



Superstability of the Cauchy equation with squares in finite-dimensional normed algebras

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Abstract. Our purpose is to provide an affirmative answer to Moszner's problem [cf. Moszner (Ann Univ Paed Crac Stud Math XI:69–94, 2012), p. 93] concerning the superstability of the Cauchy equation with squares

$$f(x + y)^2 = (f(x) + f(y))^2$$

in the class of functions mapping an Abelian semigroup into a finite-dimensional normed algebra without divisors of zero.

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1. Introduction

The source of our inspiration to deal with the superstability of the Cauchy equation with squares in finite-dimensional normed algebras without divisors of zero is Moszner's paper [10], where he proved superstability, in a sense of Baker (cf. [1]), of several functional equations in such a setting, and, primarily, the following problem that we can find therein.

Problem 1. (cf. [10, p. 93]) Let f be a function from an Abelian semigroup to a finite-dimensional normed algebra without zero divisors. Is the equation

$$f(x + y)^2 = (f(x) + f(y))^2 \tag{1}$$

superstable?

The main purpose of the present paper is to give an affirmative answer to this problem.

Before going to the details, for the sake of completeness we briefly discuss the historical background concerning the equation in question and its stability, as it has been dealt with in a number of papers.

The Cauchy equation with squares (1) was firstly considered for real-valued functions, for which it can be equivalently written in one of the following forms:

$$|f(x + y)| = |f(x) + f(y)|, \quad (2)$$

or

$$f(x + y) + f(x) + f(y) \neq 0 \implies f(x + y) = f(x) + f(y), \quad (3)$$

each of which admits further generalisation from the real case to more general structures.

The general solution of Eq. (3) is provided in [7, Theorem 8].

Stability results concerning Eq. (2) are contained in [5] for real-valued functions and, for the class of functions taking values in Riesz spaces, in [6]. There are also known results concerning the stability of the generalized Eq. (2) for functions acting into a normed space, called Fischer–Muszély functional equation

$$\|f(x + y)\| = \|f(x) + f(y)\|. \quad (4)$$

It occurs that, despite the fact that under the assumption that the norm is strictly convex Eq. (4) is equivalent to the Cauchy functional equation (cf. [8]), even if we consider \mathbb{R}^2 with the Euclidean norm as a target space of f , Eq. (4) fails to be stable in the Hyers–Ulam sense. There exists a function satisfying this equation approximately, but it is not close to any function satisfying it exactly (cf. [5, Proposition 1]). However, if we restrict ourselves to the class of surjective functions, then we have the stability (cf. [12]).

The stability of Eq. (3), in the class of functions mapping an Abelian semigroup into a Banach space, was investigated in [2] with the double perturbation approach, suitable for its conditional form, and in a more general setting in [3].

As a straightforward consequence of the main result of [2] we have the stability of Eq. (1) in the class of complex functions (cf. [2, Theorem 2]). Using this result Schwaiger proved that, for complex functions, Eq. (1) is superstable in the sense of Baker (cf. [11]), i.e., any complex function defined on an Abelian semigroup $(S, +)$ and satisfying, with some $\varepsilon \geq 0$, the inequality

$$|f(x + y)^2 - (f(x) + f(y))^2| \leq \varepsilon \quad \text{for } x, y \in S,$$

has to be either bounded or additive.

Another result related to the stability behaviour of Eq. (1) can be found in [9].

The stability of the Cauchy equation with squares (1) in the class of f -algebra-valued functions has been recently investigated in [4]. In such a setting Eq. (1) occurs to be stable in the Hyers–Ulam sense and, unlike the complex case, not superstable.

Throughout the paper, \mathbb{N} and \mathbb{R} denote the sets of all positive integers and real numbers, respectively.

2. Main results

In order to provide an answer to Problem 1 we will make use Theorem 1 from [2] and Lemma 2.5.1 from [10] that we quote here for the reader's convenience.

Theorem 1. (cf. [2], Theorem 1) *Let $(S, +)$ be an Abelian semigroup and let $(X, \|\cdot\|)$ be a Banach space. If, for some $\varepsilon_1, \varepsilon_2 \geq 0$ and all $x, y \in S$, a function $f : S \rightarrow X$ satisfies*

$$\|f(x + y) + f(x) + f(y)\| > \varepsilon_1 \implies \|f(x + y) - f(x) - f(y)\| \leq \varepsilon_2, \tag{5}$$

then there exists a unique additive function $a : S \rightarrow X$ such that

$$\|f(x) - a(x)\| \leq \max\{\varepsilon_1, \varepsilon_2\}$$

for all $x \in S$.

Lemma 1. (cf. [10, Lemma 2.5.1]) *Let A be a finite-dimensional normed algebra without zero divisors. Then for all $a_n, b_n \in A$ the conditions: $a_n b_n \rightarrow 0$ and $a_n \rightarrow a \neq 0$ imply $b_n \rightarrow 0$.*

Our main result reads as follows.

Theorem 2. *Let $(S, +)$ be an Abelian semigroup and let $(A, \|\cdot\|)$ be a finite-dimensional normed algebra without divisors of zero. If, for some $\varepsilon \geq 0$, a function $f : S \rightarrow A$ satisfies*

$$\|f(x + y)^2 - (f(x) + f(y))^2\| \leq \varepsilon \text{ for } x, y \in S \tag{6}$$

then it is either bounded or additive.

Proof. We assume that $f : S \rightarrow A$ satisfies (6) and is unbounded, and prove that then it has to be additive.

At first let us observe that there exist $\varepsilon_1, \varepsilon_2 \geq 0$ such that (5) holds true. For the indirect proof suppose the contrary. Then there exist sequences $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ in S with

$$\|f(x_n + y_n) + f(x_n) + f(y_n)\| \rightarrow +\infty$$

and

$$\|f(x_n + y_n) - f(x_n) - f(y_n)\| \rightarrow +\infty.$$

Denote $\mu_n := f(x_n + y_n) + f(x_n) + f(y_n)$. Without loss of generality we may assume that $\mu_n \neq 0$ and that $\mu_n / \|\mu_n\| \rightarrow c \neq 0$ as $\|\mu_n / \|\mu_n\|\| = 1$ and A is finite-dimensional. On the other hand, by (6), for every $n \in \mathbb{N}$ we have

$$\|(f(x_n + y_n) + f(x_n) + f(y_n))(f(x_n + y_n) - f(x_n) - f(y_n))\| \leq \varepsilon.$$

Dividing the above inequality by $\|\mu_n\|$, side by side, we obtain

$$\left\| \frac{\mu_n}{\|\mu_n\|} (f(x_n + y_n) - f(x_n) - f(y_n)) \right\| \leq \frac{\varepsilon}{\|\mu_n\|} \quad \text{for } n \in \mathbb{N}$$

which, according to Lemma 1, yields $\|f(x_n + y_n) - f(x_n) - f(y_n)\| \rightarrow 0$ and brings a contradiction.

By Theorem 1 there exists an additive mapping $a : S \rightarrow A$ with

$$\|f(x) - a(x)\| \leq \max\{\varepsilon_1, \varepsilon_2\} \quad \text{for } x \in S. \tag{7}$$

We have assumed that f is unbounded, therefore, by (7), a is nontrivial and f is of the form $f = a + b$ with some bounded function $b : S \rightarrow A$. Then, using (6) one can verify that

$$\|a(y)(b(x + y) - b(x) - b(y))\| \leq \delta(x) \quad \text{for } x, y \in S, \tag{8}$$

with some $\delta : S \rightarrow \mathbb{R}$. Since a is nontrivial, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ with $\|a(y_n)\| \rightarrow +\infty$ and $a(y_n) \neq 0$. We may assume that $\frac{a(y_n)}{\|a(y_n)\|} \rightarrow c \neq 0$, because A is finite-dimensional and $\left\| \frac{a(y_n)}{\|a(y_n)\|} \right\| = 1$. By the additivity of a we also have

$$\frac{a(y + y_n)}{\|a(y_n)\|} \rightarrow c \neq 0 \quad \text{for } y \in S. \tag{9}$$

Using (8) with $y + y_n$ in place of y and then dividing the inequality by $\|a(y_n)\|$, side by side, we obtain

$$\left\| \frac{a(y + y_n)}{\|a(y_n)\|} (b(x + y + y_n) - b(x) - b(y + y_n)) \right\| \leq \frac{\delta(x)}{\|a(y_n)\|}$$

for $x, y \in S$ and $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ in the above inequality and making use of Lemma 1 and (9) we infer that

$$b(x) = \lim_{n \rightarrow +\infty} (b(x + y + y_n) - b(y + y_n)) \quad \text{for } x, y \in S. \tag{10}$$

Therefore, for arbitrarily fixed $x, y \in S$, we have

$$\begin{aligned} b(x + y) &= \lim_{n \rightarrow +\infty} (b(x + y + y_n) - b(y_n)) \\ &= \lim_{n \rightarrow +\infty} (b(x + y + y_n) - b(y + y_n) + b(y + y_n) - b(y_n)) \\ &= \lim_{n \rightarrow +\infty} (b(x + y + y_n) - b(y + y_n)) + \lim_{n \rightarrow +\infty} (b(y + y_n) - b(y_n)) \\ &= b(x) + b(y). \end{aligned}$$

Thus $b = 0$ as a bounded and additive function, and consequently $f = a$. \square

Remark 1. The function $f : \mathbb{R} \rightarrow M_2(\mathbb{R})$, given by

$$f(x) = \begin{bmatrix} x & 0 \\ 0 & \sqrt{\varepsilon/3} \end{bmatrix} \quad \text{for } x \in \mathbb{R}$$

shows that the assumption in Theorem 2 that A has no divisors of zero is essential.

Remark 2. Theorem 2 generalises results concerning the stability of Eq. (1), especially those contained in [9, 11].

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