REWRITING SYSTEMS AND HOMOLOGY OF GROUPS

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1. INTRODUCTION

Anick, in [2], describes a resolution of a field k by free modules over an associative augmented algebra R over k. The construction involves the use of a special type of normal form for the elements of R. The theory of such normal forms and, more particularly, the special form of presentations which lead to them has been expounded by Bergman in [4] for ring theory and is available more generally as the theory of term rewriting systems. The theory has largely been developed within computer science but see, for example, Le Chenadec's book [10] for a more algebraic approach and some specific group-theoretic examples.

When the ring R is the monoid ring kG of a monoid G, the normal form required for Anick's proof can be obtained via a complete rewriting system for G (for definitions see Section 2.1). A very similar idea has been exploited, independently, by Squier [11]. Given a complete rewriting system for a monoid G, Squier constructs an exact complex of length four

$$P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{Z}$$

which, in dimension less than or equal to 2, co-incides with the well known relation sequence (see, for example, Section VI.6 of [9]).

The aim of this paper is to describe an alternative approach to the results of Anick (in fact a minor generalisation) which generalises the work of Squier. The general approach is based on that of Squier and retains what I believe is a somewhat more constructive flavour. The general technique can be described as building a 'cubical complex' based on directed graphs associated with the rewriting of certain words in the generators of a complete rewriting system. When, for example, the rewriting system is the natural one which takes as generating set the elements of a group G and as rules all pairs $ab \to c$ with $c = ab$ in G we obtain the bar resolution for G.

It is hoped that the alternative approach used here will enable some more insight into the practical aspects of calculating these resolutions. I was unaware of Anick's work for most of the time I was constructing these arguments and I thank Ken A. Brown (of Glasgow) for bringing Anick's paper to my attention. My original argument gives rise to modules which may in some cases be considerably larger than those produced by Anick's approach. I have included the (relatively minor) alterations necessary to yield the Anick resolution but have left in the original arguments which form a natural generalisation of Squier's proof. Ken S. Brown (of Cornell) has now constructed another proof [7] which has a strong topological flavour.

We give a brief map of the paper. In Section 2 we deal with some preliminaries. We describe the basic terminology of rewriting systems and directed graphs naturally associated with them. We also define the 'cubes' and 'stars' that we use in the proof and furnish the main technical facts we will require concerning them. In Section 3 we give the proof of the main result. This is proved inductively and the technical part of the inductive hypothesis can be found in Section 3.2. There are also two illustrative examples in Section 3.1. In Section 4 we make some comments regarding the technique and then give a number of different examples of the resolution in various special cases.

2. Preliminaries

2.1. Notation and rewriting systems

Throughout the paper, Σ will denote a set (of 'generators') and Σ^* will denote the free monoid (that is, semigroup with identity) on Σ. We will often refer to elements of Σ^* as words in Σ and the identity 1 of Σ^* will correspond to the empty word. A *rewriting system* (Σ, R) on Σ consists of a subset R of $\Sigma^* \times \Sigma^*$ together with Σ. The monoid presented by (Σ, R) is the monoid with presentation on generators Σ and relations all equations $l = r$ with $(l, r) \in R$. If $u, v \in \Sigma^*$ and if $u = alb, v = arb$ with $(l, r) \in R$ and $a, b \in \Sigma^*$, we will write $u \to v$. If there is a sequence

$$u = w_1, \ \ldots, \ w_n = v$$

of words $w_i \in \Sigma^*$ with $w_i \to w_{i+1}$, or $w_i = w_{i+1}$, then we write $u_1 \xrightarrow{*} u_2$ (and say 'u_1 R-reduces to u_2'). If, for some u, $u \xrightarrow{*} v$ implies $u = v$, we say that u is *irreducible.*

Definition. *A* rewriting system (Σ, R) *associated with a monoid G is complete if the following two conditions hold:*

(a) *there are no infinite chains* $u_1 \to u_2 \to \cdots \to u_n \to \cdots$;
(b) *each congruence class of the congruence defining G as a quotient of Σ^* contains exactly one irreducible.*

Thus, in a complete rewriting system, we can rewrite each word in Σ^* to a unique irreducible representing the same element of G and the irreducibles therefore provide a 'normal form' for G. In fact, we shall identify each element of G with its corresponding irreducible and so regard G as a subset of Σ^* (but not, of course, as a submonoid). If $w \in \Sigma^*$, $\overline{w}$ will denote the unique irreducible word in Σ^* which has the same image in G as w.

By a suitable 'tidying' process on (Σ, R), we can also assume that it is minimal in the sense that

(a) if $(l, r) \in R$ then r is irreducible;

(b) if $(l_1, r_1), (l_2, r_2) \in R$ then l_1 is not a subword of l_2.

This minimal rewriting system is uniquely determined by the set of irreducibles. We refer to Groves and Smith [8] or Squier [11] for details.

Given $u, v \in \Sigma^*$ with $u \xrightarrow{*} v$, there will, in general, be many ways of reducing u to v. We will choose, for each such pair u, v, a preferred reduction

$$u = w_1 \to \cdots \to w_n = v.$$

Further, we do this so that the restriction of this preferred reduction to $w_i \to \cdots \to w_j$ is itself the preferred reduction from w_i to w_j. The well-founded nature of the order deriving from $\to$ shows that this is possible. We could, for example, always choose – when a choice is required – the left-most occurrence of (the left hand side of) a rule in each w_i. There will be several occasions in the following when we will invoke this preferred reduction to resolve possible ambiguities.

For further details on rewriting systems in general, we refer to Le Chenadec's book [10].

2.2. Directed graphs and cubes

In this sub-section we will establish some terminology. The objects are fairly standard but we have not been able to find a suitable reference.

A *directed graph* Δ consists of a set $V(\Delta)$ of vertices and a set $E(\Delta)$ of edges together with initial and terminal functions $i, t\colon E(\Delta) \to V(\Delta)$. A *path* in Δ is a sequence

$$p = (e_1, \ldots, e_k) \text{ with } e_j \in E(\Delta),\ t(e_j) = i(e_{j+1}),\ (1 \leqslant j \leqslant k-1).$$

We extend i, t by defining $i(p) = i(e_1)$, $t(p) = t(e_k)$. (So if we identify an edge with a path of length one, the two possible definitions agree). We also allow, for each $v \in V(\Delta)$, an empty path with no edges and initial and terminal points v. The set of all paths in Δ will be denoted $P(\Delta)$.

We will be concerned only with two types of directed graph.

(1) Suppose that (Σ, R) is a minimal rewriting system for a monoid G. Then $\Gamma = \Gamma(\Sigma, R)$ is the directed graph with $V(\Gamma) = \Sigma^*$ and $E(\Gamma)$ the set of all instances of a single application of a rewriting rule. More precisely, for each $w \in \Sigma^*$ and each decomposition $w = ulv$ with $(l, r) \in R$ there is an edge e with $i(e) = w = ulv$ and $t(e) = urv$. When necessary we will refer to such an edge as $e = (w, u)$; since we have chosen the rewriting system to be minimal, this notation is unambiguous. If, for

example, $\Sigma = \{a\}$ and $R = \{a^2, 1\}$, then we have three edges with initial vertex a^4, namely $(a^4, 1)$, (a^4, a) and (a^4, a^2).

(2) For a natural number n, denote the set $\{1, \ldots, n\}$ by $\underline{n}$. Denote the empty set by $\underline{0}$. In either case, denote the power set of $\underline{n}$ by $2^{\underline{n}}$. We can give $2^{\underline{n}}$ the structure of a directed graph – which, abusing notation, will also be denoted $2^{\underline{n}}$. Let $V(2^{\underline{n}}) = 2^{\underline{n}}$ and

$$E(2^{\underline{n}}) = \{ (S, x) : S \in 2^{\underline{n}},\ x \in \underline{n} \setminus S \}.$$

Also define $i((S,x)) = S$, $t((S,x)) = S \cup \{x\}$. The induced ordering is then just the inclusion ordering on subsets. We can identify the vertices and edges of $2^{\underline{n}}$ with the vertices and edges of a standard n-dimensional cube (with one vertex – $\underline{0}$ – 'uppermost'). The notation simply gives a way of referring to this when technical verification is required.

Let Δ be a directed graph (in our applications it will usually be a $\Gamma(\Sigma, R)$). An *n-cube* μ in Δ consists of a pair of functions

$$\mu_V \colon V(2^{\underline{n}}) \to V(\Delta), \qquad \mu_E \colon E(2^{\underline{n}}) \to P(\Delta)$$

so that if μ_E takes an edge E to a path P, then μ_V takes the initial and terminal points of E to the initial and terminal points of P. Thus μ picks out the 1-skeleton of a geometric cube within the directed graph Δ. (For this reason the reader should be aware that a better but more clumsy name for what I call an n-cube would be '1-skeleton of an n-cube'.) Note that $2^{\underline{n}}$ is an n-cube in itself via the identity mapping. Also a 0-cube in Δ can be identified with a vertex and a 1-cube can be identified with a path.

It will be notationally convenient in the following to drop the subscript notation for the maps associated with μ. Thus we will refer to $\mu(v)$ or $\mu(e)$ for $v \in V(2^{\underline{n}})$ or $e \in E(2^{\underline{n}})$.

2.3. Faces of cubes

The resolution we build up in Section 3 will attempt to imitate the chain complex of a cubical complex. Our differential will imitate a genuine boundary map. Since we can identify our cubes with the 1-skeleta of geometric cubes we should clearly define the faces of our cubes so that they are the 1-skeleta of the geometric faces of such cubes. Thus the ideas made explicit below are straightforward even though there is some technical detail required.

For each $i \in \underline{n}$, let τ_i denote the function $\tau_i \colon \underline{n-1} \to \underline{n}$ given by

$$\tau_i(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \geqslant i. \end{cases}$$

DEFINITION. *Let* $i \in \underline{n}$.

(a) *An upper* $(n-1)$-face τ_i^+ *of* $2^{\underline{n}}$ *is the homomorphism* $\tau_i^+ : 2^{\underline{n-1}} \to 2^{\underline{n}}$ *given by*

$$\begin{aligned} (\tau_i^+)_V(S) &= \tau_i(S) && (S \in 2^{\underline{n-1}}), \\ (\tau_i^+)_E((S,x)) &= (\tau_i(S), \tau_i(x)) && (x \in \underline{n-1} \setminus S). \end{aligned}$$

(b) *A* lower $(n-1)$-face τ_i^- *of* $2^{\underline{n}}$ *is the homomorphism* $\tau_i^- : 2^{\underline{n-1}} \to 2^{\underline{n}}$ *given by*

$$\begin{aligned} (\tau_i^-)_V(S) &= \tau_i(S) \cup \{i\} && (S \in 2^{\underline{n-1}}), \\ (\tau_i^-)_E((S,x)) &= (\tau_i(S) \cup \{i\}, \tau_i(x)) && (x \in \underline{n-1} \setminus S). \end{aligned}$$

DEFINITION. *Let* ρ *be an* n*-cube in a directed graph* Δ . *Then an* upper $(n-1)$-face ρ_i^+ *of* ρ *is the* $(n-1)$*-cube* $\rho \circ \tau_i^+$. *A* lower $(n-1)$-face ρ_i^- *is the* $(n-1)$*-cube* $\rho \circ \tau_i^-$.

(The 'composites' here should be interpreted as the pair of maps obtained by taking the composites of the vertex maps and the composites of the edge maps.)

It is easy to verify that the faces of $2^{\underline{n}}$ will correspond with the 1-skeleta of the faces of a geometric cube. Further, the upper faces will be those which include the point $\underline{0}$ and the lower faces those which include the point $\underline{n}$. In particular, when $n = 1$ and a 1-cube is identified with a path, the upper 0-face can be identified with the initial point of this path and the lower 0-face with the terminal point. Note that a 0-cube has no faces.

We now have the beginnings of a 'boundary map' on n-cubes. The next three results will provide some of the technical justification for our final use of this 'boundary map'. They are routine technical exercises and we omit the proofs.

LEMMA 2.1. *Let* ρ *be an* n*-cube in a directed graph* Δ. *If* $i \in \underline{n}$, $j \in \underline{n-1}$, $i \leqslant j$, *then*

$$(\rho_i^\varepsilon)_j^\eta = (\rho_{j+1}^\eta)_i^\varepsilon \qquad (\varepsilon, \eta \in \{+,-\}).$$

COROLLARY 2.2. *Let* $\Omega^{\varepsilon,\eta}$ *denote* $\{ (\rho_i^\varepsilon)_j^\eta : i \in \underline{n},\ j \in \underline{n-1},\ \varepsilon, \eta \in \{+,-\} \}$. *Then*

(a) $\Omega^{\varepsilon,\varepsilon}$ *is the union* $\bigcup_{i \leqslant j} \{(\rho_i^\varepsilon)_j^\varepsilon, (\rho_{j+1}^\varepsilon)_i^\varepsilon\}$ *of equal pairs;*

(b) *there is a bijective correspondence, with corresponding elements equal, between* $\Omega^{\varepsilon,\eta}$ *and* $\Omega^{\eta,\varepsilon}$ *given by*

$$(\rho_i^\varepsilon)_j^\eta \longleftrightarrow (\rho_{j+1}^\eta)_i^\varepsilon \quad \text{if } i \leqslant j.$$

Although the order of the set $\underline{n}$ should not make a significant difference to our ideas, we cannot ignore it entirely. Thus we need to look at the effect of a change of this order. If ρ is an n-cube in Δ and π is a permutation of $\underline{n}$ (and, by extension, of $2^{\underline{n}}$), then $\rho \circ \pi$ is also an n-cube. We need to investigate the faces of $\rho \circ \pi$. Since, however, the 'adjacent transpositions' of the form $(j, j+1)$ generate the symmetric group S_n, it will suffice to do this in case $\pi = (j, j+1)$.

LEMMA 2.3. *Let π denote the permutation $(j, j+1)$ of $\underline{n}$ and let ρ be an n-cube. Then, for each $i \in \underline{n}$ and $\varepsilon \in \{+,-\}$,*

$$(\rho \circ \pi)_i^\varepsilon = \begin{cases} \rho_{i+1}^\varepsilon & \text{if } i = j, \\ \rho_i^\varepsilon & \text{if } i = j+1, \\ \rho_i^\varepsilon \circ \pi' & \text{if } i \neq j, j+1. \end{cases}$$

where π' is the permutation of $\underline{n-1}$ given by

$$\pi' = \begin{cases} (j, j+1) & \text{if } j+1 < i, \\ (j-1, j) & \text{if } j > i. \end{cases}$$

The proof is again a straightforward combinatorial exercise and is omitted.

2.4. Cubes in the directed graph of a rewriting system

Suppose that (Σ, R) is a minimal complete rewriting system for a monoid G and let $\Gamma = \Gamma(\Sigma, R)$ be the corresponding graph. Observe that an n-cube μ in Γ provides a description of different R-reductions – with up to n different starting points – of the word $\mu(\underline{0})$ in Σ^*. We shall frequently want to study only the beginnings of these R-reductions – occurring at the 'top layer' of the cube – and so we introduce a notation for this.

An *n-star* in Γ ($n \geqslant 0$) will consist of a vertex w and n edges e_j with $i(e_j) = w$. Within stars we also allow the possibility of an 'empty edge' with initial and terminal points equal to w. (This is strictly an empty *path* but the slight abuse of notation is convenient). If $e_j = (w, a_j)$, we will suppose the e_j ordered so that a_j is a prefix of a_{j+1}. (Empty edges may be placed arbitrarily in the order.) Thus we order from the left by occurrence of the rules. We then denote the n-star by

$$[w; e_1, \ldots, e_n].$$

If the star contains an empty edge, we shall say it is *degenerate*.

Thus 0-stars and non-degenerate 1-stars can be identified, respectively, with vertices and edges in Γ .

The fact that the vertex set of Γ is the monoid Σ^* means that we have a natural action of Σ^* on this vertex set – by multiplication on either the right or the left. It is easy to see that this extends to actions of Σ^* on $E(\Gamma)$ and so to actions of Σ^* on the directed graph Γ. From there we can easily define an action of Σ^* on stars (and also on cubes). That is, identifying vertices of Γ with elements of Σ^*, we can define the product of a 0-star with an arbitrary star. The next step is to observe that this can be generalised into a product of arbitrary stars. Thus define

$$[w_1; e_1, \dots, e_k].[w_2; e_{k+1}, \dots, e_{k+l}] = [w_1w_2; e_1w_2, \dots, e_kw_2, w_1e_{k+1}, \dots, w_1e_{k+l}].$$

The product is again a star, with edges in correct order as written. If $k = 0$ or $l = 0$ then this product agrees, after suitable identifications, with the action of Σ^* on stars. It is an easy technical verification – using the fact that Σ^* is a free monoid – that any star decomposes uniquely as a product of indecomposable stars.

When such an indecomposable star has neither empty edges nor repeated edges, it will be called *critical*. Thus a critical 0-star is an element of Σ, the critical 1-stars are in 1-1 correspondence with rules in R and the critical 2-stars can be identified with critical pairs (see, for example, [10]).

We can also make a similar definition of product for cubes. In the following, $T - k$ denotes $\{t - k : t \in T\}$.

DEFINITION. *Let σ be a k-cube in Γ and τ an l-cube. Suppose $k + l = n$ and define the* product n-cube $\sigma \times \tau$ *as follows:*

let $U \subseteq 2^{\underline{n}}$ and write $U = S \cup T$ with $S \subseteq \underline{k}$ and $T \subseteq k + \underline{l} = \{k+1, \dots, n\}$;
let $x \in \underline{n} \setminus U$; and
define

$$(\sigma \times \tau)(U) = \sigma(S)\tau(T - k),$$

$$(\sigma \times \tau)((S, x)) = \begin{cases} \sigma((S, x))\tau(T - k) & \text{if } x \in \underline{k}, \\ \sigma(S)\tau((T - k, x)) & \text{if } x \in k + \underline{l}. \end{cases}$$

We omit the routine checks that this does, indeed, define an n-cube. We note that, for $k = 0$ or $l = 0$, this agrees with the action of Σ^* on the edges of Γ. If $\rho = \sigma \times \tau$ with $k, l \geqslant 1$, then we shall say that ρ is *decomposable*.

There are no surprises in calculating the faces of a product and we again leave the following routine technical verification to the reader.

LEMMA 2.4. *Let ρ_1 be a k-cube and ρ_2 an l-cube with $k + l = n$. Then, for each $i \in \underline{n}$, $\varepsilon \in \{+, -\}$,*

$$(\rho_1 \times \rho_2)_i^\varepsilon = \begin{cases} (\rho_1)_i^\varepsilon \times \rho_2 & \text{if } i \leqslant k, \\ \rho_1 \times (\rho_2)_{i-k}^\varepsilon & \text{if } i > k. \end{cases}$$

We developed the notion of star by considering the 'top layer' of a cube and so each cube ρ has an associated star obtained by taking the edges emanating from $\rho(\underline{0})$.

Of course we must both supply empty edges and re-order, if necessary. If π is the permutation which gives the necessary re-ordering then we shall call sign(π) the *sign* of ρ. Where there is more than one choice for this sign, we assign it arbitrarily; its value will then not be important.

We now turn to the converse problem of associating an n-cube ρ with an n-star. Suppose, firstly, that $\rho = [w; e_1, \dots, e_n]$ is indecomposable. Let the edge e_i involve application of a rule for which the left hand side is the subword of w occupying positions $o(i), \dots, t(i)$. Suppose $S = \{i_1, \dots, i_m\} \subseteq \underline{n}$ with $i_1 \leqslant \dots \leqslant i_m$. Now choose $k_1 = 0$, k_2, ..., $k_l = m$ so that

$$t(i_j) \geqslant o(i_{j+1}) \text{ for } k_p < j < k_{p+1} \text{ and } t(i_{k_p}) < o(t_{k_p+1}) \text{ for } 1 < p < l.$$

Thus $\{k_1, \dots, k_l\}$ mark the largest "overlapping blocks" of subwords of w which correspond to edges e_i with $i \in S$. Let

$$w = a_1 u_1 a_2 \dots u_{l-1} a_l$$

with u_p occupying positions $o(i_{k_p+1}) \dots t(i_{k_{p+1}})$ in w. Define $\rho(S)$ to be the word

$$\rho(S) = a_1 \overline{u_1} a_2 \dots \overline{u_{l-1}} a_l$$

in Γ.

If $S \subseteq T$ there will clearly be at least one path in Γ from S to T. Let the path from w to $w_{\{i\}}$ consist of just the edge e_i and choose all the other paths required to define the n-cube by taking the preferred path in each case. In this way we define an n-cube with associated n-star equal to μ. If μ is an arbitrary n-star then it is a product of indecomposable n-stars and we define the associated n-cube to be the product of the associated indecomposable n-cubes. We shall refer to cubes obtained in this way as *canonical.*

Given any non-degenerate cube we can associate with it a star and then with this star a canonical cube. Not surprisingly, the upper faces (but only the upper faces) of a non-degenerate cube and its associated cube are closely related.

LEMMA 2.5. *Let σ be a non-degenerate n-cube with associated canonical n-cube ρ. Then each upper $(n-1)$-face σ_i^+ of σ and ρ_i^+ of ρ is non-degenerate. Further, σ_i^+ and ρ_i^+ have a common associated $(n-1)$-cube.*

PROOF. The first statement is trivial. Also the $(n-1)$-star of σ_i^+ is obtained from the n-star of σ by omitting an edge (and possibly translating). Since σ and ρ then have a common n-star and the associated canonical cube depends only on the n-star, the result follows easily.

2.5. Relationship with Anick's chains

We turn briefly to the question of the relationship of our techniques, and in particular the n-stars, with the n-chains of Anick [2]. Let G be the cyclic group of order k with generator a and one rule $a^k \to 1$. It is a simple combinatorial exercise to verify that the number of critical n-stars for this system is $(k-1)^{(n-1)}$ whereas the number of Anick's n-chains is one! (This, together with related examples, is, however, the worst case of which we know.) There is clearly some difference and we now try to explain this.

We shall say that a critical n-star $[w; e_1, \dots, e_n]$ is *non-overlapping* if for no i do the subwords involved in the edges e_{i-1} and e_{i+1} overlap. More precisely if $e_j = (w, a_j)$ and e_j involves an application of the rule (l_j, r_j) then we require that $a_{i-1}l_{i-1}$ is a subword of a_{i+1}. Note that the term 'non-overlapping' refers not to adjacent edges/rules – this would be of no interest – but to ones separated by a third one.

If we are to use these techniques with non-overlapping n-stars we will need to know that the cube associated with such a star has faces which, if non-degenerate, are also associated with non-overlapping stars. For the upper faces of a cube this is clear.

Suppose that ρ is an n-cube associated with a non-overlapping n-star. Denote $\rho(\underline{0})$ by w. Then the non-overlapping property shows that we can decompose w in the form

$$w = w_1' w_2' u_2 w_3' \dots w_{n-1}' u_{n-1} w_n'$$

where $w_i' u_i w_{i+1}'$ is the left hand side of the rule associated with the i-th edge $\rho(\underline{0}, \{i\})$. (Interpret u_1 and u_n as 1.) Consider the lower face ρ_i^- and assume it is non-degenerate. The 'apex' $\rho_i^-(\underline{0})$ is

$$w_1' \dots u_{i-1} \overline{w_i' u_i w_{i+1}'} u_{i+1} \dots w_{n-1}' u_{n-1} w_n'.$$

The edges leading from this apex are the first edges in the paths $\rho(\{i\}, \{i,j\})$ with $i \neq j$. Thus the only possible 'overlapping' occurs when the two edges correspond either to the values for j of $\{i-2, i+1\}$ or to the values of $\{i-1, i+2\}$. But the rules applied must be separated in the former case by u_{i-1} and in the latter case by u_{i+1}.

Hence we can restrict ourselves to non-overlapping critical stars – and cubes – and retain most of the previous discussion. This is still not enough, however. In the example above there are still many non-overlapping critical n-stars. We define a *right-minimal* non-overlapping n-star inductively. Suppose that $[w; e_1, \dots, e_n]$ is a non-overlapping n-star. Let w_1' be the largest prefix which is not involved in the rule corresponding to the second edge (this agrees with the notation above) and write $w = w_1' v$ and $e_i = w_1' e_i' (i > 1)$. We say that $[w; e_1, \dots, e_n]$ is *right minimal* if

(1) $[v; e_2', \dots, e_n']$ is a right-minimal non-overlapping $(n-1)$-star;
(2) for no proper suffix a of w_1' and edge e_1' is there a non-overlapping n-star of the form $[av; e_1', ae_2', \dots, ae_n']$.

An inspection of the definition of Anick's n-chains (but reversing the left-right order) should now convince the reader that these are in 1-1 correspondence with our right-minimal non-overlapping critical n-stars. For reasons which are by now becoming fairly obvious we shall abbreviate 'right-minimal non-overlapping critical' by 'special'. Observe that the underlying word of a non-overlapping n-star always contains a suffix which is a special n-star.

3. The resolution

3.1. Statement, preliminaries and examples

We begin with a brief statement of the Theorem although the essential content, being technical in nature, is in the inductive hypotheses described in Section 3.2. Let (Σ, R) be a minimal complete rewriting system for a monoid G. We continue to identify elements of G with R-irreducibles in Σ^*. Let K be a commutative ring (with 1).

Let P'_n be the free K-module with basis the set of all n-stars. Let Q_n be the submodule generated by

(1) all n-stars with an empty path;
(2) all n-stars with a repeated edge;
(3) all n-stars which are the product of a k-star with a l-star with $k, l \geqslant 1$;
(4) all expressions of the form $[u;][w; e_1, \dots, e_n][v;] - \overline{u}[w; e_1, \dots, e_n]$.

(For the latter, note that we are using the product defined for stars.)

Let P_n be the quotient module P'_n/Q_n. We shall abuse notation by identifying stars with their images in P_n. Thus a degenerate star, for example, is identified with zero. P_n has a natural structure as a KG-module via (4) above and the natural product of stars. Under this structure it is a free (left) KG-module with basis the critical n-stars. P_1 has a basis in bijective correspondence with the rules of R and P_0 a basis in bijective correspondence with Σ. Define P_{-1} to be the monoid ring KG.

THEOREM 3.1. *There is a KG-resolution of K:*

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow K.$$

There is, of course, a dimension shift, by 1, in the naming of these modules. It is very convenient for the terminology of the proof, however, to have P_n generated by n-stars rather than $(n-1)$-stars.

The proof of Theorem 3.1 will occupy the remainder of Section 3.

REMARK. We will also consider the case that P_n is generated by all *special* n-stars. We will generally not give the full proof in this case but will indicate the points at which there is substantial divergence from the main argument.

EXAMPLES

There is sufficient technical detail in the proof that it may be helpful to give some simple examples of the ideas we will use. Let $\Sigma = \{a_1, a_2, a_3, a_4\}$ and let $R_{ji}\,(1 \leqslant i < j \leqslant 4)$ be the rule $a_j a_i \to a_i a_j$. Let $R = \{R_{ji}\}$. Then (Σ, R) presents the free commutative monoid of rank 4 with basis the image of Σ.

One critical 2-star for (Σ, R) is represented by

$$\begin{tikzcd} & a_3a_2a_1 \arrow[dl] \arrow[dr] & \\ a_2a_3a_1 & & a_3a_1a_2 \end{tikzcd}$$

and we can thus produce a (unique) associated 2-cube μ as follows

$$\begin{tikzcd} & a_3a_2a_1 \arrow[dl] \arrow[dr] & \\ a_2a_3a_1 \arrow[d] & & a_3a_1a_2 \arrow[d] \\ a_2a_1a_3 \arrow[dr] & & a_1a_3a_2 \arrow[dl] \\ & a_1a_2a_3 & \end{tikzcd}$$

where, for example, the left-hand lower face is given by the unique path from $a_2a_3a_1$ to $a_1a_2a_3$. This cube has an evident boundary obtained by dividing the faces into single edges. Since, for example, the topmost right-hand edge involves a reduction $a_3a_2a_1 \to a_3a_1a_2$ we can say that this edge is 'covered' by the element of P_1 corresponding to a_3R_{21}. We then have the following expression, in P_1, for the boundary:

$$\begin{aligned} &a_3R_{21} + R_{31} + a_1R_{32} && \text{(right edge)}, \\ &-aR_{32} - a_2R_{31} - R_{21} && \text{(left edge)}. \end{aligned}$$

Note that we have used the left action of Σ^* on G but regard the right action as trivial.

The only indecomposable 3-star ρ for (Σ, R) is the one with underlying word $a_4a_3a_2a_1$ and edges corresponding to the three possible applications of rules. The vertices of the associated cube are as below and the edges between these vertices can be completed in more than one way.

$$\rho(\underline{0}) = a_4a_3a_2a_1$$

$$\rho(\{1\}) = a_3a_4a_2a_1 \qquad \rho(\{2\}) = a_4a_2a_3a_1 \qquad \rho(\{3\}) = a_4a_3a_1a_2$$

$$\rho(\{1,2\}) = a_2a_3a_4a_1 \qquad \rho(\{1,3\}) = a_3a_4a_1a_2 \qquad \rho(\{2,3\}) = a_4a_1a_2a_3$$

$$\rho(\{1,2,3\}) = a_1a_2a_3a_4$$

(The reader is advised to try drawing the full section of the directed graph – it has 24 vertices – below $a_4a_3a_2a_1$ and seeing what choices are possible).

The way to 'cover' faces with critical 2-cubes is to match up the boundary of the face with the boundary of critical 2-cubes. For the upper faces it is clear how to do this; for the lower faces a little more work is needed. For example, the complete graph below the vertex $\rho(\{1\})$ is as follows.

$$\begin{tikzcd}
 & & a_3a_4a_2a_1 \arrow[dl] \arrow[dr] & \\
 & a_3a_2a_4a_1 \arrow[dl] \arrow[d] & & a_3a_4a_1a_2 \arrow[d] \\
a_2a_3a_4a_1 \arrow[d] & a_3a_2a_1a_4 \arrow[dl] \arrow[dr] & & a_3a_1a_4a_2 \arrow[dl] \\
a_2a_3a_1a_4 \arrow[d] & & a_3a_1a_2a_4 \arrow[d] & \\
a_2a_1a_3a_4 \arrow[dr] & & a_1a_3a_2a_4 \arrow[dl] & \\
 & a_1a_2a_3a_4 & &
\end{tikzcd}$$

Thus it is reasonable to define a covering for the first lower face to be $a_3[a_4a_2a_1]+[a_3a_2a_1]$ where $[a_ia_ja_k]$ denotes the 2-star corresponding to the word $a_ia_ja_k$.

We are still using the left action of Σ^* and regarding the right action as trivial and we are ignoring the 'square' headed by $a_3a_2a_4a_1$ since it corresponds to a decomposable cube (or star). It is not difficult to verify that, repeating this for all the 6 faces we obtain

face	covering
ρ_1^+	$a_4[a_3a_2a_1]$
ρ_2^+	0
ρ_3^+	$[a_4a_3a_2]$
ρ_1^-	$a_3[a_4a_2a_1]+[a_3a_2a_1]$
ρ_2^-	$a_2[a_4a_3a_1]+[a_4a_2a_1]$
ρ_3^-	$[a_4a_3a_1]+a_1[a_4a_3a_2]$.

We will therefore define the boundary of the 3-star ρ to be the (signed) sum

$$(a_4-1)[a_3a_2a_1]-(a_3-1)[a_4a_2a_1]+(a_2-1)[a_4a_3a_1]-(a_1-1)[a_4a_3a_2].$$

The reader familiar with the usual (Koszul complex) resolution for free abelian groups or monoids will recognise this as the expression appropriate to that complex. Full details for this example are given in Section 4.

ORDERINGS

We require some discussion of orderings. There is a natural partial order on Σ^* (or Γ) which defines u to be less than or equal to v if there is a directed path in Γ from v to u or equivalently if $v \xrightarrow{*} u$. Because (Σ, R) is complete, this is a well founded partial order (that is, a partial order with no infinite descending chains). We extend this slightly, by defining, for $u, v \in \Sigma^*$,

$$u \leqslant v \text{ if } u \text{ is a prefix of an } R\text{-reduction of } v.$$

Clearly '$\leqslant$' is still a well-founded partial order on Σ^* and satisfies

$$u \leqslant v \text{ implies } au \leqslant av \text{ for } a \in \Sigma^*.$$

We can further extend '$\leqslant$' firstly to stars and secondly to G-translates of stars (within $\cup_n P_n$) by defining

$$g[v; e_1, \ldots, e_m] \leqslant h[w; f_1, \ldots, f_n] \quad \text{if} \quad gv \leqslant hw.$$

Note that, in this case, '$\leqslant$' need not be antisymmetric and so is a pre-order rather than an order. It is still, however, well founded. Finally, we can extend to the whole of $\cup_n P_n$ by defining, for $\alpha, \beta \in \cup_n P_n$, $\alpha \leqslant \beta$ if each element in the support of α is less than or equal to some element in the support of β. This relation is again a well-founded pre-order.

3.2. THE INDUCTIVE HYPOTHESIS

We fix $n \geqslant 1$ and suppose that, for each $k \leqslant n$, we have the following:

(a) a KG-module homomorphism $\delta_k \colon P_k \to P_{k-1}$;

(b) for each k-cube ρ, an element $C_k(\rho)$ of P_k.

These two functions δ_k and C_k should satisfy, for $1 \leqslant k \leqslant n$,

$(I)_k$ $\delta_{k-1} \circ \delta_k = 0$;

$(II)_k$ for each $\alpha \in \ker(\delta_{k-1})$, there exists $\beta \in P_k$ with $\beta \leqslant \alpha$ and $\delta_k(\beta) = \alpha$;

$(III)_k$ given any k-cube ρ,

(i) if ρ is non-degenerate and is associated with a k-star α then $C_k(\rho) - \operatorname{sign}(\rho)\alpha < \rho(\underline{0})$; in any case, $C_k(\rho) \leqslant \rho(\underline{0})$;

(ii)

$$\delta_k(C_k(\rho)) = \sum_{\substack{i \in \underline{k} \\ \varepsilon \in \{+,-\}}} \varepsilon(-1)^{i+1} C_{k-1}(\rho_i^\varepsilon) \qquad (= D_{k-1}(\rho), \text{ say}).$$

We comment briefly on the significance of these requirements. Firstly $(I)_k$ requires that δ_k be a differential whereas $(II)_k$ requires that the complex be exact with the extra technical requirement that $\beta \leqslant \alpha$. The functions C_k ('C' is for covering) are intended to mimic the idea of a cubical decomposition. So part (i) of $(III)_k$ gives the stars – and their associated cubes – as the basic building blocks of the decomposition. Clearly part (ii) requires that δ_k should imitate the face map on cubes.

We need to begin the induction by defining the maps δ_k, C_k for $k = 0, 1$. The ideas are standard and the material is also well covered in Squier's paper [11]; so we will be brief. The map $P_{-1} \to K$ is, of course, the augmentation map. Recall that a critical 0-star $[a;\ ]$ corresponds to an element a of Σ and a 0-cube to an element of Σ^*. If $a \in \Sigma$, define $\delta_0([a;\ ]) = \overline{a} - 1 \in KG$. If $w = a_1 \dots a_l \in \Sigma^*$ with $a_i \in \Sigma$, define

$$C_0(w) = \sum_{i=1}^{l} a_1 \dots a_{i-1}[a_i;\ ].$$

Thus C_0 is essentially the Fox differential.

If $[w; e_1]$ is a critical 1-star, then w is the left hand side of a rewriting rule. Define

$$\delta_1([w; e_1]) = C_0(w) - C_0(\overline{w}).$$

If ρ is a 1-cube, then it corresponds to a path $(e_1, \dots, e_n)$ in Γ. Suppose that $e_i = (w_i, a_i)$ and that e_i involves the application of a rule (l_i, r_i). Define

$$C_1(\rho) = \sum_{i=1}^{m} a_i[l_i; e_i']$$

where $e_i = a_i e_i'$. It is easily verified that $(III)_1$ is also satisfied. After suitable translation of notation, these definitions agree with the standard ones related to the relation sequence (see, again, the discussion in Squier [11]).

Remark. In the case that P_n is generated by all *special* n-stars the requirements are similar. We cover only cubes with non-overlapping stars and we cover them with special n-stars. Note that for $n = 0, 1$ there is no difference between 'special' and 'critical'.

3.3. The inductive step

Inductive hypothesis I

Let $\alpha \in P_{n+1}$ be a critical $(n+1)$-star and let ρ be the associated canonical $(n+1)$-cube. Define

$$\delta_{n+1}(\alpha) = D_n(\rho).$$

The verification that $\delta_n \circ \delta_{n+1} = 0$, and so that $(I)_{n+1}$ is satisfied, is now straightforward. In fact, $\delta_k(D_k(\rho)) = 0$ for any k-cube ρ and any $k \leqslant n$; use part (ii) of

$(III)_k$ to express $\delta_k(D_k(\rho))$ in terms of elements of P_{k-1} and Corollary 2.2 to show that the resulting expression is zero. We will use this fact later.

We can now obtain a more explicit expression for δ_k. Suppose that $2 \leqslant k \leqslant n+1$ and that $\alpha = [w; e_1, \ldots, e_k]$ is a critical k-star. Let $[w; e_1, \ldots, \hat{e}_i, \ldots, e_k]$ be the star obtained from α by omitting edge e_i. If $i \neq 1$ or k this will be either critical or decomposable and in the latter case it corresponds to the zero element of P_k. In case $i = 1$ we can write

$$[w; e_2, \ldots, e_k] = a[w'; e_2', \ldots, e_k']$$

where $w = aw'$ and $e_i = ae_i'$.

LEMMA 3.2. *With the notation above,*

$$\delta_k([w; e_1, \ldots, e_k]) = a[w'; e_2', \ldots, e_k'] + \sum_{i=1}^{k-1} (-1)^i [w; e_1, \ldots, \hat{e}_i, \ldots, e_k] + \beta$$

where $\beta < w$.

PROOF. Let ρ be the k-cube associated with $[w; e_1, \ldots, e_k]$. Then we must have

$$\delta_k([w; e_1, \ldots, e_k]) = D_{k-1}(\rho).$$

Let β denote $\sum_{i \in \underline{k}} (-1)^i \, C_{k-1}(\rho_i^-)$. Then, by $(III)_{k-1}$,

$$C_{k-1}(\rho_i^-) \leqslant \rho_i^-(\underline{0}) \leqslant \rho(i) < \rho(\underline{0}) = w$$

(recall that ρ is non-degenerate). Hence $\beta < w$.

Also, by Lemma 2.5, ρ_i^+ will have an associated cube which is associated with the relevant $(k-1)$-star (i.e. $a[w'; e_2', \ldots, e_k']$ if $i = 1$ and $[w; e_1, \ldots, \hat{e}_i, \ldots, e_k]$ otherwise). The result now follows easily using $(III)_{k-1}$.

REMARK. In the case that P_n is generated by all *special* n-stars the lemma becomes simpler because the terms of the alternating sum will always be zero.

INDUCTIVE HYPOTHESIS *II*

Consider the following condition on an element α of P_n.

(3-1) If $u[w; e_1, \ldots, e_n]$ is in the $\mathbb{Z}$-support of α and e_1 is associated with the rule (l, r), then every proper prefix of ul is reduced.

LEMMA 3.3. *If α satisfies* (3-1) *and* $\delta_n(\alpha) = 0$ *then* $\alpha = 0$.

PROOF. Suppose that $\alpha \neq 0$. Choose $x = u[w; e_1, \ldots, e_n]$ in the support of α in such a way that uw is $\leqslant$-maximal amongst all such elements and that, amongst all elements with the same value of uw, the term u has maximal length.

Retaining the notation of Lemma 3.2, we know that the term $x_1 = ua[w'; e'_2, \dots, e'_n]$ occurs in the support of $\delta_{n-1}(x)$ (it cannot cancel because of the description given in Lemma 3.2). By (3-1), ua is reduced and because $\delta_n(\alpha') = 0$, x_1 must also occur in the support of $\delta_n(y)$ for some $y = v[w_1; f_1, \dots, f_n]$ in P_n which is distinct from x.

Using Lemma 3.2, it follows that we can express $\delta_n(y)$ in the form

$$\delta_n(y) = vb[w'_1; f'_1, \dots, f'_n] + \sum_{i=1}^{n-1}(-1)^i v[w_1; f_1, \dots, \hat{f}_i, \dots, f_n] + \gamma$$

where $\gamma < vw_1$. Recalling the choice of x, it is clear from the maximality of the term uw of x that $uw = vw_1$ and so that x_1 does not occur in γ. Also, if x_1 were to be a term in the alternating sum above then we would have $v = ua$, contradicting the maximality of u. Thus x_1 is the 'leading' term above; that is

$$ua[w'; e'_2, \dots, e'_n] = vb[w'_1; f'_2, \dots, f'_n].$$

Hence $ua = vb$, $w' = w'_1$ and $e'_i = f'_i \quad (i = 2, \dots, n)$.

Suppose that e_1 applies the rule with left-hand-side l and f_1 the rule with left-hand-side m. Because of (3-1), every proper prefix of ul and of vm is reduced. But one of ul and vm is a prefix of the other as $uw = vw_1$. Thus $ul = vm$.

Therefore every proper suffix of l and m is reduced and one must be a suffix of the other. Thus $l = m$. Now it is easy to see that $x = y$, contrary to supposition. The proof of Lemma 3.3 is complete.

It remains to reduce to the case of the previous lemma. We do the bulk of this in the following lemma.

LEMMA 3.4. *Any $\alpha \in P_n$ can be written in the form*

$$\alpha = \alpha' + \delta_{n+1}(\beta)$$

where α' satisfies (3-1) *and $\beta \leqslant \alpha$.*

PROOF. It clearly suffices to prove this when α is a $\mathbb{Z}$-generating element of P_n: say, $\alpha = u[w; e_1, \dots, e_n]$.

Suppose that α fails to satisfy (3-1). Amongst all such α choose it so that uw is least under the ordering '$\leqslant$' and amongst all such elements with this least value of uw choose it so that u has minimal length.

Retaining the notation above, there must be decompositions $u = u_0 u_1$ and $l = l_0 l_1$ so that $u_1 l_0$ is the left hand side of a rule in R. Let e_0 be the corresponding edge. Then $[u_1 w; e_0, u_1 e_1, \dots, u_1 e_n]$ is a critical $(n+1)$-star. Further, by Lemma 3.2, we can write

$$\delta_{n+1}([u_1 w; e_0, u_1 e_1, \dots, u_1 e_n]) = u_1[w; e_1, \dots, e_n] + \gamma$$

where $\gamma \leqslant u_1 w$ and the terms of the form $a[v; \dots]$ in the support of γ which have $av = u_1 w$ must also have $a = 1$.

Thus, multiplying by u_0, we obtain

$$\delta_{n+1}(u_0[u_1 w; e_0, u_1 e_1, \dots, u_1 e_n]) = u[w; e_1, \dots, e_n] + u_0 \gamma.$$

Note that, writing $\beta = u_0[u_1 w; e_0, u_1 e_1, \dots, u_1 e_n])$, we have that β is bounded by some element in the support of α (in fact α itself) and so $\beta \leqslant \alpha$. Also, each term $\overline{u_0 a}[v; \dots]$ in the support of $u_0 \gamma$ has either $\overline{u_0 a} v \leqslant u_0 a v < uw$ or else has $a = 1$ and in this case $\overline{u_0 a}$ is a proper prefix of u.

Using the minimality of α, it follows that each element in the $\mathbf{Z}$-support of $u_0 \gamma$ must satisfy (3-1). The proof of the lemma follows.

It is clear that the required result follows from the two lemmas. If α lies in the kernel of δ_n, then write $\alpha = \alpha' + \delta_{n+1}(\beta)$ as in Lemma 3.4. Then, as $\delta_{n+1}(\beta)$ also lies in the kernel of δ_n by the inductive hypothesis $(I)_{n+1}$, it follows that $\delta_n(\alpha') = 0$. Hence $\alpha' = 0$ by Lemma 3.3 and the result follows.

Remark. In the case that P_n is generated by all *special* n-stars the proof is again somewhat simpler. The condition (3-1) should be altered to replace "every proper suffix of ul" with "the word uw_1'" (using the notation of Section 2.5). In Lemma 3.3 we do not need to worry about the terms in the alternating sum in applying Lemma 3.2 and the arguments of Anick in Lemma 1.3 of [2] can be used to replace the last paragraph of the proof. Lemma 3.4 goes through much as before except that we must decompose uw_1' in such a way as to ensure that the ensuing critical $(n+1)$-star is right-minimal (and so special).

Inductive hypothesis *III*

We will begin by showing that, for any non-degenerate $(n+1)$-cube ρ and any permutation π of $\underline{n}$, we have

$$D_n(\rho \circ \pi) - \operatorname{sign}(\pi) D_n(\rho) < \rho(\underline{0}) \qquad (= w, \text{ say}). \tag{3-2}$$

Observe that, if this is true for two permutations π then it is also true for their product. Hence it will suffice to prove it for π an 'adjacent transposition' of the form $(j, j+1)$ – since such transpositions generate the symmetric group. We therefore suppose that $\pi = (j, j+1)$.

Note that

$$C_n(\rho_i^-) \leqslant \rho_i^-(\underline{0}) = \rho(\{i\}) < \rho(\underline{0}) = w.$$

Thus we can, in the expression for D_n, ignore the lower faces of the relevant cubes.

We can now apply Lemma 2.3. Let α_i be the star of ρ_i^+. If $i \neq j, j+1$, then by Lemma 2.3 we have that $(\rho \circ \pi)_i^+ = (\rho_i^+) \circ \pi'$. By part (i) of the inductive assumption $(III)_n$ we also have that

$$C_n(\rho_i^+ \circ \pi') - \operatorname{sign}(\rho_i^+ \circ \pi')\alpha_i < w$$

and

$$C_n(\rho_i^+) - \operatorname{sign}(\rho_i^+)\alpha_i < w.$$

Noting that $\operatorname{sign}(\pi') = \operatorname{sign}(\pi)$, we therefore have that

$$C_n((\rho \circ \pi)_i^+) - \operatorname{sign}(\pi) C_n(\rho_i^+) < w$$

whenever $i \neq j, j+1$.

It is easy to deduce from Lemma 2.3 that the terms for $i = j$ or $i = j+1$ in (3-2) will cancel. Thus the proof of (3-2) follows in the case that ρ is non-degenerate.

We will now show that, if μ is a non-degenerate canonical $(n+1)$-cube which has a star α with zero image in P_{n+1}, then

$$D_n(\mu) < \mu(\underline{0}) = w. \tag{3-3}$$

If α has a repeated edge then the fact that we can have $\mu \circ \pi = \mu$ for an *odd* permutation π, together with (3-2), gives the required result.

So we need to consider (3-3) when α is a product of two stars of strictly smaller size – k and l, say, with $k + l = n + 1$ and $k, l \geqslant 1$. In this case, μ is also such a product. If $k, l > 1$, then Lemma 2.3 shows that each face of μ is a product of this type and so $C_n(\mu_i^\varepsilon) < w$ for all i, ε, by part (i) of $(III)_n$. So (3-3) is true in this case.

Suppose that $\mu = \nu \times \kappa$ with ν a 1-star (or element of Σ^*) and κ a n-star. Then the faces of μ are again decomposable as a product of factors of non-zero dimension except for the pair of faces $\nu_1^+ \times \kappa$ and $\nu_1^- \times \kappa$. But, using part (i) of $(III)_n$ and denoting the star of κ by γ, we have that

$$(\nu_1^+ \times \kappa) - \overline{\nu(\underline{0})}\gamma < w$$

and

$$(\nu_1^- \times \kappa) - \overline{\nu(\{1\})}\gamma < w.$$

Since $\overline{\nu(\underline{0})} = \overline{\nu(\{1\})}$ it follows easily that (3-3) is true in this case also.

Let α be the (image in P_{n+1} of the) star of a $(n+1)$-cube ρ, let π be a permutation that orders the initial edges of ρ into the same order as α and let μ be the cube associated with α.

Suppose that ρ is non-degenerate. We claim that

$$D_n(\rho) - \operatorname{sign}(\rho)\delta_{n+1}(\alpha) < \rho(\underline{0}) = w. \tag{3-4}$$

If α is zero this follows from the above discussion. Otherwise, we can use (3-2) to assume that the leading edges of ρ are ordered in the order in which they occur in α. We can then use Lemma 2.5 to show that $D_n(\rho) - D_n(\mu) < \rho(\underline{0}) = w$.

Finally, we can write $\alpha = u\beta v$ with $u, v \in \Sigma^*$ and β critical. There is then a corresponding decomposition $\mu = u\nu v$ of canonical cubes. Since, by construction, we have $\delta_{n+1}(\beta) = \nu$ we also have $\delta_{n+1}(\alpha) = D_n(\mu)$. Thus (3-4) follows in this case and so now in all cases.

We are now ready to define the 'covering map' C_{n+1}. If ρ is degenerate then observe that $\delta_n(D_n(\rho)) = 0$ and so, by $(II)_{n+1}$ of the inductive hypothesis, we can find γ with $\delta_{n+1}(\gamma) = D_n(\rho)$ and $\gamma \leqslant D_n(\rho) \leqslant \rho(\underline{0})$. Define $C_{n+1}(\rho) = \gamma$. The requirements of $(III)_{n+1}$ are then satisfied.

Suppose now that ρ is non-degenerate. By (3-4) we know that

$$D_n(\rho) - \operatorname{sign}(\rho)\delta_{n+1}(\alpha) < \rho(\underline{0}) = w.$$

But we also know that

$$\delta_n(D_n(\rho) - \operatorname{sign}(\rho)\delta_{n+1}(\alpha)) = 0.$$

Hence, by the inductive hypothesis $(II)_{n+1}$ (which has already been proved), we can find $\gamma \in P_{n+1}$ with

$$\gamma \leqslant D_n(\rho) - \operatorname{sign}(\rho)\delta_{n+1}(\alpha) < w$$

and

$$\delta_{n+1}(\gamma) = D_n(\rho) - \operatorname{sign}(\rho)\delta_{n+1}(\alpha).$$

Define

$$C_{n+1}(\rho) = \operatorname{sign}(\rho)\alpha + \gamma.$$

The two requirements for hypothesis III follow easily. Firstly,

$$C_{n+1}(\rho) - \operatorname{sign}(\rho)\alpha = \gamma < \rho(\underline{0});$$

and secondly,

$$\delta_{n+1}(C_{n+1}(\rho)) = \delta_{n+1}(\operatorname{sign}(\rho)\alpha + \gamma) = D_n(\rho).$$

This completes the inductive step III and with it the proof of Theorem 3.1.

4. Comments, Calculations and Examples

4.1. Monoids with a finite rewriting system

There is an immediate application of the resolution to groups (more generally monoids) with a finite complete rewriting system. If Σ and R are finite, there can clearly be only finitely many critical n-stars because each one is formed from n occurrences of rules in R. It follows that each P_n is a finitely generated KG-module. When there is such a finitely generated resolution of K, we say that G is of type FP_∞ over K. (In fact FP_∞ over $\mathbb{Z}$ implies FP_∞ over any other K.) Thus we have the following theorem.

Theorem 4.1 (Anick, Squier). *If G is a monoid (group) with a finite complete rewriting system then G is of type FP_∞.*

The theorem is easily deducible from Anick's work in [2] (although Anick's definition of normal form corresponding to a rewriting system is unnecessarily restrictive). It was made explicit by Squier [11] in the special case where FP_3 replaces FP_∞.

In Groves and Smith [8] it is shown that all soluble groups which are constructible in the sense of Bieri and Baumslag [3] have a finite complete rewriting system. The classes of soluble constructible groups and soluble groups of type FP_∞ are not currently known to differ and, in particular, are known to co-incide for metabelian (2-step soluble) groups. Thus the theorem comes close to yielding a group-theoretical characterisation of soluble groups with a finite complete rewriting system.

It may well be, however, that another productive approach is to consider regular rather than finite rewriting systems. (This should probably require that the set of left hand sides of rules should form a regular language). It is far from clear and possibly of some interest to establish what effect this has on the homological properties of the group or monoid. It is not even clear for example whether the homology groups of such monoids, taken with integer coefficients, are restricted in any way.

4.2. Inverses

A significant drawback, from the point of group theorists, of both rewriting systems and of this resolution is that it concerns monoids and not groups. Groups can, of course, be considered as a special case of monoids in which elements have inverses but most group theorists, including this author, would prefer to think of them as a separate algebraic type with two operations – of inversion as well as multiplication.

The practical consequences here are that rewriting systems for groups must contain rules of the form $aa^{-1} \to 1$ and $a^{-1}a \to 1$ for each generator a of the group whereas in a group presentation this would be taken care of automatically. (These rules can be avoided, of course, when a has finite order.) It would be very useful to have a form of rewriting for groups in which provision of inverses and their cancellation in rewriting was provided automatically.

The effect on the resolution is that it must usually contain many more generators than seems ideally necessary. Consider the submodules generated by stars all edges of which involve rules which cancel a generator with its inverse. It is a simple matter to check that these form a subcomplex and that the quotient by this subcomplex is exact. Thus we can effectively ignore such stars and will usually do so in the following. This does not seem to solve the problem completely, however, as we have not excluded stars which consist largely, but not entirely, of such edges. It would be good to have some way of removing the redundancies that appear to be still present.

There is one case in which we can solve most of these problems.

LEMMA 4.2. *Let G be a group with a submonoid M having the property that $M.M^{-1} = G$. Then*

(1) $\mathbb{Z} \otimes_{\mathbb{Z}M} \mathbb{Z}G \cong \mathbb{Z}$;
(2) *$\mathbb{Z}G$ is a flat $\mathbb{Z}M$-module;*
(3) *if $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ is a $\mathbb{Z}M$-free resolution of $\mathbb{Z}$ then $\underline{\underline{P}} \otimes_{\mathbb{Z}M} \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ is a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$.*

The proof is straightforward and is omitted.

4.3. CALCULATION – AND THE BAR RESOLUTION

In the remainder of this section, it will be convenient to adopt a new terminology. If $\alpha = [w; e_1, \ldots, e_k]$ is a critical n-star then we say that w is n-critical. The terminology is not ideal because w does not necessarily determine α but it will save a lot of space.

It is generally straightforward to list all of the generators of the resolution but it is not so straightforward to practically describe the boundary maps. This is not too surprising in that we can not expect the problem of calculating homology to become suddenly easy. We can use the 'cubical' structure to make some progress, however. In the language of this paper the problem is one of determining a covering for the faces of a canonical n-cube associated with a critical n-star. The upper faces are easily dealt with; each such face is either itself associated with a critical $(n-1)$-star or is a product with factors of non-zero dimension and so has zero covering.

We are left with the problem of covering the lower faces of the n-cube. Let μ be a non-degenerate $(n-1)$-cube and let $w = \mu(\underline{0})$. We are going to cover μ with critical $(n-1)$-stars and so we need to 'decompose' μ into the associated $(n-1)$-cubes. For each such cube its 'apex' will include a $(n-1)$-critical word. Thus, in seeking to find a covering of this cube we should look among those words which both include subwords underlying critical $(n-1)$-stars and which are R-reductions of w.

It is not, of course, sufficient to simply add the (appropriately weighted) critical $(n-1)$-stars whose underlying words are R-reductions of w. Consider, for example, the (complete) rewriting system (Σ, R) with

$$\begin{aligned}
\Sigma &= \{\, a, b, c, d, e, f \,\}, \\
R &= \{\, ba \to abf,\ ca \to ace,\ cb \to bcd, \\
&\qquad da \to ad,\ ea \to ae,\ fa \to af,\ db \to bd,\ eb \to be,\ fb \to bf, \\
&\qquad dc \to cd,\ ec \to ce,\ fc \to cf,\ ed \to de,\ fd \to df,\ fe \to ef \,\}
\end{aligned}$$

(derived from a subsemigroup of a free nilpotent group of rank three and class two).

Let $n = 2$. There is a 2-critical word cba. Drawing the ordered graph of all reductions of cba it can be seen that there is more than one choice for a covering for the lower face (path) below $bcda$ but the choice is available because there is a word $abcfed$ which contains a 2-critical subword fed. (There are also two words underlying decomposable stars.) The lower face (path) below $cabf$ has no such words and consequently a unique choice of covering.

If there were no 2-critical subwords in the R-reductions of cba we would know that any covering for a lower face is unique – for otherwise the two alternative choices of 1-critical words would bound 2-cubes corresponding to 2-critical words. This is general.

LEMMA 4.3. *Let μ be a $(n-1)$-cube with $\mu(\underline{0}) = w$. Suppose that, among the subwords of reductions of w, there is no n-critical subword. Suppose also that, the $(n-1)$-critical subwords of reductions occur in the form $b_1, \dots, b_k$ with $u_1 = a_1 b_1 c_1, \dots,$ $u_k = a_k b_k c_k$ reductions of w. Then the covering $C_{n-1}(\mu)$ of μ is unique and is an expression of the form $\varepsilon_1 a_1 \alpha_1 + \cdots + \varepsilon_n a_n \alpha_n$ with $\varepsilon_i \in \{-1, 0, 1\}$ and α_i the critical star corresponding to b_i.*

PROOF. The latter statement is clear – and covered in the preceding discussion. For the uniqueness, suppose that d_1 and d_2 are two distinct possibilities for $C_{n-1}(\mu)$. Then $\delta_{n-1}(d_1 - d_2) = 0$ (by *III* of Section 3.2) and so $d_1 - d_2$ is in the image of δ_n. Thus $d_1 - d_2 = \delta_n(e)$ with $e \leqslant d_1 - d_2$ by *II* of Section 3.2.

Now take a n-critical word u underlying a n-critical star in the support of e. Then u is bounded by some n-critical word corresponding to a n-critical star in the support of $d_1 - d_2$. Thus $u \leqslant w$. Since this contradicts the initial assumptions we have completed the proof.

In using this result, we need to make two choices. Firstly we must choose which $(n-1)$-stars to include in the covering and secondly we must then assign them a sign. It will, in fact, frequently suffice to take the (positive) sum of all such stars but this is not necessarily the case and the choice needs to be checked in each case. We will omit these checks as they involve largely routine computation.

In what follows we will calculate a number of resolutions using these techniques. We will frequently use what is often known as the *standard* rewriting system for a monoid M; that is, we take $\Sigma = M \setminus \{1\}$ and $R = \{ ab \to \overline{ab} : a, b \in \Sigma \}$ where $\overline{ab}$ denotes either the empty word or the element of Σ representing the product of a and b. We begin with the (very familiar) resolution arising from this.

The n-stars have underlying words $m_1 \dots m_{n+1}$ and the i-th edge involves an application of the rule $m_i m_{i+1} \to \overline{m_i m_{i+1}}$. Let μ be such a star with associated cube ρ. Observe that the lower faces of ρ have apex of word length n. As the rules are length reducing, no n-critical words, and only the apex itself among $(n-1)$-critical words can occur amongst the reductions of these apices. Hence the i-th lower face, with apex $m_1 \dots \overline{m_i m_{i+1}} \dots m_{n+1}$ is covered by the $(n-1)$-star with the same underlying word.

The star corresponding to the word $m_1 \dots m_{n+1}$ will be written as $[m_1 | \dots | m_{n+1}]$. (It will be convenient to extend this so that we allow the possibilty that $m_i = 1$ in which case the corresponding star is zero.) Then we have

$$\begin{aligned}
\delta_n([m_1|\dots|m_{n+1}]) &= m_1[m_2|\dots|m_{n+1}] && \text{(from the first upper face)} \\
&\quad + (-1)^{n+1}[m_1|\dots|m_n] && \text{(from the n-th upper face)} \\
&\quad + \sum_{i=1}^{n} (-1)^i [m_1|\dots \overline{m_i m_{i+1}} \dots |m_{n+1}] && \text{(from the lower faces).}
\end{aligned}$$

We thus recover the (normalised) bar resolution for a group or monoid. Had we allowed M, rather than $M \setminus \{1\}$, as the generating set, we would have obtained the unnormalised resolution.

4.4. Free and direct products

If G and H are groups with complete rewriting systems, we can form a complete rewriting system for their free product $G * H$ by combining the generating sets – Σ_G and Σ_H say – and combining the sets of rules. Thus a rule of the combined system involves generators from one only of Σ_G and Σ_H. The same therefore follows for critical n-stars and, of course, their boundaries. Thus the resolution $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ is the direct sum of two resolutions $\underline{\underline{P}}_G \twoheadrightarrow \mathbb{Z}$ and $\underline{\underline{P}}_H \twoheadrightarrow \mathbb{Z}$. Here $P_G \twoheadrightarrow \mathbb{Z}$ – for example – is the tensor product, over $\mathbb{Z}G$ and with $\mathbb{Z}(G * H)$, of the $\mathbb{Z}G$-free resolution corresponding to the rewriting system for G. The usual facts on the homology of $G * H$ (see, for example p. 220 of Hilton and Stammbach [9]) can be recovered easily.

The direct product $G \times H$ is dealt with similarly. This time, however, we must add a set of rules

$$R' = \{ hg \to gh : g \in G, h \in H \}.$$

The n-critical words are thus the juxtapositions of l-critical words in Σ_H (on the left) with m-critical words in Σ_G where $l+m+1=n$. (The extra 1 in the sum comes, of course from the rule which interchanges the right-most letter of the H-word with the left-most letter of the G-word.)

The complex obtained is just the tensor product of the complexes obtained from the individual rewriting systems for G and H.

4.5. Free products with amalgamation

The question of rewriting systems for free products with amalgamation (or for HNN-extensions) given rewriting systems for the factors is rather complicated (see [8] for a special case) and we shall not attempt to discuss anything approaching the general case. There is a simple case, however, which we describe briefly.

Let $K = G *_A H$ be the free product of G and H amalgamating A; we shall regard A as a subgroup of G and H. Let S and T be transversals for A in G and H respectively. Let (Σ_A, R_A) be the standard rewriting system for A. Then there is a natural rewriting system for K of the form

$$\begin{aligned} ab &\to \overline{ab} && (a,b\in\Sigma_A),\\ s_1s_2 &\to a(s_1,s_2)s_{12} && (s_1,s_2,s_{12}\in S, a(s_1,s_2)\in\Sigma_A),\\ t_1t_2 &\to a(t_1,t_2)t_{12} && (t_1,t_2,t_{12}\in T, a(t_1,t_2)\in\Sigma_A),\\ sa &\to b(s,a)u(s,a) && (u(s,a)\in S, b(s,a)\in\Sigma_A),\\ ta &\to b(t,a)u(t,a) && (u(t,a)\in T, b(t,a)\in\Sigma_A). \end{aligned}$$

This leads to a resolution which we leave the reader to make explicit but which can be easily described. Let $\underline{\underline{P}}'_G$ and $\underline{\underline{P}}'_H$ and $\underline{\underline{P}}'_A$ be the complexes obtained by taking the rewriting systems for G, H and A obtained from the above. Thus $(\underline{\underline{P}}'_G)_n$, for example, will be generated by all n-stars with underlying words of the form $s_1\ldots s_k a_{k+1}\ldots a_{n+1}$.

Let $\underline{\underline{P}}_G = \underline{\underline{P}}'_G \bigotimes_{\mathbb{Z}G} \mathbb{Z}K$ and define $\underline{\underline{P}}_H$ and $\underline{\underline{P}}_A$ similarly. Then $\underline{\underline{P}}_G$ and $\underline{\underline{P}}_H$ are complexes of $\mathbb{Z}K$-modules which each have a sub-complex isomorphic to $\underline{\underline{P}}_A$. The $\mathbb{Z}K$-module complex we obtain from the rewriting system above is the direct sum of $\underline{\underline{P}}_G$ and $\underline{\underline{P}}_H$ amalgamating the sub-complex $\underline{\underline{P}}_A$.

This yields a direct, although not particularly elegant, method of deriving the Mayer-Vietoris formula for the homology of an amalgamated free product (cf. Section II.7 of [6].) (Note that it is not easy to give explicit values for the differentials in this discussion – but compare with Section 4.8).

4.6. Free abelian groups

We will, in fact, deal with free abelian monoids. A free abelian group G clearly contains a free abelian monoid M with $MM^{-1} = G$. So we can apply Lemma 4.2 to recover the group case.

Let A be a free abelian monoid with basis Σ. We shall assume Σ to be a totally ordered set. The set of rules R will be the obvious set

$$R = \{\, ba \to ab : a, b \in \Sigma, a < b \,\}.$$

The n-critical words are then the set of all expressions

$$w = a_1 \dots a_{n+1} \qquad \text{with} \qquad a_1 > \dots > a_{n+1}.$$

Denoting by w_i the effect of applying the j-th rule to w we have

$$w_i = a_1 \dots a_{i-1} a_{i+1} a_i a_{i+2} \dots a_{n+1}.$$

Note that the rules do not alter the length of a word; using this and a simple combinatorial argument, it is easy to see that no n-critical word lies among the proper reductions of w. Thus there is a unique form for the covering of the lower faces which we can find by inspecting the $(n-1)$-critical subwords amongst the reductions of the w_i. It is again easy to see that the only such reductions of w_i are

$$a_{i+1}(a_1 \dots a_{i-1} a_i a_{i+2} \dots a_{n+1}) \quad \text{and} \quad (a_1 \dots a_{i-1} a_{i+1} a_{i+2} \dots a_{n+1}) a_i$$

where the $(n-1)$-critical words have been put in parentheses.

Denote the (unique) star underlying the critical word w by $[w]$. We obtain (after the necessary further checks)

$$\begin{aligned}
&\delta_n([a_1 \dots a_{n+1}]) \\
&\quad = a_1 [a_2 \dots a_{n+1}] && \text{(from the first upper face)} \\
&\quad\quad + (-1)^{n+1} [a_1 \dots a_n] && \text{(from the n-th upper face)} \\
&\quad\quad + \sum_{i=1}^{n} (-1)^i \left(a_{i+1} [a_1 \dots a_{i-1} a_i a_{i+2} \dots a_{n+1}] + [a_1 \dots a_{i-1} a_{i+1} a_{i+2} \dots a_{n+1}] \right) \\
& && \text{(from the lower faces)} \\
&\quad = \sum_{i=1}^{n+1} (-1)^{i+1} (a_i - 1) [a_1 \dots \widehat{a_i} \dots a_{n+1}].
\end{aligned}$$

We have obtained the usual complex – often referred to as the *Koszul complex* – for free abelian monoids. (See, for example, Section 6.1 of [3].)

4.7. Finite cyclic groups

The cyclic group case seems the least satisfactory application of this approach.

The approach via 'special stars' (equivalent to that of Anick) yields a complex $\underline{\underline{P}}$ which has one generator in each dimension. More precisely let $G = \langle x : x^k = 1 \rangle$ and let

$$w_n = \begin{cases} x^{ln} & \text{if } k = 2l-1, \\ x^{ln+1} & \text{if } k = 2l. \end{cases}$$

Then w_n is the unique word underlying a special (see Section 2.5) n-star and the subwords involved in the n edges w_n start at positions $1, n, n+1, 2n, \ldots$. The usual formula for the differential (see for example [6]) can then be deduced.

If, however, we take the 'full' approach using all critical n-stars, then we have many more generators. In fact, we have one critical n-star for each $(n-1)$-tuple $(l_1, \ldots, l_{n-1})$ with $0 < l_i < k$. Here the corresponding edges will involve subwords which start in positions $1, 1+l_1, 1+l_1+l_2, \ldots$. A little checking will confirm that we have the normalised bar resolution – but shifted by two dimensions.

It seems likely that this problem – of having an excessive number of generators – will recur for any rewriting system which involves elements of finite order.

4.8. Extensions

Suppose that

$$K \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$$

is an extension of groups and that (Σ_K, R_K) and (Σ_Q, R_Q) are complete rewriting systems for K and Q respectively. Then there is a complete rewriting system (Σ_G, R_G) for G with $\Sigma_G = \Sigma_K \cup \Sigma_Q$. Choose a transversal $\tau\colon Q \to G$ to π – so that $\pi \circ \tau$ is the identity on Q. It will be convenient to regard Σ_Q as a subset of both Q and G so that τ is the identity on Σ_Q.

The rules R_G are then formed from three types:

(1) R_K;
(2) rules of the form $l \to rk$ where $l \to r$ is in R_Q and $\tau(r)^{-1}\tau(l) = k \in \Sigma_K^*$;
(3) rules $ks \to sk'$ where $s \in \Sigma_Q$, $k \in \Sigma_K$, and k' is the R_K-reduced form of the element $s^{-1}ks \in K$.

(See, for example, Groves and Smith [8].)

There is then a corresponding resolution.

We shall elaborate only in the case that (Σ_K, R_K) and (Σ_Q, R_Q) are the standard rewriting systems. We denote the term $\tau(\overline{s_1 s_2})^{-1}\tau(s_1 s_2)$ occurring in rules of type (2) by $u(s_1, s_2)$.

The n-critical words are then the words of the form $w = k_1 \dots k_p s_1 \dots s_q$ with $p + q - 1 = n$. The word obtained by applying the i-th rule is

$$w_i = \begin{cases} k_1 \dots \overline{k_i k_{i+1}} \dots k_p s_1 \dots s_q & \text{if } i < p, \\ k_1 \dots k_{l-1} s_1 k_p^{s_1} s_2 \dots s_q & \text{if } i = p, \\ k_1 \dots k_l s_1 \dots \overline{s_j s_{j+1}} u(s_j, s_{j+1}) \dots s_q & \text{if } i > p,\ j = i - p. \end{cases}$$

(Here $\overline{s_j s_{j+1}}$ denotes $\tau \circ \pi(s_j s_{j+1})$. As usual we shall denote the star corresponding to a word by enclosing the word in brackets).

It is easy to check that no n-critical word occurs in the proper reductions of w. (Note that the length of w is never increased by the application of a rule and is decreased unless $i = p$.) In order to find the diferential we would need to find the $(n-1)$-critical words which occur. The problem seems much harder than in previous examples perhaps because there are two problems that have not occurred previously. Firstly, there is a possibility of an $(n-1)$-critical word occurring in two distinct lower faces and secondly such words may also occur in the cover with a negative sign.

It seems likely, however, that this resolution co-incides with one given by André in [1] and this enables us to make an intelligent guess at the appropriate coverings. The covering of the upper faces is, as usual, immediate and the covering of the i'th lower face when $i < p$ can easily be seen to be $[k_1 \dots \overline{k_i k_{i+1}} \dots k_p s_1 \dots s_q]$.

To describe our guess for the coverings of the other lower faces we turn to André's notation in [1]. To understand the next two paragraphs the reader should refer to [1] and the notation described there. We will cover the faces by stars which correspond to the summands of $\delta_n([k_1 \dots k_p s_1 \dots s_q])$ with $i \geqslant 1$. (These stars will also need to be assigned appropriate signs which we will not make explicit but which can be deduced from [1]).

Each such summand contains an entry which is a conjugate of k_p and we will assign the entry to a face according to the precise nature of this conjugate. If the entry is k_p itself, then the star also contains a unique subword $\overline{s_j s_{j+1}} u(s_j, s_{j+1})$; we assign the star to the covering of the $(p+j)$'th lower face. In general the entry is of the form

$$\rho_\beta^j(s_1, \dots, s_k)^{-1} k_p \rho_\beta^j(s_1, \dots, s_k).$$

Assign the corresponding star to the p'th face if $\beta = \lambda$ and otherwise assign it to the $p + j_{\lambda-1}$'th face.

CONJECTURE. *The above assignment, together with a suitable choice of signs, yields a covering for the faces of the canonical cube associated with* $[k_1 \dots k_p s_1 \dots s_q]$.

We can be more explicit in the case that the extension splits. For then the transversal τ can be assumed to be a homomorphism and $u(s_1, s_2)$ is always trivial. In this case the covering of the faces is easily expressed. Let $\alpha = [k_1 \dots k_p s_1 \dots s_q]$ be a critical n-star. Then

$k_1[k_2 \dots k_p s_1 \dots s_q]$	covers upper face 1,
$[k_1 \dots k_p s_1 \dots s_{q-1}]$	covers upper face n,
$[k_1 \dots \overline{k_i k_{i+1}} \dots k_p s_1 \dots s_q]$	covers lower face i when $i < p$,
$s_1[\overline{k_1^{s_1}} \dots \overline{k_p^{s_1}} s_2 \dots s_q] + [k_1 \dots k_{p-1} s_1 \dots s_q]$	covers lower face p,
$[k_1 \dots k_p s_1 \dots \overline{s_{i-p} s_{i+1-p}} \dots s_q]$	covers lower face i when $i > p$.

It is now straightforward to write down an explicit differential for the resolution. This agrees, after suitable translation of notation, with the split extension case of the resolution given by André.

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