7. Planar non-smooth dynamical systems

We provide a survey of some results for the special case of planar non-smooth dynamical systems.

7.1 Lyapunov constants

Following COLL/GASULL/PROHENS [53] we investigate the number of small periodic solutions for real-analytic planar systems by means of the so-called Lyapunov constants. For smooth systems, a main source of motivation to study the number of periodic solutions of planar equations with polynomial right-hand sides is that this was stated as an important problem by Hilbert, as part of "Hilbert's 16th problem".

We consider the system

$$\begin{array}{l} \dot{x} = f_1^+(x,y) \\ \dot{y} = f_2^+(x,y) \end{array}, \quad y > 0, \qquad \text{and} \qquad \begin{array}{l} \dot{x} = f_1^-(x,y) \\ \dot{y} = f_2^-(x,y) \end{array}, \quad y < 0, \tag{7.1}$$

where $f^\pm = (f_1^\pm, f_2^\pm)$ are real-analytic functions defined in a neighborhood of $(x,y) = (0,0)$. We write their series expansion as

$$f_1^\pm(x,y) = a^\pm + b^\pm x + c^\pm y + d^\pm x^2 + e^\pm xy + f^\pm y^2 + \dots \tag{7.2a}$$

$$f_2^\pm(x,y) = k^\pm + l^\pm x + m^\pm y + n^\pm x^2 + o^\pm xy + p^\pm y^2 + \dots \tag{7.2b}$$

and since we need $(0,0)$ to be an equilibrium for both component equations, we will throughout assume that

$$a^\pm = 0 = k^\pm.$$

Definition 7.1.1. *A component equation $\pm$ is of focus type, if*

$$(b^\pm - m^\pm)^2 + 4l^\pm c^\pm < 0 \quad \text{and} \quad l^\pm > 0. \tag{7.3}$$

We call $(0,0)$ a singularity of focus-focus type, if both equations $+$ and $-$ are of focus type.

Note that since

$$(f^{\pm})'(0,0) = \begin{pmatrix} b^{\pm} & c^{\pm} \\ l^{\pm} & m^{\pm} \end{pmatrix}$$

are the linearizations at $(0,0)$, condition (7.3) means that the eigenvalues

$$\frac{1}{2}\Big((b^{\pm} + m^{\pm}) \;+/-\; \sqrt{(b^{\pm} - m^{\pm})^2 + 4c^{\pm}l^{\pm}}\Big)$$

are complex conjugate, and close to the origin the flows locally do rotate counterclockwise, the latter due to $l^{\pm} > 0$.

Next we introduce the Poincaré return maps

$$s \mapsto h^+(s) \mapsto h^-(h^+(s))$$

that are well-defined in a neighborhood of the origin; see Fig. 7.1. Note also that therefore it is not possible for a solution of (7.1) to stick to the discontinuity line $\{(x, y) : y = 0\}$ for some time, i.e., (7.1) needs not be considered as a differential inclusion.

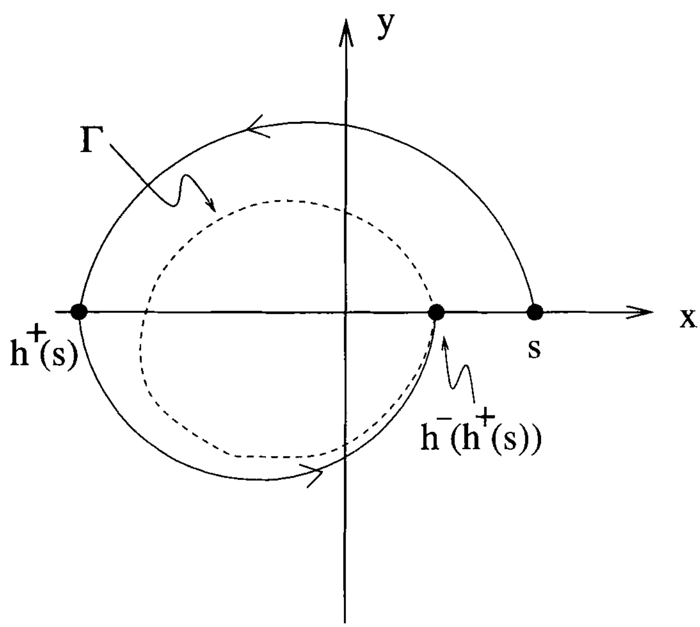

Fig. 7.1. The map $s \mapsto h^+(s) \mapsto h^-(h^+(s))$ and Γ from Ex. 7.1.1

In this situation one defines the Lyapunov constants to be the non-zero coefficients in the series expansion of the function $s \mapsto h^-(h^+(s)) - s$ for s close to zero; observe that $h^-(h^+(s)) = s$ corresponds to a periodic solution of (7.1).

Definition 7.1.2. *If* $V_k \neq 0$ *and*

$$h^-(h^+(s)) - s = V_k s^k + \mathcal{O}(s^{k+1})$$

for $s > 0$ *close to zero, then* V_k *is called the kth Lyapunov constant.*

Here we implicitly understand that V_k will only be considered in case that $V_1 = V_2 = \ldots = V_{k-1} = 0$. Thus in particular, if e.g. $V_1 = 0$, then the relations resulting from this equality can be used to simplify the expression for V_2, etc. As we are going to explain below in Ex. 7.1.1, information on the Lyapunov constants can be utilized to derive results about the number of small periodic solutions; see BAUTIN [22], BLOWS/LLOYD [26], GASULL/GUILLAMON/MAÑOSA [89], or GASULL/PROHENS [90] for more on the usage of Lyapunov constants for smooth systems.

Theorem 7.1.1. *Let* $(0,0)$ *be a singularity of focus-focus type, and*

$$\nu^\pm = \exp\left\{\frac{\pi(b^\pm + m^\pm)}{\sqrt{-[(b^\pm - m^\pm)^2 + 4l^\pm c^\pm]}}\right\}.$$

Then the first Lyapunov constant is

$$V_1 = \nu^+\nu^- - 1.$$

Proof: We will first deal with the equation $(\dot{x}, \dot{y}) = f^+(x,y)$ in the upper half-plane. It will be convenient to represent the solution starting at $(x(0), y(0)) = (s, 0)$ as

$$x(t) = R^+(\theta(t))\cos\theta(t), \quad y(t) = R^+(\theta(t))\sin\theta(t). \tag{7.4}$$

Differentiating this w.r. to t, it follows that

$$\frac{dR^+}{d\theta} = \frac{\mathrm{Re}(\bar{z}\dot{z})}{\mathrm{Im}(\bar{z}\dot{z})} R^+ \bigg|_{z=R^+e^{i\theta}}, \quad R^+(0;s) = s, \tag{7.5}$$

for the function $R^+ = R^+(\theta; s)$, with $\theta \in [0, \pi]$ and $s > 0$ small; here $z = x + iy$. Then $h^+(s) = R^+(\pi; s)$, and we are going to argue that

$$R^+(\pi; s) - s = (\nu^+ - 1)s + \mathcal{O}(s^2).$$

To see this, we expand

$$R^+(\theta; s) - s = \sum_{k=1}^{\infty} w_k^+(\theta)s^k, \quad \text{with} \quad w_k^+(0) = 0, \ k \in \mathbb{N}, \tag{7.6}$$

and derive a differential equation for w_1^+ using (7.5). We first write $(\dot{x}, \dot{y}) = f^+(x,y)$ in the complex form

$$\dot{z} = \sum_{k=1}^{\infty} F_k^+(z,\bar{z}), \quad \text{where} \quad F_k^+(z,\bar{z}) = \sum_{\substack{\alpha,\beta\in\mathbb{N}_0 \\ \alpha+\beta=k}} c_{\alpha\beta}^+ z^\alpha \bar{z}^\beta. \tag{7.7}$$

For simplicity we henceforth omit the superscript "+". Then

$$F_k(Re^{i\theta}, Re^{-i\theta}) = R^k F_k(e^{i\theta}, e^{-i\theta}),$$

and this shows that

$$\frac{\operatorname{Re}(\bar{z}\dot{z})}{\operatorname{Im}(\bar{z}\dot{z})} R\bigg|_{z=Re^{i\theta}} = \frac{\sum_{k=1}^{\infty} \operatorname{Re}(S_k(\theta))R^k}{\sum_{k=1}^{\infty} \operatorname{Im}(S_k(\theta))R^{k-1}}, \quad S_k(\theta) = e^{-i\theta} F_k(e^{i\theta}, e^{-i\theta}). \tag{7.8}$$

Next we observe that $\operatorname{Im}(S_1(\theta)) > 0$ due to (7.3). Indeed, $F_1(z,\bar{z}) = c_{10}z + c_{01}\bar{z}$ implies upon decomposing c_{10} and c_{01} in real and imaginary part, and by comparing (7.7) to $(\dot{x},\dot{y}) = f^+(x,y)$ and (7.2a), (7.2b), that

$$b^+ = \operatorname{Re}(c_{10} + c_{01}), \quad c^+ = \operatorname{Im}(c_{01} - c_{10}), \tag{7.9a}$$
$$l^+ = \operatorname{Im}(c_{10} + c_{01}), \quad \text{and} \quad m^+ = \operatorname{Re}(c_{10} - c_{01}), \tag{7.9b}$$

in particular

$$(b^+ - m^+)^2 + 4l^+c^+ = 4\Big(|c_{01}|^2 - \operatorname{Im}(c_{10})^2\Big). \tag{7.10}$$

We have $S_1(\theta) = c_{10} + c_{01}e^{-2i\theta}$, whence $\operatorname{Im}(S_1(\theta)) = \operatorname{Im}(c_{10}) + \langle c_{01}, Je^{2\theta i}\rangle$, with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the usual inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^2 \times \mathbb{R}^2$. Assume $\operatorname{Im}(c_{10}) < 0$. Then we get from (7.3), (7.9b), and (7.10) that $|\operatorname{Im}(c_{10})| = -\operatorname{Im}(c_{10}) < \operatorname{Im}(c_{01}) \le |c_{01}| < |\operatorname{Im}(c_{10})|$, a contradiction. Therefore $\operatorname{Im}(c_{10}) \ge 0$, and consequently $-\langle c_{01}, Je^{2\theta i}\rangle \le |c_{01}| < |\operatorname{Im}(c_{10})| = \operatorname{Im}(c_{10})$ implies that in fact $\operatorname{Im}(S_1(\theta)) > 0$. From this, (7.5), (7.6), and (7.8) we infer that with $R = R^+$

$$\begin{aligned}\sum_{k=1}^{\infty} w_k'(\theta)s^k &= \frac{dR}{d\theta} = \frac{\operatorname{Re}(S_1(\theta))}{\operatorname{Im}(S_1(\theta))}R + \mathcal{O}(R^2) \\ &= \frac{\operatorname{Re}(S_1(\theta))}{\operatorname{Im}(S_1(\theta))}(s + w_1(\theta)s) + \mathcal{O}(s^2),\end{aligned}$$

hence

$$w_1'(\theta) = \frac{\operatorname{Re}(S_1(\theta))}{\operatorname{Im}(S_1(\theta))}(1 + w_1(\theta)), \quad \theta \in [0, 2\pi], \quad w_1(0) = 0,$$

by comparing the coefficients of s and recalling the initial condition from (7.6). After some calculation this may be integrated to yield $w_1(\pi) = \nu^+ - 1$, thus indeed

$$R^+(\pi; s) - s = (\nu^+ - 1)s + \mathcal{O}(s^2)$$

by evaluating (7.6) at $\theta = \pi$. Similarly it can be verified that

$$R^-(\pi; s) - s = (\nu^- - 1)s + \mathcal{O}(s^2)$$

for $s < 0$ small, where R^- is analogous to R^+ for the lower half-plane, in particular $h^-(s) = R^-(\pi; s)$. We therefore get

$$\begin{aligned} h^-(h^+(s)) - s &= R^-(\pi; R^+(\pi; s)) - s = \nu^- R^+(\pi; s) + \mathcal{O}(s^2) - s \\ &= (\nu^+\nu^- - 1)s + \mathcal{O}(s^2), \end{aligned}$$

as was to be shown. □

More precise information can be obtained if in addition

$$F_1^+(z, \bar{z}) = (i + \lambda^+)z \quad \text{and} \quad F_1^-(z, \bar{z}) = (i + \lambda^-)z, \tag{7.11}$$

with $\lambda^\pm \in \mathbb{R}$ and $F_1^\pm$ as in (7.7), which means that the linear parts are in Jordan form.

Theorem 7.1.2. *If (7.11) holds and* $(0,0)$ *is a singularity of focus-focus type, then*

$$\begin{aligned} V_1 &= e^{\pi(\lambda^+ + \lambda^-)} - 1, \quad V_2 = w_2^+(\pi) + w_2^-(\pi)e^{3\lambda^+\pi}, \\ V_3 &= e^{\lambda^+\pi}w_3^+(\pi) - 2w_2^+(\pi)^2 + w_3^-(\pi)e^{5\lambda^+\pi}, \quad \textit{and} \\ V_4 &= e^{2\lambda^+\pi}w_4^+(\pi) - 5e^{\lambda^+\pi}w_2^+(\pi)w_3^+(\pi) + 5w_2^+(\pi)^3 + e^{7\lambda^+\pi}w_4^-(\pi). \end{aligned}$$

Proof: Observe that using (7.9a) and (7.9b) we may alternatively write e.g.

$$\nu^+ = \exp\left\{ \frac{\pi \operatorname{Re}(c_{10})}{\sqrt{\operatorname{Im}(c_{10})^2 - |c_{01}|^2}} \right\}.$$

According to (7.11) we have

$$\operatorname{Re}(c_{10}) = \lambda^+, \quad \operatorname{Im}(c_{10}) = 1, \quad \text{and} \quad c_{01} = 0,$$

hence $\nu^+ = e^{\pi\lambda^+}$, and similarly $\nu^- = e^{\pi\lambda^-}$. By Thm. 7.1.1 we thus have $V_1 = e^{\pi(\lambda^+ + \lambda^-)} - 1$. Concerning the higher Lyapunov constants, in principle their explicit form can be found as we did for V_1, although this requires a lot of calculation and instead of the polar coordinates from (7.4) the using of so-called generalized polar coordinates

$$x(t) = R(\theta(t))^q \operatorname{Cs}(\theta(t)), \quad y(t) = R(\theta(t))^p \operatorname{Sn}(\theta(t)).$$

Here $q, p \in \mathbb{N}$ have to be chosen appropriately, and

$$\frac{d}{d\theta}\operatorname{Cs}(\theta) = -\operatorname{Sn}(\theta)^{2p-1}, \quad \frac{d}{d\theta}\operatorname{Sn}(\theta) = \operatorname{Cs}(\theta)^{2q-1},$$
$$\operatorname{Cs}(0) = \sqrt[2q]{\frac{1}{p}}, \quad \text{and} \quad \operatorname{Sn}(0) = 0;$$

see COLL/GASULL/PROHENS [53] for more details. □

In the formulas for V_2, V_3, and V_4, the $w_k^{\pm}$ are the coefficients of $R^{\pm}$, cf. (7.6). We recall that V_2 is only relevant if $V_1 = 0$ and $V_2 \neq 0$, V_3 is used in case that $V_1 = V_2 = 0$ and $V_3 \neq 0$, and so on. The reference COLL/GASULL/PROHENS [53] also contains explicit formulas for the first four Lyapunov constants in case that one or both of the component equations do have a parabolic contact point at $(x, y) = (0, 0)$ rather than a focus.

The following example illustrates the calculation of Lyapunov constants and also their relevance for proving the existence of small periodic solutions in planar systems.

Example 7.1.1. We consider the problem

$$\dot{z} = \begin{cases} (i+\lambda)z + Az^2 + Bz\bar{z} + C\bar{z}^2 & : \quad \operatorname{Im}(z) > 0 \\ iz & : \quad \operatorname{Im}(z) < 0 \end{cases} \tag{7.12}$$

in complex form. Here

$$A = 2 + i\Big(\varepsilon_1 + \frac{1}{3}\varepsilon_2 - 2\varepsilon_3 - \frac{\sqrt{6}}{4}\Big), \quad B = 1 + i\Big(\varepsilon_3 + \frac{\sqrt{6}}{8}\Big), \quad \text{and}$$
$$C = i\Big(\varepsilon_2 - 9\varepsilon_3 - 9\frac{\sqrt{6}}{8}\Big).$$

Then for $\lambda > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 < 0$, and $\varepsilon_3 < 0$ small such that $|\lambda| \ll |\varepsilon_1| \ll |\varepsilon_2| \ll |\varepsilon_3|$ there are at least three limit cycles for (7.12).

We first calculate the corresponding Lyapunov constants. Note that (7.12) is of the form required in (7.11), with $\lambda^+ = \lambda$ and $\lambda^- = 0$. Moreover, $(0,0)$ is a singularity of focus-focus, as, writing $F_1^+(z,\bar{z}) = c_{10}^+ z + c_{01}^+ \bar{z}$ and $F_1^-(z,\bar{z}) = c_{10}^- z + c_{01}^+ - \bar{z}$, we have

$$c_{10}^+ = i + \lambda, \quad c_{01}^+ = 0, \quad c_{10}^- = i, \quad \text{and} \quad c_{01}^- = 0,$$

and this in turn yields due to (7.10) and (7.9b)

$$(b^{\pm} - m^{\pm})^2 + 4l^{\pm}c^{\pm} = -4 \quad \text{and} \quad l^{\pm} = 1.$$

Hence Thm. 7.1.2 applies to give

$$V_1 = e^{\pi\lambda} - 1, \quad V_2 = -2\varepsilon_1, \quad V_3 = -\frac{\pi}{3}\varepsilon_2, \quad \text{and}$$
$$V_4 = 8(3 + 16\varepsilon_3^2 + 6\sqrt{6}\varepsilon_3)\varepsilon_3;$$

we omitted the explicit evaluation of V_2, V_3, and V_4.

To finally verify the existence of at least three limit cycles, the argument indicated in BLOWS/LLOYD [26, p. 220] may be employed. First we consider (7.12) with $\lambda = \varepsilon_1 = \varepsilon_2 = 0$. Then the first non-zero Lyapunov constant is $V_4 < 0$, the latter for $\varepsilon_3 < 0$ small. Hence

$$h^-(h^+(s)) - s = V_4 s^4 + \mathcal{O}(s^5) \tag{7.13}$$

for $s > 0$ small implies that the origin is stable. Thus we find a region encircled by a curve Γ such that the flow for (7.12) with $\lambda = \varepsilon_1 = \varepsilon_2 = 0$ strictly points inward across Γ; rigorously such Γ can be found as a level set of an appropriate Lyapunov function that is strictly decreasing along solutions. One may also imagine constructing Γ as follows: take some $s_0 > 0$ small and let $s_1 = h^-(h^+(s_0)) < s_0$. Then draw (in counterclockwise direction) a curve from s_1 to itself which, at each point (x, y), is a little more steepened than the solution trajectory through (x, y); see Fig. 7.1 on p. 186.

Having determined Γ, we note that perturbing (7.12) with $\lambda = \varepsilon_1 = \varepsilon_2 = 0$ a little will not affect the property of Γ that the flow strictly points inward across Γ. Hence we next choose $|\varepsilon_2| \ll 1$ such that $\varepsilon_2 < 0$, instead of $\varepsilon_2 = 0$, and we keep $\lambda = \varepsilon_1 = 0$. Then $V_3 > 0$ is the first non-zero Lyapunov constant, and

$$h^-(h^+(s)) - s = V_3 s^3 + \mathcal{O}(s^4)$$

for $s > 0$ in a neighborhood of $s = 0$ that will be smaller than the one where (7.13) has been valid. Consequently, the origin has become unstable, and similarly to the foregoing we find Γ_1 inside Γ such that the flow of (7.12), with $\lambda = \varepsilon_1 = 0$ and ε_4 fixed in the previous step, is strictly outward on Γ_1. Moreover, it is strictly inward on Γ. Continuing this way, for $\lambda > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 < 0$, and $\varepsilon_3 < 0$ small such that $|\lambda| \ll |\varepsilon_1| \ll |\varepsilon_2| \ll |\varepsilon_3|$ there are curves Γ_3, Γ_2, Γ_1, and Γ, one contained inside the next, such that the flow of (7.12) is strictly inward on Γ and Γ_2, and strictly outward on Γ_1 and Γ_3. By the Poincaré–Bendixson theorem that is also valid for non-smooth planar systems, see FILIPPOV [82, Ch. 3.13, Thm. 6], we thus conclude that there is a limit cycle between Γ_3 and Γ_2, Γ_2 and Γ_1, and Γ_1 and Γ, respectively. Hence we find at least three limit cycles. $\diamondsuit$

We remark that this example also highlights a difference between smooth and non-smooth planar systems concerning Lyapunov constants, since $V_1 = V_2 = \ldots = V_{k-1} = 0$ and $V_k \neq 0$ implies k is odd for smooth systems. Choosing $\lambda = 0$ and $\varepsilon_1 \neq 0$, we however obtain that $V_1 = 0$ and $V_2 \neq 0$ for (7.12).

See COLL/GASULL/PROHENS [52] and COLL/PROHENS/GASULL [54] for further results in the same direction, mainly concerning the special case of a Liénard type system $f_1^\pm(x, y) = -y + f^\pm(x)$ and $f_2^\pm(x, y) = x$ in (7.1), where $f^\pm$ is a polynomial of degree at least two.

7.2 Hopf bifurcation

Closely related to the preceding section is the subject of Hopf bifurcation of periodic solutions for planar non-smooth systems. For simplicity we consider the problem in normal form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & -\omega^+(\lambda) \\ \omega^+(\lambda) & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + g^+(x,y,\lambda), \quad y > 0, \quad \text{and} \tag{7.14a}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & -\omega^-(\lambda) \\ \omega^-(\lambda) & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + g^-(x,y,\lambda), \quad y < 0, \tag{7.14b}$$

where $g^\pm = (g_1^\pm, g_2^\pm)$ are real-analytic functions defined in a neighborhood of $(x,y) = (0,0)$ and $\lambda = 0$ such that $|g^\pm(x,y,\lambda)| \leq C(x^2+y^2)$ in this neighborhood. The linearization matrices do have eigenvalues $\lambda \pm i\omega^+(\lambda)$ and $\lambda \pm i\omega^-(\lambda)$, respectively, and we suppose that $\omega^+(0) > 0$ as well as $\omega^-(0) > 0$; both functions $\omega^\pm(\cdot)$ are assumed to be real-valued and of class C^1 close to $\lambda = 0$.

Theorem 7.2.1. *At $\lambda = 0$, system (7.14a), (7.14b) undergoes a Hopf bifurcation. More precisely, there exist $\delta > 0$ and a unique continuous function $\lambda^* :]-\delta,\delta[\to \mathbb{R}$ such that $\lambda^*(0) = 0$, and if $s \in]-\delta,\delta[$ and $s \neq 0$, then there is a periodic orbit of (7.14a), (7.14b) with $\lambda = \lambda^*(s)$ that passes through $(s,0)$. The corresponding period is $T(s) > 0$ for a continuous function $T :]-\delta,\delta[\to]0,\infty[$ satisfying $T(0) = \frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)}$. In a neighborhood of $(x,y) = (0,0)$ and $\lambda = 0$, all periodic solutions of (7.14a), (7.14b) are obtained this way.*

Proof: The proof runs more or less analogously to the smooth case, see MARSDEN/MCCRACKEN [138]. Defining the return maps $h^\pm$ as in Sect. 7.1 and observing that both maps here do additionally depend on λ, we need to find a zero of $s \mapsto h^-(h^+(s,\lambda),\lambda) - s$. To exclude the trivial solution $s = 0$ one introduces

$$H(s,\lambda) = \begin{cases} \dfrac{h^-(h^+(s,\lambda),\lambda) - s}{s} & : \quad s \neq 0 \\ V_1(\lambda) & : \quad s = 0 \end{cases},$$

with $V_1(\lambda)$ the first Lyapunov constant. Observe that in the notation of (7.2a), (7.2b) we have

$$b^\pm = \lambda, \quad c^\pm = -\omega^\pm(\lambda), \quad l^\pm = \omega^\pm(\lambda), \quad \text{and} \quad m^\pm = \lambda,$$

whence $(0,0)$ is a singularity of focus-focus type for λ close to zero, since $(b^\pm - m^\pm)^2 + 4l^\pm c^\pm = -4\omega^\pm(\lambda)^2$; recall Definition 7.1.1. Thus $\nu^\pm = \exp(\frac{\pi\lambda}{\omega^\pm(\lambda)})$, and Thm. 7.1.1 implies the explicit form

$$V_1(\lambda) = \exp\left(\frac{\pi\lambda}{\omega^+(\lambda)} + \frac{\pi\lambda}{\omega^-(\lambda)}\right) - 1 \tag{7.15}$$

for the first Lyapunov constant.

We wish to apply the implicit function theorem to solve $H(s,\lambda) = 0$ w.r. to $\lambda = \lambda^*(s)$ in a neighborhood of $s = 0$. By definition of the first Lyapunov constant, it may be shown that $(s,\lambda) \mapsto H(s,\lambda)$ is continuous close to $(s,\lambda) = (0,0)$. Moreover, with $H_1(s,\lambda) = h^-(h^+(s,\lambda),\lambda) - s$ we note that $H_1(0,\lambda) \equiv 0$, whence $\frac{\partial H_1}{\partial\lambda}(0,\lambda) \equiv 0$, and also $V_1(\lambda) = \frac{\partial H_1}{\partial s}(0,\lambda)$. This shows that

$$\frac{\frac{\partial H_1}{\partial\lambda}(s,\lambda)}{s} = \frac{\frac{\partial H_1}{\partial\lambda}(s,\lambda) - \frac{\partial H_1}{\partial\lambda}(0,\lambda)}{s} \longrightarrow \frac{\partial^2 H_1}{\partial s\partial\lambda}(0,\lambda) = \frac{\partial^2 H_1}{\partial\lambda\partial s}(0,\lambda) = V_1'(\lambda)$$

as $s \to 0$. Thus H is found to be continuously differentiable w.r. to λ. Next, $H(0,0) = 0$ due to (7.15), and $\frac{\partial H}{\partial\lambda}(0,0) = V_1'(0) = \frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)} > 0$. Hence a unique continuous function $\lambda^* :]-\delta,\delta[\to \mathbb{R}$ with $\lambda^*(0) = 0$ and such that $H(s,\lambda^*(s)) \equiv 0$ in $]-\delta,\delta[$ can be found; see DEIMLING [62, Thm. 15.1]. By definition of H this means that $h^-(h^+(s,\lambda^*(s)),\lambda^*(s)) \equiv s$, i.e., for each $s \in]-\delta,\delta[$ the system (7.14a), (7.14b) with $\lambda = \lambda^*(s)$ has a (small) periodic orbit. The fact that $T(0) = \frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)}$ follows by explicitly writing down the return maps. An alternative argument is that the corresponding periodic solution is pieced together by "half" a periodic solution of (7.14a) and "half" a periodic solution of (7.14b), both for $\lambda = 0$. From the (smooth) Hopf bifurcation theorem we know that those periodic solutions do have periods $\frac{2\pi}{\omega^+(0)}$ and $\frac{2\pi}{\omega^-(0)}$, respectively, whence we find $T(0) = \frac{1}{2}(\frac{2\pi}{\omega^+(0)}) + \frac{1}{2}(\frac{2\pi}{\omega^-(0)}) = \frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)}$; see also Ex. 7.2.1 below. The uniqueness assertion is a consequence of the uniqueness of the function $\lambda^*(\cdot)$; see MORITZ [146] for more details of the proof. □

Note that $\lambda^*(\cdot)$ cannot be expected to be a C^1-function, since this would require that also H were C^1 w.r. to both s and λ. For the same reason, in general also T will only be continuous rather than differentiable. One may, however, formulate compatibility conditions on the coefficients of the real-analytic nonlinearities g^+ and g^- that will guarantee higher regularity of those functions.

Example 7.2.1. For illustration we investigate the simple system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\lambda & -\omega^+\\ \omega^+ & \lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} - (x^2+y^2)\begin{pmatrix}x\\ y\end{pmatrix}, \quad y > 0, \quad \text{and} \tag{7.16a}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\lambda & -\omega^-\\ \omega^- & \lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} - (x^2+y^2)\begin{pmatrix}x\\ y\end{pmatrix}, \quad y < 0, \tag{7.16b}$$

with $\omega^\pm > 0$. Introducing polar coordinates $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$, this becomes

$$\begin{pmatrix} \dot r \\ \dot\theta \end{pmatrix} = \begin{pmatrix} r\lambda - r^3 \\ \omega^+ \end{pmatrix}, \quad \sin\theta > 0, \quad \text{and} \quad \begin{pmatrix} \dot r \\ \dot\theta \end{pmatrix} = \begin{pmatrix} r\lambda - r^3 \\ \omega^- \end{pmatrix}, \quad \sin\theta < 0, \tag{7.17}$$

From Thm. 7.2.1 we know that (7.16a), (7.16b) undergoes a Hopf bifurcation at $\lambda = 0$, and we will use (7.17) to obtain more precise information in this special case. Starting the "+"-system at $(r^+(0), \theta^+(0)) = (s, 0)$, we find $\theta^+(t) = \omega^+ t$ for $t \in [0, \frac{\pi}{\omega^+}]$, and the solution for $r = r^+$ is

$$r^+(t) = s\sqrt{\frac{\lambda}{s^2 + (\lambda - s^2)e^{-2\lambda t}}} \quad (\lambda \neq 0), \quad \text{and}$$
$$r^+(t) = \frac{s}{\sqrt{2s^2 t + 1}} \quad (\lambda = 0);$$

this can be shown by deriving an ODE for $\frac{1}{r^2}$, and also note that the denominator has the sign of λ. Then we start the "$-$"-system with data $(r^-(0), \theta^-(0)) = (r^+(\frac{\pi}{\omega^+}), \pi)$ to get $\theta^-(t) = \omega^- t + \pi$ for $t \in [0, \frac{\pi}{\omega^-}]$. Since the differential equation for r^- is the same as the one for r^+, we hence can directly consider the latter up to time $T = \frac{\pi}{\omega^+} + \frac{\pi}{\omega^-}$ to obtain

$$h^-(h^+(s,\lambda),\lambda) = s\sqrt{\frac{\lambda}{s^2 + (\lambda - s^2)e^{-2\lambda T}}} \quad (\lambda \neq 0), \quad \text{and}$$
$$h^-(h^+(s,0),0) = \frac{s}{\sqrt{2s^2 T + 1}}, \quad \text{where} \quad T = \frac{\pi}{\omega^+} + \frac{\pi}{\omega^-}.$$

Note that this is well-defined for all $s \in \mathbb{R}$. To find the non-trivial zeroes of $H_1(s,\lambda) = h^-(h^+(s,\lambda),\lambda) - s$, we observe that there is no such zero, hence no periodic solutions, in case that $\lambda = 0$. For $\lambda \neq 0$ the equation $H_1(s,\lambda) = 0$ simplifies to $\lambda = s^2$, whence $\lambda < 0$ is impossible, and $s = \sqrt{\lambda}$ for $\lambda > 0$. Therefore periodic solutions do exist exactly for $\lambda > 0$, all with equal period $T = \frac{\pi}{\omega^+} + \frac{\pi}{\omega^-}$, and the function $\lambda^*(\cdot)$ from Thm. 7.2.1 which solves $H_1(s, \lambda^*(s)) \equiv 0$ is found to be $\lambda^*(s) = s^2$. From the 1D phase portrait of $\dot r = \lambda r - r^3$, or else by calculating $\frac{d}{ds} h^-(h^+(s,\lambda),\lambda)|_{\lambda = s^2} = e^{-2\lambda T} < 1$, it moreover follows that the periodic orbits are asymptotically stable. ◇

In MORITZ [146] some further results related to Hopf bifurcations are obtained, also on stability and for the case of real parts with opposite signs, i.e., when the linearizations in (7.14a) and (7.14b) are $\begin{pmatrix} \lambda & -\omega^+(\lambda) \\ \omega^+(\lambda) & \lambda \end{pmatrix}$ and $\begin{pmatrix} -\lambda & -\omega^-(\lambda) \\ \omega^-(\lambda) & -\lambda \end{pmatrix}$, respectively.

7.3 Piecewise linear planar systems

A particularly accessible special case of planar systems are piecewise linear systems of the form

$$\dot{q} = Aq + \operatorname{sgn}(w \cdot q)v, \tag{7.18}$$

with $q = (x, y) \in \mathrm{IR}^2$, A a real (2×2)-matrix, and given vectors $v, w \in \mathrm{IR}^2$ with $w \neq 0$. Such systems play a role in electrical circuits with a twin triode, ANDRONOV/VITT/KHAIKIN [9, p. 344], or in control systems with a two-point relay characteristic, LEFSCHETZ [127, p. 82]. Equation (7.18) has a discontinuity line $\{q \in \mathrm{IR}^2 : w \cdot q = 0\}$, and the problem is to classify the dynamical behaviour of the system in dependence of the 8 parameters (4 for A, and 2 for w and v, respectively). Following GIANNAKOPOULOS/KAUL/PLIETE [92] or PLIETE [179], we make the following assumptions:

$$\operatorname{trace}(A) \neq 0, \quad \operatorname{trace}(\mathrm{A})^2 < 4\det(A). \tag{7.19}$$

It is then possible to determine the number of periodic solutions of (7.18) and their stability. Some of those solutions may stick to the discontinuity for some time (sliding motion), whence sgn has to be considered as the corresponding multi-valued Sgn, i.e., $\operatorname{Sgn}(0) = [-1, 1]$ as before.

As a first step, the number of parameters can be reduced to three, since by means of a suitable transformation (7.18) is seen to be equivalent to

$$\dot{q} = A_\sigma q + \operatorname{sgn}(y)b, \quad A_\sigma = \begin{pmatrix} 0 & -\sigma \\ 1 & -1 \end{pmatrix}, \quad \sigma = \det(A)/\operatorname{trace}(\mathrm{A})^2 > 1/4, \tag{7.20}$$

where

$$b_1 = -[(Aw^\perp) \cdot v^\perp]/\operatorname{trace}(\mathrm{A})^2, \quad b_2 = -(w \cdot v)/\operatorname{trace}(\mathrm{A}),$$

with $w^\perp = (-w_2, w_1)$. In addition, this transformation rotates the discontinuity line to $\{(x, y) \in \mathrm{IR}^2 : y = 0\}$. In (7.20) there are only three parameters left, σ, b_1, and b_2.

Then an analysis can be carried out for the resulting semiflow to find the critical points, i.e., such $q_0 = (x_0, y_0)$ with $(0, 0) \in A_\sigma q_0 + \operatorname{Sgn}(y_0)b$, in all the different possible cases; see GIANNAKOPOULOS/KAUL/PLIETE [92]. Concerning the existence of periodic solutions, $b_2 \geq 0$ is a necessary condition, as may be seen from the phase portraits. For $b_2 > 0$ the closed orbits that cross the discontinuity line transversely (no sticking) can be found as fixed points of the return map $h = h^- \circ h^+$, where e.g. h^+ maps a point $(x_0, 0)$ along a trajectory that is contained in $\{(x, y) : y > 0\}$ to the first return point $(x_1, 0)$ to the x-axis. Under certain circumstances, it may be verified that h, considered as a map on the x-axis, is strictly increasing and concave, and hence has at most two fixed points. Thus there are at most two closed orbits that do not stick to the discontinuity line. With some more effort, the precise number of periodic solutions can be detected, and also their stability. Moreover, the existence of periodic solutions with sliding motion and of homoclinic orbits can be studied in detail; cf. GIANNAKOPOULOS/KAUL/PLIETE [92] for the complete results. In particular, it turns out that there are at most three periodic solutions for (7.20).