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Calculus of Variations

Positive ground states for a system of Schrödinger equations with critically growing nonlinearities

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Abstract We study the following problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-2}v & \text{in} \quad \Omega, \\ -\Delta v = \mu v^{2^*-1} + u^{2^*-1} & \text{in} \quad \Omega, \\ u > 0, v > 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 4$, $2^* = 2N/(N-2)$, $\lambda \in \mathbb{R}$ and $\mu \geq 0$ and we obtain existence and nonexistence results, depending on the value of the parameters λ and μ.

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1 Introduction

In the last years, nonlinear elliptic systems have been intensively studied by many authors and results, also for semiclassical states and in the singularly perturbed settings, have been

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obtained (see, for instance, [2,3,7–9,15,18,20,21,23–27,30] and references therein). This kind of systems appears if we look for solitary waves of suitable time-dependent nonlinear Schrödinger systems which arise in many physical problems, especially in nonlinear optics (see e.g. [1]) and in Hartree–Fock theory (see e.g. [16]).

In this paper we deal with the problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-2}v & \text{in} \quad \Omega, \\ -\Delta v = \mu v^{2^*-1} + u^{2^*-1} & \text{in} \quad \Omega, \\ u > 0, v > 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 4$, $2^* = 2N/(N-2)$, $\lambda \in \mathbb{R}$ and $\mu \geq 0$.

If $\mu = 0$, problem $\mathcal{P}$ is an N-dimensional variant of the critical problem studied in [5], where the authors, following the *classical* approach in the Schrödinger–Poisson or in the Klein–Gordon–Maxwell systems (see [5] and references therein), use the so-called *reduction method*, namely, the second equation has a unique solution for a given u and it is possible to put it in the first equation, reducing the system to a single *nonlocal* equation. In [5], the energy functional has the Mountain Pass geometry and the classical approach due to Brezis–Nirenberg [10] can be adopted.

However, if $\mu > 0$, the reduction argument can be no longer applied since the map $H_0^1(\Omega) \ni u \mapsto v_u \in H_0^1(\Omega)$, where v_u is a solution to the problem

$$\begin{cases} -\Delta v = \mu |v|^{2^*-2}v + |u|^{2^*-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

is not necessarily well-defined. Recall, indeed, that if $u \not\equiv 0$ and $\mu > 0$, then (1.1) may have at least two solutions (see [29]) or no solution (see [14,22,32]).

We look for solutions of $\mathcal{P}$ as critical points of the C^1-functional $\mathcal{J} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{J}(u, v) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 + \frac{1}{2(2^*-1)}\int_\Omega |\nabla v|^2$$
$$-\frac{\mu}{2^*(2^*-1)}\int_\Omega |v|^{2^*} - \frac{1}{2^*-1}\int_\Omega |u|^{2^*-1}v.$$

We are interested in *nontrivial* solutions of $\mathcal{P}$, namely solutions $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with both $u \not\equiv 0$ and $v \not\equiv 0$. Actually, in this kind of system, one can consider also the so-called *semi-trivial* solutions, i.e. solutions $(u, 0)$ with $u \not\equiv 0$ or $(0, v)$ with $v \not\equiv 0$. We observe that, for our problem $\mathcal{P}$, in the first case the second equation of $\mathcal{P}$ implies that $u \equiv 0$, while, in the second case, our system $\mathcal{P}$ reduces to the well-known equation

$$-\Delta v = \mu v^{2^*-1}, \quad \mu > 0 \tag{1.2}$$

and existence results to (1.2), under topology requirements on Ω, have been given, see [6,10].

In particular, we are interested in positive *ground states* of $\mathcal{P}$, namely solutions that minimize J on the Nehari manifold

$$\mathcal{N} := \{(u, v) \in (H_0^1(\Omega) \times H_0^1(\Omega))\backslash\{(0, 0)\} \mid \mathbf{G}(u, v) = (0, 0)\}, \tag{1.3}$$

where

$$\mathbf{G}(u, v) = \left(\|\nabla u\|_2^2 - \lambda\|u\|_2^2 - \int_\Omega |u|^{2^*-1}v, \|\nabla v\|_2^2 - \mu\|v\|_{2^*}^{2^*} - \int_\Omega |u|^{2^*-1}v\right)$$

and $\|\cdot\|_p$ stands for the standard norm in $L^p(\Omega)$.

Let

$$\mathbb{I}_N = \begin{cases} [0, \sqrt{6}/9] & \text{if} \quad N = 4, \\ [0, \mu^*] & \text{if} \quad N = 5, \\ [0, 1] & \text{if} \quad N = 6, \\ [0, +\infty[& \text{if} \quad N \geq 7, \end{cases}$$

where $\mu^* > 0$ is defined in Theorem 2.10. Our principal aim is to prove the following result.

Theorem 1.1 *If $\mu \in \mathbb{I}_N$ and $\lambda \in (0, \lambda_1(\Omega))$, then problem $\mathcal{P}$ has a ground state solution.*

Due to the presence of two critical terms in the functional $\mathcal{J}$, whose sum may change sign, there are some difficulties in estimation of the Mountain Pass level for which Palais–Smale sequences are convergent. Therefore the classical approach by Brezis and Nirenberg in [10], seems to be difficult to adopt. Moreover employing the Nehari manifold technique for a system of equations like e.g. in [8,11–13,15,21,27,30] one might expect that for any nontrivial (u, v), there are unique $s_0, t_0 > 0$ such that $\mathcal{J}(s_0u, t_0v) = \max_{s,t\geq 0} \mathcal{J}(su, tv)$. However not all functions can be projected on $\mathcal{N}$ due to the sign-changing nonlinearity. Thus, in order to obtain Theorem 1.1, we proceed as follows. First of all, in Sect. 2, we consider the *limit* case ($\Omega = \mathbb{R}^N$ and $\lambda = 0$), which, as usual, plays a crucial role in comparison of the ground state levels and we construct ground states for this last problem by means of the Aubin–Talenti instantons [4,28]. In Sect. 2.1 we provide results concerning the limiting case for $N = 4$, in Sect. 2.2 we consider the remaining cases $N \geq 5$. Then, in Sect. 3, we restrict our considerations to a set $\mathcal{A}$ of *admissible* pairs (see (3.1)) such that any function in $\mathcal{A}$ can be projected onto $\mathcal{N}$. Next we observe that almost all elements of a Palais–Smale sequence of $\mathcal{J}$ are admissible and can be projected on the appropriate Nehari manifold of the limiting problem. This enable us to compare the ground state level with the Mountain Pass level of $\mathcal{P}$ using Lemma 3.2. Finally we get a nontrivial weak limit of the Palais–Smale sequence, in which $\mathcal{J}$ attains its ground state level. We note that obtaining the positivity of solutions to $\mathcal{P}$ is not straightforward since $\mathcal{J}(u, v) \neq \mathcal{J}(|u|, |v|)$. Moreover, a standard procedure based on replacing u and v by the positive parts u_+ and v_+ in the nonlinear terms in $\mathcal{J}$ does not work since the obtained functional is not of C^1-class. These difficulties are overcome at the end of Sect. 3 by defining a suitable C^1-functional $\mathcal{J}_+$ (see (3.6)) and replacing Palais–Smale sequences by nonnegative ones.

Finally Sect. 4 is devoted to the following nonexistence results, which shows, in a certain sense, the optimality of the hypotheses in Theorem 1.1. Let

$$\mu_N := \begin{cases} \dfrac{2(N-2)}{N+2}\left(\dfrac{6-N}{N+2}\right)^{\frac{6-N}{2(N-2)}} & \text{if} \quad N = 4, 5, \\ 1 & \text{if} \quad N = 6. \end{cases} \tag{1.4}$$

We have

Theorem 1.2 *Problem $\mathcal{P}$ has no solution provided that one of the following conditions holds:*

(1) $\mu > \mu_N$ *and* $\lambda \leq 0$ ($N = 4, 5, 6$)*;*
(2) $\mu \in \mathbb{R}$ *and* $\lambda \geq \lambda_1(\Omega)$*;*
(3) $\mu \in \mathbb{R}$, $\lambda \leq 0$ *and Ω is smooth and starshaped.*

In the paper C denotes a generic positive constant which can change from line to line.

2 The limit problem

First of all, let us recall some well known facts. Let S be the best constant such that

$$S\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{2/2^*}\leq\int_{\mathbb{R}^N}|\nabla u|^2\quad\text{for all } u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\tag{2.1}$$

and let us consider the Aubin–Talenti instantons

$$U_{\varepsilon,y}(x)=[N(N-2)]\left(\frac{\varepsilon}{\varepsilon^2+|x-y|^2}\right)^{(N-2)/2},$$

with $\varepsilon>0$, $y\in\mathbb{R}^N$ (see [4,28]). It is well known that the functions $U_{\varepsilon,y}\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ are solutions of

$$-\Delta u=|u|^{2^*-2}u\text{ in }\mathbb{R}^N,\tag{2.2}$$

satisfy

$$\int_{\mathbb{R}^N}|\nabla U_{\varepsilon,y}|^2=\int_{\mathbb{R}^N}|U_{\varepsilon,y}|^{2^*}=S^{N/2}$$

and $\{U_{\varepsilon,y}\in\mathcal{D}^{1,2}(\mathbb{R}^N)|,\ \varepsilon>0,\ y\in\mathbb{R}^N4\}$ consists of all positive solutions of (2.2).

In order to estimate the energy levels of $\mathcal{J}$, in this section we consider the *limit* system

$$\begin{cases}-\Delta u=|u|^{2^*-3}uv & \text{on}\quad\mathbb{R}^N,\\ -\Delta v=\mu|v|^{2^*-2}v+|u|^{2^*-1} & \text{on}\quad\mathbb{R}^N,\\ u,v\in\mathcal{D}^{1,2}(\mathbb{R}^N)\end{cases}\tag{2.3}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)=\{u\in L^{2^*}(\mathbb{R}^N)\mid|\nabla u|\in L^2(\mathbb{R}^N)\}$, equipped with the norm $(\int_{\mathbb{R}^N}|\nabla\cdot|^2)^{1/2}$. We look for nontrivial solutions of (2.3) as critical points of the functional

$$\begin{aligned}\mathcal{J}_0(u,v)=&\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+\frac{1}{2(2^*-1)}\int_{\mathbb{R}^N}|\nabla v|^2\\ &-\frac{\mu}{2^*(2^*-1)}\int_{\mathbb{R}^N}|v|^{2^*}-\frac{1}{2^*-1}\int_{\mathbb{R}^N}|u|^{2^*-1}v\end{aligned}$$

defined in $\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathcal{D}^{1,2}(\mathbb{R}^N)$. In particular, we are interested to ground state solutions of (2.3) of the form $(kU_{\varepsilon,y},lU_{\varepsilon,y})$ with $k,l>0$. So we consider

$$\mathcal{N}_0:=\{(u,v)\in(\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathcal{D}^{1,2}(\mathbb{R}^N))\backslash\{(0,0)\}\mid\mathbf{G}_0(u,v)=(0,0)\}$$

where

$$\mathbf{G}_0(u,v)=\left(\int_{\mathbb{R}^N}|\nabla u|^2-\int_{\mathbb{R}^N}|u|^{2^*-1}v,\int_{\mathbb{R}^N}|\nabla v|^2-\mu\int_{\mathbb{R}^N}|v|^{2^*}-\int_{\mathbb{R}^N}|u|^{2^*-1}v\right)$$

and

$$\mathcal{N}_0':=\{(u,v)\in(\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathcal{D}^{1,2}(\mathbb{R}^N))\backslash\{(0,0)\}\mid H_0(u,v)=0\}$$

where

$$H_0(u,v)=\int_{\mathbb{R}^N}|\nabla u|^2+\frac{1}{2^*-1}\int_{\mathbb{R}^N}|\nabla v|^2-\frac{2^*}{2^*-1}\int_{\mathbb{R}^N}|u|^{2^*-1}v-\frac{\mu}{2^*-1}\int_{\mathbb{R}^N}|v|^{2^*}.$$

Of course $\mathcal{N}_0$ and $\mathcal{N}_0'$ are C^1-manifolds since, for all $(u, v) \in \mathcal{N}_0$,

$$\mathbf{G}_0'(u, v)[u, v] = \left((2 - 2^*) \int_{\mathbb{R}^N} |\nabla u|^2, (2 - 2^*) \int_{\mathbb{R}^N} |\nabla v|^2\right) \neq (0, 0)$$

and, for all $(u, v) \in \mathcal{N}_0'$,

$$H_0'(u, v)[u, v] = (2 - 2^*) \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2 - 2^*}{2^* - 1} \int_{\mathbb{R}^N} |\nabla v|^2 \neq 0.$$

Let us define

$$A := \inf_{(u,v)\in\mathcal{N}_0} \mathcal{J}_0(u, v) \quad \text{and} \quad A' := \inf_{(u,v)\in\mathcal{N}_0'} \mathcal{J}_0(u, v).$$

Since $\mathcal{N}_0 \subset \mathcal{N}_0'$, then

$$A' \leq A.$$

In the next subsections we find ground state for (2.3), we show that $A = A'$ and we evaluate exactly the ground state level.

2.1 The limit problem for $N = 4$

In this subsection focus on the case $N = 4$.

Lemma 2.1 *If* $0 \leq \mu \leq 2\sqrt{3}/9$, *then* $\mathcal{N}_0 \neq \emptyset$.

Proof Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^4)$, $u > 0$ and $\bar{m}$ be a strictly positive solution of

$$m^3 - m + \mu = 0$$

which exists since, by $0 \leq \mu \leq 2\sqrt{3}/9$, the function

$$f(m) = m^3 - m + \mu \tag{2.4}$$

satisfies $f(0) = \mu \geq 0$ and, in the minimum point $\sqrt{3}/3$, $f(\sqrt{3}/3) = \mu - 2\sqrt{3}/9 \leq 0$. Then

$$\left(\sqrt{\bar{m}\left(\int_{\mathbb{R}^4} |\nabla u|^2\right)\left(\int_{\mathbb{R}^4} u^4\right)^{-1}} u, \sqrt{\left(\int_{\mathbb{R}^4} |\nabla u|^2\right)\left(\bar{m}\int_{\mathbb{R}^4} u^4\right)^{-1}} u\right) \in \mathcal{N}_0.$$

□

To state a condition that allows to get $A' = A$, we need the following technicalities.

Lemma 2.2 *The system*

$$\begin{cases} kl = 1, \\ \mu l^3 + k^3 = l \\ k, l > 0 \end{cases} \tag{2.5}$$

has a solution if and only if $\mu \leq 2\sqrt{3}/9$. *In particular, if* $\mu \leq 0$ *or* $\mu = 2\sqrt{3}/9$, *then the solution is unique and, if* $0 < \mu < 2\sqrt{3}/9$, *then system* (2.5) *has two different solutions.*

Proof We argue as in the Proof of Lemma 2.1. Indeed the function f has a unique strictly positive zero if $\mu \leq 0$ or $\mu = 2\sqrt{3}/9$ and two strictly positive zeros if $0 < \mu < 2\sqrt{3}/9$. Then, denoted with $\bar{m}$ such zeros, we have that $(k, l) = (\sqrt{\bar{m}}, 1/\sqrt{\bar{m}})$ satisfy system (2.5).
□

Lemma 2.3 *Let $k, l > 0$ satisfy*

$$k^2 + \frac{1}{3}l^2 \le \frac{4}{3}k^3 l + \frac{\mu}{3}l^4. \tag{2.6}$$

(1) *If $\mu = 0$ then*

$$\frac{4}{3} \le k^2 + \frac{1}{3}l^2.$$

(2) *If $\mu \in (0, \sqrt{6}/9)$ then*

$$k_2^2 + \frac{1}{3}l_2^2 = \min_{i=1,2}\left\{k_i^2 + \frac{1}{3}l_i^2\right\} \le k^2 + \frac{1}{3}l^2 \tag{2.7}$$

and

$$k_2^2 + \frac{1}{3}l_2^2 < \frac{1}{3\mu},$$

where (k_i, l_i) are the solutions of the system (2.5), $k_1 < k_2$ and $l_2 < l_1$.

(3) *If $\mu = \sqrt{6}/9$ then*

$$k_2^2 + \frac{1}{3}l_2^2 = \frac{1}{3\mu} \le k^2 + \frac{1}{3}l^2.$$

(4) *If $\mu \in (\sqrt{6}/9, 2\sqrt{3}/9]$ then*

$$\frac{1}{3\mu} < k^2 + \frac{1}{3}l^2.$$

Proof Let us fix $k, l > 0$ satisfying (2.6) and

$$\bar{k} = k\sqrt{\frac{3k^2 + l^2}{l(4k^3 + \mu l^3)}} \quad \text{and} \quad \bar{l} = \sqrt{\frac{l(3k^2 + l^2)}{4k^3 + \mu l^3}}.$$

We have that

$$0 < \bar{k} \le k, \quad 0 < \bar{l} \le l, \tag{2.8}$$

$$\frac{k}{l} = \frac{\bar{k}}{\bar{l}},$$

$$\bar{k}^2 + \frac{1}{3}\bar{l}^2 = \frac{4}{3}\bar{k}^3\bar{l} + \frac{\mu}{3}\bar{l}^4 \tag{2.9}$$

By (2.8) we have that

$$\bar{k}^2 + \frac{1}{3}\bar{l}^2 \le k^2 + \frac{1}{3}l^2.$$

So it is sufficient to prove (2.7) for $(\bar{k}, \bar{l})$.

We notice that, since the system

$$\begin{cases} \bar{k} = m\bar{l} \\ 3\bar{k}^2 + \bar{l}^2 = 4\bar{k}^3\bar{l} + \mu\bar{l}^4 \\ \bar{l}, \bar{k} > 0 \end{cases}$$

admits a unique solution

$$\bar{k} = m\sqrt{\frac{3m^2+1}{4m^3+\mu}}, \quad \bar{l} = \sqrt{\frac{3m^2+1}{4m^3+\mu}}, \tag{2.10}$$

for every $m > 0$, the curve given by (2.9) (for $\bar{k}, \bar{l} > 0$), can be parametrized by m using (2.10). Thus we consider

$$\bar{k}^2 + \frac{1}{3}\bar{l}^2 = \frac{(3m^2+1)^2}{3(4m^3+\mu)} =: \psi(m), \quad m > 0. \tag{2.11}$$

If $\mu = 0$ then the function ψ has a unique global minimum point (on the positive halfline) in 1 and $\psi(1) = 4/3$.

If $0 < \mu < \frac{\sqrt{6}}{9}$ then the function ψ admits two critical points $m_1 < m_2$ which solve the equation $m^3 - m + \mu = 0$ and the global minimum m_2 (on the positive halfline) satisfies

$$\psi(m_2) < \frac{1}{3\mu} = \lim_{m\to 0^+} \psi(m).$$

Moreover, if we take

$$k_i := m_i\sqrt{\frac{3m_i^2+1}{4m_i^3+\mu}} = \sqrt{m_i} \quad \text{and} \quad l_i := \sqrt{\frac{3m_i^2+1}{4m_i^3+\mu}} = \frac{1}{\sqrt{m_i}}, \quad i = 1, 2,$$

we have that (k_i, l_i) solve system (2.5), $k_1 < k_2, l_2 < l_1$ and

$$\psi(m_2) = k_2^2 + \frac{1}{3}l_2^2.$$

If $\mu = \frac{\sqrt{6}}{9}$ then, for any $m > 0$,

$$\psi(m_2) = \lim_{m\to 0^+} \psi(m) = \frac{1}{3\mu}.$$

If $\frac{\sqrt{6}}{9} < \mu \le \frac{2\sqrt{3}}{9}$ then

$$\psi(m) > \lim_{m\to 0^+} \psi(m) = \frac{1}{3\mu}.$$

□

Before we prove the main results of this section, we show the following preliminary properties.

Proposition 2.4 *Let* $\mu \in [0, \sqrt{6}/9)$.

(1) *If* $\mu = 0$, $\mathcal{N}_0'$ *does not contain semitrivial couples.*
(2) *If* $\mu \in (0, \sqrt{6}/9)$, $\mathcal{N}_0'$ *does not contain semitrivial couples* $(u, 0)$ *and*

$$A' < \inf_{(0,v)\in\mathcal{N}_0'} \mathcal{J}_0(0, v). \tag{2.12}$$

Proof Statement (1) and the first part of (2) are obvious. So it remains to prove (2.12). We notice that $\mathcal{N}_0'$ contains couples $(0, v)$ with $v \in \mathcal{D}^{1,2}(\mathbb{R}^4)$: it is sufficient to take $v = \mu^{-1/2}U_{\varepsilon,y}$.

Let $(0, v) \in \mathcal{N}_0'$. We have that

$$H_0(0, v) = \frac{1}{3}\int_{\mathbb{R}^4} |\nabla v|^2 - \frac{\mu}{3}\int_{\mathbb{R}^4} v^4 = 0$$

and so

$$\mathcal{J}_0(0, v) = \frac{1}{12}\int_{\mathbb{R}^4} |\nabla v|^2.$$

Moreover, for every $s > 0$,

$$(t(s)sv, t(s)v) \in \mathcal{N}_0' \quad \text{with } t(s) := \left[\frac{(3s^2+1)\mu}{4s^3+\mu}\right]^{1/2}$$

and then

$$A' \leq \mathcal{J}_0(t(s)sv, t(s)v) = \frac{1}{12}\frac{(3s^2+1)^2\mu}{4s^3+\mu}\int_{\mathbb{R}^4} |\nabla v|^2.$$

Thus, passing to the infimum,

$$A' \leq \frac{(3s^2+1)^2\mu}{4s^3+\mu} \inf_{(0,v)\in\mathcal{N}_0'} \mathcal{J}_0(0, v)$$

and we conclude observing that, since $\mu \in (0, \sqrt{6}/9)$,

$$\left.\frac{(3s^2+1)^2\mu}{4s^3+\mu}\right|_{s=2/(9\mu)} < 1.$$

□

Corollary 2.5 *If* $\mu \in [0, \sqrt{6}/9)$ *and* A' *is attained for some* $(u, v) \in \mathcal{N}_0'$*, then* $u \neq 0$ *and* $v \neq 0$.

Using the notations introduced before we are ready to prove the following results.

Theorem 2.6 *If* $\mu \in (0, \sqrt{6}/9)$*, then, for every* $\varepsilon > 0$ *and* $y \in \mathbb{R}^4$*, we have that* $(k_2 U_{\varepsilon,y}, l_2 U_{\varepsilon,y})$ *is a ground state solution of* (2.3) *and*

$$\mathcal{J}_0(k_2 U_{\varepsilon,y}, l_2 U_{\varepsilon,y}) = A = A' = \frac{1}{4}\left(k_2^2 + \frac{1}{3}l_2^2\right)S^2.$$

Proof Let $\mu \in (0, \sqrt{6}/9)$. Since (k_2, l_2) satisfies the system (2.5), then it can be easily shown that for every $y \in \mathbb{R}^4$ we have that $\mathcal{J}_0'(k_2 U_{\varepsilon,y}, l_2 U_{\varepsilon,y}) = 0$ and so $(k_2 U_{\varepsilon,y}, l_2 U_{\varepsilon,y}) \in \mathcal{N}_0$. Hence

$$A' \leq A \leq \mathcal{J}_0(k_2 U_{\varepsilon,y}, l_2 U_{\varepsilon,y}) = \frac{1}{4}\left(k_2^2 + \frac{1}{3}l_2^2\right)S^2.$$

Let $\{(u_n, v_n)\} \subset \mathcal{N}_0'$ be a minimizing sequence, i.e. such that $\mathcal{J}_0(u_n, v_n) \to A'$. We notice that we can assume $u_n \neq 0$ and $v_n \neq 0$. Indeed, if $(u_n, v_n) \in \mathcal{N}_0'$, as observed before (see (2) of Proposition 2.4), $v_n \neq 0$ and the existence of a subsequence such that $u_n = 0$ contradicts (2.12).

Since

$$S\left[\left(\int_{\mathbb{R}^4} u_n^4\right)^{1/2}+\frac{1}{3}\left(\int_{\mathbb{R}^4} v_n^4\right)^{1/2}\right]\le\int_{\mathbb{R}^4}|\nabla u_n|^2+\frac{1}{3}\int_{\mathbb{R}^4}|\nabla v_n|^2=\frac{4}{3}\int_{\mathbb{R}^4}u_n^3v_n+\frac{\mu}{3}\int_{\mathbb{R}^4}v_n^4$$
$$\le\frac{4}{3}\left(\int_{\mathbb{R}^4}u_n^4\right)^{3/4}\left(\int_{\mathbb{R}^4}v_n^4\right)^{1/4}+\frac{\mu}{3}\int_{\mathbb{R}^4}v_n^4,$$

then

$$\left[\frac{1}{\sqrt{S}}\left(\int_{\mathbb{R}^4}u_n^4\right)^{1/4}\right]^2+\frac{1}{3}\left[\frac{1}{\sqrt{S}}\left(\int_{\mathbb{R}^4}v_n^4\right)^{1/4}\right]^2$$
$$\le\frac{4}{3}\left[\frac{1}{\sqrt{S}}\left(\int_{\mathbb{R}^4}u_n^4\right)^{1/4}\right]^3\left[\frac{1}{\sqrt{S}}\left(\int_{\mathbb{R}^4}v_n^4\right)^{1/4}\right]+\frac{\mu}{3}\left[\frac{1}{\sqrt{S}}\left(\int_{\mathbb{R}^4}v_n^4\right)^{1/4}\right]^4.$$

Thus, by (2) of Lemma 2.3 we get

$$k_2^2+\frac{1}{3}l_2^2\le\frac{1}{S}\left[\left(\int_{\mathbb{R}^4}u_n^4\right)^{1/2}+\frac{1}{3}\left(\int_{\mathbb{R}^4}v_n^4\right)^{1/2}\right].$$

Therefore

$$A'+o_n(1)=\mathcal{J}_0(u_n,v_n)=\frac{1}{4}\left(\int_{\mathbb{R}^4}|\nabla u_n|^2+\frac{1}{3}\int_{\mathbb{R}^4}|\nabla v_n|^2\right)$$
$$\ge\frac{S}{4}\left[\left(\int_{\mathbb{R}^4}u_n^4\right)^{1/2}+\frac{1}{3}\left(\int_{\mathbb{R}^4}v_n^4\right)^{1/2}\right]$$
$$\ge\frac{1}{4}\left(k_2^2+\frac{1}{3}l_2^2\right)S^2$$

and thus

$$A'=\frac{1}{4}\left(k_2^2+\frac{1}{3}l_2^2\right)S^2.$$

□

Proceeding as in the proof of Theorem 2.6 and applying (1) of Proposition 2.4 and (1) of Lemma 2.3 we can prove

Theorem 2.7 *If $\mu=0$, then, for every $\varepsilon>0$ and $y\in\mathbb{R}^4$, we have that $(U_{\varepsilon,y},U_{\varepsilon,y})$ is a ground state solution of* (2.3) *and*

$$\mathcal{J}_0(U_{\varepsilon,y},U_{\varepsilon,y})=A=A'=\frac{S^2}{3}.$$

Moreover we have

Theorem 2.8 *If $\mu\in[\sqrt{6}/9,2\sqrt{3}/9]$, then for every $\varepsilon>0$ and $y\in\mathbb{R}^4$, $(0,\frac{1}{\sqrt{\mu}}U_{\varepsilon,y})$ is a ground state solution of* (2.3) *and*

$$\mathcal{J}_0\left(0,\frac{1}{\sqrt{\mu}}U_{\varepsilon,y}\right)=A'=A=\frac{1}{12\mu}S^2.$$

Moreover if $\mu\in(\sqrt{6}/9,2\sqrt{3}/9]$, then any minimizer (u,v) of $\mathcal{J}_0$ on $\mathcal{N}_0'$ is semitrivial, i.e. $u=0$.

Proof It is simple to verify that $(0, \frac{1}{\sqrt{\mu}}U_{\varepsilon,y})$ solves (2.3) and

$$\mathcal{J}_0\left(0, \frac{1}{\sqrt{\mu}}U_{\varepsilon,y}\right) = \frac{1}{12\mu}S^2.$$

Thus

$$A' \leq A \leq \frac{1}{12\mu}S^2.$$

Now we consider a minimizing sequence Let $\{(u_n, v_n)\} \subset \mathcal{N}_0'$ (such that $\mathcal{J}_0(u_n, v_n) \to A'$) and we distinguish two cases: if $u_n \neq 0$ we can proceed as in the proof of Theorem 2.6 getting, by (3) and (4) of Lemma 2.3,

$$\frac{1}{3\mu} \leq \frac{1}{S}\left[\left(\int_{\mathbb{R}^4} u_n^4\right)^{1/2} + \frac{1}{3}\left(\int_{\mathbb{R}^4} v_n^4\right)^{1/2}\right].$$

Thus

$$\mathcal{J}_0(u_n, v_n) \geq \frac{1}{12\mu}S^2. \tag{2.13}$$

If $u_n = 0$, since $(0, v_n) \in \mathcal{N}_0'$ and v_n satisfies (2.1), we obtain

$$\|v_n\|_4^2 \geq \frac{1}{\mu}S$$

and so the estimate (2.13) holds too. Thus the first part of the statement is proved.

Finally, suppose by contradiction that there exists a minimizer $(u, v) \in \mathcal{N}_0'$ of $\mathcal{J}_0$ on $\mathcal{N}_0'$ with $u \neq 0$. Then $v \neq 0$ and similarly as above, by (4) of Lemma 2.3,

$$\frac{1}{3\mu} < \frac{1}{S}\left[\left(\int_{\mathbb{R}^4} u^4\right)^{1/2} + \frac{1}{3}\left(\int_{\mathbb{R}^4} v^4\right)^{1/2}\right].$$

Thus

$$A' = \mathcal{J}_0(u, v) \geq \frac{S}{4}\left[\left(\int_{\mathbb{R}^4} u^4\right)^{1/2} + \frac{1}{3}\left(\int_{\mathbb{R}^4} v^4\right)^{1/2}\right] > \frac{1}{12\mu}S^2 = A'$$

and we get a contradiction. □

Finally, by (3) of Lemma 2.3 and Theorem 2.8 we have

Theorem 2.9 *If $\mu = \sqrt{6}/9$, then, for every $\varepsilon > 0$ and $y \in \mathbb{R}^4$, we have that $(k_2 U_{\varepsilon,y}, l_2 U_{\varepsilon,y})$ is a nontrivial ground state of* (2.3).

2.2 The limit problem for $N \geq 5$

In this subsection we study the limit problem for a general $N \geq 5$. We notice that in the previous subsection the key points consist of the existence of a zero of the function f in (2.4) (to prove that $\mathcal{N}_0$ is nonempty), the solutions of the system (2.5), the condition (2.6), and the global minimum of the function ψ in (2.11). For a general N, the mentioned issues take the following form

$$f_N(m) = m^{2^*-1} - m^{2^*-3} + \mu, \quad m > 0, \tag{2.14}$$

$$\begin{cases} k^{2^*-3}l = 1, \\ \mu l^{2^*-1} + k^{2^*-1} = l \\ k, l > 0, \end{cases} \tag{2.15}$$

$$k^2 + \frac{1}{2^*-1}l^2 \leq \frac{2^*}{2^*-1}k^{2^*-1}l + \frac{\mu}{2^*-1}l^{2^*}, \tag{2.16}$$

and

$$\psi_N(m) = \frac{((2^*-1)m^2+1)^{\frac{2^*}{2^*-2}}}{(2^*-1)(2^* m^{2^*-1}+\mu)^{\frac{2}{2^*-2}}} \quad m > 0. \tag{2.17}$$

If $N = 5$, the function f_5 in (2.14) has the same geometry of the function f in (2.4). Thus we can repeat the similar arguments used in the Sect. 2.1 and we have

Theorem 2.10 *There exists $\mu^* \in (0, 6/(7\sqrt[6]{7}))$ such that:*

- *if $\mu = 0$, then, for every $\varepsilon > 0$ and $y \in \mathbb{R}^5$, we have that $(U_{\varepsilon,y}, U_{\varepsilon,y})$ is a ground state solution of (2.3) and*

$$\mathcal{J}_0(U_{\varepsilon,y}, U_{\varepsilon,y}) = A = A' = \frac{2}{7}S^{5/2};$$

- *if $\mu \in (0, \mu^*)$, then, for every $\varepsilon > 0$ and $y \in \mathbb{R}^5$, we have that $(k_2U_{\varepsilon,y}, l_2U_{\varepsilon,y})$ is a ground state solution of (2.3) and*

$$\mathcal{J}_0(k_2U_{\varepsilon,y}, l_2U_{\varepsilon,y}) = A = A' = \frac{1}{5}\left(k_2^2 + \frac{3}{7}l_2^2\right)S^{5/2}$$

where (k_i, l_i) are the solutions of the system (2.15), $k_1 < k_2$ and $l_2 < l_1$;
- *if $\mu = \mu^*$, then, for every $\varepsilon > 0$ and $y \in \mathbb{R}^5$, we have that $(k_2U_{\varepsilon,y}, l_2U_{\varepsilon,y})$ is a nontrivial ground state of (2.3);*
- *if $\mu \in [\mu^*, 6/(7\sqrt[6]{7})]$, then for every $\varepsilon > 0$ and $y \in \mathbb{R}^5$, $(0, \frac{1}{\mu^{3/4}}U_{\varepsilon,y})$ is a ground state solution of (2.3) and*

$$\mathcal{J}_0\left(0, \frac{1}{\mu^{3/4}}U_{\varepsilon,y}\right) = A' = A = \frac{3}{35\mu^{3/2}}S^{5/2}.$$

Moreover if $\mu \in (\sqrt{6}/9, 2\sqrt{3}/9]$, then any minimizer (u, v) of $\mathcal{J}_0$ on $\mathcal{N}_0'$ is semitrivial, i.e. $u = 0$.

We notice that, the upper bound on μ to obtain that $\mathcal{N}_0 \neq \emptyset$ for $N = 4, 5$ is given by μ_N in (1.4). For $N \geq 6$ the geometry of function f_N is different and allows us to prove the following results.

Theorem 2.11 *If $N = 6$ and $\mu \in [0, 1]$ then, for every $\varepsilon > 0$ and $y \in \mathbb{R}^6$, we have that $(\sqrt{1-\mu}U_{\varepsilon,y}, U_{\varepsilon,y})$ is a ground state solution of (2.3) and*

$$\mathcal{J}_0(\sqrt{1-\mu}U_{\varepsilon,y}, U_{\varepsilon,y}) = A = A' = \frac{1}{6}\left(\frac{3}{2} - \mu\right)S^3.$$

Proof In this case it is easy to prove that, if $u \in \mathcal{D}^{1,2}(\mathbb{R}^6)$, $u > 0$, then $(\sqrt{1-\mu}u, u) \in \mathcal{N}_0$ and the couple $(\sqrt{1-\mu}, 1)$ is the unique solution of the system (2.15). Thus, arguing as in Lemma 2.3 we can prove that (2.16) implies

$$k^2 + \frac{1}{2}l^2 \geq \frac{3}{2} - \mu.$$

Hence, similarly as in Theorem 2.6 we conclude. □

Theorem 2.12 *If $N \geq 7$ and $\mu \geq 0$ then, for every $\varepsilon > 0$ and $y \in \mathbb{R}^N$, we have that $(\bar{m}^{\frac{1}{2^*-2}} U_{\varepsilon,y}, \bar{m}^{\frac{3-2^*}{2^*-2}} U_{\varepsilon,y})$ is a ground state solution of (2.3) and*

$$\mathcal{J}_0(\bar{m}^{\frac{1}{2^*-2}} U_{\varepsilon,y}, \bar{m}^{\frac{3-2^*}{2^*-2}} U_{\varepsilon,y}) = A = A' = \frac{1}{N}\left(\tilde{k}^2 + \frac{1}{2^*-1}\tilde{l}^2\right) S^{N/2}$$

where $(\tilde{k}, \tilde{l})$ is the unique solution of system (2.15).

Proof Take any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u > 0$. In this case the function f_N in (2.14) is strictly increasing and satisfies

$$\lim_{m\to 0^+} f(m) = -\infty \quad \text{and} \quad \lim_{m\to+\infty} f(m) = +\infty.$$

Thus it admits a unique nontrivial zero $\bar{m}$ and then

$$\left(\left[\bar{m}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{-1}\right]^{\frac{1}{2^*-2}} u,\right.$$
$$\left.\left[\bar{m}^{3-2^*}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{-1}\right]^{\frac{1}{2^*-2}} u\right) \in \mathcal{N}_0$$

and system (2.15) has a unique solution

$$(\tilde{k}, \tilde{l}) = (\bar{m}^{\frac{1}{2^*-2}}, \bar{m}^{\frac{3-2^*}{2^*-2}}).$$

As before we can prove that (2.16) implies

$$k^2 + \frac{1}{2^*-1} l^2 \geq \tilde{k}^2 + \frac{1}{2^*-1}\tilde{l}^2 = \frac{\bar{m}^{\frac{2(3-2^*)}{2^*-2}}}{2^*-1}((2^*-1)\bar{m}^2 + 1)$$

considering the function ψ_N in (2.17) which has a global minimum point at the zero of the function f_N. Hence, arguing as in Theorem 2.6 we conclude. □

3 Positive ground states for $\mathcal{P}$

In this section we investigate the existence of ground states for our problem $\mathcal{P}$ and we prove our main result. First of all we notice that $\mathcal{N}$, defined in (1.3), is a C^1-manifold since

$$\mathbf{G}'(u, v)[u, v] = ((2-2^*)(\|\nabla u\|_2^2 - \lambda\|u\|_2^2), (2-2^*)\|\nabla v\|_2^2) \neq (0, 0)$$

for all $(u, v) \in \mathcal{N}$.

Lemma 3.1 *If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \in \mathbb{I}_N$, then $\mathcal{N} \neq \emptyset$.*

Proof We proceed as before. Let us take $u \in H_0^1(\Omega)$, $u > 0$ and $\bar{m}$ be a strictly positive solution of

$$m^{2^*-1} - \sigma m^{2^*-3} + \mu = 0, \quad \sigma := \frac{\|\nabla u\|_2^2}{\|\nabla u\|_2^2 - \lambda\|u\|_2^2}$$

whose existence can be obtained arguing as in Lemma 2.1 and using that $\sigma > 1$ for $N = 4, 5$, as in Theorem 2.11 for $N = 6$ or as in Theorem 2.12 for $N \geq 7$.

Then

$$((\bar{m}\bar{\sigma})^{\frac{1}{2^*-2}}u, (\bar{m}^{3-2^*}\bar{\sigma})^{\frac{1}{2^*-2}}u) \in \mathcal{N}, \quad \bar{\sigma} := \frac{\|\nabla u\|_2^2 - \lambda\|u\|_2^2}{\|u\|_{2^*}^{2^*}}.$$

□

We note that, if $N = 4, 5$, arguing in the same way, we can prove that $\mathcal{N} \neq \emptyset$ for $\mu \in [0, \mu_N]$, where μ_N is given by (1.4).

Now, let

$$\mathcal{B} := \inf_{w\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(w(t))$$

where $\Gamma := \{w \in C([0, 1], H_0^1(\Omega) \times H_0^1(\Omega)) |\ w(0) = (0, 0),\ \mathcal{J}(w(1)) < 0\}$. We have

Lemma 3.2 *If* $\lambda > 0$ *and* $\mu \in \mathbb{I}_N$, *then* $\mathcal{B} < A$.

Proof Without loss of generality we can assume that $0 \in \Omega$. Then there exists $R > 0$ such that $\bar{B}_R(0) \subset \Omega$. Let $\chi \in C_0^1(\Omega)$ be a nonnegative function such that $\chi \equiv 1$ on $\bar{B}_R(0)$. For every $\varepsilon > 0$ let us define $U_\varepsilon = \chi U_{\varepsilon,0}$. By [10], see also [31], we have that

$$\|\nabla U_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \quad \|U_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N)$$

and

$$\|U_\varepsilon\|_2^2 \geq C\varphi_N(\varepsilon) + O(\varepsilon^{N-2}) \text{ for } N \geq 5,$$

for some $C > 0$, where

$$\varphi_N(\varepsilon) = \begin{cases} \varepsilon^2|\log\varepsilon| & \text{if} \quad N = 4, \\ \varepsilon^2 & \text{if} \quad N \geq 5. \end{cases}$$

Let $(k, l) \in \mathbb{R}^2$, $k, l > 0$ such that $(kU_{\varepsilon,y}, lU_{\varepsilon,y})$ is a ground state of the limit problem (2.3). and consider $(u_\varepsilon, v_\varepsilon) = (kU_\varepsilon, lU_\varepsilon)$. We have that

$$\|\nabla u_\varepsilon\|_2^2 = k^2 S^{N/2} + O(\varepsilon^{N-2}), \quad \|\nabla v_\varepsilon\|_2^2 = l^2 S^{N/2} + O(\varepsilon^{N-2}),$$
$$\|v_\varepsilon\|_{2^*}^{2^*} = l^{2^*} S^{N/2} + O(\varepsilon^N), \quad \int_\Omega u_\varepsilon^{2^*-1} v_\varepsilon = k^{2^*-1} l S^{N/2} + O(\varepsilon^N)$$

and

$$\|u_\varepsilon\|_2^2 \geq C\varphi(\varepsilon) + O(\varepsilon^{N-2}).$$

Then, since (k, l) satisfies

$$k^2 + \frac{1}{2^*-1} l^2 = \frac{2^*}{2^*-1} k^{2^*-1} l + \frac{\mu}{2^*-1} l^{2^*},$$

we get

$$\begin{aligned}\mathcal{J}(tu_\varepsilon, tv_\varepsilon) &= \frac{1}{2}t^2\|\nabla u_\varepsilon\|_2^2 - \frac{\lambda}{2}t^2\|u_\varepsilon\|_2^2 + \frac{1}{2(2^*-1)}t^2\|\nabla v_\varepsilon\|_2^2 - \frac{\mu}{2^*(2^*-1)}t^{2^*}\|v_\varepsilon\|_{2^*}^{2^*} \\ &\quad - \frac{1}{2^*-1}t^{2^*}\int_\Omega u_\varepsilon^{2^*-1}v_\varepsilon \\ &\le \frac{1}{2}t^2\left(\left(k^2 + \frac{1}{2^*-1}l^2\right)S^{N/2} - \lambda C\varphi(\varepsilon) + O(\varepsilon^{N-2})\right) \\ &\quad - \frac{1}{2^*}t^{2^*}\left(\left(k^2 + \frac{1}{2^*-1}l^2\right)S^{N/2} + O(\varepsilon^N)\right) \\ &= \frac{1}{2}t^2(NA - \lambda C\varphi(\varepsilon) + O(\varepsilon^2)) - \frac{1}{2^*}t^{2^*}(NA + O(\varepsilon^N)).\end{aligned}$$

Let us denote

$$A_\varepsilon = NA - \lambda C\varphi(\varepsilon) + O(\varepsilon^{N-2}), \quad B_\varepsilon = NA + O(\varepsilon^N)$$

and consider

$$f(t) := \frac{A_\varepsilon}{2}t^2 - \frac{B_\varepsilon}{2^*}t^{2^*}.$$

We have that

$$\max_{t>0} f(t) = \frac{1}{N}\left(\frac{A_\varepsilon}{B_\varepsilon^{(N-2)/N}}\right)^{N/2} < A$$

for $\varepsilon > 0$ sufficiently small. Thus

$$\mathcal{B} \le \max_{t>0} \mathcal{J}(tu_\varepsilon, tv_\varepsilon) < A.$$

□

Let us consider

$$\mathcal{N}' = \{(u, v) \in (H_0^1(\Omega) \times H_0^1(\Omega))\backslash\{(0, 0)\} \mid H(u, v) = 0\}$$

where

$$H(u, v) = \|\nabla u\|_2^2 - \lambda\|u\|_2^2 + \frac{1}{2^*-1}\|\nabla v\|_2^2 - \frac{\mu}{2^*-1}\|v\|_{2^*}^{2^*} - \frac{2^*}{2^*-1}\int_\Omega |u|^{2^*-1}v$$

and

$$\mathcal{A} := \left\{(u, v) \in (H_0^1(\Omega) \times H_0^1(\Omega)) \mid \mu\|v\|_{2^*}^{2^*} + 2^*\int_\Omega |u|^{2^*-1}v > 0\right\} \tag{3.1}$$

the set of *admissible* pairs. Note that, if $\lambda \in (0, \lambda_1(\Omega))$, we have that $\mathcal{N}'$ is a C^1-manifold being, for all $(u, v) \in \mathcal{N}'$,

$$H'(u, v)[u, v] = (2 - 2^*)\left(\|\nabla u\|_2^2 - \lambda\|u\|_2^2 + \frac{1}{2^*-1}\|\nabla v\|_2^2\right) \neq 0.$$

Moreover

$$\mathcal{N} \subset \mathcal{N}' \subset \mathcal{A}$$

and, in view of the Hölder inequality and the Sobolev embeddings,

$$H(u, v) \geq \|(u, v)\|^2 - C\|(u, v)\|^{2^*} \tag{3.2}$$

for some constant $C > 0$, where

$$\|(u, v)\|^2 := \|\nabla u\|_2^2 - \lambda\|u\|_2^2 + \frac{1}{2^* - 1}\|\nabla v\|_2^2.$$

We have

Proposition 3.3 *If* $\lambda \in (0, \lambda_1(\Omega))$ *and* $\mu \in \mathbb{I}_N$, *then*

$$\inf_{(u,v)\in\mathcal{N}'} \mathcal{J}(u, v) = \inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} \mathcal{J}(tu, tv) = \mathcal{B} > 0.$$

Proof Let $(u, v) \in \mathcal{A}$ and

$$\bar{t} = \left[\left(\|\nabla u\|_2^2 - \lambda\|u\|_2^2 + \frac{1}{2^* - 1}\|\nabla v\|_2^2\right)\left(\frac{\mu}{2^* - 1}\|v\|_{2^*}^{2^*} + \frac{2^*}{2^* - 1}\int_\Omega |u|^{2^*-1}v\right)^{-1}\right]^{\frac{1}{2^*-2}}.$$

Observe that $(\bar{t}u, \bar{t}v) \in \mathcal{N}'$ and so

$$\mathcal{J}(\bar{t}u, \bar{t}v) \geq \inf_{(u,v)\in\mathcal{N}'} \mathcal{J}(u, v).$$

Moreover $\bar{t}$ is the unique strictly positive real number such that

$$\mathcal{J}(\bar{t}u, \bar{t}v) = \max_{t\geq 0} \mathcal{J}(tu, tv).$$

If $(u, v) \in \mathcal{N}'$, then $\bar{t} = 1$ and, since $\mathcal{N}' \subset \mathcal{A}$, we get

$$\inf_{(u,v)\in\mathcal{N}'} \mathcal{J}(u, v) \geq \inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} \mathcal{J}(tu, tv).$$

Moreover, since if $(u, v) \in \mathcal{A}$, then there is $t > 0$ such that $\mathcal{J}(tu, tv) < 0$,

$$\inf_{(u,v)\in\mathcal{A}} \max_{t\geq 0} \mathcal{J}(tu, tv) \geq \mathcal{B}.$$

Let now $w = (w_1, w_2) \in \Gamma$. Since, for t small, $H(w(t)) > 0$ and

$$H(w(1)) = 2\mathcal{J}(w(1)) - \frac{2}{N(2^* - 1)}\left(\mu\|w_2(1)\|_{2^*}^{2^*} + 2^*\int_\Omega |w_1(1)|^{2^*-1}w_2(1)\right) < 0,$$

we have that there exists $t_1 > 0$ such that $H(w(t_1)) = 0$, namely $w(t_1) \in \mathcal{N}'$.
Then

$$\mathcal{B} \geq \inf_{(u,v)\in\mathcal{N}'} \mathcal{J}(u, v).$$

Finally, note that if $\mathcal{J}(u_n, v_n) \to 0$ and $(u_n, v_n) \in \mathcal{N}'$ then $\|(u_n, v_n)\| \to 0$ which contradicts the inequality (3.2). Thus

$$\inf_{(u,v)\in\mathcal{N}'} \mathcal{J}(u, v) > 0.$$

□

We notice that in this last proof we only need that $\mathcal{N} \neq \emptyset$. Then, if $N = 4, 5$ we can assume $\mu \in [0, \mu_N]$.

Remark 3.4 In the study of elliptic problems involving the Mountain Pass geometry, usually one expects that a Nehari manifold is homeomorphic to the unit sphere (see e.g. [31, Lemma 4.1]). However, due to the sing-changing nonlinearities, this no longer holds in our case, but we have that the map $(u, v) \mapsto (\bar{t}u, \bar{t}v)$, where $\bar{t}$ is given by (3), defines a homeomorphism from $\mathcal{A} \cap S^1$ into $\mathcal{N}'$, with $S^1 := \{(u, v) \in (H_0^1(\Omega) \times H_0^1(\Omega)) |\ \|(u, v)\| = 1\}$.

Before we prove the main result of this section, we show the following preliminary property.

Proposition 3.5 *Let* $\lambda \in (0, \lambda_1(\Omega))$ *and* $\mu \geq 0$. *If a ground state* (u, v) *of* $\mathcal{P}$ *exists, then* (u, v) *nontrivial.*

Proof Let $(u, v) \in \mathcal{N}$ be such that

$$\mathcal{J}(u, v) = \inf_{\mathcal{N}} \mathcal{J}.$$

If $v = 0$, then $\langle \mathcal{J}'(u, 0), (u, 0)\rangle = 0$ implies $u = 0$. Now suppose that $u = 0$. If $\mu = 0$, then we get easily that $v = 0$. Let $\mu > 0$ and then v is a nontrivial solution to

$$\begin{cases} -\Delta v = \mu |v|^{2^*-2} v & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Observe that

$$\begin{aligned} &\inf\{\mathcal{J}(0, w)|\ w \in H_0^1(\Omega)\backslash\{0\},\ \|\nabla w\|_2^2 = \mu \|w\|_{2^*}^{2^*}\} \\ &\quad \leq \mathcal{J}(0, v) = \inf_{\mathcal{N}} \mathcal{J} \\ &\quad \leq \inf\{\mathcal{J}(0, w)|\ w \in H_0^1(\Omega)\backslash\{0\},\ \|\nabla w\|_2^2 = \mu \|w\|_{2^*}^{2^*}\} \end{aligned}$$

and

$$\begin{aligned} &\inf\{\mathcal{J}(0, w)|\ w \in H_0^1(\Omega)\backslash\{0\},\ \|\nabla w\|_2^2 = \mu \|w\|_{2^*}^{2^*}\} \\ &\quad = \frac{1}{N(2^* - 1)} \inf\{\|\nabla w\|_2^2|\ w \in H_0^1(\Omega)\backslash\{0\},\ \|\nabla w\|_2^2 = \mu \|w\|_{2^*}^{2^*}\} \\ &\quad = \frac{1}{N(2^* - 1))\mu^{(N-2)/2}} \inf\{\|\nabla w\|_2^N \mid w \in H_0^1(\Omega),\ \|w\|_{2^*} = 1\}. \end{aligned}$$

Then

$$\bar{v} = \left(\frac{\mu}{\|\nabla v\|_2^2}\right)^{1/2^*} v$$

satisfies $\|\bar{v}\|_{2^*} = 1$ and

$$\|\nabla \bar{v}\|_2^N = N(2^* - 1)\mu^{(N-2)/2} \mathcal{J}(0, v) = \inf\{\|\nabla w\|_2^N \mid w \in H_0^1(\Omega),\ \|w\|_{2^*} = 1\},$$

which is a contradiction (see [31, Proposition 1.43]). □

Now we are ready to prove the following

Theorem 3.6 *If* $\lambda \in (0, \lambda_1(\Omega))$, $\mu \in \mathbb{I}_N$, *then there exists a ground state* (u, v) *of* $\mathcal{J}$ *such that*

$$\mathcal{J}(u, v) = \inf_{\mathcal{N}} \mathcal{J} = \inf_{\mathcal{N}'} \mathcal{J} = \mathcal{B}.$$

Proof The functional $\mathcal{J}$ satisfies the geometrical assumptions of the Mountain Pass Theorem. Indeed, obviously, $\mathcal{J}(0,0)=0$. Using the Poincaré and the Sobolev inequalities we have that

$$\mathcal{J}(u,v)\geq C(\|\nabla u\|_2^2+\|\nabla v\|_2^2-\|\nabla v\|_2^{2^*}-\|\nabla u\|_2^{2^*-1}\|\nabla v\|_2)\geq\alpha$$

for some $\alpha>0$ and $\rho=\sqrt{\|\nabla u\|_2^2+\|\nabla v\|_2^2}$ sufficiently small. Moreover if $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega))$ satisfies

$$\mu\|v\|_{2^*}^{2^*}+2^*\int_\Omega|u|^{2^*-1}v\,dx>0,$$

then

$$\begin{aligned}\mathcal{J}(tu,tv)&=\frac{t^2}{2}\left(\|\nabla u\|_2^2-\lambda\|u\|_2^2+\frac{1}{2^*-1}\|\nabla v\|_2^2\right)\\&\quad-\frac{t^{2^*}}{2^*-1}\left(\frac{\mu}{2^*}\|v\|_{2^*}^{2^*}+\int_\Omega|u|^{2^*-1}v\,dx\right)\to-\infty\end{aligned}$$

as $t\to+\infty$. Then there exists a $(PS)_{\mathcal{B}}$-sequence $\{(u_n,v_n)\}\in H_0^1(\Omega)\times H_0^1(\Omega)$ for $\mathcal{J}$ at level $\mathcal{B}$, i.e. a sequence such that $\mathcal{J}(u_n,v_n)\to\mathcal{B}$ and $\mathcal{J}'(u_n,v_n)\to 0$. Since for some constant $C>0$

$$\begin{aligned}C(\|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2)&\leq\mathcal{J}(u_n,v_n)-\frac{1}{2^*}\langle\mathcal{J}'(u_n,v_n),(u_n,v_n)\rangle\\&\leq\mathcal{B}+1+\sqrt{\|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2},\end{aligned}$$

we have that the sequence $\{(u_n,v_n)\}$ is bounded. Therefore, up to a subsequence, we may assume that there exists $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ such that

$$\begin{array}{llll}u_n\rightharpoonup u & \text{in } H_0^1(\Omega), & v_n\rightharpoonup v & \text{in } H_0^1(\Omega),\\ |u_n|^{2^*-1}\rightharpoonup|u|^{2^*-1} & \text{in } L^{2^*/(2^*-1)}(\Omega), & |v_n|^{2^*-2}v_n\rightharpoonup|v|^{2^*-2}v & \text{in } L^{2^*/(2^*-1)}(\Omega),\\ |u_n|^{2^*-3}u_nv_n\rightharpoonup|u|^{2^*-3}uv & \text{in } L^{2^*/(2^*-1)}(\Omega), & u_n\to u & \text{in } L^2(\Omega),\\ u_n\to u & \text{a.e. on } \Omega, & v_n\to v & \text{a.e. on } \Omega.\end{array}$$

Hence, for every $(\xi,\eta)\in H_0^1(\Omega)\times H_0^1(\Omega)$, we have

$$\begin{aligned}&|\langle\mathcal{J}'(u_n,v_n),(\xi,\eta)\rangle-\langle\mathcal{J}'(u,v),(\xi,\eta)\rangle|\\&=\bigg|\int_\Omega(\nabla u_n-\nabla u)\nabla\xi-\lambda\int_\Omega(u_n-u)\xi+\frac{1}{2^*-1}\int_\Omega(\nabla v_n-\nabla v)\nabla\eta\\&\quad-\frac{1}{2^*-1}\int_\Omega(|u_n|^{2^*-1}-|u|^{2^*-1})\eta-\frac{\mu}{2^*-1}\int_\Omega(|v_n|^{2^*-2}v_n-v^{2^*-2}v)\eta\\&\quad-\int_\Omega(|u_n|^{2^*-3}u_nv_n-|u|^{2^*-3}uv)\xi\bigg|\to 0.\end{aligned}$$

Thus $\mathcal{J}'(u,v)=0$.

We claim that $(u,v)\neq(0,0)$. Indeed, suppose by contradiction that $(u,v)=(0,0)$ and so

$$u_n\to 0\text{ in }L^2(\Omega).\tag{3.3}$$

Since $\mathcal{J}$ is continuous and $\mathcal{J}(u_n,v_n)\to\mathcal{B}>0$, then (u_n,v_n) cannot converge to $(0,0)$ in $H_0^1(\Omega)\times H_0^1(\Omega)$. So, up to a subsequence, we may assume that $(u_n,v_n)\neq(0,0)$ and

$\|(u_n, v_n)\| \geq C > 0$ and, moreover, that $(u_n, v_n) \in \mathcal{A}$ for all $n \in \mathbb{N}$. Indeed, if there exists a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ in $(H_0^1(\Omega) \times H_0^1(\Omega)) \cap \mathcal{A}^c$, then

$$\langle \mathcal{J}'(u_{n_k}, v_{n_k}), (u_{n_k}, v_{n_k})\rangle \geq \|(u_{n_k}, v_{n_k})\|^2$$

and, since

$$\langle \mathcal{J}'(u_{n_k}, v_{n_k}), (u_{n_k}, v_{n_k})\rangle \to 0 \text{ as } k \to +\infty,$$

we get a contradiction.

Hence, if we take

$$t_n = \left[((2^* - 1)\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2) \left(\mu \|v_n\|_{2^*}^{2^*} + 2^* \int_\Omega |u_n|^{2^*-1} v_n \right)^{-1} \right]^{\frac{1}{2^*-2}}$$

and we denote in the same way the functions in $H_0^1(\Omega)$ and their extensions in $\mathbb{R}^N$ putting the function equal to zero in $\mathbb{R}^N \setminus \Omega$, we have that $(t_n u_n, t_n v_n) \in \mathcal{N}_0'$ and so

$$\langle \mathcal{J}_0'(t_n u_n, t_n v_n), (t_n u_n, t_n v_n)\rangle = 0. \tag{3.4}$$

Moreover, using (3.3),

$$\langle \mathcal{J}_0'(u_n, v_n), (u_n, v_n)\rangle = \langle \mathcal{J}'(u_n, v_n), (u_n, v_n)\rangle + o(1) = o(1). \tag{3.5}$$

Thus, combining (3.4) and (3.5) we get that $t_n \to 1$. Hence, taking into account Lemma 3.2, Theorems 2.6, 2.7, and 2.9 for $N = 5$, or corresponding results from Sect. 2.2 for $N \geq 6$, we have

$$\mathcal{B} < A = A' \leq \lim_n \mathcal{J}(t_n u_n, t_n v_n) = \mathcal{B}$$

getting a contradiction.

Hence $(u, v) \neq (0, 0)$ and $(u, v) \in \mathcal{N} \subset \mathcal{N}'$. Similarly as above, we find $t_n \to 1$ such that $(t_n u_n, t_n v_n) \in \mathcal{N}'$. In view of Proposition 3.3 we get

$$\inf_{\mathcal{N}} \mathcal{J} \leq \mathcal{J}(u, v) \leq \lim_{n\to\infty} \mathcal{J}(t_n u_n, t_n v_n) = \mathcal{B} = \inf_{\mathcal{N}'} \mathcal{J} \leq \inf_{\mathcal{N}} \mathcal{J}$$

and we conclude. □

To prove that our solutions are positive, let us write $u = u_+ + u_-$, where u_+ and u_- are respectively the positive and the negative part of u and let us consider the following functional

$$\begin{aligned} \mathcal{J}_+(u, v) = &\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 + \frac{1}{2(2^*-1)}\int_\Omega |\nabla v|^2 \\ &- \frac{\mu}{2^*(2^*-1)}\int_\Omega v_+^{2^*} - \frac{1}{2^*-1}\int_\Omega u_+^{2^*-1} v \end{aligned} \tag{3.6}$$

which is of C^1 class on $H_0^1(\Omega) \times H_0^1(\Omega)$, with

$$\begin{aligned} \langle \mathcal{J}_+'(u, v), (\xi, \eta)\rangle = &\int_\Omega \nabla u \nabla \xi - \lambda \int_\Omega u\xi - \int_\Omega u_+^{2^*-2} v \xi + \frac{1}{2^*-1}\int_\Omega \nabla v \nabla \eta \\ &- \frac{\mu}{2^*-1}\int_\Omega v_+^{2^*-1}\eta - \frac{1}{2^*-1}\int_\Omega u_+^{2^*-1}\eta. \end{aligned}$$

We have

Lemma 3.7 *Suppose that $\{(u_n, v_n)\}$ is a $(PS)_c$-sequence for $\mathcal{J}_+$, with $c > 0$. Then $\{(u_n, v_n)\}$ is bounded and $\{((u_n)_+, (v_n)_+)\}$ is also a $(PS)_c$-sequence for $\mathcal{J}_+$.*

Proof Let $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $\mathcal{J}_+$. There exists $C > 0$ such that

$$C(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2) \le \mathcal{J}_+(u_n, v_n) - \frac{1}{2^*}\langle \mathcal{J}_+'(u_n, v_n), (u_n, v_n)\rangle \le c + 1 + \sqrt{\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2}$$

and so $\{(u_n, v_n)\}$ is bounded. Moreover

$$\begin{aligned} o(1) &= \langle \mathcal{J}_+'(u_n, v_n), ((u_n)_-, (v_n)_-)\rangle \\ &= \|\nabla (u_n)_-\|_2^2 - \lambda\|(u_n)_-\|_2^2 + \frac{1}{2^*-1}\|\nabla (v_n)_-\|_2^2 - \frac{1}{2^*-1}\int_\Omega (u_n)_+^{2^*-1}(v_n)_- \\ &\ge C(\|\nabla (u_n)_-\|_2^2 + \|\nabla (v_n)_-\|_2^2), \end{aligned}$$

and then $((u_n)_-, (v_n)_-) \to (0, 0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and

$$\int_\Omega (u_n)_+^{2^*-1}(v_n)_- \to 0.$$

Thus

$$\begin{aligned} \mathcal{J}_+(u_n, v_n) - \mathcal{J}_+((u_n)_+, (v_n)_+) &= \frac{1}{2}(\|\nabla (u_n)_-\|_2^2 - \lambda\|(u_n)_-\|_2^2) + \frac{1}{2(2^*-1)}\|\nabla (v_n)_-\|_2^2 \\ &\quad - \frac{1}{2^*-1}\int_\Omega (u_n)_+^{2^*-1}(v_n)_- \\ &\to 0. \end{aligned}$$

Finally, since $\{(u_n)_+\}$ is bounded and $((u_n)_-, (v_n)_-) \to (0, 0)$ in $H_0^1(\Omega)$, then for every $(\xi, \eta) \in H_0^1(\Omega) \times H_0^1(\Omega)$ we have

$$\begin{aligned} &|\langle \mathcal{J}_+'(u_n, v_n) - \mathcal{J}_+'((u_n)_+, (v_n)_+), (\xi, \eta)\rangle| \\ &= \left| \int_\Omega \nabla (u_n)_- \nabla \xi - \lambda \int_\Omega (u_n)_- \xi + \frac{1}{2^*-1}\int_\Omega \nabla (v_n)_- \nabla \eta - \int_\Omega (u_n)_+^{2^*-2}(v_n)_- \xi \right| \\ &\le C(\|\nabla (u_n)_-\|_2 \|\nabla \xi\|_2 + \|\nabla (v_n)_-\|_2 \|\nabla \eta\|_2 + \|\nabla (u_n)_+\|_2^{2^*-2}\|\nabla (v_n)_-\|_2 \|\nabla \xi\|_2) \\ &\le C(\|\nabla (u_n)_-\|_2 + (1 + \|\nabla (u_n)_+\|_2^{2^*-2})\|\nabla (v_n)_-\|_2)(\|\nabla \xi\|_2^2 + \|\nabla \eta\|_2^2)^{\frac{1}{2}} \end{aligned}$$

and then

$$\|\mathcal{J}_+'(u_n, v_n) - \mathcal{J}_+'((u_n)_+, (v_n)_+)\| \to 0.$$

□

Proof of Theorem 1.1 As in the proof of Theorem 3.6 we can show that the functional $\mathcal{J}_+$ satisfies the geometrical assumptions of the Mountain Pass Theorem. Then there exists a $(PS)_{\mathcal{B}}$-sequence $\{(u_n, v_n)\} \in H_0^1(\Omega) \times H_0^1(\Omega)$ for $\mathcal{J}_+$ at level $\mathcal{B}$ In view of Lemma 3.7 we may assume that $u_n = (u_n)_+$ and $v_n = (v_n)_+$ and $\{(u_n, v_n)\}$ is bounded. Note that $\mathcal{J}(u_n, v_n) = \mathcal{J}_+(u_n, v_n)$ and thus we can conclude following the arguments given in proof of Theorem 3.6, getting a ground state (u, v) of $\mathcal{J}$ such that $u, v \ge 0$. Finally the Strong Maximum Principle (see [17, Theorem 8.19]) implies that $u, v > 0$. □

4 Nonexistence result

Proof of (1) of Theorem 1.2 Let $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be a nontrivial solution to $\mathcal{P}$ for $\lambda \leq 0$ and $\mu > \mu_N$. In system $\mathcal{P}$ multiply the fist equation by v, the second equation by u and this leads to

$$\int_\Omega u(u^{2^*-1} - u^{2^*-3}v^2 + \mu v^{2^*-1} - \lambda v) = 0 \tag{4.1}$$

Considering the function f_N in (2.14), we get that $u^{2^*-1} - u^{2^*-3}v^2 + \mu v^{2^*-1} > 0$ in Ω for $\mu > \mu_N$ and this is in a contradiction with (4.1). □

Proof of (2) of Theorem 1.2 Suppose that $\lambda \geq \lambda_1(\Omega)$ and $\mu \in \mathbb{R}$. We proceed similarly as in [10, Remark 1.1] arguing only on the first equation of $\mathcal{P}$. Let $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be a nontrivial solution to $\mathcal{P}$ and φ_1 the eigenfunction of $-\Delta$ with Dirichlet boundary conditions corresponding to $\lambda_1(\Omega)$. Multiplying the first equation of $\mathcal{P}$ by φ_1 we have

$$-\int_\Omega \Delta u \varphi_1 = \lambda \int_\Omega u\varphi_1 + \int_\Omega u^{2^*-2} v\varphi_1.$$

On the other hand

$$-\int_\Omega \Delta u\varphi_1 = -\int_\Omega u\Delta\varphi_1 = \lambda_1(\Omega)\int_\Omega u\varphi_1$$

and so if $\lambda \geq \lambda_1(\Omega)$ we reach a contradiction. □

Proof of (3) of Theorem 1.2 Here we adopt Pohožaev type arguments (see e.g. [19] or [31, Appendix B]). Let $\Omega \subset \mathbb{R}^N$ be a star shaped domain and $(u, v) \in H^2(\bar{\Omega}) \times H^2(\bar{\Omega})$ be nontrivial solution of $\mathcal{P}$. If we multiply the first equation of $\mathcal{P}$ by $x \cdot \nabla u$ and the second one by $x \cdot \nabla v$ we have that

$$\begin{aligned}
0 &= (\Delta u + \lambda u + u^{2^*-2}v)(x\cdot\nabla u)\\
&= \operatorname{div}[(\nabla u)(x\cdot\nabla u)] - |\nabla u|^2 - x\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right) + \frac{\lambda}{2}\left[\operatorname{div}(xu^2) - Nu^2\right]\\
&\quad + \frac{1}{2^*-1}[\operatorname{div}(xu^{2^*-1}v) - Nu^{2^*-1}v - u^{2^*-1}(x\cdot\nabla v)]\\
&= \operatorname{div}\left[(\nabla u)(x\cdot\nabla u) - x\frac{|\nabla u|^2}{2} + \frac{\lambda}{2}xu^2 + \frac{1}{2^*-1}xu^{2^*-1}v\right] + \frac{N-2}{2}|\nabla u|^2 - \frac{N\lambda}{2}u^2\\
&\quad - \frac{N}{2^*-1}u^{2^*-1}v - \frac{1}{2^*-1}u^{2^*-1}(x\cdot\nabla v)
\end{aligned}$$

and

$$\begin{aligned}
0 &= (\Delta v + \mu v^{2^*-2}v + u^{2^*-1})(x\cdot\nabla v)\\
&= \operatorname{div}[(\nabla v)(x\cdot\nabla v)] - |\nabla v|^2 - x\cdot\nabla\left(\frac{|\nabla v|^2}{2}\right) + \frac{\mu}{2^*}\left[\operatorname{div}(xv^{2^*}) - Nv^{2^*}\right] + u^{2^*-1}(x\cdot\nabla v)\\
&= \operatorname{div}\left[(\nabla v)(x\cdot\nabla v) - x\frac{|\nabla v|^2}{2} + \frac{\mu}{2^*}xv^{2^*}\right] + \frac{N-2}{2}|\nabla v|^2 - \frac{N\mu}{2^*}v^{2^*} + u^{2^*-1}(x\cdot\nabla v).
\end{aligned}$$

Integrating on Ω and using the boundary conditions on u and v we obtain

$$0 = \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \mathbf{n}}\right|^2 x\cdot\mathbf{n} + \frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N\lambda}{2}\|u\|_2^2 - \frac{N}{2^*-1}\int_\Omega u^{2^*-1}v - \frac{1}{2^*-1}\int_\Omega u^{2^*-1}(x\cdot\nabla v) \tag{4.2}$$

and

$$0 = \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial v}{\partial \mathbf{n}}\right|^2 x\cdot\mathbf{n} + \frac{N-2}{2}\|\nabla v\|_2^2 - \frac{N\mu}{2^*}\|v\|_{2^*}^{2^*} + \int_\Omega u^{2^*-1}(x\cdot\nabla v), \tag{4.3}$$

where $\mathbf{n}$ is the unit exterior normal to $\partial\Omega$.

Moreover, multiplying the equations of $\mathcal{P}$ respectively by u and by v we get

$$\|\nabla u\|_2^2 = \lambda\|u\|_2^2 + \int_\Omega u^{2^*-1}v \tag{4.4}$$

and

$$\|\nabla v\|_2^2 = \mu\|v\|_{2^*}^{2^*} + \int_\Omega u^{2^*-1}v. \tag{4.5}$$

Hence, combining (4.2), (4.3), (4.4) and (4.5), we have

$$-\lambda\|u\|_2^2 + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \mathbf{n}}\right|^2 x\cdot\mathbf{n} + \frac{1}{2(2^*-1)}\int_{\partial\Omega}\left|\frac{\partial v}{\partial \mathbf{n}}\right|^2 x\cdot\mathbf{n} = 0 \tag{4.6}$$

Then, if $\lambda < 0$ we get a contradiction.

If $\lambda = 0$, from (4.6) we have

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega$$

and so, using the positivity of u and v and the first equation of $\mathcal{P}$ we get a contradiction. □

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