi}(x)|^2 \mathrm{e}^{-|s-u||x|^\alpha}\,duds\,dx,$$

so the assumptions of Lemma 6.3 are satisfied (we use (6.20)).

Proof of Theorem 3.5 For ψ of the form (3.6), we denote by ξ_ψ^T the random variable defined by (1.1) with $\mathbf{1}_{[0,t]}$ replaced by ψ.

To prove the theorem it suffices to show that

$$\lim_{T\to\infty}|E\mathrm{e}^{i\xi_\psi^T} - E\mathrm{e}^{i\langle Y,\psi\rangle}| = 0 \qquad (6.23)$$

for any ψ of the form (3.6). Indeed, from (6.23) and Theorem 3.2, we infer convergence of finite-dimensional distributions, and from (6.21), we obtain tightness in $C([0,\tau])$ for each $\tau > 0$ (see [6], Thm. 12.3; note that the constant C_1 in (6.21) does not depend on T).

Fix any $f \in \mathcal{F}$ and denote $\psi_\delta = \psi * f_\delta$, $\delta > 0$, where f_δ is given by (2.15). Let

$$\xi_\varphi^{T,\varepsilon,f} = \frac{1}{T}\sum_{\substack{j,k \\ j\neq k}}\sigma_j\sigma_k\langle \Lambda_\varepsilon^f(x_j+\rho^j, x_k+\rho^k; T), \varphi\rangle, \quad \varepsilon > 0, \varphi \in \mathcal{S}, \qquad (6.24)$$

which is well defined by Lemma 6.2.

Using the estimate $|E\mathrm{e}^{i\eta_1} - E\mathrm{e}^{i\eta_2}| \leq \frac{1}{2}E|\eta_1 - \eta_2|^2$, valid for centered random variables η_1, η_2, it is easy to see that (6.23) will be proved if we show

$$\lim_{\delta\to 0}\sup_{T\geq 1} E|\xi_\psi^T - \xi_{\psi_\delta}^T|^2 = 0, \qquad (6.25)$$

$$\lim_{\varepsilon\to 0}\sup_{T\geq 1}\sup_{0<\delta\leq 1} E|\xi_{\psi_\delta}^T - \xi_{\psi_\delta}^{T,\varepsilon,f}|^2 = 0, \qquad (6.26)$$

$$\lim_{T\to\infty}\sup_{0<\delta\leq 1} E|\langle :X_T\otimes X_T:, \Phi_{\varepsilon,\psi_\delta}^f\rangle - \xi_{\psi_\delta}^{T,\varepsilon,f}| = 0, \quad \varepsilon > 0, \qquad (6.27)$$

$$\langle :X_T\otimes X_T:, \Phi_{\varepsilon,\psi_\delta}^f\rangle \Rightarrow \langle :X\otimes X:, \Phi_{\varepsilon,\psi_\delta}^f\rangle \text{ as } T\to\infty, \quad \varepsilon > 0, \delta > 0, \qquad (6.28)$$

$$\lim_{\varepsilon\to 0}\sup_{0<\delta\leq 1} E|\langle :X\otimes X:, \Phi_{\varepsilon,\psi_\delta}^f\rangle - \langle Y,\psi_\delta\rangle|^2 = 0, \qquad (6.29)$$

$$\lim_{\delta\to 0} E|\langle Y,\psi_\delta\rangle - \langle Y,\psi\rangle|^2 = 0. \qquad (6.30)$$

Using (6.18), (6.17), (6.15) and then (6.20), we have

$$
\begin{aligned}
E|\xi_\psi^T - \xi_{\psi_\delta}^T|^2 &= \frac{2}{T^2} \int\limits_{[0,T]^4} \int\limits_{\mathbb{R}^2} |\widehat{\psi}(x+y)|^2 |1 - \widehat{f}(\delta(x+y))|^2 \\
&\quad \times e^{-|s-u||x|^\alpha} e^{-|r-v||y|^\alpha} dxdydsdudrdv \\
&\leq 8 \int\limits_{\mathbb{R}^2} |\widehat{\psi}(x+y)|^2 |1 - \widehat{f}(\delta(x+y))|^2 |x|^{-\alpha} |y|^{-\alpha} dxdy. \quad (6.31)
\end{aligned}
$$

Hence, (6.25) follows by (2.16) and since $\frac{1}{2} < \alpha < 1$.

Next, we apply Lemma 6.2 to

$$
F(x+\rho^1, y+\rho^2) = \langle \Lambda(x+\rho^1, y+\rho^2; T) - \Lambda_\varepsilon^f(x+\rho^1, y+\rho^2; T), \psi_\delta \rangle,
$$

and by (6.18) we obtain

$$
\begin{aligned}
E|\xi_{\psi_\delta}^T - \xi_{\psi_\delta}^{T,\varepsilon,f}|^2 &\leq \frac{2}{T^2} \int\limits_{\mathbb{R}^2} E(\langle (\Lambda - \Lambda_\varepsilon^f)(x+\rho^1, y+\rho^2; T), \psi_\delta \rangle)^2 dxdy \\
&\leq \frac{C}{T^2} \int\limits_{[0,T]^4} \int\limits_{\mathbb{R}^2} |\widehat{\psi}(x+y)|^2 |\widehat{f}(\varepsilon y) - 1|^2 e^{-|s-u||x|^\alpha} e^{-|r-v||y|^\alpha} dxdydsdrdudv,
\end{aligned}
$$

by (6.16) and (2.16). By (6.20), (6.7) and (2.16) we obtain (6.26).

To prove (6.27), observe that random variables almost identical to $\xi_{\psi_\delta}^{T,\varepsilon,f}$ and $\langle :X_T \otimes X_T :, \Phi_{\varepsilon,\psi_\delta}^f \rangle$ have already appeared in [11] with different notation (and different scaling). Hence, by (4.1), (8.8), (8.14) and the two subsequent formulas in [11], we have

$$
\begin{aligned}
A(T,\varepsilon,\delta) &:= E|\langle :X_T \otimes X_T :, \Phi_{\varepsilon,\psi_\delta}^f \rangle - \xi_{\psi_\delta}^{T,\varepsilon,f}|^2 \\
&= \frac{1}{T^2} \int\limits_{[0,T]^4} \int\limits_{\mathbb{R}} E\Phi_{\varepsilon,\psi_\delta}^f(x+\rho_s, x+\rho_u) \Phi_{\varepsilon,\psi_\delta}^f(x+\rho_r, x+\rho_v) dxdvdrduds.
\end{aligned}
$$

$$(6.32)$$

It is easy to see that since the support of ψ_δ is contained in a compact set which is independent of δ, we have (see (3.2) and (3.6))

$$
|\Phi_{\varepsilon,\psi_\delta}^f(x,y)| \leq C(\varepsilon,f)\phi(x)\phi(y), \quad (6.33)
$$

where

$$
\phi(x) = \frac{1}{1+|x|^2}.
$$

Let $(\mathcal{T}_t)_t$ denote the α-stable semigroup and G its potential, $G\varphi = \int_0^\infty \mathcal{T}_t \varphi dt$. From (6.32), (6.33), and the Markov property, we get

$$A(T, \varepsilon, \delta)$$

$$\leq 4! C^2(\varepsilon, f) \frac{1}{T^2} \int_\mathbb{R} \int_0^T \int_s^T \int_u^T \int_r^T \mathcal{T}_s(\phi(\mathcal{T}_{u-s}\phi(\mathcal{T}_{r-u}(\phi\mathcal{T}_{v-r}\phi))))(x) dx dv dr du ds$$

$$\leq C_1(\varepsilon, f) \frac{1}{T^2} \int_0^T \int_\mathbb{R} \phi(x) G(\phi G(\phi G\phi))(x) dx ds$$

$$\leq C_2(\varepsilon, f) \frac{1}{T}.$$

In the last estimate, we used the fact that $G\phi$ is bounded. Hence, (6.27) follows.

(6.28) follows from (6.22) since $\Phi_{\varepsilon,\varphi_\delta}^f \in \mathcal{S}$.

In the proof of (6.29), we use (3.3) and Lemma 3.1, which yield

$$E|\langle : X \otimes X :, \Phi_{\varepsilon,\psi_\delta}^f \rangle - \langle Y, \psi_\delta \rangle|^2$$

$$= \frac{1}{\pi^2} \int_{\mathbb{R}^2} |\widehat{\psi}_\delta(x + y)|^2 |\widehat{f}(\varepsilon y) - 1|^2 |x|^{-\alpha} |y|^{-\alpha} dx dy$$

$$+ \frac{1}{\pi^2} \int_{\mathbb{R}^2} |\widehat{\psi}_\delta(x + y)|^2 (\widehat{f}(\varepsilon y) - 1)(\overline{\widehat{f}(\varepsilon x) - 1}) |x|^{-\alpha} |y|^{-\alpha} dx dy$$

$$\leq \frac{2}{\pi^2} \int_{\mathbb{R}^2} |\widehat{\psi}(x + y)|^2 |\widehat{f}(\varepsilon y) - 1|^2 |x|^{-\alpha} |y|^{-\alpha} dx dy.$$

Hence, (6.29) follows by (6.7) and (2.16).

Finally, from (3.3) and Lemma 3.1, it is easy to see that

$$E|\langle Y, \psi_\delta \rangle - \langle Y, \psi \rangle|^2 = \frac{2}{\pi^2} \int_\mathbb{R} |\widehat{\psi}(x + y)|^2 |\widehat{f}(\delta(x + y)) - 1|^2 |x|^{-\alpha} |y|^{-\alpha} dx dy,$$

which tends to 0 as $\delta \to 0$ (cf (6.31)). □

Proof of Theorem 4.1 Fix $0 \leq s < t$. By (2.1) and (2.2), we know that

$$E e^{i[z_1(\xi_t - \xi_s) + z_2(\xi_{t+\tau} - \xi_{s-\tau})]} = \exp\left\{ \frac{1}{2} \sum_{k=2}^\infty \frac{(2i\sigma)^k}{k} \widetilde{R}_k(z_1, z_2, s, t, \tau) \right\}, \quad (6.34)$$

where $\widetilde{R}_k(z_1, z_2, s, t, T)$ is the right-hand side of (2.2) with

$$\psi(x) = z_1 \mathbb{1}_{(s,t]} + z_2 \mathbb{1}_{(s+\tau,t+\tau]}, \tag{6.35}$$

and formula (6.34) holds for $z_1, z_2 \in \mathbb{R}$ and $\tau > 0$ such that the series in (6.34) converges.

To continue with the proof, we need the following lemma and corollary.

Lemma 6.5 *There exists $C(s,t) > 0$ independent of τ such that*

$$|\widetilde{R}_k(z_1, z_2, s, t, \tau)| \le (|z_1| + |z_2|)^k (C(s,t))^k, \quad k = 2, 3, \ldots. \tag{6.36}$$

An immediate consequence of this lemma is

Corollary 6.6 *For all $0 < s < t$ there exists a neighborhood $U(s,t)$ of 0 in $\mathbb{R}^2$ such that (6.34) holds for $(z_1, z_2) \in U(s,t)$ and all $\tau > 0$.*

Proof of lemma 6.5 Consider an integral

$$I_k(\varepsilon_1, \ldots, \varepsilon_k) = \int\limits_{s+\varepsilon_1\tau}^{t+\varepsilon_1\tau} \cdots \int\limits_{s+\varepsilon_k\tau}^{t+\varepsilon_k\tau} |x_1 - x_2|^{H-1} \ldots |x_{k-1} - x_k|^{H-1} |x_k - x_1|^{H-1} dx_1 \ldots dx_k, \tag{6.37}$$

where $(\varepsilon_1, \ldots, \varepsilon_k) \in \{0, 1\}^k$. For each j, we substitute $x'_j = x_j - \varepsilon_j \tau$ and use

$$|y - x + \tau|^{H-1} \le |y - x|^{H-1} \quad \text{for} \quad x, y \in [s, t], \tau > \tau_0(s, t).$$

Hence, similarly as in formula (13) in [36],

$$I_k(\varepsilon_1, \ldots, \varepsilon_k) \le (C(s,t))^k. \tag{6.38}$$

The same inequality holds for $\tau \le \tau_0(s, t)$.

By (6.35), (2.2),

$$\widetilde{R}_k(z_1, z_2, s, t, \tau) = z_1^k I_k(0, \ldots, 0) + \widetilde{\widetilde{R}}_k(z_1, z_2, s, t, \tau) + z_2^k I_k(0, \ldots, 0), \tag{6.39}$$

where

$$\widetilde{\widetilde{R}}_k(z_1, z_2, s, t, \tau) = \sum_{j=1}^{k-1} z_1^{k-j} z_2^j \sum_{\substack{(\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k \\ \varepsilon_1 + \ldots + \varepsilon_k = j}} I_k(\varepsilon_1, \ldots, \varepsilon_k). \tag{6.40}$$

Hence, (6.38) implies

$$|\widetilde{R}_k(z_1, z_2, s, t, \tau)| \le \sum_{j=1}^{k} \binom{k}{j} |z_1|^{k-j} |z_2|^j (C(s,t))^k = (|z_1| + |z_2|)^k (C(s,t))^k, \tag{6.41}$$

which proves the lemma. □

We return to the proof of the theorem.

Let $U(s,t)$ be as in Corollary 6.6. It is now clear that for (z_1, z_2) in $U(s,t)$, the function $D_\tau^\xi(z_1, z_2, s, t)$ given by (4.3) is well defined for all $\tau > 0$.

Moreover, by (4.3), (4.2), (6.34) and (6.39),

$$D_\tau^\xi(z_1, z_2, s, t) = \left| \frac{1}{2} \sum_{k=2}^{\infty} \frac{(2i\sigma)^k}{k} \widetilde{\widetilde{R}}_k(z_1, z_2, s, t, \tau) \right| \tag{6.42}$$

It is not difficult to see that due to the circular form of the integrand in each I_k in (6.40), after the substitution $x_j' = x_j - \varepsilon_j \tau$ (see (6.37)), there appear at least two factors of the form $|y - x + \tau|^{H-1}$, $y, x \in [s,t]$, and for some $(\varepsilon_1, \ldots, \varepsilon_k)$, there are exactly two such factors. Hence, for each $\gamma < 2 - 2H$,

$$\lim_{\tau \to \infty} \tau^\gamma \widetilde{\widetilde{R}}_k(z_1, z_2, s, t, \tau) = 0,$$

and for some z_1, z_2

$$\lim_{\tau \to \infty} \tau^{2-2H} \widetilde{\widetilde{R}}_u(z_1, z_2, s, t, \tau) \neq 0.$$

Hence, the theorem follows by Lemma 6.5.

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