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Convergence in measure and in category, similarities and differences

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Abstract. Convergence in category of a sequence of real-valued functions has been introduced by E. Wagner as an analogue of convergence in measure. In the paper it is shown that in some circumstances both kinds of convergence behave similarly, but sometimes the behaviour is different.

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In the paper $\mathcal{L}$ will denote the σ-algebra of Lebesgue measurable sets on the real line $\mathbb{R}$, $\mathcal{B}$—the σ-algebra of sets having the Baire property, λ—the Lebesgue measure and $\mathcal{K}$—the σ-ideal of sets of the first category. If $B \in \mathcal{B}$, then $\tilde{\mathcal{B}}$ will denote the regular open part of B, i.e. $\tilde{B}$ is the unique regular open set such that $B = \tilde{B} \triangle P$, where P is of the first category. As usual $A + x = \{t + x : t \in A\}$. Recall that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable real-valued functions defined on $[0, 1]$ converges in measure to a function $f : [0, 1] \to \mathbb{R}$ if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $f_{n_{m_p}} \underset{p\to\infty}{\to} f(x)$ almost everywhere (a classical theorem of F. Riesz, see for example [5], Th. 9.2.1, p. 226). Following Wagner ([12]) we say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued functions defined on $[0, 1]$ having the Baire property converges in category to a function $f : [0, 1] \to \mathbb{R}$ if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $f_{n_{m_p}}(x) \underset{p\to\infty}{\to} f(x)$ except on a set of the first category.

We say that a sequence $\{B_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{B}$ converges in category to $B \in \mathcal{B}$ ($B_n \underset{n\to\infty}{\to} B$ in category) if and only if the sequence $\{\chi_{B_n}\}_{n\in\mathbb{N}}$ of characteristic functions converges in category to χ_B. Similarly we say that a sequence $\{B_n\}_{n\in\mathbb{N}}$ of measurable sets converges in measure to $B \in \mathcal{L}$ ($B_n \underset{n\to\infty}{\to} B$

Birkhäuser

in measure) if and only if the sequence $\{\chi_{B_n}\}_{n\in\mathbb{N}}$ converges in measure to χ_B. Since in both cases $\{B_n\}_{n\in\mathbb{N}}$ converges to B if and only if $\{B_n \triangle B\}_{n\in\mathbb{N}}$ converges to the empty set, in the sequel we shall consider mainly convergence to $\emptyset$. Obviously $A_n \underset{n\to\infty}{\to} \emptyset$ in measure if and only if $\lambda(A_n) \underset{n\to\infty}{\to} 0$. We shall restrict ourselves to subsets of the unit interval.

There exists a vast literature concerning translations of sets and interrelations between the Steinhaus property, automatic continuity of homomorphisms, shift-compactness, regular and slow variation and Tauberian conditions (see [2,3,7–9,11]).

The aim of the paper is to compare several properties of convergence in measure versus convergence in category. Recall that if $\{r_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence of real numbers convergent to zero, then for each measurable set $B \subset [0,1]$, the sequence $\{(B + r_n) \cap [0,1]\}_{n\in\mathbb{N}}$ converges in measure to B (see for ex. [1] p. 901, [13] or [6], Chapter XII, §62, p. 271) and for each set $B \subset [0,1]$ having the Baire property the sequence $\{(B+r_n)\cap[0,1]\}_{n\in\mathbb{N}}$ converges in category to B (indeed convergence is off a set of the first category). In the case of measure there exists a set B and a sequence $\{r_n\}_{n\in\mathbb{N}}$ for which the above sequence does not converge almost everywhere (see again [13]). Hence the category case is sometimes better than the measure case. The word "sometimes" is justified by the fact that the theorem of Egoroff does not have a category analogue (see, for example [10], p. 38). Below we shall show that the worse behaviour of the category case appears also in different circumstances.

The following propositions follow immediately from the definition and from the fact that if measurable functions (defined on $[0,1]$) in the sequence are commonly bounded, then convergence in measure is equivalent to convergence in L^1.

Proposition 1. *If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of measurable subsets of $[0,1]$ such that $A_n \underset{n\to\infty}{\to} \emptyset$ in measure, then the sequence $f_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i}$ converges to 0 in measure. If a sequence $\{\chi_{A_n}\}_{n\in\mathbb{N}}$ converges to zero almost everywhere, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ also converges to zero almost everywhere.*

Proposition 2. *If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of measurable subsets of $[0,1]$ such that $A_n \underset{n\to\infty}{\to} \emptyset$ in measure and $\{r_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers convergent to zero, then $(A_n + r_n) \cap [0,1] \underset{n\to\infty}{\to} \emptyset$ in measure.*

Proposition 3. *If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of measurable subsets of $[0,1]$ such that the sequence $f_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i}$, $n \in \mathbb{N}$, converges in measure to 0, and $\{r_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers convergent to 0, then the sequence $g_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{B_i}$, where $B_n = (A_n + r_n) \cap [0,1]$, $n \in \mathbb{N}$ also converges in measure to 0.*

To study convergence in category we shall need the following lemma:

Key Lemma. *Let* $\{B_n\}_{n\in\mathbb{N}}$ *be a sequence of subsets of* $[0,1]$ *having the Baire property. The following conditions are equivalent:*

1. $B_n \underset{n\to\infty}{\longrightarrow} \emptyset$ in category,

2. for every nonempty interval $[a,b] \subset [0,1]$ there exist a number $\epsilon > 0$ and a positive integer n_0 such that for every $n \geq n_0$ there exists an interval $(c_n, d_n) \subset [a,b]$ with the properties $|d_n - c_n| > \epsilon$ and $(c_n, d_n) \cap \tilde{B}_n = \emptyset$,

3. for every nonempty interval of the form $[\frac{p-1}{2^k}, \frac{p}{2^k}]$, where $k \in \mathbb{N}$ and $p \in \{1, 2, \ldots, 2^k\}$ there exist a number $\epsilon > 0$ and a positive integer n_0 such that for every $n \geq n_0$ there exists an interval $(c_n, d_n) \subset [\frac{p-1}{2^k}, \frac{p}{2^k}]$ with the properties that $|d_n - c_n| > \epsilon$ and $(c_n, d_n) \cap \tilde{B}_n = \emptyset$.

Proof. (1) $\implies$ (2). Suppose that (2) does not hold. There exists a closed interval $I \subset [0,1]$ such that for every positive integer $k > \frac{1}{\lambda(I)}$ there exists an increasing sequence $\{n_m^k\}_{m\in\mathbb{N}}$ of positive integers such that for each open interval $J \subset I$ for which $\lambda(J) > \frac{1}{k}$ we have $J \cap \tilde{B}_{n_m^k} \neq \emptyset$ for all $m \in \mathbb{N}$. Choose a sequence $\{m_k\}_{k > \frac{1}{\lambda(I)}}$ such that $\{n_{m_k}^k\}_{k>\frac{1}{\lambda(I)}}$ is an increasing sequence. If $n_k = n_{m_k}^k$ for $k > \frac{1}{\lambda(I)}$, then for each subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k>\frac{1}{\lambda(I)}}$ and for each $q \in \mathbb{N}$ the set $(\bigcup_{p\geq q} \tilde{B}_{n_{k_p}}) \cap I$ is dense and open in I, so 1) does not hold.

(2) $\implies$ (3). This is obvious.

(3) $\implies$ (1). Let $I_{2^k+p-1} = [\frac{p-1}{2^k}, \frac{p}{2^k}]$, for $k \in \mathbb{N} \cup \{0\}$, $p \in \{1, 2, \ldots, 2^k\}$.

Take an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers. We shall construct a subsequence $\{n_{m_p}\}_{m\in\mathbb{N}}$ of $\{n_m\}_{m\in\mathbb{N}}$ for which $\chi_{\tilde{B}_{n_{m_p}}} \underset{p\to\infty}{\longrightarrow} \chi_\emptyset$ except on the set of the first category, which yields $B_n \underset{n\to\infty}{\longrightarrow} B$ in category.

From the assumption it follows that there exist $\epsilon_1 > 0$ and a number $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ there exists an open interval $(c_m, d_m) \subset I_1$ such that $d_m - c_m > \epsilon_1$ and $(c_m, d_m) \cap \tilde{B}_m = \emptyset$. Choose a subsequence $\{n_{m_j}\}_{j\in\mathbb{N}}$ for which there exists $\lim_{j\to\infty} c_{m_j} = c^1$. Then we have $\lim_{j\to\infty} \chi_{\tilde{B}_{n_{m_j}}}(x) = 0$ for $x \in J_1 = (c^1, c^1 + \epsilon_1) \subset I_1$.

For convenience we shall denote the sequence $\{n_{m_j}\}_{j\in\mathbb{N}}$ by $\{n_m^1\}_{m\in\mathbb{N}}$.

Similarly for I_2 we choose a subsequence $\{n_m^2\}_{m\in\mathbb{N}}$ of $\{n_m^1\}_{m\in\mathbb{N}}$ for which there exists a nondegenerated open interval $J_2 \subset I_2$ such that $\lim_{m\to\infty} \chi_{\tilde{B}_{n_m^2}}(x) = 0$ for $x \in J_2$.

By induction we define a sequence of subsequences $\{\{n_m^i\}_{m\in\mathbb{N}}\}_{i\in\mathbb{N}}$ such that $\{n_m^{i+1}\}_{m\in\mathbb{N}}$ is a subsequence of $\{n_m^i\}_{m\in\mathbb{N}}$ and $\lim_{m\to\infty} \chi_{\tilde{B}_{n_m^i}}(x) = 0$ for $x \in J_i$, where $J_i \subset I_i$ is a nondegenerated open interval.

If $\{n_{m_p}\}_{p\in\mathbb{N}}$ is a diagonal sequence for $\{\{n_m^i\}_{m\in\mathbb{N}}\}_{i\in\mathbb{N}}$, then $\lim_{p\to\infty} \chi_{\tilde{B}_{n_{m_p}}}(x) = 0$ for $x \in \bigcup_{i=1}^{\infty} J_i$. Since $\bigcup_{i=1}^{\infty} J_i$ is an open set dense in $[0,1]$, we even obtain convergence to zero off a nowhere dense set. $\square$

The proof of the part (1) $\implies$ (2) is quite similar to the proof of the implication (vi) $\implies$ (vii) of Th. 2.2.2 in [4], pp. 22–25.

It is worth mentioning that a similar condition (omitting intervals of length ϵ) appears when studying sets of strong measure zero (see [7], Lemma 3.5.1 and [8])

Now we shall show that only the analogues of Proposition 2 and the second part of Proposition 1 are true in the category case.

Theorem 1. *If $\{B_n\}_{n\in\mathbb{N}}$ is a sequence of subsets of $[0,1]$ having the Baire property such that $B_n \underset{n\to\infty}{\to} \emptyset$ in category and $\{r_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers convergent to zero, then $(B_n + r_n) \cap [0,1] \underset{n\to\infty}{\to} \emptyset$ in category.*

Proof. Take $[a,b] \subset [0,1]$, $[a,b]$—a nondegenerated interval. From the condition (2) of the lemma it follows that there exist $\epsilon > 0$ and $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$ there exists an interval $(c_n, d_n) \subset [a,b]$ with the properties that $d_n - c_n > \epsilon$ and $(c_n, d_n) \cap \tilde{B}_n = \emptyset$. Since $r_n \underset{n\to\infty}{\to} 0$, there exists $n_2 \in \mathbb{N}$ such that $|r_n| < \frac{\epsilon}{2}$ for $n \geq n_2$. Put $(e_n, f_n) = (c_n + r_n, d_n)$ if $r_n > 0$ and $(e_n, f_n) = (c_n, d_n + r_n)$ if $r_n < 0$. Then we have $(e_n, f_n) \subset [a,b] \cap (c_n + r_n, d_n + r_n)$ and $|f_n - e_n| > \frac{\epsilon}{2}$ for $n \geq n_0 = \max(n_1, n_2)$. Since $(c_n + r_n, d_n + r_n) \cap (\tilde{B}_n + r_n) = \emptyset$, we have $(e_n, f_n) \cap (\tilde{B}_n + r_n) = \emptyset$ for $n \geq n_0$. From the lemma it follows immediately that $(B_n + r_n) \cap [0,1] \underset{n\to\infty}{\to} \emptyset$ in category. $\square$

Theorem 2. *There exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of subsets of $[0,1]$ having the Baire property such that $B_n \underset{n\to\infty}{\to} \emptyset$ in category but the sequence $f_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{B_i}$ does not converge to 0 in category.*

Proof. Put $n_0 = 0$ and $n_j = \sum_{i=1}^{j} 4^i$ for $j \in \mathbb{N}$. The definition of B_n will proceed in step-intervals of size 4^k (thus motivating the use of the numbers n_j). As a preliminary we define the sets A_k^i which are 4^k contiguous subsets of $(0, \frac{1}{2^{k-1}})$ of equal lengths. Let $A_k^i = (\frac{i}{4^k.2^{k-1}}, \frac{i+1}{4^k.2^{k-1}})$ for $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, 4^k - 1\}$. Put $C_k^i = \bigcup_{j=0}^{2^{k-1}-1}(A_k^i + \frac{j}{2^{k-1}})$ for $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, 4^k - 1\}$. Each C_k^i, the union of the 2^k shifts of A_k^i, has measure 4^{-k}.

Observe that for fixed $k \in \mathbb{N}$ the sets C_k^i are pairwise disjoint and $[0,1] \setminus \bigcup_{i=0}^{4^k-1} C_k^i$ is a finite set.

Put

$$B_n = [0,1] \setminus C_1^n = [0,1] \setminus A_1^n \quad \text{for } n \in \{0,1,2,3\} = \{n_0, n_0+1, n_0+2, n_1 - 1\}$$

and

$$B_n = [0,1] \setminus (C_1^{i_1} \cup C_2^{i_2}) \quad \text{for } n \in \{n_1, n_1 + 1, \ldots, n_2 - 1\},$$

where

$$i_1 \equiv (n - n_1) \pmod{4^1}$$

and

$$i_2 \equiv (n - n_1) \pmod{4^2}.$$

Generally, let $B_n = [0,1] \setminus \bigcup_{m=1}^{j} C_m^{i_m}$ for $n \in \{n_{j-1}, n_{j-1}+1, \ldots, n_j - 1\}$, where $i_m \equiv (n - n_{j-1}) (\bmod 4^m)$ for $m \in \{1, 2, \ldots, j\}$. The set B_n (where $n \in \{n_{j-1}, n_{j-1}+1, \ldots n_j - 1\}$) omits intervals with sum-length $4^{-1}+4^{-2}+\cdots+4^{-j}$.

Observe that the sequence $\{B_n\}_{n\in\mathbb{N}\cup\{0\}}$ of closed sets fulfills the condition 3) from the lemma. Indeed, for each interval $[\frac{p-1}{2^{k-1}}, \frac{p}{2^{k-1}}]$, and for each positive $\epsilon < \frac{1}{4^k \cdot 2^{k-1}}$ and for each $n \geq n_{k-1}$ there exists an open subinterval of length exactly $\frac{1}{4^k \cdot 2^{k-1}}$ disjoint with $\tilde{B}_n = \operatorname{Int} B_n$. Hence $B_n \underset{n\to\infty}{\to} \emptyset$ in category. Simultaneously for each $k \in \mathbb{N}$ and for each $x \in [0,1]$ we have by reference to the extreme case

$$n_k f_{n_k}(x) = \sum_{n=1}^{n_k} \chi_{B_n}(x) \geq n_k - n_k \left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{k-1}} \right) > \frac{2}{3} n_k,$$

so $\frac{1}{n_k} \sum_{n=1}^{n_k} \chi_{B_n}(x) \geq \frac{2}{3}$ for each $x \in [0,1]$ and each $k \in \mathbb{N}$, which means that the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not converge to 0 in category. □

Theorem 3. *If a sequence $\{\chi_{A_n}\}_{n\in\mathbb{N}}$ converges to zero except on a set of the first category, then the sequence $\{f_n\}_{n\in\mathbb{N}}$, where $f_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i}$ also converges to zero except on a set of the first category.*

Proof. Put $E = \{x \in [0,1] : \lim_{n\to\infty} \chi_{A_n}(x) = 0\}$. Obviously $E = \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_n$ and E is residual in $[0,1]$. Observe that if $x \in E$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{A_i}(x) = 0$, since there exists $n \in \mathbb{N}$ such that $\chi_{A_m}(x) = 0$ for $m \geq n$. Hence the set $\{x \in [0,1] : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i}(x) = 0\}$ is also residual in $[0,1]$. □

Theorem 4. *There exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets having the Baire property such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i}(x) = 0$ except on a set of the first category but $\{\frac{1}{n}\sum_{i=1}^{n} \chi_{A_i+3^{-i}}(x)\}_{n\in\mathbb{N}}$ does not converge to 0 with respect to $\mathcal{I}$.*

Proof. Put $r_n = 3^{-n}$ for $n \in \mathbb{N}$ and

$$D_k = \bigcup_{i=1}^{2^{k-1}} \left(\frac{i}{2^{k-1}} - r_{2^k}, \frac{i}{2^{k-1}} \right) \quad \text{for} \quad k \in \mathbb{N}.$$

Finally let $A_n = (D_k - r_n) \cap (0,1)$ for $n \in \{2^{k-1}, \ldots, 2^k - 1\}$, $k \in \mathbb{N}$. Observe that for each $k \in \mathbb{N}$ the components of D_k are narrow enough that 2^{k-1} translates of A_n for $2^{k-1} \leq n < 2^k$ are disjoint. Hence for each $x \in (0,1)$ we have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i}(x) = 0$. At the same time $A_n + r_n = D_k$ for $n \in \{2^{k-1}, \ldots, 2^k - 1\}$ and the set $D = \limsup_{k\to\infty} D_k$ is a dense G_δ set. If $x \in D$, then for infinitely many k (referring to $n = 2^k$) we have $x \in D_k$ and then $x \in \bigcap_{n=2^{k-1}}^{2^k-1} (A_n + r_n)$, hence $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{A_i+r_i}(x) \geq \frac{1}{2}$. That means that $\{\frac{1}{n}\sum_{i=1}^{n}\chi_{A_i+r_i}\}_{n\in\mathbb{N}}$ does not converge to 0 except on a set of the first category. To prove that $\{\frac{1}{n}\sum_{i=1}^{n}\chi_{A_i+3^{-i}}\}_{n\in\mathbb{N}}$ does not converge to 0 in category, take an arbitrary increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive numbers. For each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that $2^{k_m-1} \leq m \leq 2^{k_m} - 1$. If $n_m < 2^{k_m-1} + 2^{k_m-2}$, then $\frac{1}{n_m}\sum_{i=1}^{n_m}\chi_{A_i+r_i}(x) > \frac{1}{4}$ for $x \in D_{k_m-1}$ and if $n_m \geq 2^{k_m-1} + 2^{k_m-2}$, then $\frac{1}{n_m}\sum_{i=1}^{n_m}\chi_{A_i+r_i}(x) > \frac{1}{4}$ for $x \in D_{k_m}$. Take $p_m = k_m - 1$ in the first case and $p_m = k_m$ in the second; then the $\frac{1}{4}$ lower bound holds for $x \in \limsup_{n\to\infty} D_{p_m}$, a dense G_δ set. From the arbitrariness of $\{n_m\}_{m\in\mathbb{N}}$ the claim follows immediately. $\square$

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