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Isometric deformations of surfaces preserving the third fundamental form

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Abstract We study the following problem: To what extend is a surface in the Euclidean space $\mathbb{R}^4$ determined by the third fundamental form? We prove the existence of families of surfaces in $\mathbb{R}^4$ which allow isometric deformations with isometric but not congruent Gaussian images. In particular, we provide a method which gives locally all surfaces in $\mathbb{R}^4$ with conformal Gauss map that allow such deformations. As a consequence, we have a way for constructing non-spherical pseudoumbilical surfaces in $\mathbb{R}^4$.

Keywords Third fundamental form · Isometric deformations

Mathematics Subject Classification (2000) Primary 53A05; Secondary 53C40 · 53C42

1 Introduction

There has been much interest in classifying submanifolds in terms of given geometric data. Since the beginning of differential geometry the Gauss map has played a major role in surface theory. A natural generalization of this classical map for an isometric immersion $f : M^n \to \mathbb{R}^m$ of an n-dimensional Riemannian manifold M^n into the Euclidean m-space $\mathbb{R}^m$ is defined by assigning to each point $x \in M^n$ the space $df_x(T_xM^n)$ in the Grassmannian $G_{n,m}$ of n-planes in $\mathbb{R}^m$. The Gauss map $g : M^n \to G_{n,m}$ defined in this way has been extensively studied. In particular, Dajczer and Gromoll [4] raised the following question: *To what extend is a Euclidean submanifold determined by its Gauss map?*

In [4], they gave a complete answer to the above question by proving that the isometric immersions with congruent Gauss maps can be described locally in terms of circular immersions. Relevant results were obtained by Hoffman and Osserman [10],

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while Dajczer and Vergasta [6,15] studied the above question for conformal immersions.

A geometric invariant that figures in the study of submanifolds is the so-called *third fundamental form III* which is the pull back of the canonical metric on the Grassmannian $G_{n,m}$ via g (cf. [13]). In an attempt to generalize their theory, Dajczer and Gromoll [5] treated the following question: *To what extend is a Euclidean submanifold determined by its third fundamental form?*

In [5], they answered this question for the codimension-1 case (for the conformal case see [16]). They proved that if $f, \widetilde{f} : M^n \to \mathbb{R}^{n+1}$ are non-congruent hypersurfaces with the same third fundamental form, then f is minimal and $\widetilde{f}$ belongs locally to the associated family of f.

In this paper we initiate the study of the case of higher codimension. It turns out that additional interesting phenomena arise in this case.

The results in [5] motivate the study of *III-deformations*, a notion that may be viewed as an generalization of the associated family of minimal surfaces. A *III*-deformation of an isometric immersion $f : M^n \to \mathbb{R}^m$ is a one-parameter family of isometric immersions $f_t : M^n \to \mathbb{R}^m, t \in (-\varepsilon, \varepsilon), \varepsilon > 0$, having the same third fundamental form and satisfying $f_0 = f$. The deformation is *trivial* if for each t there exists a rigid motion τ_t such that $f_t = \tau_t \circ f$. An immersion f is called *III-deformable* if there exists a non-trivial *III*-deformation; f is said to be *locally III- deformable* if each point of M^n has a neighborhood U such that f is *III*-deformable on U.

It turns out that *III*-deformations preserve the mean curvature. So *III*-deformable immersions may be viewed as an extension of Bonnet surfaces in $\mathbb{R}^3$, i.e., surfaces that allow non-trivial deformations which preserve the mean curvature, or equivalently the principal curvatures (cf. [3,11]).

The aim of this paper is to study locally *III*-deformable surfaces in $\mathbb{R}^4$. Obviously, non-totally geodesic minimal surfaces in $\mathbb{R}^4$ or in a 3-sphere are locally *III*-deformable, and the *III*-deformations are given by the associated family. As opposed to the codimension-1 case, we prove the existence of families of locally *III*-deformable surfaces in $\mathbb{R}^4$ which are neither minimal nor minimal in a 3-sphere. By a result in [10], these locally *III*-deformable surfaces do not allow isometric deformations with congruent Gaussian images.

In fact, we show that surfaces lying fully in $\mathbb{R}^4$ are locally *III*-deformable if the first normal space is one-dimensional, i.e., if the Gauss map is not regular.

Furthermore, we provide a method that gives all locally *III*-deformable pseudoumbilical surfaces in $\mathbb{R}^4$. To the best of our knowledge, this method yields the first examples of pseudoumbilical surfaces which are neither minimal in $\mathbb{R}^4$ nor minimal in 3-spheres. The classification is based on the fact that a certain quadratic differential is holomorphic for locally *III*-deformable pseudoumbilical surfaces in $\mathbb{R}^4$. It is worth mentioning that such surfaces are isothermic surfaces in the sense of Palmer [14].

Moreover, we obtain some global results for locally *III*-deformable surfaces, and provide examples that justify the necessity of global assumptions.

2 Preliminaries

Let M^2 be an oriented two-dimensional Riemannian manifold and $f : M^2 \to \mathbb{R}^4$ an isometric immersion equipped with the induced metric $\langle , \rangle$. In the sequel, all manifolds under consideration are assumed to be connected. The normal bundle of f carries an orientation which is the one induced by that of $\mathbb{R}^4$. The second fundamental form B of f is given by the Gauss formula

$$\overline{\nabla}_X df(Y) = df(\nabla_X Y) + B(X,Y),$$

where $\overline{\nabla}$ is the connection of the induced bundle $f^*(T\mathbb{R}^4)$ arising from the usual connection in $\mathbb{R}^4$, ∇ is the Levi–Civitá connection of M^2 and X, Y are arbitrary tangent vector fields. The shape operator A_ξ associated with a section ξ of the normal bundle of f is given by the Weingarten formula

$$\overline{\nabla}_X \xi = -df(A_\xi X) + \nabla^\perp_X \xi,$$

where $\nabla^\perp$ stands for the normal connection and X is a tangent vector field.

Let $\{e_1, e_2, e_3, e_4\}$ be a local orthonormal frame field such that e_1, e_2 are tangent to M^2 and e_3, e_4 are normal to f. The dual forms ω_j and the connection forms ω_{jk} satisfy the structure equations

$$\begin{aligned} d\omega_j &= \sum_{k=1,2} \omega_{jk} \wedge \omega_k, \quad j = 1,2, \\ d\omega_{jk} &= \sum_{l=1}^{4} \omega_{jl} \wedge \omega_{lk}, \quad 1 \le j,k \le 4. \end{aligned} \tag{2.1}$$

The shape operators A_3, A_4 with respect to e_3, e_4 are given by

$$A_3 = \omega_{13} \otimes e_1 + \omega_{23} \otimes e_2, \quad A_4 = \omega_{14} \otimes e_1 + \omega_{24} \otimes e_2.$$

The Gaussian curvature K and the normal curvature K_n are defined, respectively, by

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 \quad \text{and} \quad d\omega_{34} = -K_n \omega_1 \wedge \omega_2.$$

Then the structure equations imply that

$$K = \det A_3 + \det A_4, \tag{2.2}$$

$$K_n = -\langle [A_3, A_4]\, e_1, e_2 \rangle. \tag{2.3}$$

Moreover, the mean curvature vector field is defined by

$$\overrightarrow{H} = \frac{1}{2}(\mathrm{trace} A_3)\, e_3 + \frac{1}{2}(\mathrm{trace} A_4)\, e_4.$$

The mean curvature H is given by $H := |\overrightarrow{H}|$. The first normal space N_1 of f at a point $x \in M^2$ is the vector space generated by the set $\{B(X,Y) | X, Y \in T_x M^2\}$.

According to Obata [13], the third fundamental form of f is given by

$$III(X,Y) = \left\langle A_3^2 X, Y \right\rangle + \left\langle A_4^2 X, Y \right\rangle, \tag{2.4}$$

or equivalently

$$III(X,Y) = 2\left\langle A_{\overrightarrow{H}} X, Y \right\rangle - K \langle X, Y \rangle, \tag{2.5}$$

where $A_{\overrightarrow{H}}$ is the shape operator associated with $\overrightarrow{H}$ and X, Y are arbitrary tangent vector fields of M^2.

The immersion f is called pseudoumbilical if there exists a function λ such that $A_{\overrightarrow{H}} X = \lambda X$ for any tangent vector field X. It is known (cf. [9]) that f is pseudoumbilical if and only if its Gauss map is conformal which is equivalent to $III = (2H^2 - K)\langle , \rangle$.

3 Auxiliary results

Let $f : M^2 \to \mathbb{R}^4$ be an isometric immersion of an oriented two-dimensional Riemannian manifold M^2 into $\mathbb{R}^4$. Hereafter, we suppose that the mean curvature is non-zero at each point. Then we may choose a global orthonormal frame field $\{e_3, e_4\}$ in the normal bundle such that e_3 is the normalized mean curvature vector field, i.e., $e_3 = \overrightarrow{H}/H$ and denote by A_3 and A_4 the corresponding shape operators. Furthermore, we choose a local orthonormal frame field $\{e_1, e_2\}$ in the tangent bundle such that

$$\omega_{13} = k_1\omega_1 \quad \text{and} \quad \omega_{23} = k_2\omega_2, \tag{3.1}$$

where $k_1 + k_2 = 2H$. Since trace$A_4 = 0$, there exist functions μ and ρ such that

$$\omega_{14} = \mu\omega_1 + \rho\omega_2 \quad \text{and} \quad \omega_{24} = \rho\omega_1 - \mu\omega_2. \tag{3.2}$$

It is obvious that the first normal space satisfies $\dim N_1 \geq 1$. Moreover, $\dim N_1 = 1$ at a point $x \in M^2$ if and only if A_4 vanishes at x.

Let $\widetilde{f} : M^2 \to \mathbb{R}^4$ be another isometric immersion. Corresponding quantities for $\widetilde{f}$ are denoted by the same symbol with tilde. We follow the above-mentioned notation throughout the paper.

Appealing to (2.5), we verify that $\widetilde{III} = III$ if and only if $\widetilde{A}_{\overrightarrow{H}} = A_{\overrightarrow{H}}$. Suppose now that $\widetilde{III} = III$. Since trace$A_{\overrightarrow{H}} = 2H$ and trace$\widetilde{A}_{\overrightarrow{H}} = 2\widetilde{H}$, we immediately get $\widetilde{H} = H$. Hence, $\widetilde{f}$ has non-zero mean curvature too and we may choose $\widetilde{e}_3$ to be parallel to mean curvature vector field of $\widetilde{f}$. Consequently, we have $\widetilde{A}_3 = A_3$. Bearing in mind (2.4), we deduce that $\widetilde{A}_4^2 = A_4^2$. If furthermore, $\dim N_1 = 2$, then $\det A_4 = -\mu^2 - \rho^2 < 0$, and we may define a tensor field T by $T := A_4^{-1} \circ \widetilde{A}_4$. From $\widetilde{A}_4^2 = A_4^2$, we readily verify that T is orthogonal. On account of $\widetilde{A}_3 = A_3$ and (2.2), we obtain $\det \widetilde{A}_4 = \det A_4$. Thus T is orientation preserving, and we have proved the following:

Lemma 1 *Let $f, \widetilde{f} : M^2 \to \mathbb{R}^4$ be isometric immersions of an oriented two-dimensional Riemannian manifold M^2.*

(i) *$f, \widetilde{f}$ have the same third fundamental form if and only if $\widetilde{A}_{\overrightarrow{H}} = A_{\overrightarrow{H}}$.*
(ii) *If $f, \widetilde{f}$ have the same third fundamental form, then they have the same mean curvature.*
(iii) *If f has nowhere zero mean curvature,* $\dim N_1 = 2$ *and $f, \widetilde{f}$ have the same third fundamental form, then there exists an orientation preserving orthogonal tensor field T such that $\widetilde{A}_4 = A_4 \circ T$.*

Now suppose that $\dim N_1 = 2$. According to Lemma 1(iii), we may write

$$T = \cos\varphi I + \sin\varphi J$$

for some function φ, where I is the identity map of the tangent bundle TM^2 and J is the complex structure. Then we have

$$\widetilde{\omega}_{13} = \omega_{13}, \quad \widetilde{\omega}_{23} = \omega_{23}, \tag{3.3}$$

$$\widetilde{\omega}_{14} = \cos\varphi\omega_{14} + \sin\varphi\omega_{24} \quad \text{and} \quad \widetilde{\omega}_{24} = -\sin\varphi\omega_{14} + \cos\varphi\omega_{24}. \tag{3.4}$$

Comparing the structure equations (2.1) of f and $\widetilde{f}$, for $j=1,2$, $k=3$, and using (3.3) and (3.4), we find that

$$\widetilde{\omega}_{34}=\cos\varphi\,\omega_{34}-\sin\varphi*\omega_{34}, \tag{3.5}$$

where $*$ denotes the Hodge star operator. Similarly, comparing the structure equations (2.1) of f and $\widetilde{f}$, for $j=1,2,\ k=4$, and using (3.5), we get

$$d\varphi=\frac{k_1-k_2}{\mu^2+\rho^2}\sin\varphi\Big((\mu\cos\varphi+\rho\sin\varphi)\omega_{34}+(\rho\cos\varphi-\mu\sin\varphi)*\omega_{34}\Big). \tag{3.6}$$

Again comparing the structure equations (2.1) for $j=3$ and $k=4$, we obtain

$$\sin\varphi d\varphi\wedge\omega_{34}+\cos\varphi d\varphi\wedge*\omega_{34}+\sin\varphi d*\omega_{34}=-\mu(k_1-k_2)\omega_1\wedge\omega_2,$$

which in view of (3.6) is equivalently written as

$$d*\omega_{34}+\mu(k_1-k_2)\left(1+\frac{|\omega_{34}|^2}{\mu^2+\rho^2}\right)\omega_1\wedge\omega_2=0. \tag{3.7}$$

Setting

$$\sigma:=\cot\varphi,$$

we deduce that (3.6) is equivalent to

$$d\sigma=\Omega+\sigma*\Omega,$$

where Ω is the 1-form given by

$$\Omega:=\frac{k_1-k_2}{\mu^2+\rho^2}\left(-\rho\omega_{34}+\mu*\omega_{34}\right).$$

From (2.3), (3.1) and (3.2), we easily get

$$K_n=\rho(k_1-k_2)\quad\text{and}\quad\text{trace}(A_3A_4)=\mu(k_1-k_2). \tag{3.8}$$

Bearing in mind (3.7) and (3.8), we have the following:

Lemma 2 *Let $f:M^2\to\mathbb{R}^4$ be an isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature and* $\dim N_1=2$. *If there exists another isometric immersion of M^2 into $\mathbb{R}^4$, geometrically distinct from f, having the same third fundamental form, then the function σ satisfies*

$$d\sigma=\Omega+\sigma*\Omega, \tag{3.9}$$

where

$$\Omega=\frac{K_n}{\det A_4}\omega_{34}-\frac{\text{trace}(A_3A_4)}{\det A_4}*\omega_{34}$$

and the normal connection form ω_{34} satisfies

$$d*\omega_{34}+\text{trace}(A_3A_4)\left(1-\frac{|\omega_{34}|^2}{\det A_4}\right)\omega_1\wedge\omega_2=0. \tag{3.10}$$

Conversely, if M^2 is simply connected and f fulfills (3.10)*, then for any function σ that satisfies* (3.9) *there exists an isometric immersion of M^2 into $\mathbb{R}^4$, geometrically distinct from f, whose third fundamental coincides with that of f.*

The converse part of Lemma 2 follows from the fundamental theorem of submanifolds. In fact, for a solution σ of Eq. (3.9), we consider the 1-forms

$$\widetilde{\omega}_{13} = \omega_{13}, \quad \widetilde{\omega}_{23} = \omega_{23}, \quad \widetilde{\omega}_{14} = \cos\varphi\omega_{14} + \sin\varphi\omega_{24},$$
$$\widetilde{\omega}_{24} = -\sin\varphi\omega_{14} + \cos\varphi\omega_{24} \quad \text{and} \quad \widetilde{\omega}_{34} = \cos\varphi\omega_{34} - \sin\varphi * \omega_{34},$$

where φ is defined by $\varphi := \operatorname{arccot}\sigma$. Then using (3.9) and (3.10), we readily verify that $\omega_1, \omega_2, \omega_{12}, \widetilde{\omega}_{jk}, 1 \leq j,k \leq 4$, satisfy the structure equations. According to the fundamental theorem of submanifolds there exists an immersion of M^2 into $\mathbb{R}^4$ whose third fundamental form coincides with that of f.

Now, we are ready to give necessary and sufficient conditions for a surface in $\mathbb{R}^4$ to be locally III-deformable.

Proposition 3 *Let $f : M^2 \to \mathbb{R}^4$ be an isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature and* $\dim N_1 = 2$.

(i) *Then f is locally III-deformable if and only if* (3.10) *is fulfilled and*

$$d * \Omega = 0, \tag{3.11}$$
$$d\Omega + \Omega \wedge *\Omega = 0. \tag{3.12}$$

(ii) *If f is locally III-deformable, then, away from pseudoumbilical points and points where the normal connection form ω_{34} vanishes, the conformal metric*

$$\langle , \rangle_* = -\frac{(k_1 - k_2)^2 |\omega_{34}|^2}{\det A_4} \langle , \rangle$$

has Gaussian curvature $K_* = -1$.

Proof If f is locally III-deformable, then Eq. (3.9) admits infinitely many solutions, which implies (3.11) and (3.12).

Conversely, we assume that (3.10), (3.11) and (3.12) are fulfilled. Since $*\Omega$ is closed, on any simply connected subset U we may write $*\Omega = dF$, for some function F. Then (3.12) is equivalent to $\Delta F = |\nabla F|^2$, where ΔF and ∇F denote the Laplacian and gradient of F, respectively. The 1-form $\mathrm{e}^{-F} * dF$ is closed and so we may define the function σ by

$$\sigma(x) = -\mathrm{e}^{F(x)} \int_{x_0}^{x} \mathrm{e}^{-F} * dF, \quad x, x_0 \in U.$$

It is easy to verify that $\sigma_t := t\mathrm{e}^F + \sigma$ is a solution of Eq. (3.9) for any $t \in \mathbb{R}$. This according to Lemma 2 gives rise to a non-trivial III-deformation of f on U and therefore f is locally III-deformable.

Now suppose that f is locally III-deformable and consider the 1-forms $\omega_1^* := *\Omega$ and $\omega_2^* := -\Omega$. Then $\langle , \rangle_* = \left(\omega_1^*\right)^2 + \left(\omega_2^*\right)^2$, and (3.11) and (3.12) imply that $\omega_{12}^* = \omega_2^*$, where ω_{12}^* is the connection form of $\langle , \rangle_*$. Since the Gaussian curvature of $\langle , \rangle_*$ is given by $d\omega_{12}^* = -K_* \omega_1^* \wedge \omega_2^*$, (3.12) yields $K_* = -1$. □

4 Surfaces satisfying $\dim N_1 = 1$

In this section, we consider surfaces in $\mathbb{R}^4$ whose first normal space N_1 is one-dimensional. It is shown that such surfaces are locally *III*-deformable, provided that they lie fully in $\mathbb{R}^4$. This is a consequence of the following local classification of surfaces in $\mathbb{R}^4$ that satisfy $\dim N_1 = 1$.

Proposition 4 *Let $f : M^2 \to \mathbb{R}^4$ be an isometric immersion of a two-dimensional Riemannian manifold M^2 such that* $\dim N_1 = 1$. *Then locally*

(i) *$f(M^2)$ lies on an affine three-dimensional subspace of $\mathbb{R}^4$,*
(ii) *$f(M^2)$ is part of a cylinder erected over a curve lying fully on an affine three-dimensional subspace of $\mathbb{R}^4$,*
(iii) *$f(M^2)$ is part of a cone shaped over a curve lying fully on a 3-sphere, or*
(iv) *$f(M^2)$ is part of the tangent surface of a curve lying fully in $\mathbb{R}^4$.*

Proof We choose a local orthonormal frame field $\{e_3, e_4\}$ in the normal bundle such that e_3 spans N_1 and a local orthonormal frame $\{e_1, e_2\}$ in the tangent bundle such that $\omega_{13} = k_1\omega_1$ and $\omega_{23} = k_2\omega_2$. Since $\omega_{14} = \omega_{24} = 0$, the structure equations (2.1) for $j = 1, 2$, and $k = 3$ yield

$$e_1(k_2) = (k_1 - k_2)\omega_{12}(e_2) \quad \text{and} \quad e_2(k_1) = (k_1 - k_2)\omega_{12}(e_1). \tag{4.1}$$

Similarly, from (2.1) for $j = 1, 2$, and $k = 4$, we get

$$k_1\omega_1 \wedge \omega_{34} = k_2\omega_2 \wedge \omega_{34} = 0. \tag{4.2}$$

We distinguish the following cases.

Case 1 Assume that $k_1k_2 \neq 0$ on a connected open subset U. Then (4.2) implies that $\omega_{34} = 0$, which means that e_4 is parallel in the normal bundle. Since $A_4 = 0$, the Weingarten formula implies that e_4 is constant on U. A standard argument shows that $f(U)$ lies on an affine three-dimensional subspace of $\mathbb{R}^4$ which is perpendicular to e_4.

Case 2 Assume that $k_1k_2 = 0$ and $(k_1, k_2) \neq (0, 0)$ on a connected open subset U. Without loss of generality, we suppose that $k_2 = 0$. Then (4.1) gives $\omega_{12}(e_2) = 0$ and consequently the integral curves of e_2 are geodesics. Using the Gauss formula, we see that these geodesics are mapped via f into straight lines. In addition, (4.2) yields $\omega_{34}(e_2) = 0$, and the Weingarten formula shows that e_3 and e_4 are constant along the integral curves of e_2. This means that the normal space and consequently the tangent space is a fixed plane along each integral curve of e_2. We choose local coordinates (x, y) such that $\frac{\partial}{\partial x}$ is parallel to e_1 and $\frac{\partial}{\partial y}$ is parallel to e_2. The induced metric has the form $\langle , \rangle = Edx^2 + Gdy^2$ and the connection form ω_{12} is given by

$$\omega_{12} = -\frac{(\sqrt{E})_y}{\sqrt{G}}dx + \frac{(\sqrt{G})_x}{\sqrt{E}}dy.$$

From $\omega_{12}(e_2) = 0$, we deduce that G depends only on y. We introduce new coordinates (u, v), where $v = \int \sqrt{G(y)}dy$ is the arc length of the integral curves of e_2, and $u = x$. Furthermore, we set $\gamma := f - vdf(e_2)$ and $r := df(e_2)$. Using the Gauss formula, we get $d\gamma(e_2) = 0$, and so γ depends only on u. Moreover, in view of Weingarten formula, we infer that r depends only on u. Consequently

$$f(u,v) = \gamma(u) + vr(u), \tag{4.3}$$

and $f(U)$ is a ruled surface. Furthermore, we note that $\langle \gamma', r \rangle = 0$.

If r is constant, say a, then we easily get $e_2(\langle \gamma, a \rangle) = 0$. Thus the curve γ lies on an affine 3-space perpendicular to a, and (4.3) shows that $f(U)$ is part of a cylinder erected over γ.

Now suppose that r is not constant. Since the tangent plane remains constant along the rulings, we deduce that $r' = p\gamma' + qr$, for suitable functions p, q. Using $\langle \gamma', r \rangle = 0$, we get $q = 0$ and consequently $r' = p\gamma'$.

If p is constant, then $r = p\gamma + c_0$, where c_0 is constant. Hence, (4.3) shows that $f(U)$ is a cone with vertex at $-\frac{1}{p}c_0$.

If p is not constant, we consider the curve $\overline{\gamma}(u) := f(u, -1/p(u))$. It is obvious that $\overline{\gamma}'(u) = \frac{p'(u)}{p(u)} r(u)$. Then (4.3) shows that $f(U)$ is part of the tangent surface of the curve $\overline{\gamma}$. □

Theorem 5 *Every full isometric immersion $f : M^2 \to \mathbb{R}^4$ of a two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature and* $\dim N_1 = 1$ *is locally III-deformable.*

Proof According to Proposition 4, we consider the following cases.

Case 1 Assume that $f(M^2)$ is part of a cylinder erected over a curve lying fully on an affine three-dimensional subspace of $\mathbb{R}^4$. Without loss of generality we assume that $f(u,v) = (\gamma(u), v)$, where γ is a unit speed curve in $\mathbb{R}^3$ with curvature $k > 0$ and torsion $\tau \neq 0$. The immersion f has induced metric $\langle , \rangle = du^2 + dv^2$, and third fundamental form $III = k^2 du^2$. Let $\gamma_t, t \in \mathbb{R}$, be a family of unit speed curves in $\mathbb{R}^3$ with curvature $k_t = k$, torsion $\tau_t = (1-t)\tau$ and $\gamma_0 = \gamma$. Then $f_t, t \in \mathbb{R}$, given by $f_t(u,v) = (\gamma_t(u), v)$, is a non-trivial III-deformation of f.

Case 2 Assume that $f(M^2)$ is part of a cone shaped over a curve lying fully on a 3-sphere S^3. Without loss of generality we assume that $f(u,v) = v\gamma(u), v > 0$, where γ is a unit speed curve in S^3 with curvature $k > 0$ and torsion $\tau \neq 0$. The immersion f has induced metric $\langle , \rangle = v^2 du^2 + dv^2$, and third fundamental form $III = \frac{k^2}{v^2} du^2$. Let $\gamma_t, t \in \mathbb{R}$, be a family of unit speed curves in S^3 with curvature $k_t = k$, torsion $\tau_t = (1-t)\tau$ and $\gamma_0 = \gamma$. Then $f_t, t \in \mathbb{R}$, given by $f_t(u,v) = v\gamma_t(u)$, is a non-trivial III-deformation of f.

Case 3 Assume that $f(M^2)$ is part of the tangent surface of a curve lying fully in $\mathbb{R}^4$. This means that $f(u,v) = \gamma(u) + v\gamma'(u), v > 0$, where γ is a unit speed curve in $\mathbb{R}^4$ with curvatures $k_1 > 0, k_2$ and $k_3 \neq 0$. The immersion f has induced metric $\langle , \rangle = (1 + k_1^2 v^2) du^2 + 2dudv + dv^2$, and third fundamental form $III = k_2^2 du^2$. Let $\gamma_t, t \in \mathbb{R}$, be a family of unit speed curves in $\mathbb{R}^4$ with curvatures $k_{1,t} = k_1, k_{2,t} = k_2$, $k_{3,t} = (1-t)k_3$ and $\gamma_0 = \gamma$. Then $f_t, t \in \mathbb{R}$, given by $f_t(u,v) = \gamma_t(u) + v\gamma_t'(u)$, is a non-trivial III-deformation of f. □

We note that Theorem 5 is not true without the assumption of fullness since by [5] the only locally III-deformable surfaces in $\mathbb{R}^3$ are the minimal ones.

5 Pseudoumbilical surfaces

In this section, we consider pseudoumbilical surfaces in $\mathbb{R}^4$ which are locally III-deformable and provide a method for producing all such surfaces. Let $f : M^2 \to \mathbb{R}^4$ be a pseudoumbilical isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature H. In the notation of Sect. 3, this is equivalent to $k_1 = k_2 = H$. The umbilical points of f, i.e., points where $H^2 = K$, are precisely the points where $\dim N_1 = 1$.

From (2.3) we get $K_n = 0$. Since $\mathrm{trace}(A_3A_4) = 0$, the 1-form Ω vanishes on $M^2 \backslash M_0$, where M_0 is the set of umbilical points. Then from Lemma 2 and Proposition 3, we conclude that if there exists locally a geometrically different isometric immersion $\widetilde{f}$ with the same third fundamental form, then f is locally III-deformable. Moreover, in this case the only solutions to Eq. (3.9) are the constant ones. Thus locally III-deformable pseudoumbilical surfaces in $\mathbb{R}^4$ allow a sort of associated family like minimal surfaces.

It is easy to see that round 2-spheres in $\mathbb{R}^4$ are not locally III-deformable. Hence, if f is locally III-deformable, then the set of non-umbilical points $M^2 \backslash M_0$ is dense in M^2.

Proposition 3 yields the following simple criterion for a pseudoumbilical surface to be locally III-deformable.

Lemma 6 *Let $f : M^2 \to \mathbb{R}^4$ be an umbilic-free pseudoumbilical isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature. Then f is locally III-deformable if and only if the normal connection form ω_{34} is co-closed, i.e., $d * \omega_{34} = 0$.*

On account of $k_1 = k_2 = H$, (3.1) and (3.2), the structure equations (2.1) for $j = 1, 2$ and $k = 3$ are equivalent to

$$e_1(H) = -\mu\omega_{34}(e_1) - \rho\omega_{34}(e_2),$$
$$e_2(H) = -\rho\omega_{34}(e_1) + \mu\omega_{34}(e_2).$$

From these and using $H^2 - K = \mu^2 + \rho^2$, we easily get

$$|\nabla H|^2 = (H^2 - K)|\omega_{34}|^2, \tag{5.1}$$

where ∇H stands for the gradient of the mean curvature.

Hence if a point $x \in M^2$ is not a critical point of the mean curvature H, then x is not an umbilical point of f, and the normal connection form ω_{34} does not vanish at x.

By means of isothermal coordinates, M^2 may be viewed as a Riemann surface. Let $TM^2 \otimes \mathbb{C}$ be the complexified tangent bundle. For a local complex coordinate $z = x + iy$ we set as usual

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Proposition 7 *Let $f : M^2 \to \mathbb{R}^4$ be a pseudoumbilical isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature. Assume that f is locally III-deformable. Then*

(i) *The complex valued function $\phi := 4(\omega_{34}(\frac{\partial}{\partial z}))^2$ is holomorphic, where $z = x + iy$ is a local complex coordinate and ω_{34} is linearly extended to $TM^2 \otimes \mathbb{C}$. Moreover, the holomorphic quadratic form $\Phi = \phi dz^2$ is defined globally on M^2;*
(ii) *Either the normal connection form ω_{34} vanishes at isolated points only, or ω_{34} vanishes identically and f is minimal in a 3-sphere of $\mathbb{R}^4$;*
(iii) *Away from critical points of the mean curvature, the metric $\langle , \rangle_* = \frac{|\nabla H|^2}{H^2 - K} \langle , \rangle$ is flat.*

Proof According to Lemma 6, $*\omega_{34}$ is closed. Because of $K_n = 0$, the form ω_{34} is closed too. Then around each point we may write

$$\omega_{34} = dv \quad \text{and} \quad *\,\omega_{34} = -du,$$

for some functions u, v. Let $z = x+iy$ be a local complex coordinate. Then the induced metric has the form $\langle , \rangle = E(dx^2 + dy^2)$. From the above equations, we deduce that the complex valued function $u + iv$ satisfies the Cauchy–Riemann equations and consequently ϕ is given by

$$\phi = v_x^2 - v_y^2 - 2iv_x v_y.$$

Now it is easy to verify that ϕ satisfies the Cauchy–Riemann equations, and so it is holomorphic.

To show that ϕdz^2 is defined globally, let $\tilde{z} = \tilde{x} + i\tilde{y}$ be another local complex coordinate. Then on the common coordinate neighborhood we have

$$\frac{\partial}{\partial z} = \frac{d\tilde{z}}{dz}\frac{\partial}{\partial \tilde{z}} \quad \text{and} \quad \omega_{34}\left(\frac{\partial}{\partial z}\right)dz = \omega_{34}\left(\frac{\partial}{\partial \tilde{z}}\right)d\tilde{z}.$$

Hence, $\omega_{34}(\frac{\partial}{\partial z})dz$ is invariant under change of coordinates and $\Phi = \phi dz^2$ is defined globally on M^2.

On the other hand, we have $|\phi|^2 = E^2|\omega_{34}|^4$. Moreover, the holomorphicity of ϕ implies that either the normal connection form ω_{34} vanishes at isolated points only, or ω_{34} vanishes identically. In the latter case, it is easy to prove that f is minimal in a 3-sphere of $\mathbb{R}^4$ (cf. [1]).

Now assume that the mean curvature has no critical points. Then (5.1) implies that $|\omega_{34}|^2 > 0$ or equivalently $|\phi|^2 > 0$. From the holomorphicity of ϕ, we get $\Delta \log |\phi|^2 = 0$, where Δ is the Laplacian of the induced metric. Therefore, $\Delta \log |\omega_{34}|^2 = -\Delta \log E$. Since $\Delta \log E = -2K$, we finally get $\Delta \log |\omega_{34}|^2 = 2K$. Hence, by virtue of (5.1), the metric $\langle , \rangle_* = \frac{|\nabla H|^2}{H^2 - K} \langle , \rangle$ is flat. □

For the local classification of *III*-deformable pseudoumbilical surfaces in $\mathbb{R}^4$ we need the following auxiliary lemmas.

Lemma 8 ([2]) *Let $u : U \to \mathbb{R}$ be a smooth function such that $\Delta_0 u = P(u)$ and $|\nabla_0 u|^2 = Q(u) > 0$, where U is an open subset of $\mathbb{R}^2, P, Q : \mathbb{R} \to \mathbb{R}$ are smooth functions, and Δ_0, ∇_0 denote the Euclidean Laplacian and gradient operators, respectively. Then the level curves of u are parallel lines or concentric circles.*

Lemma 9 ([7]) *Let M^2 be a two-dimensional Riemannian manifold and $u : M^2 \to \mathbb{R}$ a smooth function such that $\Delta u = P(u)$ and $|\nabla u|^2 = Q(u)$, for smooth functions $P, Q : \mathbb{R} \to \mathbb{R}$, where ∇u denotes the gradient of u. Then on the open set $\{x \in M^2 : \nabla u(x) \neq 0\}$ the Gaussian curvature K satisfies*

$$2KQ + (2P - Q')(P - Q') + Q(2P' - Q'') = 0.$$

Now we are ready to give the local description of pseudoumbilical surfaces in $\mathbb{R}^4$ which are locally *III*-deformable.

Theorem 10 *Let $f : M^2 \to \mathbb{R}^4$ be a locally III-deformable pseudoumbilical isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere* *zero mean curvature H. Then around each point, which is not a critical point of the mean curvature, there exist local coordinates (x, y) such that the induced metric is given by $\langle , \rangle = e^{g(H)}(dx^2 + dy^2)$ for a real valued function g. Moreover, one of the following alternatives holds*:

(i) $g(H) = -\log(c^2 - H^2)$, *where $c > 0$ is a constant, and H satisfies*

$$\Delta_0 H + \frac{2H}{c^2 - H^2}|\nabla_0 H|^2 = -H. \tag{5.2}$$

(ii) *Away from points where $g''(H) - (g'(H))^2 - 2e^{g(H)} = 0$, g is a solution of the fourth-order differential equation*

$$(2P - Q')(P - Q') + Q(2P' - Q'') = 0 \tag{5.3}$$

with $2P - Q' \neq 0$ and $Q > 0$, where

$$P(H) := \frac{-Hg''(H) + 2He^{g(H)} + 2H^2e^{g(H)}g'(H)}{g''(H) - (g'(H))^2 - 2e^{g(H)}},$$
$$Q(H) := \frac{Hg'(H) - 2H^2e^{g(H)}}{g''(H) - (g'(H))^2 - 2e^{g(H)}}.$$

Furthermore, the mean curvature has the form $H(x, y) = h^{-1}(\sqrt{x^2 + y^2})$, where h^{-1} is the inverse of the function h defined by

$$h(H) := \frac{2\sqrt{Q(H)}}{|2P(H) - Q'(H)|}.$$

(iii) *Away from points where $g''(H) - (g'(H))^2 - 2e^{g(H)} = 0$, g is a solution of the third-order differential equation $2P - Q' = 0$ with $Q > 0$, and H is a function of x that satisfies*

$$\left(\frac{dH}{dx}\right)^2 = Q(H). \tag{5.4}$$

Conversely, given a simply connected open subset U of $\mathbb{R}^2$ and functions g and $H : U \to \mathbb{R}^+$ as in (i), (ii) *or* (iii)*, then there exists a locally III-deformable pseudoumbilical isometric immersion of the manifold $\left(U, e^{g(H)}(dx^2 + dy^2)\right)$ into $\mathbb{R}^4$ with mean curvature H.*

Proof Let U be a simply connected neighborhood which is free of critical points of the mean curvature, and consequently umbilic-free by (5.1). Then, we have $\Phi = \phi(w)dw^2 \neq 0$ on U. By virtue of Proposition 7, we can pick a branch of $\sqrt{\phi(w)}$ and choose complex coordinate $z = z(w) = x + iy$, determined up to an additive constant, by integrating the 1-form

$$dz := i\sqrt{\Phi} = i\sqrt{\phi(w)}dw.$$

In these coordinates, we have $\Phi = -dz^2$. The induced metric has the form $\langle , \rangle = E(dx^2 + dy^2)$. As in the proof of Proposition 7, we pick functions u, v on U such that $\omega_{34} = dv$ and $*\omega_{34} = -du$. Then $u + iv$ is holomorphic and

$$\Phi = (v_x^2 - v_y^2 - 2iv_xv_y)dz^2.$$

Thus we get $u_x = v_y = \pm 1$ and $u_y = -v_x = 0$. Without loss of generality, we may assume that $v = y$ and $u = x$. Hence, we obtain

$$\omega_{34} = dy \quad \text{and} \quad *\,\omega_{34} = -dx. \tag{5.5}$$

We choose an orthonormal frame field $\{e_1, e_2\}$ in the tangent bundle such that

$$e_1 = \frac{1}{\sqrt{E}}\frac{\partial}{\partial x} \quad \text{and} \quad e_2 = \frac{1}{\sqrt{E}}\frac{\partial}{\partial y}.$$

From (5.5), we get $\omega_{34}(e_1) = 0$ and $\omega_{34}(e_2) = 1/\sqrt{E}$. Using

$$\omega_1 = \sqrt{E}dx, \quad \omega_2 = \sqrt{E}dy, \quad \omega_{12} = -\frac{1}{2}(\log E)_y\, dx + \frac{1}{2}(\log E)_x\, dy$$

and (5.5), we readily verify that structure equations (2.1) for $j = 1, 2$ and $k = 3, 4$ are equivalent to

$$\mu = H_y, \quad \rho = -H_x \tag{5.6}$$
$$-\mu_y + \rho_x + \rho\,(\log E)_x - \mu\,(\log E)_y = H, \tag{5.7}$$
$$\mu_x + \rho_y + \rho\,(\log E)_y + \mu\,(\log E)_x = 0. \tag{5.8}$$

Combining (5.8) with (5.6), we get $dH \wedge d\log E = 0$. This means that $\log E$ depends on H, i.e., $E = \mathrm{e}^{g(H)}$, for a function g. Thus

$$\langle , \rangle = \mathrm{e}^{g(H)}(dx^2 + dy^2).$$

On account of (5.6), (5.7) is transformed to

$$\Delta_0 H + g'(H)|\nabla_0 H|^2 = -H. \tag{5.9}$$

From (2.2) we find that the Gaussian curvature K is given by $K = H^2 - \mu^2 - \rho^2$, or equivalently by virtue of (5.6), $K = H^2 - |\nabla_0 H|^2$. Moreover, since

$$K = -\frac{\Delta_0 \log E}{2E} \quad \text{and} \quad E = \mathrm{e}^{g(H)},$$

we finally obtain

$$g'(H)\Delta_0 H + \left(g''(H) - 2\mathrm{e}^{g(H)}\right)|\nabla_0 H|^2 = -2H^2\mathrm{e}^{g(H)}. \tag{5.10}$$

Conversely, let $H : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a positive solution of the overdetermined system of the partial differential equations (5.9) and (5.10), where U is simply connected. Then there exists a pseudoumbilical isometric immersion of the Riemannian manifold $\left(U, \mathrm{e}^{g(H)}(dx^2 + dy^2)\right)$ into $\mathbb{R}^4$ with mean curvature H which is locally III-deformable. In fact, we consider the 1-forms

$$\omega_1 = \mathrm{e}^{\frac{g(H)}{2}}dx, \quad \omega_2 = \mathrm{e}^{\frac{g(H)}{2}}dy, \;\; \omega_{12} = -\tfrac{1}{2}g'(H)\left(H_y dx - H_x dy\right),$$
$$\omega_{13} = H\omega_1, \quad \omega_{23} = H\omega_2,$$
$$\omega_{14} = H_y\omega_1 - H_x\omega_2, \;\; \omega_{24} = -H_x\omega_1 - H_y\omega_2 \quad \text{and} \quad \omega_{34} = dy.$$

Using (5.9) and (5.10), one can easily verify that the 1-forms $\omega_1, \omega_2, \omega_{jk}$, $1 \le j, k \le 4$, fulfill the structure equations. According to the fundamental theorem of submanifolds there exists an immersion with mean curvature H which, in view of Lemma 6, is locally III-deformable. Thus the local classification of pseudoumbilical locally

III-deformable surfaces in $\mathbb{R}^4$ is reduced to the study of the overdetermined system of equations (5.9) and (5.10).

We proceed with the study of this system and view it as a linear system with unknowns $\Delta_0 H$ and $|\nabla_0 H|^2$. Its determinant is $g''(H) - (g'(H))^2 - 2e^{g(H)}$. If the determinant is identically zero, then (5.9) and (5.10) imply that $g'(H) = 2He^{g(H)}$, and consequently $g(H) = -\log(c^2 - H^2)$, where c is a positive constant. Moreover, the system of equations (5.9) and (5.10) is equivalent to (5.2).

Now assume that $g''(H) - (g'(H))^2 - 2e^{g(H)} \neq 0$. Then (5.9) and (5.10) are equivalent to

$$\Delta_0 H = P(H) \quad \text{and} \quad |\nabla_0 H|^2 = Q(H) > 0, \tag{5.11}$$

where

$$P(H) := \frac{-Hg''(H) + 2He^{g(H)} + 2H^2 e^{g(H)} g'(H)}{g''(H) - (g'(H))^2 - 2e^{g(H)}},$$
$$Q(H) := \frac{Hg'(H) - 2H^2 e^{g(H)}}{g''(H) - (g'(H))^2 - 2e^{g(H)}}.$$

This means that H is an isoparametric function and appealing to Lemma 5.2, we see that g satisfies (5.3). According to Lemma 8, we have either $H = H(r)$, where $r = \sqrt{x^2 + y^2}$, or $H = H(x)$.

At first, we suppose that $H = H(r)$. Then the Eqs. (5.11) are transformed to

$$\frac{dH}{dr} = \varepsilon\sqrt{Q(H)} \quad \text{and} \quad \frac{1}{r}\frac{dH}{dr} + \frac{d^2H}{dr^2} = P(H),$$

where $\varepsilon = \pm 1$. These imply that

$$r = \frac{2\varepsilon\sqrt{Q(H)}}{2P(H) - Q'(H)} = h(H).$$

The function h is invertible since by (5.3) we get

$$h'(H) = \frac{\varepsilon}{\sqrt{Q(H)}} \neq 0.$$

Thus we obtain $H(x, y) = h^{-1}(r)$.

Conversely, let $g(t)$ be a function that satisfies (5.3) with $2P - Q' \neq 0$ and $Q > 0$. Using (5.3), we find that

$$h'(t) = \frac{\varepsilon}{\sqrt{Q(t)}},$$

where $\varepsilon = 1$, if $2P - Q' > 0$ and $\varepsilon = -1$, if $2P - Q' < 0$. This implies that $s = h(t)$ is invertible. In addition, we have

$$\frac{dh^{-1}}{ds} = \varepsilon\sqrt{Q(h^{-1}(s))} \quad \text{and} \quad \frac{d^2h^{-1}}{ds^2} = \frac{1}{2}Q'(h^{-1}(s)).$$

We consider the function $H(x, y) = h^{-1}(r)$. Then a direct computation shows that

$$|\nabla_0 H|^2 = \left(\frac{dH}{dr}\right)^2 = Q(H)$$

and

$$\Delta_0 H = \frac{1}{r}\frac{dh^{-1}}{ds}(r) + \frac{d^2h^{-1}}{ds^2}(r) = \frac{1}{r}\varepsilon\sqrt{Q(h^{-1}(r))} + \frac{1}{2}Q'(h^{-1}(r)) = P(H).$$

Hence, H fulfills (5.11) and so H is a solution of the overdetermined system of equations (5.9) and (5.10).

Now assume that $H = H(x)$. Then the second of (5.11) is equivalent to (5.4). Since

$$\Delta_0 H = \frac{d^2H}{dx^2},$$

by virtue of the first equation of (5.11), we have

$$\frac{d^2H}{dx^2} = P(H).$$

Moreover, using

$$\left(\frac{dH}{dx}\right)^2 = Q(H),$$

we get $2P - Q' = 0$.

Conversely, let $g(t)$ be a function that satisfies $2P - Q' = 0$ with $Q > 0$ and $H = H(x)$ be a positive solution of Eq. (5.4). Then we readily verify that $\Delta_0 H = P(H)$ and so H satisfies (5.11). Consequently, H is a positive solution of the system of equations (5.9) and (5.10). This completes the proof. □

Remark 1 We recall that a surface in a Euclidean space is called isothermic in the sense of [14], if there exist locally a pair of harmonic functions u_1, u_2 such that the lines of curvature of each section of the normal bundle are contained in a level set $u_i = const$. It is obvious from the proof of Theorem 10 that a locally III-deformable pseudoumbilical surface in $\mathbb{R}^4$ is isothermic.

Remark 2 It is worth noticing that there exist flat locally III-deformable pseudoumbilical surfaces. In fact, we consider the functions $g(t) = -2\log t + c, t > 0$, where c is a real constant, and $H(x) = e^{\pm x}, x \in \mathbb{R}$. It is easy to see that g and H satisfy the assumptions of Theorem 10(iii) and give rise to a locally III-deformable umbilic-free pseudoumbilical isometric immersion of the flat Riemannian manifold $\left(\mathbb{R}^2, \mathrm{e}^{c\pm 2x}(dx^2 + dy^2)\right)$ into $\mathbb{R}^4$.

6 Global results

In this section, we prove some results about locally III-deformable surfaces in $\mathbb{R}^4$ under global assumptions.

Theorem 11 *Let $f : M^2 \to \mathbb{R}^4$ be a pseudoumbilical isometric immersion of a compact oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature.*

(i) *If f is locally III-deformable, then M^2 is homeomorphic to the 2-sphere with $g \geq 1$ handles.*
(ii) *If M^2 has non-negative Gaussian curvature, then f is not locally III-deformable, unless $f(M^2)$ is the Clifford torus in a 3-sphere of $\mathbb{R}^4$.*

Proof We suppose that f is locally III-deformable. Moreover, assume in the contrary that M^2 is homeomorphic to the sphere S^2. From Proposition 7(i), we know that the global differential form $\Phi = \phi dz^2$ is holomorphic. Appealing to the Riemann–Roch theorem, we deduce that $\Phi \equiv 0$. Then the normal connection form vanishes identically, and according to Proposition 7(ii), f is minimal in a 3-sphere of $\mathbb{R}^4$. Since M^2 is homeomorphic to S^2, a well-known result shows that $f(M^2)$ is a standard round 2-sphere. This is a contradiction, since round 2-spheres in $\mathbb{R}^4$ are not locally III-deformable.

Now we assume that M^2 has non-negative Gaussian curvature. By the Gauss Bonnet theorem, M^2 is homeomorphic to the sphere S^2 or to the torus $S^1 \times S^1$. Moreover, we suppose that f is locally III-deformable. Then M^2 is homeomorphic to the torus and the Gaussian curvature is zero. Appealing to the Riemann–Roch theorem, we deduce that $\Phi = \phi dz^2$ either vanishes identically or is nowhere zero. Bearing in mind the fact that the zeros of Φ are precisely the points where ω_{34} vanishes, from (5.1) and the compactness of M^2, we infer that Φ cannot be everywhere non-zero. Thus Φ is identically zero. According to Proposition 7(ii), f is minimal in a 3-sphere of $\mathbb{R}^4$. A result due to Lawson [12] shows that $f(M^2)$ is the Clifford torus in a 3-sphere. □

Theorem 12 *Let $f : M^2 \to \mathbb{R}^4$ be a locally III-deformable isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with nowhere zero mean curvature and $\dim N_1 = 2$. If M^2 is compact, then f is minimal in a 3-sphere and M^2 is homeomorphic to the torus $S^1 \times S^1$.*

Proof We assume that f is locally III-deformable. From Proposition 3, we know that the globally defined 1-form Ω (see Lemma 2) satisfies $d\Omega + \Omega \wedge *\Omega = 0$ on M^2. Integrating and using Stokes' theorem, we get

$$\int_{M^2} |\Omega|^2 dA = 0,$$

where dA is the volume element of M^2. Hence $\Omega = 0$ everywhere. Since

$$|\Omega|^2 = -\frac{(k_1 - k_2)^2}{\det A_4} |\omega_{34}|^2,$$

we deduce that $(k_1 - k_2)^2 |\omega_{34}|^2 = 0$ on M^2. We claim that f is pseudoumbilical. Assume in the contrary that $k_1 \neq k_2$ on an open subset U. Then ω_{34} vanishes on U, and so $K_n = 0$ on U. By virtue of (3.8), we get $\rho = 0$ on U. Moreover, (3.10) yields $\mathrm{trace}(A_3 A_4) = 0$, or equivalently in view of (3.8), $\mu = 0$ on U. This contradicts the assumption $\dim N_1 = 2$. Therefore, f is pseudoumbilical. In addition, f is umbilic-free since $\dim N_1 = 2$. Hence, the shape operator A_4 associated with e_4 is not proportional to the identity transformation. This implies that M^2 allows a smooth one-dimensional distribution by choosing at each point $x \in M^2$ the set of all vectors in $T_x M^2$ which are eigenvectors of A_4 corresponding to the largest eigenvalue. Consequently, M^2 is homeomorphic to the torus $S^1 \times S^1$. Then the Riemann–Roch theorem implies that the holomorphic differential form $\Phi = \phi dz^2$ either vanishes identically or is nowhere zero. On account of (5.1), the compactness of M^2, and the fact that f is umbilic-free, we infer that $\Phi = \phi dz^2$ can not be everywhere non-zero. Thus $\Phi = \phi dz^2$ vanishes identically or equivalently ω_{34} is identically zero. In view of Proposition 7(ii), f is minimal in a 3-sphere. □

The following is an immediate consequence of Theorem 12.

Corollary 13 *Let $f : M^2 \to \mathbb{R}^4$ be an isometric immersion of an oriented two-dimensional Riemannian manifold M^2 with $\dim N_1 = 2$. If M^2 is compact with non-negative Gaussian curvature, then f is not locally III-deformable, unless $f(M^2)$ is the Clifford torus in a 3-sphere of $\mathbb{R}^4$.*

Proof We claim that the mean curvature H is nowhere zero. In fact, if $H(x) = 0$ at a point $x \in M^2$, then (2.2) yields $K(x) \leq 0$. So, $K(x) = 0$ and f is totally geodesic at x. This contradicts the assumption $\dim N_1 = 2$. Now assume that f is locally *III*-deformable. Theorem 12 shows that f is pseudoumbilical and Theorem 11(ii) implies that $f(M^2)$ is the Clifford torus in a 3-sphere. □

The following examples justify the necessity of compactness in Theorem 12.

Example 1 Let $(x(s), y(s))$, $s \in I \subseteq \mathbb{R}$, be a unit speed plane curve. We consider the immersion $f : I \times \mathbb{R} \to \mathbb{R}^4$ defined by

$$f(s,t) = (x(s)\cos(ct), x(s)\sin(ct), y(s)\cos t, y(s)\sin t),$$

where $c > 0$ and $c^2x^2(s) + y^2(s) > 0$, for any $s \in I$. The induced metric is $\langle , \rangle = ds^2 + \left(c^2x^2(s) + y^2(s)\right) dt^2$. We consider the orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ given by

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{\sqrt{c^2x^2(s) + y^2(s)}} \frac{\partial}{\partial t},$$

$$e_3 = (\dot{y}(s)\cos(ct), \dot{y}(s)\sin(ct), -\dot{x}(s)\cos t, -\dot{x}(s)\sin t),$$

$$e_4 = \frac{(y(s)\sin(ct), -y(s)\cos(ct), -cx(s)\sin t, cx(s)\cos t)}{\sqrt{c^2x^2(s) + y^2(s)}},$$

where dot denotes differentiation with respect to s. Obviously, e_1, e_2 are tangent, e_3, e_4 are normal, and $\omega_1 = ds$, $\omega_2 = \sqrt{c^2x^2(s) + y^2(s)}dt$. A straightforward computation shows that e_3 is parallel to the mean curvature vector and

$$\omega_{13} = k_1\omega_1, \quad \omega_{23} = k_2\omega_2, \quad \omega_{14} = \mu\omega_1 + \rho\omega_2, \quad \omega_{24} = \rho\omega_1 - \mu\omega_2,$$

where

$$k_1 = \dot{x}\ddot{y} - \ddot{x}\dot{y}, \quad k_2 = \frac{\dot{x}y - c^2x\dot{y}}{c^2x^2 + y^2}, \quad \mu = 0 \quad \text{and} \quad \rho = \frac{c(x\dot{y} - \dot{x}y)}{c^2x^2 + y^2}.$$

Furthermore, we have

$$\omega_{34} = -\frac{c(x\dot{x} + y\dot{y})}{c^2x^2 + y^2}\omega_2.$$

The immersion f satisfies $\dim N_1 = 2$ if and only if $x\dot{y} - \dot{x}y \neq 0$. It is obvious that f fulfills (3.10). Moreover, the 1-form Ω is given by

$$\Omega = \frac{k_2 - k_1}{\rho}\omega_{34}$$

and satisfies (3.11). Hence, f is locally III-deformable if and only if (3.12) is fulfilled or equivalently

$$\left(\frac{\dot{x}x+\dot{y}y}{\sqrt{c^2x^2+y^2}}\frac{k_2-k_1}{\rho}\right)^{\cdot}=\frac{c(\dot{x}x+\dot{y}y)^2}{(c^2x^2+y^2)^{\frac{3}{2}}}\left(\frac{k_2-k_1}{\rho}\right)^2. \tag{6.1}$$

Viewing the curve $(x(s),y(s))$ as a graph of a function $y(x)$ over x, the differential equation (6.1) is transformed to

$$\left\{\frac{(x+yy')\sqrt{c^2x^2+y^2}}{(xy'-y)\sqrt{1+(y')^2}}\left(\frac{y-c^2xy'}{c^2x^2+y^2}+\frac{y''}{1+(y')^2}\right)\right\}' = \frac{(x+yy')^2\sqrt{c^2x^2+y^2}}{(xy'-y)^2\sqrt{1+(y')^2}}\left(\frac{y-c^2xy'}{c^2x^2+y^2}+\frac{y''}{1+(y')^2}\right)^2, \tag{6.2}$$

where prime denotes differentiation with respect to x. The condition $\dim N_1=2$ is now equivalent to $xy'-y\neq 0$. Equation (6.2) is written in the form $y'''=F(x,y,y',y'')$, for some continuous function F defined on a sufficiently small open subset of $\mathbb{R}^4$. According to the existence theorem of ordinary differential equations, any solution of (6.2) with appropriate initial conditions gives rise to a non-pseudoumbilical locally III-deformable surface in $\mathbb{R}^4$ with $\dim N_1=2$.

Example 2 Let $(x(s),y(s)), s\in I\subseteq\mathbb{R}$, be a unit speed plane curve. We consider the immersion $f: I\times\mathbb{R}\to\mathbb{R}^4$ defined by

$$f(s,t)=(x(s),ct,y(s)\cos t,y(s)\sin t),$$

where $c>0$. The induced metric is $\langle,\rangle=ds^2+\left(y^2(s)+c^2\right)dt^2$. We consider the orthonormal frame field $\{e_1,e_2,e_3,e_4\}$ given by

$$e_1=\frac{\partial}{\partial s},\quad e_2=\frac{1}{\sqrt{y^2(s)+c^2}}\frac{\partial}{\partial t},$$
$$e_3=(-\dot{y}(s),0,\dot{x}(s)\cos t,\dot{x}(s)\sin t),\quad e_4=\frac{(0,-y(s),-c\sin t,c\cos t)}{\sqrt{y^2(s)+c^2}},$$

where dot denotes differentiation with respect to s. Obviously, e_1,e_2 are tangent, e_3,e_4 are normal, and $\omega_1=ds$, $\omega_2=\sqrt{y^2(s)+c^2}dt$. A straightforward computation shows that e_3 is parallel to the mean curvature vector and

$$\omega_{13}=k_1\omega_1,\quad \omega_{23}=k_2\omega_2,\quad \omega_{14}=\mu\omega_1+\rho\omega_2,\quad \omega_{24}=\rho\omega_1-\mu\omega_2,$$

where

$$k_1=\dot{x}\ddot{y}-\dddot{x}y,\quad k_2=-\frac{\dot{x}y}{y^2+c^2},\quad \mu=0\quad\text{and}\quad \rho=\frac{c\dot{y}}{y^2+c^2}.$$

In addition, we get

$$\omega_{34}=\frac{c\dot{x}}{y^2+c^2}\omega_2.$$

The immersion f satisfies $\dim N_1 = 2$ if and only if $\dot{y} \neq 0$. It is clear that f fulfills (3.10). Moreover, the 1-form Ω is given by

$$\Omega = \frac{k_2 - k_1}{\rho}\omega_{34}$$

and satisfies (3.11). Hence, f is locally *III*-deformable if and only if (3.12) is fulfilled or equivalently

$$\left(\frac{\dot{x}\,(k_2 - k_1)}{\rho\sqrt{y^2 + c^2}}\right)^{\cdot} + c\sqrt{y^2 + c^2}\left(\frac{\dot{x}\,(k_2 - k_1)}{\rho\,(y^2 + c^2)}\right)^2 = 0. \tag{6.3}$$

Viewing the curve $(x(s), y(s))$ as a graph of a function $y(x)$ over x, the differential equation (6.3) is transformed to

$$\begin{aligned}&\left\{\frac{\sqrt{c^2 + y^2}}{y'\sqrt{1 + (y')^2}}\left(\frac{y}{c^2 + y^2} + \frac{y''}{1 + (y')^2}\right)\right\}'\\ &\quad = \frac{\sqrt{c^2 + y^2}}{c(y')^2\sqrt{1 + (y')^2}}\left(\frac{y}{c^2 + y^2} + \frac{y''}{1 + (y')^2}\right)^2,\end{aligned} \tag{6.4}$$

where prime denotes differentiation with respect to x. The condition $\dim N_1 = 2$ is equivalent to $y' \neq 0$. As in Example 1, any solution of (6.4) with appropriate initial conditions gives rise to a non-pseudoumbilical locally *III*-deformable surface in $\mathbb{R}^4$ with $\dim N_1 = 2$.

Remark 3 We do not know whether Examples 1 and 2 are the only non-pseudoumbilical locally *III*-deformable surfaces in $\mathbb{R}^4$ with $\dim N_1 = 2$.

Remark 4 Chen [1] introduced the notion of allied mean curvature. In the notation of Sect. 3, the allied mean curvature $\mathcal{A}(\overrightarrow{H})$ of an isometric immersion $f : M^2 \to \mathbb{R}^4$ with nowhere zero mean curvature is given by

$$\mathcal{A}(\overrightarrow{H}) = \frac{H}{2}\mathrm{trace}(A_3A_4)e_4.$$

The immersion f is called $\mathcal{A}$-immersion if the allied mean curvature $\mathcal{A}(\overrightarrow{H})$ vanishes identically. The immersions given in Examples 1 and 2 are indeed non-pseudoumbilical $\mathcal{A}$-immersions and were inspired by [8].

Remark 5 Let $f : M^2 \to \mathbb{R}^4$ be an isometric immersion with nowhere zero mean curvature and $\dim N_1 = 2$. From Proposition 3 it follows that, if M^2 is compact and the allied mean curvature $\mathcal{A}(\overrightarrow{H})$ is nowhere zero, then f is not locally *III*-deformable.

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