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Multi-self-similar Markov processes on $\mathbb{R}_+^n$ and their Lamperti representations

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Abstract. A classical result, due to Lamperti, establishes a one-to-one correspondence between a class of strictly positive Markov processes that are self-similar, and the class of one-dimensional Lévy processes. This correspondence is obtained by suitably time-changing the exponential of the Lévy process. In this paper we generalise Lamperti's result to processes in n dimensions. For the representation we obtain, it is essential that the same time-change be applied to all coordinates of the processes involved. Also for the statement of the main result we need the proper concept of self-similarity in higher dimensions, referred to as multi-self-similarity in the paper.

The special case where the Lévy process ξ is standard Brownian motion in n dimensions is studied in detail. There are also specific comments on the case where ξ is an n-dimensional compound Poisson process with drift.

Finally, we present some results concerning moment sequences, obtained by studying the multi-self-similar processes that correspond to n-dimensional subordinators.

1. Introduction and main results

Consider $(B_u + \nu u)_{u\geq 0}$, a one-dimensional Brownian motion (BM) with drift $\nu \geq 0$ started at 0. Lamperti's [15] representation of $(\exp(B_u + \nu u))_{u\geq 0}$ as

$$\exp(B_u + \nu u) = R^{(\nu)}_{\int_0^u dv\, \exp 2(B_v+\nu v)} \quad (u \geq 0) \tag{1}$$

where $\left(R_t^{(\nu)}\right)_{t\geq 0}$ is a Bessel process (BES) of index ν or 'dimension' $2\nu + 2$ started at 1, has proved to be a powerful tool in the study of the exponential functional $\left(\int_0^u dv\, \exp 2(B_v + \nu v)\right)_{u\geq 0}$ which plays an important role for a number of questions in mathematical finance (e.g. Dufresne [6], Geman and Yor [7]; see also

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the collection of papers: Yor [21]), in the study of hyperbolic Brownian motion (e.g. Gruet [8], Ikeda and Matsumoto [10]) and Brownian motion in random media (e.g. Hu, Shi and Yor [9], Comtet and Monthus [3], Comtet, Monthus and Yor [4] and Kawazu and Tanaka [13]).

Lamperti's original representation is not (1) but the squared version

$$\exp 2\left(B_u + \nu u\right) = R^{(\nu)^2}_{\int_0^u dv\ \exp 2(B_v + \nu v)} \qquad (u \geq 0) \tag{2}$$

where $S = R^{(\nu)^2}$ is a squared Bessel process (BESQ) of 'dimension' $2\nu + 2$, i.e. S satisfies the SDE

$$dS_t = (2\nu + 2)\ dt + 2\sqrt{S_t}\, dW_t \tag{3}$$

with W a standard BM(1). Here the point of the representation (2) rather than (1) is that $R^{(\nu)^2}$ is the diffusion with the *self-similarity* (or semi-stability) property used by Lamperti [15] in his main result, Theorem 4.1, part of which may informally be stated as follows: any 1-self-similar strictly positive and 'nice' Markov process is a time-change of the exponential of a Lévy process; see (5) below.

For the discussion of (1) and (2) we assumed that the Brownian motion B should have drift $\nu \geq 0$ which ensures that $R^{(\nu)}_t$ and S_t are well defined for all $t \geq 0$. Throughout the paper we shall work under conditions so that the random time-changes we consider map the time axis $[0, \infty[$ *onto* itself. Note however that Lamperti's Theorem 4.1 in [15] in particular contains a version of (2) also when $\nu < 0$ but with S_t defined only up to the finite killing time $\int_0^\infty dv\ \exp 2\left(B_v + \nu v\right)$. Our main result, Theorem 1, can be generalised similarly, but we do not pursue this generalisation here.

In a recent paper, studying some concrete examples of multidimensional diffusions, Jacobsen [12] found an n-dimensional analogy to (2) when the one-dimensional Brownian motion with drift is replaced by an n-dimensional Gaussian Lévy process $G = \left(G^i\right)_{1 \leq i \leq n}$ (Brownian motion in n dimensions with some drift vector and some covariance matrix) and $R^{(\mu)^2}$ is replaced by a certain n-dimensional diffusion $S = \left(S^i\right)_{1 \leq i \leq n}$, referred to as the *multi-self-similar diffusion* below (see (17) for the precise definition of S), and the same time-change is applied to all coordinates. More precisely, (2) in its n-dimensional form becomes

$$\exp G^i_u = S^i_{\int_0^u dv\ \exp \bar{G}_v} \qquad (u \geq 0) \tag{4}$$

provided G is such that the one-dimensional scaled Brownian motion $\bar{G} := \sum_i G^i$ has drift ≥ 0, a condition equivalent to the requirement that $\int_0^\infty dv\ \exp \bar{G}_v = \infty$ a.s., cf. (9) below.

Since Lamperti's representation (2) holds with Brownian motion with drift replaced by any one-dimensional Lévy process ξ such that $\int_0^\infty dv\ \exp \xi_v = \infty$ a.s. with the resulting counterpart of $R^{(\nu)^2}$ a 1-self-similar Markov process X, i.e.

$$\exp \xi_u = X_{\int_0^u dv\ \exp \xi_v}, \tag{5}$$

it seemed natural to search for a general version of (4), where G is replaced by an n-dimensional Lévy process $\xi = \left(\xi^i\right)_{1 \le i \le n}$ and S is replaced by an n-dimensional Markov process, self-similar in a suitable sense. Note that the representation is required to hold coordinatewise with the same time-change used on all coordinates.

Notation. Below, $\mathbb{R}_+$ denotes the *open* interval $]0, \infty[$ while $\mathbb{R}_0$ is the interval $[0, \infty[$. If Y is a process starting from a given state y, $Y_0 = y$ a.s., we write $Y^{(y)}$ to emphasize the starting value. If ξ is a Lévy process in n dimensions it is always understood that $\xi_0 = \mathbf{0} = (0, \dots, 0)$ a.s. and if $a \in \mathbb{R}^n$, $\xi^{(a)} := \xi + a$ is the same Lévy process started from a, but always defined on the same probability space as ξ. If X is a Markov process, then $X^{(x)}$ denotes X starting from the given state x, with $X^{(x)}$ defined on some probability space – only in special cases (such as (10) below) is there a natural construction of all $X^{(x)}$ for x arbitrary on the same probability space. Conversely, if $\left(X^{(x)}\right)_{x \in E}$ for a state space E, is a family of processes (on the same or different probability spaces), with $X^{(x)}$ starting at x, and each $X^{(x)}$ enjoying the Markov property with the same Markov transition semigroup, we shall say that $\left(X^{(x)}\right)_{x \in E}$ is a *Markovian family.* In particular, if $X^{(x)}$ is for fixed x a Lévy process such that the convolution semigroup is the same for all x, we shall say that $\left(X^{(x)}\right)_{x \in \mathbb{R}^n \text{ or } \mathbb{R}^n_+}$ is a *Lévy family.* For the coordinate processes of $\xi^{(a)}$ and $X^{(x)}$, where $a = (a_i)$, $x = (x_i)$, we write $\xi^{i,(a_i)}$ and $X^{i,(x_i)}$ respectively.

In order to formulate the multidimensional Lamperti representation we need the appropriate concept of self-similarity for n-dimensional processes. In the literature (e.g. Kiu [14], Definition 1, Sato [19], Definition 13.4) one often sees just a verbatim copy of the basic definition in dimension one, i.e. an $\mathbb{R}^n$-valued Markov process $X = \left(X^i\right)$ is *α-self-similar* if for every $c > 0$ and every initial state x it holds that

$$\left(c^\alpha X_t^{(x/c^\alpha)}\right)_{t \ge 0} \overset{\text{(d)}}{=} \left(X_{ct}^{(x)}\right)_{t \ge 0}. \tag{6}$$

For our purposes this is however not the correct concept and instead we require (corresponding to the case $\alpha = 1$) the following definition that appears to be new:

Definition 1. *An n-dimensional Markov family $\left(X^{(x)}\right)_{x \in \mathbb{R}^n_+}$ with state space $\mathbb{R}^n_+$ is* multi-self-similar *if for all scaling factors $c_i > 0$ and all initial states $x = (x_i)$ it holds that*

$$\left(c_i X_t^{i,(x_i/c_i)}\right)_{1 \le i \le n, t \ge 0} \overset{\text{(d)}}{=} \left(X_{ct}^{(x)}\right)_{t \ge 0}, \tag{7}$$

where $c = \prod_1^n c_i$.

If $\left(X^{(x)}\right)_{x \in \mathbb{R}^n_+}$ is multi-self-similar we shall also refer to each member of the family as a multi-self-similar process.

The important difference with (6) is of course that while we still require the same time-change to apply to all the coordinate processes, we permit different scalings of each of the coordinates. Taking all $c_i = c_0 > 0$ we see in particular that if (7) holds, then $X^{(x)}$ is $1/n$-self-similar in the traditional sense, cf. (6).

Definition 1 corresponds to the case of **1**-multi-self-similarity. A natural generalisation is to call a Markov family $\left(X^{(x)}\right)_{x\in\mathbb{R}_+^n}$ *α-multi-self-similar* (where $\alpha = (\alpha_i)_{1\le i\le n}$ with all $\alpha_i > 0$) if

$$\left(c_i^{\alpha_i} X_t^{i,\left(x_i/c_i^{\alpha_i}\right)}\right)_{1\le i\le n, t\ge 0} \overset{(\mathrm{d})}{=} \left(X_{ct}^{(x)}\right)_{t\ge 0}. \tag{8}$$

This definition connects with Definition 1 in a simple manner: if $\left(Y^{(y)}\right)$ is multi-self-similar in the sense of Definition 1, then the family $\left(\tilde{Y}^{(y)}\right)$ defined by $\tilde{Y}^{i,(y_i)} = \left(Y^{i,\left(y_i^{1/\alpha_i}\right)}\right)^{\alpha_i}$ is α-multi-self-similar.

Our main result is now the following:

Theorem 1. (The multidimensional Lamperti representation).

(a) Let $\xi = \left(\xi^i\right)_{1\le i\le n}$ be an n-dimensional Lévy process starting from $\mathbf{0}$, *right-continuous with left limits and satisfying*

$$\int_0^\infty dv\, \exp\bar{\xi}_v = \infty \text{ a.s.} \tag{9}$$

where $\bar{\xi} := \sum_1^n \xi^i$. Let $x = (x_i) \in \mathbb{R}_+^n$ and define implicitly the n-dimensional process $X^{(x)}$ by

$$X^{i,(x_i)}_{\int_0^u dv\, \exp\bar{\xi}_v^{(\bar{a})}} = \exp\xi_u^{i,(a_i)} \quad (1\le i\le n, u\ge 0), \tag{10}$$

where $a_i = \log x_i$ and $\bar{a} = \sum_1^n a_i$. Then the family $\left(X^{(x)}\right)_{x\in\mathbb{R}_+^n}$ is strongly Markovian and has the multi-self-similarity property (7), with each process $X^{(x)}$ right-continuous with left limits and initial state x. Furthermore it holds that

$$\int_0^\infty ds\, \frac{1}{Z_s^{(z)}} = \infty \text{ a.s.} \tag{11}$$

where $Z^{(z)} = \prod_1^n X^{i,(x_i)}$, $z = \prod_1^n x_i$.

(b) If conversely $\left(X^{(x)}\right)_{x\in\mathbb{R}_+^n}$ is a strong Markov family with each $X^{(x)}$ right-continuous with left limits, that satisfies the multi-self-similarity property (7) and is such that (11) holds for some, and then automatically for all initial states $x \in \mathbb{R}_+^n$ with $z = \prod_1^n x_i$, then the processes $\xi^{(a)} = \left(\xi^{i,(a_i)}\right)_{1\le i\le n}$, where $\xi_0^{(a)} = a$ for all a, defined implicitly by $a_i = \log x_i$ and

$$\xi^{i,(a_i)}_{\int_0^t ds\, 1/Z_s^{(z)}} = \log X_t^{i,(x_i)} \quad (1\le i\le n, t\ge 0)$$

form a Lévy family $\left(\xi^{(a)}\right)_{a\in\mathbb{R}^n}$.

The proof of the theorem is given in Section 2 below, where we also discuss some further properties of the multi-self-similar processes, that are extensions of results from Bertoin and Yor [2]. One such result (see Proposition 1 in [2]) is

Theorem 2. *Assume that ξ is an n-dimensional subordinator with Lévy exponent $\Phi\,(p)$, i.e.*

$$\mathbb{E}\exp - \langle p, \xi_u \rangle = \exp - u\Phi\,(p) \quad (p = (p_i)_i \in \mathbb{R}_0^n). \tag{12}$$

Then for every $p \in \mathbb{R}_0^n$ there exists a probability measure ρ_p on $\mathbb{R}_0$ such that

$$\mathbb{E}\prod_{i=1}^{n}\left(X_t^{i,(1)}\right)^{-p_i} = \int_0^\infty \rho_p(dx)\, e^{-tx}, \quad (t \geq 0)$$

where $X^{(\mathbf{1})}$ is the multi-self-similar process starting from $\mathbf{1} = (1, \dots, 1)$ determined by (10) using ξ itself. The probability ρ_p is characterized by its integral moments,

$$\int_0^\infty x^k\,\rho_p(dx) = \Phi\,(p)\,\Phi\,(p + \mathbf{1})\cdots\Phi\,(p + (\mathbf{k} - \mathbf{1}))\,, \quad (k = 1, 2, \dots) \tag{13}$$

where we write $\mathbf{j} = (j, \dots, j) \in \mathbb{R}^n$.

Notation. In (12), $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product.

Theorem 2 permits the following generalisation that, as will be shown in Section 2, is obtained quite easily from Theorem 2 itself and Theorem 1:

Corollary 3. *Let $q \in \mathbb{R}_+^n$ and let ξ be an n-dimensional subordinator with Lévy exponent $\Phi\,(p)$. Then the equation*

$$\exp \xi_u^i = {}^{(q)}X^i_{\int_0^u dv\, \exp\langle q, \xi_v \rangle} \quad (1 \leq i \leq n, u \geq 0)$$

defines a process ${}^{(q)}X$ with initial state $\mathbf{1}$, which is α-multi-self-similar in the sense defined in (8) with $\alpha_i = \frac{1}{q_i}$. Furthermore, for every $p \in \mathbb{R}_0^n$ there exists a probability $\rho_{p,q}$ on $\mathbb{R}_0$ such that

$$\mathbb{E}\prod_{i=1}^{n}\left({}^{(q)}X_t^{i}\right)^{-p_i} = \int_0^\infty \rho_{p,q}(dx)\, e^{-tx}, \quad (t \geq 0)\,.$$

The probability $\rho_{p,q}$ is characterized by its integral moments,

$$\int_0^\infty x^k\,\rho_{p,q}(dx) = \Phi\,(p)\,\Phi\,(p + q)\cdots\Phi\,(p + (k - 1)\,q)\,, \quad (k = 1, 2, \dots). \tag{14}$$

Finally it also holds for any $p, q \in \mathbb{R}_0^n$ that the sequence

$$\frac{k!}{\Phi\,(p + q)\cdots\Phi\,(p + kq)}, \quad (k = 1, 2, \dots) \tag{15}$$

is the sequence of moments for a probability measure on $\mathbb{R}_0$ and that this probability is unique provided $\Phi\,(p) > 0$.

Corollary 3 in particular exhibits two types of moment sequences for probabilities on $\mathbb{R}_0$. While our arguments are probabilistic, Berg and Duran [1] obtain similar results by analytic methods.

In Section 3 we focus on $\xi = G$ being Gaussian, cf. (4) above and in particular study in depth the case where $\xi = B$ is BM(n), standard Brownian motion in n dimensions:

Theorem 4. *In the standard Brownian case, the multidimensional Lamperti representation*

$$\exp B_u^i = S^i_{\int_0^u dv\, \exp \bar{B}_v} \quad (1 \le i \le n, u \ge 0)$$

holds with the n-dimensional diffusion $S = \left(S^i\right)$ with initial state $\mathbf{1}$ described as follows: define $\left(C_u^i\right)_{1 \le i \le n, u \ge 0}$ as the Gaussian process independent of $\bar{B}$ such that

$$B_u^i = \tfrac{1}{n} \bar{B}_u + C_u^i.$$

Then there is a 2-dimensional Bessel process $(R_v)_{v \ge 0}$ starting from 1 such that S admits the skew-product representation

$$S_t^i = \left(R_{\frac{nt}{4}}\right)^{\frac{2}{n}} \exp\left(C^i_{\frac{4}{n}\int_0^{nt/4} dh\, 1/R_h^2}\right). \tag{16}$$

The initial values $\mathbf{0}$ for B and $\mathbf{1}$ for S were omitted from the notation used in the theorem. Of course we write $\bar{B} = \sum_i^n B^i$.

For S still the diffusion in Theorem 4, we also in Section 3 derive some explicit formulas for the transition semigroup, using known results on BES and BESQ processes.

In the case of a general Gaussian Lévy process G starting at $\mathbf{0}$ with drift vector $\nu = (\nu_i)_{1 \le i \le n}$ and covariance matrix $\Gamma = \left(\Gamma_{ij}\right)_{1 \le i,j \le n}$ (possibly singular, but $\neq 0$), the diffusion $S = \left(S^i\right)$ determined by (4) starts at $\mathbf{1}$ and satisfies the SDE

$$dS_t^i = \frac{\nu_i + \frac{1}{2}\Gamma_{ii}}{Z_{\setminus i,t}}\, dt + \sqrt{\frac{S_t^i}{Z_{\setminus i,t}}}\, dB_t^{\Gamma,i} \tag{17}$$

where $Z_{\setminus i} = \prod_{j:j \neq i} S^j$ and $B^\Gamma = \left(B^{\Gamma,i}\right)_{1 \le i \le n}$ is n-dimensional Brownian motion with drift 0, covariance Γ. This result was shown in Jacobsen [12] and prompted the investigation that led to the present paper. Note that (9) holds for $\xi = G$ if and only if $\bar{\nu} = \sum_1^n \nu_i \ge 0$, and that (3) corresponds to the 1-dimensional special case of (17) where G is Brownian motion with drift 2ν and variance 4 ($= \Gamma$ for $n = 1$).

In view of its importance we shall briefly indicate the direct argument that leads from the diffusion S solving (17), to the Brownian motion G, cf. Theorem 1(b): trusting that when $\bar{\nu} \ge 0$ all S_t^i are strictly positive (as may be argued by showing that $Z = \prod_{i=1}^n S^i$ is a one-dimensional diffusion and then verifying that $Z_t > 0$ always), take logarithms in (17) and use Itô's formula to arrive at

$$d \log S_t^i = \frac{\nu_i}{Z_t}\, dt + \frac{1}{\sqrt{Z_t}}\, dB_t^\Gamma,$$

from which it is clear that a time-change through $\left(\int_0^t ds\ 1/Z_s\right)_{t\geq 0}$ leads from S to G.

The multi-self-similarity property of the diffusion S is also argued easily: take $c_i > 0$, define $\tilde{S}_t^i = c_i S_t^i$ for $1 \leq i \leq n$, $t \geq 0$ and verify from (17) that

$$d\tilde{S}_t^i = \tilde{c}\frac{\nu_i + \frac{1}{2}\Gamma_{ii}}{\tilde{Z}_{\setminus i,t}}\,dt + \sqrt{\tilde{c}}\sqrt{\frac{\tilde{S}_t^i}{\tilde{Z}_{\setminus i,t}}}\,dB_t^{\Gamma,i},$$

where $\tilde{c} = \prod_{i=1}^n c_i$ and $\tilde{Z}_{\setminus i} = \prod_{j:j\neq i} \tilde{S}^j$.

To supplement the treatment of continuous processes in Section 3, we finally consider in Section 4 the simplest case with jumps, i.e. ξ is an n-dimensional compound Poisson process with drift in which case the process X obtained by the Lamperti representation becomes a piecewise deterministic Markov process in the sense of M. Davis [5].

2. The multi-self-similarity property; proofs of Theorems 1, 2 and Corollary 3

Suppose that $\left(X^{(x)}\right)$ is a right-continuous left limit Markov family which has the multi-self-similarity property (7). Taking $c_i = x_i$ in (7) corresponding to a scaling of t by $z = \prod_1^n x_i$ we see that for all $x_i > 0$

$$X^{(\mathbf{1})} \overset{\text{(d)}}{=} \left(\frac{1}{x_i} X_{zt}^{i,(x_i)}\right)_{1\leq i\leq n, t\geq 0}, \tag{18}$$

a fact we shall use frequently below.

A second useful consequence of (7) is that if $P_t\,(x,\cdot)$ denotes the transition function for X,

$$P_t\,(x,\cdot) = \mathbb{P}\,(X_{s+t} \in \cdot\,|X_s = x\,)\,,$$

then for, say, any bounded and measurable $f : \mathbb{R}_+^n \to \mathbb{R}$,

$$\int_{\mathbb{R}_+^n} P_t\,(x, dy)\ f(y) = \int_{\mathbb{R}_+^n} P_{t/z}\,(\mathbf{1}, dy)\ f\left((x_i y_i)_i\right) \tag{19}$$

where $(x_i y_i)_i$ denotes the vector with coordinates $x_i y_i$, $1 \leq i \leq n$. Thus the transition function $P_t\,(x,\cdot)$ is completely determined from the transitions $P_s\,(\mathbf{1},\cdot)$ from the state $\mathbf{1}$. Furthermore, if $f(y)$ depends on y only through the product $\prod y_i$, i.e. $f(y) = g\left(\prod y_i\right)$, we may write the integral on the right of (19) as

$$\begin{aligned}\int_{\mathbb{R}_+^n} P_{t/z}\,(\mathbf{1}, dy)\ g\left(z\prod y_i\right) &= \mathbb{E}g\left(zZ_{t/z}^{(1)}\right)\\ &= \int_{\mathbb{R}_+} \tilde{P}_t\,(z, d\tilde{y})\ g(\tilde{y})\end{aligned} \tag{20}$$

where $Z^{(1)} = \prod_1^n X^{i,(1)}$ as usual, and $\tilde{P}_t\,(z, d\tilde{y})$ is the well understood transition function for the one-dimensional 1-semi-stable Markov process Z resulting from the one-dimensional Lamperti representation of the Lévy process $\bar{\xi} = \sum \xi^i$, cf. the discussion of the agglomeration property in Corollary 5 below. But then for any general f as in (19),

$$\begin{aligned}\int_{\mathbb{R}_+^n} P_t(x, dy)\ f(y) &= \mathbb{E}\left[\mathbb{E}\left(f\left(X_t^{(x)}\right)\Big|Z_t^{(z)}\right)\right] \\ &= \int_{\mathbb{R}_+} \tilde{P}_t(z, d\tilde{z})\ \mathbb{E}\left[f\left(X_t^{(x)}\right)\Big|Z_t^{(z)} = \tilde{z}\right] \\ &= \int_{\mathbb{R}_+} \tilde{P}_t(z, d\tilde{z})\ \mathbb{E}\left[f\left(\left(x_i X_{t/z}^{i,(1)}\right)_i\right)\Big|Z_{t/z}^{(1)} = \tilde{z}/z\right] \\ &= \int_{\mathbb{R}_+} \tilde{P}_{t/z}(1, d\tilde{z})\ \mathbb{E}\left[f\left(\left(x_i X_{t/z}^{i,(1)}\right)_i\right)\Big|Z_{t/z}^{(1)} = \tilde{z}\right]. \qquad (21)\end{aligned}$$

Thus, in general, the transition function for X may be found from the knowledge of the transition functions from state 1 in the one-dimensional case *and* an understanding of the conditional law of $X_s^{(\mathbf{1})}$ given $Z_s^{(1)}$ for all s.

Note that by Dynkin's criterion (see e.g. Pitman and Rogers [16]) the discussion leading to (20) shows that $Z = \prod_1^n X^i$ is in fact a Markov process with respect to the filtration generated by X.

We proceed now with the proofs of the main results, beginning with

Proof of Theorem 1. (a) [*From ξ to X*] . Note first that because (9) is assumed to hold, also

$$\int_0^\infty dv\ \exp\left(\bar{\xi}_v^{(\bar{a})}\right) = e^{\bar{a}} \int_0^\infty dv\ \exp\left(\bar{\xi}_v\right) = \infty \text{ a.s.},$$

i.e. (10) determines $X^{(x)}$ uniquely from $\xi^{(a)}$ through time-substitution with the strictly increasing and continuous additive functional

$$\mathcal{A}_u^{(a)} = \int_0^u dv\ \exp\left(\bar{\xi}_v^{(\bar{a})}\right)$$

In particular $X^{(x)}$ is therefore cadlag and strong Markov.

Note next that all the processes $X^{(x)}$ for x arbitrary are defined on the same probability space, viz. the space where ξ and all the $\xi^{(a)}$ are defined.

Let $\left(\mathcal{F}_u^\xi\right)$ denote the filtration generated by ξ and introduce the $\mathcal{F}_u^\xi$-stopping times determining the inverse of $\mathcal{A}^{(a)}$,

$$H_t^{(a)} := \inf\left\{u \geq 0 : \mathcal{A}_u^{(a)} > t\right\} \equiv \inf\left\{u \geq 0 : \mathcal{A}_u^{(a)} = t\right\}$$

and finally write $\mathcal{G}_t^{(a)} = \mathcal{F}_{H_t^{(a)}}^\xi$. Of course

$$X_t^{i,(x_i)} = \exp \xi_{H_t^{(a)}}^{i,(a_i)} \quad (1 \leq i \leq n, t \geq 0) \qquad (22)$$

and $X^{(x)}$ is $\mathcal{G}_t^{(a)}$-adapted.

From the identity $\mathcal{A}^{(a)}_{H^{(a)}_t} = t$ it follows that $\frac{d}{dt} H^{(a)}_t = \exp -\bar{\xi}^{(\bar{a})}_{H^{(a)}_t} = 1/Z^{(z)}_t$, i.e.

$$H^{(a)}_t = \int_0^t ds\, \frac{1}{Z^{(z)}_s}.$$

Since a.s. $\mathcal{A}^{(a)}$ increases from 0 to ∞, so does the inverse $H^{(a)}$ and (11) follows.

To prove the multi-self-similar property for X, we shall in fact show the pathwise identity

$$c_i X^{i,(x_i/c_i)}_{t/c} = X^{i,(x_i)}_t \quad (1 \le i \le n, t \ge 0) \tag{23}$$

between processes for arbitrary choices of $x = (x_i) \in \mathbb{R}^n_+$ and $c_i > 0$ with $c = \prod c_i$. But obviously

$$H^{(a)}_t = \inf\left\{u : e^{\bar{a}} \int_0^u dv\, \exp \bar{\xi}_v = t\right\} = H_{te^{-\bar{a}}},$$

writing H as short for $H^{(0)}$, and therefore by (22), since $z = e^{\bar{a}}$, $x_i = e^{a_i}$,

$$X^{i,(x_i)}_t = x_i \exp \xi^i_{H_{t/z}}.$$

Using this expression with x_i replaced by x_i/c_i we also get

$$c_i X^{i,(x_i/c_i)}_{t/c} = c_i \left(\frac{x_i}{c_i} \exp \xi^i_{H_{(t/c)/(z/c)}}\right).$$

Thus (23) follows and the multi-self-similar property is proved.

It remains to show that all the processes $X^{(x)}$ share the same transition function. More specifically, defining the Markov kernels

$$P_t(x, \cdot) = \mathbb{P}\left(X^{(x)}_t \in \cdot\right)$$

we claim that for all $s, t \ge 0$ and all x,

$$\mathbb{P}\left(X^{(x)}_{t+s} \in \cdot \,\middle|\, \mathcal{G}^{(a)}_t,\ X^{(x)}_t = y\right) = P_s(y, \cdot).$$

But

$$X^{i,(x_i)}_{t+s} = \exp \xi^{i,(a_i)}_{H^{(a)}_{t+s}} = \exp \xi^{i,(a_i)}_{H^{(a)}_t + \tilde{H}_s}$$

where

$$\begin{aligned} \tilde{H}_s &= \inf\left\{u \ge 0 : \int_{H^{(a)}_t}^{H^{(a)}_t + u} dv\, \exp \bar{\xi}^{(\bar{a})}_v = s\right\} \\ &= \inf\left\{u \ge 0 : \exp\left(\bar{\xi}^{(\bar{a})}_{H^{(a)}_t}\right) \int_0^u dv\, \exp\left(\bar{\xi}_{H^{(a)}_t + v} - \bar{\xi}_{H^{(a)}_t}\right) = s\right\} \end{aligned} \tag{24}$$

where under the integral we may obviously write $\bar{\xi}$ instead of $\bar{\xi}^{(\bar{a})}$. Using that $\xi^{(a)}$ is strong Markov and Lévy we see that the conditional law of $X_{t+s}^{(x)}$ given $\mathcal{G}_t^{(a)}$, $X_t^{(x)} = y$, is the same as the conditional law of

$$\left(\exp\left(\xi^{i,(a_i)}_{H_t^{(a)}} + \left(\xi^i_{H_t^{(a)}+\tilde{H}_s} - \xi^i_{H_t^{(a)}}\right)\right)\right)_{1\le i\le n} = \left(X_t^{i,(x_i)} U^i\right)_{1\le i\le n}$$

and here, referring to (24), using that $\exp\left(\bar{\xi}^{(\bar{a})}_{H_t^{(a)}}\right) = Z_t^{(z)}$ and recalling that ξ itself corresponds to $X^{(\mathbf{1})}$, it follows that

$$\left(\left(U^i\right)_{1\le i\le n} \middle| \mathcal{G}_t^{(a)}, X_t^{(x)} = y\right) \stackrel{(\mathrm{d})}{=} X^{(\mathbf{1})}_{s/\prod y_i}.$$

Thus

$$\begin{aligned}\mathbb{P}\left(X_{t+s}^{(x)} \in \cdot \middle| \mathcal{G}_t^{(a)},\ X_t^{(x)} = y\right) &= \mathbb{P}\left(\left(y_i X^{i,(1)}_{s/\prod y_j}\right)_i \in \cdot\right)\\ &= \mathbb{P}\left(X_s^{(y)} \in \cdot\right)\end{aligned}$$

as desired, using the multi-self-similar property for the last equality.

(b) [*From X to* ξ]. With $\left(X^{(x)}\right)_{x\in\mathbb{R}_+^n}$ a multi-self-similar and strong Markov family, consider $X^{(x)}$ for an arbitrary initial state x. From (18) it follows that $Z^{(z)} = \prod_1^n X^{i,(x_i)}$ satisfies

$$\left(\tfrac{1}{z} Z_{zt}^{(z)}\right) \stackrel{(\mathrm{d})}{=} Z^{(1)} \tag{25}$$

where $Z^{(1)} = \prod_1^n X^{i,(1)}$. In particular the law of $Z^{(z)}$ depends only on z, not on the individual x_i.

Note that (25) also shows that if (11) holds for some $z > 0$, it holds for all z.

(25) shows that the $\mathbb{R}_+$-valued process Z is 1-self-similar. Hence by Lamperti's original result [15] there exists a one-dimensional Lévy process $\bar{\xi}$ such that

$$\exp \bar{\xi}_u^{(\bar{a})} = Z^{(z)}_{\int_0^u dv\ \exp \bar{\xi}_v^{(\bar{a})}} \quad (u \ge 0) \tag{26}$$

where $\bar{a} = \log z$, $\bar{\xi}^{(\bar{a})} = \bar{\xi} + \bar{a}$.

Letting $A_u^{(\bar{a})} = \int_0^u dv\ \exp \bar{\xi}_v^{(\bar{a})}$ and arguing as in the proof of (a), one finds that the inverse

$$H_t^{(\bar{a})} = \inf\left\{u \ge 0 : A_u^{(\bar{a})} = t\right\}$$

satisfies

$$H_t^{(\bar{a})} = \int_0^t ds\ \frac{1}{Z_s^{(z)}}.$$

Therefore (11) implies that $\lim_{u\to\infty} A_u^{(\bar{a})} = \infty$ a.s.

Now define the n-dimensional process $\xi^{(a)} = \left(\xi^{i,(a_i)}\right)$, where $a_i = \log x_i$, by

$$\exp \xi_u^{i,(a_i)} = X^{i,(x_i)}_{A_u^{(\bar{a})}} \quad (1 \le i \le n, u \ge 0),$$

in particular, see (26),

$$\bar{\xi}^{(\bar{a})} = \sum_{i=1}^{n} \xi^{i,(a_i)}.$$

Introducing $(\mathcal{G}_t)$ to be the filtration generated by $X^{(x)}$, we note that for each u, $A_u^{(\bar{a})}$ is a $\mathcal{G}_t$-stopping time and that $\xi^{(a)}$ is $\mathcal{F}_u$-adapted, where $\mathcal{F}_u = \mathcal{G}_{A_u^{(\bar{a})}}$. We can therefore complete the proof by showing that for all $u \ge 0$, $h > 0$ it holds that $\xi_{u+h}^{(a)} - \xi_u^{(a)}$ is independent of $\mathcal{F}_u$ with a law that depends on a, u and h through h only. We shall achieve this by identifying the conditional joint law of

$$\left(\exp\left(\xi_{u+h}^{i,(a_i)} - \xi_u^{i,(a_i)}\right)\right)_{1 \le i \le n} = \left(\frac{X^{i,(x_i)}_{A_{u+h}^{(\bar{a})}}}{X^{i,(x_i)}_{A_u^{(\bar{a})}}}\right)_{1 \le i \le n} \tag{27}$$

given $\mathcal{G}_{A_u^{(\bar{a})}}$, $X^{i,(x_i)}_{A_u^{(\bar{a})}} = x_i^\circ$, $1 \le i \le n$ for an arbitrary $x^\circ = \left(x_i^\circ\right) \in \mathbb{R}_+^n$.

First note that

$$A_{u+h}^{(\bar{a})} = A_u^{(\bar{a})} + \inf\left\{t \ge 0 : \int_0^t ds / Z^{(z)}_{A_u^{(\bar{a})}+s} = h\right\}$$

so by the strong Markov property for $X^{(x)}$, the conditional law from (27) is that of

$$\left(\frac{1}{x_i^\circ} X_\tau^{i,(x_i^\circ)}\right)_{1 \le i \le n} \tag{28}$$

with τ the stopping time for $X^{(x^\circ)}$ given by

$$\tau = \inf\left\{t \ge 0 : \int_0^t ds / Z_s^{(z^\circ)} = h\right\}$$

where of course $Z^{(z^\circ)} = \prod_1^n X^{i,(x_i^\circ)}$, $z^\circ = \prod_1^n x_i^\circ$. (The reader is reminded that $X^{(x^\circ)}$ is just the name for a process with the relevant distribution, viz. that of the multi-self-similar process X starting at x°. $X^{(x^\circ)}$ is not an object defined on the probability space where $X^{(x)}$ and $\xi^{(a)}$ are defined).

To prepare for the use of the multi-self-similar property of X in our argument, we now observe that by an elementary calculation

$$\tau = z^\circ \tau' \tag{29}$$

where

$$\tau' = \inf\left\{t' \ge 0 : \int_0^{t'} ds' / \left(\frac{1}{z^\circ} Z^{(z^\circ)}_{z^\circ s'}\right) = h\right\}.$$

Inserting (29) into (28) and using (18) we finally see that the conditional law from (27) is the marginal law of $X^{(\mathbf{1})}_{\tau^\circ}$ where τ° is the stopping time for $X^{(\mathbf{1})}$ given by

$$\tau^\circ = \inf\left\{t^\circ \geq 0 : \int_0^{t^\circ} ds^\circ / Z^{(1)}_{s^\circ} = h\right\}.$$

Since the result neither depends on $\mathcal{F}_u$ nor $\bar{a}$ nor u, the proof is complete. □

An easy consequence of Theorem 1 is the following *agglomeration property* of the multi-self-similar processes.

Corollary 5. *Suppose that $\left(X^{(x)}\right)_{x\in\mathbb{R}^n_+}$ is an n-dimensional Markov family, multi-self-similar in the sense of Definition 1 and defined in terms of one n-dimensional Lévy process ξ as in (10). Let*

$$\{1, \dots, n\} = \bigcup_{k=1}^{n'} I_k$$

where the I_k are non-empty and disjoint and define for $y_k \in \mathbb{R}_+$, $1 \leq k \leq n$ and arbitrary $x_i \in \mathbb{R}_+$ such that $\prod_{I_k} x_i = y_k$ for all k,

$$Y^{k,(y_k)} = \prod_{I_k} X^{i,(x_i)}. \tag{30}$$

Then $\left(Y^{(y)}\right)_{y\in\mathbb{R}^n_+}$ is a multi-self-similar strong Markov family with values in $\mathbb{R}^{n'}_+$.

Note. Of course $Y^{(y)} = \left(Y^{k,(y_k)}\right)_{1\leq k\leq n}$ with $y = (y_k)$. That the definition (30) is unambiguous is clear from (23) and also from the first line of the proof.

Proof. Using (10) we find

$$Y^{k,(y_k)}_{\int_0^u dv\, \exp \bar{\eta}^{(\bar{b})}_v} = \exp\left(\eta^{k,(b_k)}_u\right)$$

where $\eta^{(b)} = \eta + b$ with $b = (b_k)$ given by $b_k = \sum_{I_k} a_i$ (so that $\bar{b} = \bar{a}$), and where $\eta = \left(\eta^k\right)$ is the n'-dimensional Lévy process given by $\eta^k = \sum_{I_k} \xi^i$ (so that $\bar{\eta} = \bar{\xi}$). Now use Theorem 1(a). □

The special case of Corollary 5 with ξ Gaussian was given in Jacobsen [12].

If we take $n' = 1$, $I_1 = \{1, \dots, n\}$ we see that if X is n-dimensional multi-self-similar Markov with all $X^i > 0$, then $Z = \prod X^i > 0$ is one-dimensional 1-self-similar: $\left(cZ^{(z/c)}_t\right)_{t\geq 0} \stackrel{(\mathrm{d})}{=} \left(Z^{(z)}_{ct}\right)_{t\geq 0}$ for all $c > 0$.

Remark 1. Corollary 5 states that our multi-self-similar processes have a multiplicative agglomeration property, which is deduced easily from the (trivial) additive agglomeration property of Lévy processes. More precisely we shall say that a class $\mathcal{L}$ of laws of processes where the members of the class must correspond to processes in different dimensions, has the *additive,* resp. *multiplicative, agglomeration property* if for all $U = (U^i)_{1 \le i \le n} \overset{(d)}{\in} \mathcal{L}$ of dimension $n \ge 2$ and all disjoint partitionings $\{1, \ldots, n\} = \bigcup_{k=1}^{n'} I_k$ with the $I_k \neq \emptyset$, it holds that $\tilde{U} = \left(\tilde{U}^k\right)_{1 \le k \le n'} \overset{(d)}{\in} \mathcal{L}$ where

$$\tilde{U}^k = \begin{cases} \sum_{i \in I_k} U^i & \text{(additive case)} \\ \prod_{i \in I_k} U^i & \text{(multiplicative case).} \end{cases}$$

An instance of a class of non-Lévy processes with the additive agglomeration property is provided by the family of multivariate Jacobi diffusions in Jacobsen [12], Example 5. Taking the exponential of each coordinate of such a diffusion (without a time-change) yields a class of diffusions with the multiplicative agglomeration property, that is not multi-self-similar.

Returning to the proofs of the main results, we next give

Proof of Theorem 2. Consider for $x \in \mathbb{R}_+^n$ the functional

$$\left(\prod_{i=1}^n x_i^{p_i}\right) \int_0^\infty ds \prod_1^n \left(X_s^{i,(x_i)}\right)^{-p_i - 1}. \tag{31}$$

(This random variable is not only finite but has a finite expectation as will be argued below). By (18) the law of (31) equals the law of

$$\begin{aligned} &\left(\prod_{i=1}^n x_i^{p_i}\right) \int_0^\infty ds \prod_1^n \left(x_i X_{s/z}^{i,(1)}\right)^{-p_i - 1} \\ &= \int_0^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i - 1}. \end{aligned} \tag{32}$$

But for $t \ge 0$, the Markov property for $X^{(\mathbf{1})}$ implies that the conditional distribution of

$$V_t = \left(\prod_{i=1}^n \left(X_t^{i,(1)}\right)^{p_i}\right) \int_t^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i - 1} \tag{33}$$

given $\left(X_s^{(\mathbf{1})}\right)_{0 \le s \le t}$, $X_t^{(\mathbf{1})} = x$ is precisely the law of (31). Since by (32) that law depends neither on x nor t, we deduce that V_t is independent of $\left(X_s^{(\mathbf{1})}\right)_{0 \le s \le t}$ with a law the same as that of (32). Consequently

$$\begin{aligned} \mathbb{E} \int_t^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i - 1} &= \mathbb{E} \prod_1^n \left(X_t^{i,(1)}\right)^{-p_i} \mathbb{E} V_t \\ &= \mathbb{E} \prod_1^n \left(X_t^{i,(1)}\right)^{-p_i} \mathbb{E} \int_0^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i - 1}, \end{aligned} \tag{34}$$

whether the expectations are finite or not.

Now, with $H_t = \int_0^t ds\, 1/Z_s^{(1)}$ we have $X_s^{i,(1)} = \exp \xi^i_{H_s}$ and hence

$$\mathbb{E}\int_0^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i-1} = \mathbb{E}\int_0^\infty dH_s \, \exp\left(-\langle p, \xi_{H_s}\rangle\right)$$
$$= \int_0^\infty du \, \mathbb{E}\exp\left(-\langle p, \xi_u\rangle\right)$$
$$= \frac{1}{\Phi(p)}, \tag{35}$$

in particular the expectation is finite. We have shown that (34) may be written

$$\mathbb{E}\int_t^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i-1} = \frac{1}{\Phi(p)}\mathbb{E}\prod_1^n \left(X_t^{i,(1)}\right)^{-p_i} \tag{36}$$

with both expectations finite: that on the left is $\leq 1/\Phi(p)$ by (35).

Differentiating with respect to t in (36) gives

$$-\mathbb{E}\prod_1^n \left(X_t^{i,(1)}\right)^{-p_i-1} = \frac{1}{\Phi(p)}\frac{\partial}{\partial t}\mathbb{E}\prod_1^n \left(X_t^{i,(1)}\right)^{-p_i}.$$

Again by (36) the expression on the left equals

$$-\Phi(p+\mathbf{1})\mathbb{E}\int_t^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i-2},$$

and repeated differentiation now yields the formula

$$\frac{\partial^k}{\partial t^k}\mathbb{E}\prod_1^n \left(X_t^{i,(1)}\right)^{-p_i}$$
$$= (-1)^k\, \Phi(p)\, \Phi(p+\mathbf{1}) \cdots \Phi(p+\mathbf{k})\, \mathbb{E}\int_t^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i-k-1}, \tag{37}$$

valid for $k = 0, 1, \ldots$. From this it follows in particular that $t \mapsto \mathbb{E}\prod_1^n \left(X_t^{i,(1)}\right)^{-p_i}$ is a completely monotone function of t, hence by Bernstein's theorem there is a probability ρ_p on $\mathbb{R}_0$ such that

$$\mathbb{E}\prod_1^n \left(X_t^{i,(1)}\right)^{-p_i} = \int_0^\infty \rho_p(dx)\, e^{-tx}. \tag{38}$$

Finally, the formula (13) for the moments of ρ_p follows from (37) (for $t = 0$) and (35). □

Remark 2. Writing $(\mathcal{F}_t)$ for the filtration generated by $X^{(\mathbf{1})}$ we see from the fact that V_t given by (33) is independent of $\mathcal{F}_t$ with a law equal to that of (32), and from (35) that

$$\mathbb{E}\left[\int_0^\infty ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i-1} |\mathcal{F}_t\right]$$
$$= \int_0^t ds \prod_1^n \left(X_s^{i,(1)}\right)^{-p_i-1} + \prod_1^n \left(X_t^{i,(1)}\right)^{-p_i} \frac{1}{\Phi(p)}$$

defines a uniformly integrable $\mathcal{F}_t$-martingale. The same fact follows using a time-change on the exponential functional Lévy martingale

$$\mathbb{E}\left[\int_0^\infty dv \, \exp - \langle p, \xi_v \rangle \, |\mathcal{G}_u\right]$$
$$= \int_0^u dv \, \exp - \langle p, \xi_v \rangle + \frac{1}{\Phi(p)} \exp - \langle p, \xi_u \rangle \, .$$

Remark 3. Note that by (36), (37) may be written

$$\frac{\partial^k}{\partial t^k} \mathbb{E} \prod_1^n \left(X_t^{i,(1)}\right)^{-p_i}$$
$$= (-1)^k \, \Phi(p) \, \Phi(p+\mathbf{1}) \cdots \Phi(p+\mathbf{k}-\mathbf{1}) \, \mathbb{E} \prod_1^n \left(X_t^{i,(1)}\right)^{-p_i-k}$$

or, see (38),

$$\int_0^\infty \rho_p(dx) \, x^k e^{-tx} = \Phi(p) \, \Phi(p+\mathbf{1}) \cdots \Phi(p+\mathbf{k}-\mathbf{1}) \int_0^\infty \rho_{p+\mathbf{k}}(dx) \, e^{-tx}$$

which implies that $\rho_{p+\mathbf{k}} \ll \rho_p$ with Radon-Nikodym derivative

$$\frac{d\rho_{p+\mathbf{k}}}{d\rho_p}(x) = \frac{x^k}{\Phi(p) \, \Phi(p+\mathbf{1}) \cdots \Phi(p+\mathbf{k}-\mathbf{1})}.$$

Proof of Corollary 3. It suffices to apply Theorem 2 to the n-dimensional subordinator $\tilde{\xi}$ defined by

$$\tilde{\xi}_u^i = q_i \xi_u^i$$

which, by the Lamperti representation, has the associated multi-self-similar process $\left(\tilde{X}^i\right)_{1 \le i \le n}$ defined implicitly by

$$\exp\left(q_i \xi_u^i\right) = \tilde{X}^i_{\int_0^u dv \, \exp\langle q, \xi_v \rangle}.$$

Defining ${}^{(q)}X_t^i = \left(\tilde{X}_t^i\right)^{1/q_i}$ we then have

$$\mathbb{E}\left[\prod_{i=1}^{n}\left({}^{(q)}X_t^i\right)^{-p_i}\right] = \mathbb{E}\left[\prod_{i=1}^{n}\left(\tilde{X}_t^i\right)^{-p_i/q_i}\right]$$

and since the Lévy exponent $\tilde{\Phi}$ of $\tilde{\xi}$ is given by $\tilde{\Phi}(p) = \Phi\left((p_i q_i)_i\right)$ formula (14) follows since $\rho_{p,q}$ is obviously equal to $\tilde{\rho}_{(p_i/q_i)_i}$ and, for $j \in \mathbb{N}$,

$$\tilde{\Phi}\left(\left(\frac{p_i}{q_i}\right)_i + \mathbf{j}\right) = \Phi\left((p_i + jq_i)_i\right) = \Phi(p + jq).$$

It remains to establish (15). To this end, let ζ be a one-dimensional subordinator with Lévy exponent φ_ζ. Then, letting $a \geq 0$ and defining $I_a = \int_0^{\mathbf{e}_a} du\, e^{-\zeta_u}$, where $\mathbf{e}_a$ is independent of ζ and exponential at rate a, it holds that

$$\mathbb{E}(I_a)^k = \frac{k!}{\left(a + \varphi_\zeta(1)\right)\left(a + \varphi_\zeta(2)\right)\cdots\left(a + \varphi_\zeta(k)\right)}. \tag{39}$$

To see this, write the expectation as

$$\mathbb{E}(I_a)^k = k!\int_0^\infty du_1 \int_{u_1}^\infty du_2 \cdots \int_{u_{k-1}}^\infty du_k\, \mathbb{E}\exp\left(-\textstyle\sum_{j=1}^k \zeta_{u_j}\right)\displaystyle\prod_{j=1}^k 1_{(\mathbf{e}_a > u_j)}. \tag{40}$$

But here (writing $u_0 = 0$),

$$\begin{aligned}
&\mathbb{E}\exp\left(-\textstyle\sum_{j=1}^k \zeta_{u_j}\right)\displaystyle\prod_{j=1}^k 1_{(\mathbf{e}_a > u_j)} \\
&\quad= \mathbb{E}\exp\left(-\textstyle\sum_{j=1}^k (k+1-j)\left(\zeta_{u_j} - \zeta_{u_{j-1}}\right)\right) 1_{(\mathbf{e}_a > u_k)} \\
&\quad= e^{-au_k}\prod_{j=1}^k \exp\left(-\left(u_j - u_{j-1}\right)\varphi_\zeta(k+1-j)\right)
\end{aligned}$$

and since $u_k = \sum_1^k \left(u_j - u_{j-1}\right)$ it is now easy to perform the integrations in (40) and arrive at (39).

To proceed, consider an arbitrary n-dimensional subordinator η with Lévy exponent Φ_η. Applying the preceding to the one-dimensional subordinator $\langle q, \eta\rangle$ shows that

$$\frac{k!}{\left(a + \Phi_\eta(q)\right)\left(a + \Phi_\eta(2q)\right)\cdots\left(a + \Phi_\eta(kq)\right)}$$

defines a moment sequence. Applying this with $a = \Phi(p)$ and $\Phi_\eta = \Phi(p + \cdot) - \Phi(p)$ (which is the Lévy exponent for the Esscher transform of ξ determined by the local change of measure

$$\frac{d\widetilde{\mathbb{P}}_{|\mathcal{F}_u^\xi}}{d\mathbb{P}_{|\mathcal{F}_u^\xi}} = \exp\left(-\langle p, \xi_u\rangle + u\Phi(p)\right)$$

for any $u \geq 0$) finally shows (15) to be a sequence $(m_k)_{k\geq 1}$ of moments. That this sequence determines a unique probability on $\mathbb{R}_0$ if $\Phi(p) > 0$ follows from the simple observation that the power series $\sum_{k=0}^{\infty} m_k \frac{h^k}{k!} \leq \sum_{k=0}^{\infty} \left(\frac{h}{\Phi(p)}\right)^k$ converges for $0 \leq h < \Phi(p)$. □

3. The case with ξ standard Brownian motion

3.1. Some facts about Bessel processes

In this subsection we gather the notation and results we need about Bessel processes. As already mentioned in the introduction, the Bessel process with *index* ν (denoted BES(ν)) occurs in the one-dimensional Lamperti representation of Brownian motion with drift $\nu \geq 0$ as, see (1)

$$\exp(B_u + \nu u) = R^{(\nu)}_{A_u^{(\nu)}} \tag{41}$$

where

$$A_u^{(\nu)} = \int_0^u dv \, \exp 2\left(B_v + \nu v\right).$$

We shall call $d = 2(1+\nu)$ the *'dimension'* of the Bessel process. Thus $R^{(\nu)}$ is an $\mathbb{R}_+$-valued diffusion with infinitesimal generator $\mathcal{L}^\nu$ given by

$$\mathcal{L}^\nu f(x) = \tfrac{1}{2} f''(x) + \frac{2\nu+1}{2x} f'(x) \quad \left(f \in C_b^2(\mathbb{R}_+)\right).$$

For any $\nu \geq 0$ we denote by $\mathbb{P}_a^\nu$ the law on $C(\mathbb{R}_0, \mathbb{R}_+)$ of $R^{(\nu)}$ when starting from a. We write $(\mathrm{R}_u)_{u\geq 0}$ for the canonical process on $C(\mathbb{R}_0, \mathbb{R}_+)$ and we denote by $\mathcal{R}_t = \sigma\{\mathrm{R}_s; 0 \leq s \leq t\}$ for $t \geq 0$ the canonical filtration.

From the Cameron-Martin relationship between the laws of $(B_u + \nu u)_{u\geq 0}$ and $(B_u)_{u\geq 0}$, we deduce by time-changing and with the help of (41) that

$$\mathbb{P}^\nu_{a|\mathcal{R}_t} = \left(\frac{\mathrm{R}_t}{a}\right)^\nu \exp\left(-\frac{\nu^2}{2}\int_0^t ds \, \frac{1}{\mathrm{R}_s^2}\right) \cdot \mathbb{P}^0_{a|\mathcal{R}_t}, \tag{42}$$

a denoting the initial state: $\mathrm{R}_0 \equiv a$ a.s. under both probabilities.

The following formula, which expresses negative moments of a Bessel process will also be useful:

$$\mathbb{E}_1^\nu\left[\frac{1}{(\mathrm{R}_t)^{2b}}\right] = \frac{1}{\Gamma(b)} \int_0^{1/2t} dr \, e^{-r} r^{b-1} (1 - 2tr)^{\nu - b} \tag{43}$$

for $b \in \mathbb{C}$ with Re $b > 0$ (see e.g. Yor [20], Proposition 6.4 or Yor [21], p.67).

3.2. A proof of Theorem 4, and a characteristic function determining the semigroup of the multi-self-similar diffusion defined by standard Brownian motion

With $B = \left(B^k\right)_{1\le k\le n}$ a standard BM(n)-process and $\bar{B} = \sum_1^n B^k$, from the one-dimensional Lamperti representation

$$\exp \bar{B}_u = Z_{\int_0^u dv\, \exp \bar{B}_v}$$

with Z a 1-self-similar diffusion, we deduce by e.g. writing Itô's formula for $\exp \bar{B}$ and time-changing with the inverse of $\left(\int_0^u dv\, \exp \bar{B}_v\right)_{u\ge 0}$, or just using (2) for $\nu = 0$, that Z satisfies

$$\left(Z_{\frac{t}{n}}\right)_{t\ge 0} \overset{(\mathrm{d})}{=} \left(\left(R_{\frac{t}{4}}\right)^2\right)_{t\ge 0} \tag{44}$$

with R a 2-dimensional Bessel process starting from 1. In the sequel, just use the notation $R_t = \sqrt{Z_{\frac{4t}{n}}}$ for all t.

Note. In this subsection we label the coordinates of a process ^{k} rather than ^{i}, since below i will denote the complex unit.

We next consider the orthogonal decomposition of B with respect to $\bar{B}$, i.e. we define the process $C = \left(C^k\right)_{1\le k\le n}$ by

$$B_t^k = \tfrac{1}{n}\bar{B}_t + C_t^k \quad (t \ge 0, 1 \le k \le n)\,. \tag{45}$$

Then C is a mean 0, n-dimensional Gaussian Lévy process, independent of $\bar{B}$, where the covariance matrix for its increments is given by

$$\mathbb{E}\left[\left(C_{s+t}^k - C_s^k\right)\left(C_{s+t}^\ell - C_s^\ell\right)\right] = \mathbb{E}\left[C_t^k C_t^\ell\right] = \begin{cases} \left(1-\frac{1}{n}\right)t & (k=\ell)\,, \\ -\frac{1}{n}t & (k\ne\ell)\,. \end{cases} \tag{46}$$

By (10) in Theorem 1, the multi-self-similar diffusion $S = S^{(\mathbf{1})} = \left(S^k\right)_{1\le k\le n}$ starting from $\mathbf{1}$, determined by B satisfies

$$S^k_{\int_0^u dv\, \exp \bar{B}_v} = \exp B_u^k \tag{47}$$

and hence by time-changing and using (44)

$$S_t^k = \left(R_{\frac{nt}{4}}\right)^{\frac{2}{n}} \exp\left(C^k_{\frac{4}{n}\int_0^{nt/4} dh\, 1/R_h^2}\right)$$

which establishes (16) and completes the proof of Theorem 4.

Using some of the results from Subsection 3.1, we can now give an explicit formula for the characteristic function of $\left(\log S_t^k\right)_{1\le k\le n}$. Since $S_0 \equiv \mathbf{1}$ this gives the characteristic function for the transition probabilities $P_t(\mathbf{1},\cdot)$ from state $\mathbf{1}$ of the diffusion S which, by the discussion at the beginning of Section 2, is enough to determine the transition probabilities from any state, see (19). Furthermore (use

(21) with $x = \mathbf{1}$), $P_t(\mathbf{1}, \cdot)$ is determined by the law of Z_t (the transition probability from the state 1 for the product process $Z = \prod_{k=1}^n S^k$), which is known from (44) as that of $\left(R_{nt/4}\right)^2$, and the conditional law of S_t given Z_t. In Proposition 6 below we describe this conditional law together with the characteristic function for the transition probabilities of $\left(\log S^k\right)_k$.

We begin by deriving a first expression for the characteristic function of $\left(\log S_t^k\right)$. Let $\lambda = (\lambda_k)_{1\le k\le n} \in \mathbb{R}^n$, write $\bar{\lambda} = \sum_1^n \lambda_k$ and $T = \frac{nt}{4}$ and use that C is independent of R to obtain,

$$\mathbb{E}\left[\prod_{k=1}^n \exp\left(i\lambda_k \log S_t^k\right)\right] = \mathbb{E}\left[\prod_{k=1}^n \left(S_t^k\right)^{i\lambda_k}\right]$$
$$= \mathbb{E}\left[(R_T)^{2i\frac{\bar{\lambda}}{n}} \exp\left(-\left(\frac{2}{n}\int_0^T dh\, \frac{1}{R_h^2}\right)\varphi_n(\lambda)\right)\right] \quad (48)$$

where

$$\varphi_n(\lambda) = \mathbb{E}\left(\sum_{k=1}^n \lambda_k C_1^k\right)^2$$
$$= \left(1 - \frac{1}{n}\right)\left(\sum_{k=1}^n \lambda_k^2 - \frac{1}{n-1}\sum_{1\le k,k'\le n, k\ne k'} \lambda_k\lambda_{k'}\right). \quad (49)$$

With the help of the absolute continuity relationship (42) (for $a = 1$) we can write the last term in (48) as

$$\mathbb{E}^{\nu_n(\lambda)}\left[(R_T)^{2i\frac{\bar{\lambda}}{n} - \nu_n(\lambda)}\right]$$

with

$$\nu_n(\lambda) = \frac{2}{\sqrt{n}}\sqrt{\varphi_n(\lambda)}.$$

Now we apply (43) with $\nu = \nu_n(\lambda)$ and $b = \frac{1}{2}\nu_n(\lambda) - i\frac{\bar{\lambda}}{n}$ (whence $\nu - b = \frac{1}{2}\nu_n(\lambda) + i\frac{\bar{\lambda}}{n}$) and obtain,

$$\mathbb{E}\left[\prod_{k=1}^n \left(S_t^k\right)^{i\lambda_k}\right]$$
$$= \frac{1}{\Gamma\left(\frac{1}{2}\nu_n(\lambda) - i\frac{\bar{\lambda}}{n}\right)}\int_0^{\frac{1}{2T}} dr\, \frac{e^{-r}}{r}\left(r(1-2Tr)\right)^{\frac{1}{2}\nu_n(\lambda)}\left(\frac{1-2Tr}{r}\right)^{i\frac{\bar{\lambda}}{n}}$$

which is the first expression for the desired characteristic function. An alternative way of writing this is obtained by observing that since $\sum_1^n C_t^k \equiv 0$ (which implies

$\varphi_n(\lambda) = \varphi_n(\lambda + c\mathbf{1})$ for any $c \in \mathbb{R}$) we may as well write λ in the form $\theta + c\mathbf{1}$ for $\theta = (\theta_k)_{1\le k\le n}$ with $\bar{\theta} = \sum_1^n \theta_k = 0$ and thus arrive at

$$\begin{aligned}
&\mathbb{E}\left[\prod_{k=1}^n \left(S_t^k\right)^{i(\theta_k+c)}\right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\nu_n(\theta) - ic\right)} \int_0^{\frac{1}{2T}} dr\, \frac{e^{-r}}{r} \left(r\left(1-2Tr\right)\right)^{\frac{1}{2}\nu_n(\theta)} \left(\frac{1-2Tr}{r}\right)^{ic}. \qquad (50)
\end{aligned}$$

The form (50) of the characteristic function allows the following partial inversion of the Fourier transform: introducing

$$\Pi_t^{(\theta)} = \prod_{k=1}^n \left(S_t^k\right)^{i\theta_k}$$

and writing $Z_t = \prod_{k=1}^n S_t^k$ as before, (50) becomes when taking the gamma value to the left

$$\begin{aligned}
&\mathbb{E}\left[\Pi_t^{(\theta)} (Z_t)^{ic}\right] \int_0^\infty dx\, x^{\frac{1}{2}\nu_n(\theta)-1-ic} e^{-x} \\
&= \int_0^\infty dx\, x^{\frac{1}{2}\nu_n(\theta)-1} e^{-x}\, \mathbb{E}\left[\Pi_t^{(\theta)} \left(\frac{Z_t}{x}\right)^{ic}\right] \\
&= \int_0^{\frac{1}{2T}} dr\, \frac{e^{-r}}{r} \left(r\left(1-2Tr\right)\right)^{\frac{1}{2}\nu_n(\theta)} \left(\frac{1-2Tr}{r}\right)^{ic}
\end{aligned}$$

which, essentially by Fourier inversion with c varying freely, allows us to identify the measures

$$\begin{aligned}
f &\mapsto \int_0^\infty dx\, x^{\frac{1}{2}\nu_n(\theta)-1} e^{-x}\, \mathbb{E}\left[\Pi_t^{(\theta)} f\left(\frac{Z_t}{x}\right)\right], \\
f &\mapsto \int_0^{\frac{1}{2T}} dr\, \frac{e^{-r}}{r} \left(r\left(1-2Tr\right)\right)^{\frac{1}{2}\nu_n(\theta)} f\left(\frac{1-2Tr}{r}\right).
\end{aligned}$$

Changing the order of integration in the first integral and then making the substitution $x = yZ_t$ there, together with the substitution $\frac{1-2Tr}{r} = \frac{1}{s}$ in the second integral leads to the identity

$$\begin{aligned}
&\int_0^\infty dy\, f\left(\frac{1}{y}\right) \mathbb{E}\left[Z_t \Pi_t^{(\theta)} \left(yZ_t\right)^{\frac{1}{2}\nu_n(\theta)-1} e^{-yZ_t}\right] \\
&= \int_0^\infty ds\, f\left(\frac{1}{s}\right) s^{\frac{1}{2}\nu_n(\theta)-1} \frac{e^{-s/(2Ts+1)}}{(2Ts+1)^{\nu_n(\theta)+1}}.
\end{aligned}$$

This being true for, say all bounded Borel functions f, allows us to identify the two integrands, i.e. we have the formula

$$\mathbb{E}\Pi_t^{(\theta)} (Z_t)^{\frac{1}{2}\nu_n(\theta)} e^{-sZ_t} = \frac{e^{-s/(2Ts+1)}}{(2Ts+1)^{\nu_n(\theta)+1}} \quad (s \ge 0) \qquad (51)$$

valid for all $\theta \in \mathbb{R}^n$ with $\bar{\theta} = 0$. But the expression on the right of (51) may be recognized as the Laplace transform for the transition probability of a squared Bessel process: if Q_x^δ denotes the law of a BESQ(δ)-process X° of 'dimension' $\delta = 2(\nu + 1)$ starting from $x \geq 0$, then, see e.g. Revuz and Yor [18], Chapter XI,

$$Q_x^\delta\left(e^{-\mu X_{t'}^\circ}\right) = \frac{1}{(2\mu t' + 1)^{\frac{\delta}{2}}} \exp\left(-\frac{\mu x}{2\mu t' + 1}\right) \quad \left(\mu \geq 0, t' \geq 0\right)$$

and thus, if $q_{t'}^\delta(\cdot, \cdot)$ denotes the transition density

$$q_{t'}^\delta\left(x, x'\right) dx' = Q_x^\delta\left(X_{t'}^\circ \in dx'\right),$$

(51) implies that for all bounded Borel functions g,

$$\mathbb{E}\Pi_t^{(\theta)}\left(Z_t\right)^{\frac{1}{2}\nu_n(\theta)} g\left(Z_t\right) = \int_0^\infty dz\, q_T^{\delta_n(\theta)}(1, z)\, g(z) \tag{52}$$

where $\delta_n(\theta) = 2\left(\nu_n(\theta) + 1\right)$. But either from (44) or (52) for $\theta = 0$ (in which case $\Pi^{(\theta)} \equiv 1$, $\nu_n(\theta) = 0$) we know that

$$\mathbb{P}\left(Z_t \in dz\right) = q_T^2(1, z)\, dz, \tag{53}$$

hence (52) shows that

$$\mathbb{E}\left[\Pi_t^{(\theta)} | Z_t = z\right] = \frac{q_T^{\delta_n(\theta)}(1, z)}{q_T^2(1, z)} z^{-\frac{1}{2}\nu_n(\theta)}. \tag{54}$$

Finally, letting $\lambda = (\lambda_k) \in \mathbb{R}^n$ and using (54) wih $\theta_k = \lambda_k - \frac{1}{n}\bar{\lambda}$, where as usual $\bar{\lambda} = \sum_1^n \lambda_k$, we obtain the conditional characteristic function for $\left(\log S_t^k\right)_k$,

$$\mathbb{E}\left[\prod_{k=1}^n \left(S_t^k\right)^{i\lambda_k} | Z_t = z\right] = \frac{q_T^{\delta_n(\lambda)}(1, z)}{q_T^2(1, z)} z^{-\frac{1}{2}\nu_n(\lambda) + i\frac{\bar{\lambda}}{n}}. \tag{55}$$

We summarise our findings in the following result, where (56) is obtained from (55) inserting the known explicit forms for the q_T^δ (see e.g. Revuz and Yor [18], Chapter XI) and (57) follows taking expectations in (55), using (18), (53) and the explicit form of q_T^δ. Recall that $T = \frac{nt}{4}$.

Proposition 6. *For S the multi-self-similar diffusion starting from $\mathbf{1}$, determined by the multidimensional Lamperti representation of n-dimensional standard Brownian motion as in (47), it holds for any $t > 0$ that $Z_t = \prod_{k=1}^n S_t^k$ has density $q_{nt/4}^2(1, \cdot)$ and that the characteristic function of $\left(\log S_t^k\right)_k$ given $Z_t = z$ is given by the expression*

$$\mathbb{E}\left[\prod_{k=1}^n \left(S_t^k\right)^{i\lambda_k} | Z_t = z\right] = \left(\frac{I_{\nu_n(\lambda)}}{I_0}\right)\left(\frac{4\sqrt{z}}{nt}\right) z^{i\frac{\bar{\lambda}}{n}} \tag{56}$$

for all $z > 0$ and all $\lambda = (\lambda_k)_{1 \le k \le n} \in \mathbb{R}^n$. Finally, the transition probabilities $P_t(x, \cdot)$ for S are determined by

$$\int_{\mathbb{R}_+^n} P_t(x, dy) \prod_{k=1}^n (y_k)^{i\lambda_k} = \frac{2}{nt} \prod_{k=1}^n \left(\frac{x_k}{z^{1/n}} \right)^{i\lambda_k} \int_{\mathbb{R}_+} d\tilde{z}\, e^{-\frac{2}{nt}(z+\tilde{z})} I_{\nu_n(\lambda)} \left(\frac{4}{nt} \sqrt{z\tilde{z}} \right) \tilde{z}^{i\frac{\bar{\lambda}}{n}} \tag{57}$$

for $x \in \mathbb{R}_+^n$, $\lambda \in \mathbb{R}^n$, writing $z = \prod_{k=1}^n x_k$.

Note that from (56) it follows that if $\bar{\lambda} = 0$, then the characteristic function of $\left(\log S_t^k - \frac{1}{n} \log Z_t\right)_k$ given $Z_t = z$ is $\mathbb{R}$-valued, i.e. for any $\mu = (\mu_k)_k \in \mathbb{R}^n$ it holds that the conditional law of

$$\sum_{k=1}^n \mu_k \left(\log S_t^k - \frac{1}{n} \log Z_t \right)$$

given $Z_t = z$ is symmetric (around 0) for any $z > 0$.

3.3. The two-dimensional case

We note that the contents of Theorem 4 in the case $n = 2$ are clearly related to the conformal invariance of planar Brownian motion. Indeed, first starting with $B^1 + iB^2$ a $\mathbb{C}$-valued standard Brownian motion and noting the fact that

$$\beta_u = \frac{1}{\sqrt{2}} \left(B_u^1 + B_u^2 \right), \quad \gamma_u = \frac{1}{\sqrt{2}} \left(B_u^1 - B_u^2 \right)$$

are two independent standard Brownian motions, we get

$$\exp B_u^1 \equiv \exp \left(\frac{1}{2} \left(B_u^1 + B_u^2 \right) + \frac{1}{2} \left(B_u^1 - B_u^2 \right) \right) = \exp \left(\frac{1}{\sqrt{2}} (\beta_u + \gamma_u) \right),$$
$$\exp B_u^2 \equiv \exp \left(\frac{1}{2} \left(B_u^1 + B_u^2 \right) - \frac{1}{2} \left(B_u^1 - B_u^2 \right) \right) = \exp \left(\frac{1}{\sqrt{2}} (\beta_u - \gamma_u) \right)$$

so that

$$S_t^1 = \sqrt{Z_t} \exp \left(\frac{1}{\sqrt{2}} \gamma_t \right), \quad S_t^2 = \sqrt{Z_t} \exp \left(-\frac{1}{\sqrt{2}} \gamma_t \right)$$

with $u = \int_0^t dh \, \frac{1}{Z_h}$ and $\left(\sqrt{Z_t}\right)_{t \ge 0} \overset{(d)}{=} \left(R_{\frac{t}{2}} \right)_{t \ge 0}$ as in (44) above a process independent of γ.

Another equivalent presentation of the process $\left(S^1, S^2\right)$ is that

$$L_t := \log S_t^1 + i \log S_t^2 \quad (t \ge 0)$$

is a conformal martingale. More precisely it may be written as

$$L_t = \xi_{\int_0^t dh\, 1/S_h^1 S_h^2} \quad (t \ge 0)$$

with ξ a standard two-dimensional Brownian motion.

3.4. A change of variables

In Subsection 3.2, we explained how the law of the process $\left(S_t^k\right)_{1\le k\le n, t\ge 0}$ could be expressed in terms of that of

$$\left(R_t, \int_0^t dh\, \frac{1}{R_h^2}\right)_{t\ge 0}$$

where $R_t = \sqrt{Z_{\frac{4t}{n}}}$ is a two-dimensional Bessel process starting from 1. In the present Subsection 3.4, we show how to compute the law of $\left(S_t^k\right)_{k,t}$, in terms of that of $\left(R_t, \int_0^t ds\, R_s^2\right)_{t\ge 0}$ via the definition and study of the process

$$Y_t^k = \int_0^t Z_{\setminus k,s}\, dS_s^k \quad (1 \le k \le n, t \ge 0)\,, \tag{58}$$

where as before $Z_{\setminus k,s} = Z_s / S_s^k = \prod_{\ell \ne k} S_s^\ell$.

In the sequel, rather than developing some tedious computations, we shall refer to the following (implicit) description of the multidimensional marginals of $\left(R_t, \int_0^t ds\, R_s^2\right)_{t\ge 0}$, which, thanks to the Markov property of R, may be reduced to the description of the one-time t-marginals; this may be done via the following formula, which should be attributed to Lévy (see e.g. Pitman and Yor [17] and Yor [20] for many further developments): for $a = \alpha + i\beta, \alpha \ge 0,\ \beta \in \mathbb{R}$ and $b \ge 0$ one has

$$\begin{aligned}
&\mathbb{E}^r \left[\exp\left(-aR_t^2 - \frac{b^2}{2}\int_0^t ds\, R_s^2\right)\right] \\
&\quad = \left(\cosh(bt) + \frac{2a}{b}\sinh(bt)\right)^{-1} \exp\left(-\frac{r^2 b}{2}\frac{1+\frac{2a}{b}\coth(bt)}{\coth(bt)+\frac{2a}{b}}\right).
\end{aligned} \tag{59}$$

Here is now the description of the process $\left(Y^k\right)_k$:

Proposition 7. *i) The processes Y^k and Z satisfy the equations*

$$dY_t^k = \tfrac{1}{2}\, dt + \sqrt{Z_t}\, dB_t^k, \quad dZ_t = \sum_{k=1}^n dY_t^k = \tfrac{n}{2}\, dt + \sqrt{Z_t}\, d\bar{B}_t. \tag{60}$$

ii) The vector-valued process $\left(Y^k\right)_{1\le k\le n}$ satisfies

$$Y_t^k = \tfrac{1}{2}t + \tfrac{1}{n}\left(Z_t - 1 - \tfrac{n}{2}t\right) + \hat{C}^k\left(\int_0^t ds\, Z_s\right) \quad (1 \le k \le n, t \ge 0) \tag{61}$$

where the process $\left(\hat{C}_u^k\right)_{1\le k\le n, u\ge 0}$ is distributed as $\left(C_u^k\right)_{1\le k\le n, u\ge 0}$ (see formulas (45) and (46)), and is independent of Z.

Proof. (i) As a particular case of (17) we have

$$dS_t^k = \frac{1}{2}\frac{dt}{Z_{\setminus k,t}} + \sqrt{\frac{S_t^k}{Z_{\setminus k,t}}}\,dB_t^k$$

and (60) now follows from Itô's formula.

(ii) (61) follows from (60), once we use the decomposition of $\left(B^k\right)_k$ in terms of $\bar{B}$ and $\left(C^k\right)_k$, see (45). Then conditioning on $\bar{B}$ or Z (these two processes have the same filtration), we may express the vector-valued process $\left(\int_0^t \sqrt{Z_s}\,dC_s^k\right)_{1\le k\le n}$ evaluated at time t, as $\left(\hat{C}_u^k\right)$ evaluated at $u = \int_0^t ds\, Z_s$.

With the help of formulas (59) and (61) we are now able to write down the joint characteristic function for $\left(Y_t^k\right)_k$. We consider for $\theta \in \mathbb{R}^n$, writing A_t as short for $\int_0^t ds\, Z_s$,

$$\langle \theta, Y_t\rangle = \tfrac{1}{2}\bar{\theta}t + \tfrac{\bar{\theta}}{n}\left(Z_t - 1 - \tfrac{nt}{2}\right) + \left\langle \theta, \hat{C}_{A_t}\right\rangle. \tag{62}$$

The Gaussian variable $\left\langle \theta, \hat{C}_u\right\rangle$ is centered and has variance

$$\mathbb{E}\left[\left\langle \theta, \hat{C}_u\right\rangle^2\right] = \left(\sum_{k=1}^n \theta_k^2 - \tfrac{1}{n-1}\sum_{k,k':k\neq k'} \theta_k\theta_{k'}\right) u = \varphi_n(\theta)\, u, \tag{63}$$

cf. (49). Thus from (62) and (63) we obtain

$$\mathbb{E}\left[\exp i\,\langle \theta, Y_t\rangle\right] = \exp\left(i\tfrac{1}{2}\bar{\theta}t\right)\mathbb{E}\left[\exp\left(i\tfrac{\bar{\theta}}{n}\left(Z_t - 1 - \tfrac{nt}{2}\right) - \tfrac{1}{2}\varphi_n(\theta)\int_0^t ds\, Z_s\right)\right]$$

and, since $Z_t = R^2_{\frac{nt}{4}}$, we obtain

$$\begin{aligned}&\mathbb{E}\left[\exp i\,\langle \theta, Y_t\rangle\right]\\ &\quad = \exp\left(i\tfrac{1}{2}\bar{\theta}t\right)\mathbb{E}\left[\exp\left(i\tfrac{\bar{\theta}}{n}\left(R^2_{\frac{nt}{4}} - 1 - \tfrac{nt}{2}\right) - 2\frac{\varphi_n(\theta)}{n}\int_0^{nt/4} ds\, R_s^2\right)\right]\end{aligned}$$

which can be computed with the help of formula (59). □

4. The case with ξ compound Poisson

While in Section 3 we treated the most important case of the multivariate Lamperti representation when the Lévy process ξ is continuous, viz. ξ standard Brownian motion, we in this section shall focus on the simplest situation where ξ has jumps, i.e. we shall assume that ξ is an n-dimensional compound Poisson process with drift. The one-dimensional case (with no drift) was treated briefly by Lamperti [15], the example p. 218.

The compound Poisson process with drift (starting at $\mathbf{0}$) is given by

$$\xi_u = \beta u + \sum_{\ell=1}^{N_u} \eta_\ell$$

where $\beta = (\beta_i)_{1 \leq i \leq n}$ is the drift vector, $N = (N_u)_{u \geq 0}$ is a homogeneous Poisson process with intensity $\kappa > 0$, and $(\eta_\ell)_{\ell \geq 1}$ is a sequence of iid random variables with values in $\mathbb{R}^n \backslash 0$, independent of N. Thus in particular, writing π for the distribution of the η_ℓ, the Lévy measure for ξ is the bounded measure $\nu = \kappa\pi$ on $\mathbb{R}^n \backslash 0$.

In order to proceed we need (9) to hold, which will be assumed from now on. Note however that since $\bar{\xi} = \sum_{i=1}^n \xi^i$ is a one-dimensional compound Poisson process with drift $\bar{\beta} = \sum_{i=1}^n \beta^i$ and Lévy measure $\bar{\nu}$ the restriction to $\mathbb{R} \backslash 0$ of the measure $\nu \circ \sigma^{-1}$, where $\sigma : \mathbb{R}^n \backslash 0 \to \mathbb{R}$ is the transformation $\sigma(y) = \sum_{i=1}^n y_i$, it follows that if $\mathbb{E}\eta_1$ is well defined (i.e. $\mathbb{E}(\eta_1 \vee 0) < \infty$ or $-\mathbb{E}(\eta_1 \wedge 0) < \infty$), then (9) holds if and only if

$$\bar{\beta} + \kappa \mathbb{E}\eta_1 \geq 0.$$

Consider now the multi-self-similar Markov family $\left(X^{(x)}\right)_{x \in \mathbb{R}_+^n}$ determined by the Lamperti representation,

$$X^{i,(x_i)}_{\int_0^u dv \, \exp \bar{\xi}_v^{(\bar{a})}} = \exp \xi_u^{i,(a_i)}, \tag{64}$$

see Theorem 1, where we remind the reader that $\xi_u^{i,(a_i)} = \xi_u^i + a_i$ with $a_i = \log x_i$.

It is clear from (64) and the structure of ξ, that $X^{(x)}$ is a piecewise deterministic Markov process (PDMP) in the sense of M. Davis [5] – we shall refer to $\left(X^{(x)}\right)_{x \in \mathbb{R}_+^n}$ as a PDMP-family. In particular $X^{(x)}$ for any x has finitely many jumps on finite time intervals and all randomness for $X^{(x)}$ is contained in the jump times and the nature of the jumps.

From M. Davis [5] or Jacobsen [11] it is known that a general class of PDMP-families is obtained by considering families $\left(\tilde{X}^{(x)}\right)_x$ of piecewise continuous processes

$$\tilde{X}_t^{(x)} = \phi_{t - \tilde{T}_{\tilde{N}_t}} \left(\tilde{Y}_{\tilde{N}_t}\right) \tag{65}$$

of the following form: $0 \equiv \tilde{T}_0 < \tilde{T}_1 < \cdots \leq \infty$ are the jump times for $\tilde{X}^{(x)}$, $\tilde{N}_t = \max\left\{\ell : \tilde{T}_\ell \leq t\right\}$ is the number of jumps on $[0, t]$, while $\tilde{Y}_0 \equiv x$ and $\tilde{Y}_\ell = \tilde{X}_{\tilde{T}_\ell}$ (defined only if $\tilde{T}_\ell < \infty$) for $\ell \geq 1$ denotes the state reached by $\tilde{X}^{(x)}$ at the time of the ℓ'th jump. A structure sufficient for $\left(\tilde{X}^{(x)}\right)_x$ to be a strong Markov family is then that the $\phi_t(y)$ (apart from being continuous in t) must satisfy the semigroup property

$$\phi_{t+s}(y) = \phi_t(\phi_s(y)), \quad \phi_0(y) = y \tag{66}$$

while for the marked point process $\left(\tilde{T}_\ell, \tilde{Y}_\ell\right)_{\ell\geq 1}$ it should hold that (for $\ell = 0, 1, \ldots$),

$$\mathbb{P}\left(\tilde{T}_{\ell+1} > t \,|\mathcal{G}_\ell\right) = \exp\left(-\int_0^{t-\tilde{T}_\ell} ds\, q\left(\phi_s\left(\tilde{Y}_\ell\right)\right)\right) \tag{67}$$

on the set $\left(\tilde{T}_\ell < \infty\right)$ for $t \geq \tilde{T}_\ell$, while

$$\mathbb{P}\left(\tilde{Y}_{\ell+1} \in \cdot \,\middle|\mathcal{G}_\ell, \tilde{T}_{\ell+1}\right) = p\left(\phi_{\tilde{T}_{\ell+1}-\tilde{T}_\ell}\left(\tilde{Y}_\ell\right), \cdot\right) \tag{68}$$

on the set $\left(\tilde{T}_{\ell+1} < \infty\right)$; in (67) and (68), $\mathcal{G}_\ell$ is the σ-algebra generated by $\left(\tilde{T}_{\ell'}, \tilde{Y}_{\ell'}\right)_{1\leq\ell'\leq\ell}$; in (67), $q\,(y)$ is the intensity for a jump to occur from state y, and in (68), p is a Markov kernel on the state space with $p\,(y, \cdot)$ the distribution of the destination for a jump from state y.

In the one result of this section that we shall now present, we show that the PDMP-family determined by (64) has the structure described by (65), (66), (67) and (68), and we identify ϕ, q and p. In the statement of the proposition T_ℓ denotes the time of the ℓ'th jump of $X^{(x)}$ (which is a.s. finite for any ℓ) and $Y_\ell = X^{(x)}_{T_\ell}$ the state reached by that jump.

Proposition 8. *The PDMP-family* $\left(X^{(x)}\right)_{x\in\mathbb{R}^n_+}$ *determined by* (64) *from* ξ, *the compound Poisson process with drift, has the form* (65) *for any* $x \in \mathbb{R}^n_+$ *with the* $\phi_t\,(y) = \left(\phi^i_t\,(y)\right)_{1\leq i\leq n}$ *satisfying* (66) *and given by, writing* $z = \prod_{i=1}^n y_i$,

$$\phi^i_t\,(y) = \begin{cases} y_i\left(1 + \frac{\bar{\beta}}{z}t\right)^{\beta_i/\bar{\beta}} & \textit{if } \bar{\beta} \neq 0, \\ y_i \exp\left(\frac{\beta_i}{z}t\right) & \textit{if } \bar{\beta} = 0, \end{cases} \tag{69}$$

and the distribution of $(T_\ell, Y_\ell)_{\ell\geq 1}$ *given by* (67) *and* (68) *with* $Y_0 \equiv x$ *and*

$$q\,(y) = \frac{\kappa}{z}, \tag{70}$$

$$p\,(y, \cdot) = \textit{the law of } \left(y_i e^{\eta^i_1}\right)_{1\leq i\leq n}. \tag{71}$$

Proof. From Theorem 1 we know $\left(X^{(x)}\right)_x$ to be a strong Markov family and by the strong Markov property it therefore suffices to consider, for a given arbitrary initial state x, the behaviour of $X^{(x)}$ on the interval $[0, T_1]$ only. But then, if τ_1 is the time of the first jump for ξ, by (64) we have

$$T_1 = \int_0^{\tau_1} dv\, \exp\bar{\xi}^{(\bar{a})}_v$$

and since on $[0, \tau_1[$, ξ is deterministic, $\xi_u = \beta u$, therefore also

$$T_1 = F\,(\tau_1)\,, \tag{72}$$

$$X_t^{i,(x_i)} = \exp\left(a_i + \beta_i u\right) \quad \left(t < T_1, u = F^{-1}(t)\right)$$

with F the function

$$F(u) = \int_0^u dv \exp\left(\bar{a} + \bar{\beta} v\right) = \begin{cases} e^{\bar{a}} \frac{1}{\bar{\beta}} \left(e^{\bar{\beta} u} - 1\right) & \text{if } \bar{\beta} \neq 0, \\ u e^{\bar{a}} & \text{if } \bar{\beta} = 0. \end{cases}$$

Consequently $\phi_t^i(x) = \exp\left(a_i + \beta_i F^{-1}(t)\right)$ proving (69) (since $e^{\bar{a}} = \prod_1^n x_i$) and (66) may then be verified directly. (70) follows from (72) since $\mathbb{P}(\tau_1 > u) = e^{-\kappa u}$. Finally (71) is clear from the identities

$$\Delta X_{T_1}^{i,(x_i)} = \Delta \exp \xi_{\tau_1}^{i,(a_i)} = \exp \xi_{\tau_1 -}^{i,(a_i)} \left(e^{\eta_1^i} - 1\right) = X_{T_1 -}^{i,(x_i)} \left(e^{\eta_1^i} - 1\right),$$

where we use the standard notation Δ to denote jump sizes. □

With $\left(X^{(x)}\right)_x$ the PDMP-family described in Proposition 8, it follows from the general theory for piecewise deterministic Markov processes, M. Davis [5] or Jacobsen [11], that the infinitesimal generator has the form, writing $z = \prod_{i=1}^n x_i$, $z_{\setminus i} = \prod_{j:j \neq i} x_j$,

$$\mathcal{A} f(x) = \sum_{i=1}^n \frac{\beta_i}{z_{\setminus i}} \partial_{x_i} f(x) + \frac{\kappa}{z} \int_{\mathbb{R}^n \setminus 0} \pi(dy) \left[f\left(\left(x_i e^{y_i}\right)_{1 \leq i \leq n}\right) - f(x)\right].$$

Note that for $\bar{\beta} < 0$, $\phi_t^i(x)$ is strictly positive (as it has to be) only for $t < -z/\bar{\beta}$, hence for (67) to make sense we must have that the first jump for $X^{(x)}$ occurs before time $-z/\bar{\beta}$ with probability 1. That this is indeed the case follows from the observation that $\int_0^{-z/\bar{\beta}} ds\, q(\phi_s(x)) = \infty$ with ϕ as in (69) and q given by (70).

From the multiplicative agglomeration property (or from the one-dimensional Lamperti representation of $Z^{(z)} = \prod_{i=1}^n X^{i,(x_i)}$) we know that the product processes $\left(Z^{(z)}\right)_{z \in \mathbb{R}_+}$ also form a PDMP-family. The semigroup $\psi_t(z)$ of functions determining the deterministic behaviour of this family is quite simple, viz.

$$\psi_t(z) = \prod_{i=1}^n \phi_t^i(x_i) = z + \bar{\beta} t$$

so that $Z^{(z)}$ is always piecewise linear, and if $\bar{\beta} = 0$ it is seen that $Z^{(z)}$ is a Markov chain (piecewise constant) with state space $\mathbb{R}_+$.

As a final comment and curiosity we mention that if the Lévy measure ν for ξ is such that $\nu\left\{y : \sum_{i=1}^n y_i \neq 0\right\} = 0$ (which for $n \geq 2$ is entirely possible with a non-degenerate ν), then $\bar{\xi} \equiv 0$ and $Z^{(z)}$ is trivial, $Z_t^{(z)} \equiv z + \bar{\beta} t$.

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