INTERPRETATIONS AND THE MODEL THEORY OF THE CLASSICAL GEOMETRIES*

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The main goal of this article is to show that, for suitable choices of primitives, projective geometry is the model completion of affine geometry. Thus the geometers' predilection for a projective setting is analogous to the preference for working over the algebraic or real closure of an initially given field. Of course, we have to specify and qualify to make our thesis precise, but the choices which work are natural ones in a model theoretic investigation of projective closure.

We start from sufficient conditions for an inverse pair of interpretations to transfer model completeness from one theory to another, (1.2-3), and check whether traditional constructions among geometries and fields satisfy these (2-3.1). For example, a projective plane with collinearity relation is model complete iff the underlying (skew) field is model complete. Then we study the relations of projective closure and hyperplane removal to affine embeddings, with conclusions about existential undefinability of affine parallelism (3.2) and model companions (3.4-9); using a finer, algebraic, characterization of affine embeddings (3.10), we treat amalgamation and model completions (3.9-12). Model companions of affine spaces over ordered fields would have to be projective betweenness spaces, which we study in §4, reducing to questions about ordered fields with additional primitives. A discussion of our choice of primitives in §5 leads into rather puzzling philosophical questions about the semantics of algebraic geometry.

For basic definitions and properties of model completeness, model companions and completions, we refer to Macintyre's Handbook article [M]. We develop the theory for spaces of arbitrary fixed finite dimension $n \geq 2$ over infinite skew (i.e., not necessarily commutative) fields, giving a parallel treatment of the ordered and non-ordered cases. Commutativity isn't missed, but there are some differences between dimension 2 and higher dimensions. For example, collinearity embeddings between n-dimensional spaces need not have n-dimensional image for $n>2$. To recover this desirable feature one must instead take co-hyperplanarity as a primitive (§3.13). This is at variance with usage in the literature on logical foundations of geometry ([T], [Sch]), which has been primarily concerned with questions of axiomatizability and decidability; problems concerning morphisms are more sensitive to choices of primitive notions.

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1. MUTUAL INTERPRETABILITY AND TRANSFER OF MODEL COMPLETENESS.

The mutual interpretabilities between the classical geometries and field theories have not been used to transfer model completeness; indeed, Diller [D] pointed out the failure of model completeness for affine betweenness planes over real-closed fields, and Szczerba [Sz3] denies that transfer of model completeness arises from mutual interpretability, giving a counterexample constructed by P. Tuschik. Nonetheless, most classical geometries are model complete over model complete field theories; and for suitable choices of primitive notions . We give sufficient conditions on a mutual interpretability to guarantee transfer of model completeness; these may be seen to account for model completeness when it occurs among classical geometries.

1.1. To establish notation and a semantic point of view, we recall the notion of interpretation, following e.g. [Sz3], [Sz4]. Given a σ_1-structure $\underline{M}$, satisfying a theory T_{12} to be described, we construct a σ_2-structure $I_{12}(\underline{M})$: (i) The domain is given, for some fixed n, by σ_1-formula $\psi_d(x_1 \ldots x_n)$, and $\exists\bar{x}\psi_d(x)$ is in T_{12}; or rather, by the quotient of the preceding by σ_1-formula $\psi_=(\bar{x},\bar{y})$, with "$\psi_=$ defines an equivalence relation" in T_{12}; or there could be more than one domain of this kind (with $\exists\bar{x}\psi_d(\bar{x})$ in T_{12} for at least one); (ii) Each k-ary symbol R in σ_2 is given an interpretation on the domain(s) by σ_1-formula $\psi_R(\bar{x}_1,\ldots,\bar{x}_k)$; and "$\psi_=$ defines a congruence relation w.r.t. ψ_R" is in T_{12}; (iii) all σ_1-formulas in the above may have extra free variables $\bar{y}$; then the construction, indicated as $I_{12}(\underline{M},\bar{p})$, is determined by $\underline{M}$ together with a choice of parameters $\bar{p}$ from $\underline{M}$ for $\bar{y}$; the admissible parameter choices are given by σ_1-formula $\psi_p(\bar{y})$, with $\exists\bar{y}\psi_p(\bar{y})$ in T_{12}; the preceding requirements on $\underline{M}$ must be relativised to ψ_p in this case, e.g. we want $\forall\bar{y}[\psi_p(\bar{y}) \to \exists\bar{x}\psi_d(\bar{x},\bar{y})]$ in T_{12}.

Thus the various formulas ψ together determine a model theoretic operation

$$I_{12}\colon \mathrm{Mod}_{\sigma_1}(T_{12}) \to \mathrm{Str}(\sigma_2),$$

as well as the familiar syntactic translation operation

$$I^*_{12}\colon \mathrm{Form}(\sigma_2) \to \mathrm{Form}(\sigma_1);$$

for any $\underline{M} \in \mathrm{Mod}_{\sigma_1}(T_{12})$, $\varphi \in \mathrm{Form}(\sigma_2)$ and parameters $\bar{p}$ from $\underline{M}$, these satisfy

$$I_{12}(\underline{M},\bar{p}) \models \varphi \iff (\underline{M},\bar{p}) \models I^*_{12}(\varphi).$$

We need a few elementary observations. If $I_{12}\colon \mathrm{Mod}_{\sigma_1}(T_{12}) \to \mathrm{Str}(\sigma_2)$ and $I_{23}\colon \mathrm{Mod}_{\sigma_2}(T_{23}) \to \mathrm{Str}(\sigma_3)$ are interpretations, so is the composite interpretation

$I_{12} \circ I_{23}$: $\mathrm{Mod}_{\sigma_1}(T_{13}) \to \mathrm{Str}(\sigma_3)$ given by translating I_{23}, e.g.

$$T_{13} = T_{12} \cup \{\forall \bar{y}[\psi_p^{12}(\bar{y}) \to I_{12}^*(T_{23})]\},$$

$$\psi_p^{13}(\bar{y},\bar{z}_1,\dots,\bar{z}_t) =_{\mathrm{def}} \psi_p^{12}(\bar{y}) \wedge I_{12}^*(\psi_p^{23}(z_1,\dots,z_t)), \text{ etc.}$$

If j: $\underline{M} \to \underline{M}'$ is an elementary embedding and I_{12} an interpretation, as above, then for any fixed $\bar{p}$ from $\underline{M}$ we have an induced elementary embedding

$$I_{12}(j,\bar{p}): I_{12}(\underline{M},p) \to I_{12}(\underline{M}',j(\bar{p}))$$

and this action respects composition $j_1 \circ j_2$ and identity map. For $I_{12}(\underline{M},\bar{p})$ is obtained from $(\underline{M},\bar{p})$ by σ_1-definable constructions, so given that j is σ_1-elementary, these constructions act in the same way on (sequences of) elements of $(\underline{M},\bar{p})$ as on their j-images, (sequences of) elements of $(j(\underline{M}),j(\bar{p}))$ within $(\underline{M}',j(\bar{p}))$. If $T_1 \supseteq T_{12}$ is a σ_1-theory, I_{12} is a <u>Δ-interpretation in T_1</u> iff the domain and parameter formulas of I_{12} are equivalent to existential formulas in T_1, and all other formulas of I_{12} are equivalent in T_1 both to an existential and to a universal formula. By the same reasoning as just above, we have that if I_{12} is a Δ interpretation in T_1, then any σ_1-embedding j: $\underline{M} \to \underline{M}'$ of models of T_1 induces a σ_2-embedding $I_{12}(j,\bar{p})$: $I_{12}(\underline{M},\bar{p}) \to I_{12}(\underline{M}',j(\bar{p}))$. The composition of Δ interpretations is a Δ interpretation. As a general notion, if i: $\underline{M} \to \underline{M}'$ and j: $\underline{N} \to \underline{N}'$ are σ-embeddings, i and j are <u>elementarily equivalent embeddings</u> iff $\underline{M}'$ and $\underline{N}'$ are elementarily equivalent in the language σ augmented by a unary predicate designating the image of i resp. j. Now we have

1.2. <u>Theorem</u>. Let I_{12}: $\mathrm{Mod}_{\sigma_1}(T_{12}) \to \mathrm{Str}(\sigma_2)$, I_{21}: $\mathrm{Mod}_{\sigma_2}(T_{21}) \to \mathrm{Str}(\sigma_1)$ be interpretations, $T_1 \supseteq T_{12}$, $T_2 \supseteq T_{21}$ a σ_1 resp. σ_2-theory, such that $I_{12}(\mathrm{Mod}(T_1)) \subseteq \mathrm{Mod}(T_2)$ and T_2 is model complete. Then T_1 is model complete if

(i) I_{12} is a Δ interpretation in T_1, and

(ii) for any embedding j: $\underline{M} \to \underline{M}'$ of models of T_1 which under the composite interpretation $I_{12} \circ I_{21}$: $\mathrm{Str}(\sigma_1) \to \mathrm{Str}(\sigma_1)$ induces an embedding of the images for some choice of parameters from $\underline{M}$, some such embedding is elementarily equivalent to j.

Proof: We must show that any embedding j: $\underline{M} \to \underline{M}'$ of models of T_1 is elementary. As I_{12} is Δ in T_1, it induces a σ_2-embedding j': $I_{12}(\underline{M},\bar{p}) \to I_{12}(\underline{M}',j(\bar{p}))$. As these structures satisfy the model complete T_2, j' is in fact elementary. But then j' induces an elementary embedding under I_{21} (with possibly further parameters from $I_{12}(\underline{M},\bar{p})$, which may be traced back to $\underline{M}$). By (ii), this elementary embedding is elementarily equivalent to j, which therefore must also be elementarily. $\square$

1.3. Note that condition (i) is necessary, (ii) possibly not. In our applications,

(ii) will always arise from a much stronger condition, if it holds at all. Consider

$$(*)\quad \begin{cases} \text{For any } \underline{M} \models T_1\text{, any parameters } \bar{p} \text{ for } I_{12} \text{ from } \underline{M}\text{, there exist parameters } \bar{q} \\ \text{for } I_{21} \text{ from } I_{12}(\underline{M},\bar{p})\text{, tracing back to } \bar{p}' \text{ on } \underline{M} \text{ via } I_{12}\text{, such that} \\ \qquad I_{21}(I_{12}(\underline{M},\bar{p}),\bar{q}) \simeq \underline{M} \\ \text{and such an isomorphism is explicitly definable on } \underline{M} \text{ with parameters} \\ \bar{p}'' \supseteq \bar{p},\bar{p} \text{ by a formula } \psi_{\simeq}(x,\bar{x},\bar{y}'') \text{ independent of } \underline{M},\bar{p}''. \end{cases}$$

This makes sense, as by the definition of interpretation, any isomorphism as in (*) is indeed a relation on $\underline{M}$; by Beth's theorem (*) therefore follows from the existence on $Mod(T_1)$ of at most one such isomorphism of $\underline{M}$ with prescribed behavior on some sufficiently long finite sequence of parameters $\bar{p}''$. But in applications, we can easily write down $\psi_{\simeq}$ directly. When (*) holds, we say that (I_{12},I_{21}) is a <u>definably inverse pair of interpretations (on T_1)</u>.

However, (*) does not entail condition (ii), even given the other hypotheses of theorem 1.2.(2.6.2.). Rather, we must require that $\psi_{\simeq}$ be existential, and give an existential σ_1-formula $\psi_{p''}(\bar{y}'')$ such that

$$(**)\qquad T_1 \models \exists\bar{y}''\ \psi_{p''}(\bar{y}'') \wedge \forall\bar{y}''[\psi_{p''}(\bar{y}'') \to \text{"}\psi_{\simeq}\text{: } \underline{M} \simeq I_{21}(I_{12}(\underline{M},\bar{p}),\bar{q})\text{"}].$$

Then the isomorphism defined by $\psi_{\simeq}$ commutes with the embedding $j\colon \underline{M} \to \underline{M}'$ of models of T_1, which entails (ii); we say that (I_{12},I_{21}) is an <u>$\exists$-definably inverse pair of interpretations on T_1</u>.

These strong definability conditions give a syntactically explicit transfer of model completeness: Assuming the conditions of theorem 1.2, we have, for any σ_2-formula φ

$$T_2 \models I_{21}^*(\varphi) \leftrightarrow \chi_\forall,\chi_\exists$$

by model completeness; as I_{12} maps $Mod(T_1)$ into $Mod(T_2)$, this gives

$$T_1 \models I_{12}^*(I_{21}^*(\varphi)) \leftrightarrow I_{12}^*(\chi_\forall),\ I_{12}^*(\chi_\exists).$$

Because (I_{12},I_{21}) are definably inverse (with parameter formula satisfying **), $\varphi(x_1\ldots x_t)$ is equivalent in T_1 to both of

$$\forall\bar{y}''\ \forall\bar{x}_1\ldots\bar{x}_t[(\psi_{p''}(\bar{y}'') \wedge \bigwedge_i \psi_{\simeq}(x_i,\bar{x}_i,\bar{y}'')) \to I_{12}^*(I_{21}^*(\varphi))],$$

$$\exists\bar{y}''\ \exists\bar{x}_1\ldots\bar{x}_t[\psi_{p''}(\bar{y}'') \wedge \bigwedge_i \psi_{\simeq}(x_i,\bar{x}_i,\bar{y}'') \wedge I_{12}^*(I_{21}^*(\varphi))].$$

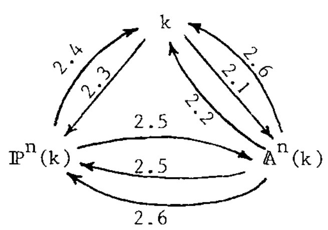
As I_{12} is Δ in T_1 and $\psi_{p''},\psi_{\simeq}$ are existential, substitution of $I_{12}^*(\chi_\forall)$ resp. $I_{12}^*(\chi_\exists)$ in these expressions give a universal resp. existential equivalent of φ in T_1.

1.4. We indicate a generalisation. Let a <u>$\forall_n$-embedding</u> be an embedding preserving

$\forall_n$-formulas, a Δ_n-interpretation in T be one in which domain and parameter formulas are $\exists_n$ and all others have both $\exists_n$ and $\forall_n$ equivalents in T; and let T have prefix dimension $\leqslant n$ iff every $\forall_n$-formula is equivalent to an $\exists_n$-formula in T iff every $\forall_n$-embedding of models of T is elementary. The proof of theorem 2.1 shows: If T_2 has prefix dimension $\leqslant m$, (ii) holds at least for $\forall_m$-embeddings j under $I_{12} \circ I_{21}$, and I_{12} is a Δ_n-interpretation, then T_1 has prefix dimension $\leqslant m+n-1$. Again, the transfer of prefix bound is explicit if this version of (ii) is replaced by (*,**) for $\exists_{n+m-1}$-formulae $\psi_{\simeq}, \psi_{p''}$.

2. SOME CLASSICAL GEOMETRICAL CONSTRUCTIONS.

We review a number of constructions relating rings and classical geometries as indicated in the diagram, to settle terminology and notation, but mainly to obtain the information about syntactic form of the constructions needed for later applications. For a parallel discussion (of Euclidean geometry), giving full detail, see [Sch]§3.25-67. (The reader might prefer to go directly to §3, referring back to the items here as needed.)



Here k is a (not necessarily commutative) field, $\mathbb{A}^n(k)$ affine space, $\mathbb{P}^n(k)$ projective space over k. We give parallel treatments for (a) k in language of rings with 1, called linear case; (b) k in language of ordered rings with 1, called ordered case, i.e. here we deal with additional geometrical primitives to correspond to the order on the field.

2.1. Affine n-space $\mathbb{A}^n(k)$, k (skew) field, in the language $\{0,1,+,\cdot\}$ of rings, $n \geqslant 2$.

2.1.1. Linear case: domain k^n. By quantifier free solvability criteria (Bourbaki, [Bo], §6.10), we have the relations, for $\bar{x}_0, \dots, \bar{x}_{n+1} \in k^n$, and any $m \leqslant n$: $C_m \bar{x}_0, \dots \bar{x}_{m+1}$ iff $\bar{x}_0, \dots \bar{x}_{m+1}$ lie in an m-dimensional linear left subspace.
Of course, $C_0 \bar{x}_0 \bar{x}_1$ iff $\bar{x}_0 = \bar{x}_1$; C_n holds universally, and C_1 is the familiar relation of collinearity. $C_1, \dots, C_{n-1}$ are interdefinable; in particular it will suf-

fice to take C_{n-1} as primitive, for we have, for $m \leqslant n-2$ and distinct $x_0, \ldots x_{m+1}$

$$C_m x_0 \ldots x_{m+1} \quad \text{iff} \quad \forall x_{m+2}\, C_{m+1} x_0 \ldots x_{m+1} x_{m+2}$$
$$\text{iff} \quad \exists yz[\sim C_{m+1} x_0 \ldots x_m yz \wedge C_{m+1} x_0 \ldots x_{m+1} y \wedge C_{m+1} x_0 \ldots x_{m+1} z] .$$

Conversely, there is a <u>positive</u> existential definition of C_{i+1} from C_i for $0 < i < n-1$: let π range over permutations of $\{0, \ldots, i+2\}$;

$$C_{i+1} x_0 \ldots x_{i+2} \quad \text{iff} \quad \exists y \exists \pi\, [C_i\, x_{\pi(0)} \ldots x_{\pi(i)} y \wedge C_1 y x_{\pi(i+1)} x_{\pi(i+2)}];$$

but for $m > 1$, C_m is <u>not</u> definable by a universal formula in $C_1, \ldots, C_{m-1}$ (3.13). Because of this collinearity would not be suited to our purposes as a unique primitive for $n > 2$; unless explicitly noted otherwise, the primitive of $\mathbb{A}^n(k)$ is C_{n-1}. This gives a Δ interpretation in any case.

2.1.2. Occasionally, parallelism is used as a primitive in $\mathbb{A}^n(k)$, cf. [Sz4] for $n = 2$:

$$\bar{x}\bar{y} \parallel \bar{u}\bar{v} \quad \text{iff} \quad (x_i - y_i)(u_j - v_j) = (x_j - y_j)(u_i - v_i) \text{ for each } 1 \leqslant i < j \leqslant n.$$

From this, C_1 is definable

$$C_1 xyz \quad \text{iff} \quad xy \parallel xz,$$

but for $n \geqslant m \geqslant 2$, C_m is not definable from $\parallel$ by a universal formula.
We will often use $\parallel$ as a defined expression, for $n = 2$ deleting C_2,

$$xy \parallel uv \text{ iff } (C_1 xyu \wedge C_1 xyv) \vee \{C_2 xyuv \wedge \forall z[C_2 xyuz \rightarrow \sim(C_1 xyz \wedge C_1 uvz)]\}$$

(coplanar non-intersection); but $\parallel$ is not definable from C_{n-1} by an existential formula (3.2).

2.1.3. Ordered case: k in language of ordered rings $\{0,1,+,\cdot,\leqslant\}$. As primitive, add the ternary betweenness relation,

$$B\bar{x}\bar{y}\bar{z} \quad \text{iff} \quad C_1 \bar{x}\bar{y}\bar{z} \wedge \forall j \leqslant n\ 0 \leqslant x_j - y_j \leqslant x_j - z_j \text{ or } 0 \geqslant x_j - y_j \geqslant x_j - z_j .$$

So this remains a Δ interpretation. For $n = 2$, B may replace C_1 as a primitive. (In the Polish tradition, B is taken as the unique primitive for all n. Cf. [ST], [K]. For us, this is as inadequate as using only C_1 in 2.1.1.)

2.1.4. The theory Th($\{\mathbb{A}^n(k)$: k skew field$\}$) turns out to be the theory of Desarguean affine n-spaces; axioms could be worked out from the axioms of Blumenthal [Blu], Chs. IV, V. (Given for $n = 2$, these are: there is a unique line through any two given points, a unique parallel to a given line through a given point, and a four-point: a quadruple of points no 3 of which are collinear; and two universal axioms which are affine versions of the projective Desargues, cf. [G]§3.1. Adequacy of the axioms is shown by carrying out coordinate ring construction (2.2.1), co-ordinatisation (2.2.4) and verifying that a skew field and an isomorphism arise.) Additional axioms on B in the ordered case: see [HD], pp. 20, 40, 149-150.

2.2. Affine Coordinate ring.

2.2.1. linear case. parameters $0, e_1, e_2$.

$\psi_p(x,y,z)$ iff $\sim C_1xyz$, $\psi_d(x)$ iff C_1x0e_1, $\psi_=(x,y)$ iff $x=y$;

$\psi_0(x)$ iff $x=0$, $\psi_1(x)$ iff $x=e_1$;

$\psi_+(x,y,z)$ iff $\exists u\ xu \parallel 0e_2 \wedge e_2u \parallel 0e_1 \wedge pe_2 \parallel uz$;

$\psi_.(x,y,z)$ iff $\exists u\ xu \parallel e_1e_2 \wedge C_10e_2u \wedge e_2y \parallel uz$.

As the point u in the last two formulas is uniquely determined in Desarguean affine n-spaces by the conditions given, these formulas have universal equivalents. Thus we have a Δ interpretation (in Desarguean affine n-spaces), assuming $\parallel$ as a primitive. Assuming C_{n-1} as primitive, this gives a Δ_2 interpretation. For these spaces, with $n \geqslant 2$ throughout, the construction gives a skew field, see [Bl]. (It is shown there that a minor generalisation of the construction already gives a $\{+,\cdot\}$-algebra without the Desargues axioms; but then most skew field identities may fail.)

2.2.2. ordered case. As above, but we also recover the field order from B, by defining the positive elements

$\psi_P(x)$ iff $B0xe_1 \vee B0e_1x$.

So still a Δ resp. Δ_2 interpretation; applied to an ordered Desarguean affine geometry it gives an ordered skew field.

2.2.3. coordinate isomorphism.
Starting from an ordered or unordered skew field, form affine n-space, and then construct the affine coordinate ring, with parameters $0=\bar{0}$, $e_1=(1,0\ldots0)$, $e_2=(0,1,0\ldots0)$. The resulting ring isomorphism, $x \mapsto (x,0\ldots0)$ is evidently uniformly $\exists$-definable on the original skew field, i.e. we have a $\exists$-definably inverse pair of interpretations. Note: the interpretations themselves are not both Δ!

2.2.4. coordinatisation isomorphism.
Starting from an ordered or unordered Desarguean affine n-space $\underline{A}$, execute the coordinate ring construction and then the affine n-space construction. We uniformly define an isomorphism between $\underline{A}$ and the result, with parameters $0,e_1,\ldots,e_n$ ($0,e_1,e_2$ coinciding with those of the coordinate ring construction) such that $\sim C_{n-1}0e_1\ldots e_n$; $\psi_{\simeq}(x,x_1\ldots x_n,y_0,\ldots,y_n)$ iff $\forall i$ "x_i is obtained from x by success-

ive projections parallel to $y_0y_1,\ldots,y_0y_n$, but not y_0y_i -ultimately giving a point on y_0y_i- followed by a projection to y_0y_1 parallel to y_iy_1."
As the intermediate points in the construction are uniquely determined by quantifier free conditions (in $\|, C_{n-1}, C_1$), we have an $\exists$-definition in terms of $\{C_{n-1}, \|\}$; in terms of C_{n-1} alone, this becomes $\exists_2$. So this is a $\exists$ resp. $\exists_2$-definably inverse pair of interpretations.

2.3. Projective n-space $\mathbb{P}^n(k)$ over (ordered; skew) field k, $n \geqslant 1$.

2.3.1. linear case. Compose the interpretation $k \to \mathbb{A}^{n+1}(k)$ with the following construction $\mathbb{A}^{n+1}(k) \to \mathbb{P}^n(k)$: Domain $\mathbb{A}^{n+1}$, subject to restriction $\psi_d(\bar{x})$ iff $\bar{x} \neq \bar{0}$ and reduction modulo (left) linear equivalence.

$$\psi_=(x,y) \quad \text{iff} \quad \bar{x} \neq 0 \wedge \bar{y} \neq \bar{0} \wedge C_1\bar{0}\bar{x}\bar{y}.$$

As relational primitive: C_{n-1}

$$\psi_{C_{n-1}}(\bar{x}_0,\ldots,\bar{x}_n) \quad \text{iff} \quad C_n\ \bar{x}_0,\ldots,\bar{x}_n,\bar{0}$$

for which $\psi_=$ defines a congruence. This gives a Δ interpretation. The comments under 2.1.1. concerning the relations $C_0,\ldots,C_n$ continue to hold in $\mathbb{P}^n(k)$.

2.3.2. ordered case. We add the quaternary relation of collinear separation S: Sxyuv iff x,y,u,v are collinear, and the pairs (x,y),(u,v) of points separate each other on the line. A line of $\mathbb{P}^n$ is a plane through $\bar{0}$ in $\mathbb{A}^{n+1}$, its points are lines through zero in that plane, for the intersection points of these lines with any line in the plane not through zero or parallel to the four given ones, "Bxuy $\wedge$ Buyx or any cyclic permutation" gives S.
This is a Δ definition, as there are always such intersection points, any such set will work, and all give the same result.

2.3.3. The theory Th($\{\mathbb{P}^n(k)$: k skewfield$\}$) turns out to be the theory of Desarguean projective n-spaces cf. [G]§6.1-2. for axioms in the linear case, and [HD], p. 150 for the order axioms.

2.3.4. These constructions work for any Desarguean affine (n+1)-space $\underline{A}^{n+1}$: Replace $\bar{0}$ by a parameter $p \in \underline{A}^{n+1}$ (as we didn't otherwise refer to the coordinates).

2.4. Projective coordinate ring

2.4.1. linear case.

Parameters $0,e_1,e_2$, u_1,u_2 : $\psi_p(\sim)$ iff "$C_1 0e_1e_2 \wedge \forall i[C_1 0e_iu_i \wedge 0\neq u_i\neq e_i]$; $\psi_d(x)$ iff $C_1 0e_1x \wedge x\neq u_1$. From here on, the construction coincides with the affine coordinate ring construction, replacing everywhere

'$ab \parallel cd$' by "ab and cd have a common intersection with u_1u_2".

Thus the interpretation becomes Δ in C_{n-1}: any such intersection is uniquely determined by a,b,c,d,u_1,u_2, and may hence be introduced interchangeably by universal or existential quantification. For very clear analysis, see Garner [G], §3.3-3.4. The construction gives a skew field iff applied to a Desarguean projective space; the the skew field is unique up to isomorphism; indeed, for fixed $0,e_1,u_1$ the skew field is uniquely determined regardless of the choice of e_2,u_2 (satisfying ψ_p).

2.4.2. ordered case. As above, but recover the skew field order by defining the positive elements by

$$\psi_{P_{os}}(x) \quad \text{iff} \quad S0xe_1u_1 \vee S0e_1xu_1,$$

so again a Δ interpretation in Desarguean projective space.

2.4.3. coordinate ring isomorphism.
Entirely as in the affine case (2.2.3) we obtain an $\exists$-definable ring isomorphism by a suitable choice of parameters in composing projective n-space construction with projective coordinate ring construction, so this pair is $\exists$-definably inverse.

2.4.4. coordinatisation isomorphism.
The construction is described in [G], p. 93 (for $n=2$). The isomorphism itself is a Δ-interpretation; it makes (proj. coord. ring, $\mathbb{P}^n$) into an $\exists$-definably inverse pair.

2.5. <u>Projective closure and hyperplane removal</u>.

2.5.1. projective closure of affine n-space $\mathbb{A}^n = \mathbb{A}^n(k)$.
Adjoining the hyperplane $\mathbb{P}^{n-1}(k)$ of directions in $\mathbb{A}^n(k)$ (2.3.1) to $\mathbb{A}^n$, with as new hyperplanes (i) the $\mathbb{P}^{n-1}$ adjoined and (ii) the hyperplanes of $\mathbb{A}^n$ with the directions of all lines there in adjoined, we obtain a structure which is canonically isomorphic to $\mathbb{P}^n(k)$ as constructed in (2.3.1) from $\mathbb{A}^{n+1}(k)$.

2.5.2. Given $\mathbb{P}^n = \mathbb{P}^n(k)$, we may take the C_{n-1}-substructure obtained by removing any hyperplane from $\mathbb{P}^n$. The result is isomorphic to $\mathbb{A}^n(k)$.

2.5.3. These operations are definably inverse interpretations; the definitions are easily worked out from those of the constructions. If we adjoin a hyperplane to $\mathbb{A}^n$ and then remove a different one, $\psi_{\simeq}$ must describe on $\mathbb{A}^n$ an automorphism of $\mathbb{P}^n$ taking one hyperplane to the other.

2.5.4. The ordered case gives no further difficulties, see [HD], p. 150.

2.6. <u>General affine coordinate ring and coordinatisation.</u>

2.6.1. linear case.
Given a Desarguean affine n-space $\mathbb{A}^n = \mathbb{A}^n(k)$, we could execute the projective coordinate ring construction in the projective closure $\mathbb{P}^n(k)$. If none of the parameters $0,e_1e_2u_1u_2$ lie in the infinite hyperplane, the construction restricts to a Δ interpretation on $\mathbb{A}^n$. This was developed (for the ordered case) by Szczerba [Sz1], [ST]. We do not recover k, for one point is missing in $\mathbb{A}^n(k)$ (the infinite point on $0e_1$). But k can be recovered by then applying the additive analog of the quotient ring construction ("algebra of differences"), a Δ interpretation without parameters.

Szczerba gives a Δ construction ψ of coordinates in $\mathbb{A}^n$, again the restriction of projective coordinatisation, which is an isomorphic embedding of $\mathbb{A}^n$ in $\mathbb{P}^n(k)$, that is, with properties analogous to $\psi_{\simeq}$ in (1.3. *,**), except that the embedding is onto the complement of a hyperplane in $\mathbb{P}^n(k)$; the construction uses a system of parameters $0,e_1,\ldots,e_n, u_1,\ldots,u_n$ as for projective co-ordinatisation. See [ST], p. 166-167 (only $n=2$ is explicitly given). The arguments require that lines contain sufficiently many points; perhaps $\geqslant 5$ would do, we will simply require lines to be infinite.

2.6.2. Using additional parameters which characterise the missing hyperplane, and applying the appropriate projective transformation η (definable over k from these parameters) to the coordinates produced by ψ, we obtain the commuting diagram

$$\begin{tikzcd} \mathbb{A}^n \arrow[r, "2.6.1."] \arrow[rr, bend right, "\eta\circ\psi"'] & k \arrow[r, "2.1.1."] & \mathbb{A}^n(k) \end{tikzcd}$$

which satisfies (1.3. *), and $\eta\cdot\psi$ is an isomorphism defined by a Δ condition. Nonetheless, (1.3. **) may not be satisfied in this way: Given an affine embedding $\mathbb{A}^n(k) \to \mathbb{A}^n(K)$, the parameters needed for η in $\eta\circ\psi(\mathbb{A}^n(K))$ need not belong to $\mathbb{A}^n(k)$ at all; this precludes the existence of an $\exists$-formula $\psi_{\bar{p}''}$ with the properties in (1.3. **). This occurs, for example, for the embedding of (3.3), regardless of whether k,K are model complete.

2.6.3. ordered case.

Using B as in (2.2.2), the general affine coordinate ring construction, gives the underlying ordered (skew) field k. But now Szczerba and Tarski realise a great gain: The field is already obtained from its positive elements, which are the points between 0 and u_1; and all arithmetic structure on these is determined by line constructions (the ordinary projective ones w.r.t. $0, e_1 e_2, u_1 u_2$) which remain entirely within the triangle $0u_1u_2$. So this Δ coordinate ring construction reconstructs k from any convex subset of $\mathbb{A}^n$. (That is, containing any points of $\mathbb{A}^n$ between contained points.) See [ST], §2, or [Sz1]; and also [Sp].

Szczerba defines a theory WGA_n, weak general (n-dimensional desarguean) ordered affine geometry, in which the coordinate ring construction is defined, and in which (the restriction of) projective coordinatisation determines an embedding of any model of WGA_n with coordinate ring k onto a nonempty convex open n-dimensional subset of $\mathbb{P}^n(k)$, preserving C_{n-1} and pairwise collinear separation S (which may be defined from B). We define the <u>general projective closure</u> of a model of WGA_n to be this embedding; it agrees with the ordinary projective closure on $\mathbb{A}^n(k)$. It may be observed that the general projective closure is a Δ interpretation.

Both [ST] and [Sz1] include in WGA_n the extension axiom

$$\forall xy \exists z [z \neq y \wedge Bxyz].$$

But all steps with existential import in general affine coordinate ring and coordinatisation constructions require the determination of intersections (C_1, C_{n-1}) within the closed convex hull of the coordinatisation parameters and point being co-ordinatised; these constructions remain Δ in WGA_n without the extension axiom. What fails, of course, are the openness statements in the representation theorems, but we will have no real use for these. For the reader's convenience, we stick to WGA_n; but it will be understood how the results are to be modified to accommodate the weaker theory.

3. MODEL THEORETIC CONSEQUENCES.

3.1. From this syntactic information about interpretations and inverse pairs of interpretations we have some immediate consequences. Let us set, for any theory T of skew fields, $\mathbb{P}^n(T) = Th(\{\mathbb{P}^n(k) : k \models T\})$, $\mathbb{A}^n(T)$ similarly, where the primitives are C_{n-1}, together with the relevant order primitive in the ordered case. Let $n \geq 2$.

<u>Proposition</u> (i) $\mathbb{P}^n(T)$ is model complete iff T is; (2.4.3)

(ii) $\mathbb{A}^n(T)$ is model complete iff T is, <u>if the quaternary parallelism</u>

relation is added to the language;

(iii) If T is model complete, $\mathbb{A}^n(T)$ has prefix dimension 2. (2.2.4)

Here (iii) follows directly from (ii), as we must simply substitute a universal formula for each occurrence of '$\|$' in the equivalent universal and existential formulas of (ii).

3.2. We show that in any infinite affine n-space, the relation $\|$ has no nonempty existentially definable subset; so (2.2) and 3.1 (ii) - (iii) are best possible. Existential undefinability of $\|$ was already shown by Diller [D] in the ordered commutative case, cf. (3.3).

Suppose in $\underline{A}^n$, ab $\|$ cd and these satisfy $\exists\bar{x}\phi(\bar{x},\bar{y})$, ϕ quantifierfree and $\underline{A}^n \models \forall\bar{y}[\exists\bar{x}\phi(\bar{x},\bar{y}) \to y_1y_2 \| y_3y_4]$. Then in the projective closure $\underline{P}^n$ of $\underline{A}^n$, we have an intersection e of ab with cd, and $\bar{x}$ such that $\phi(\bar{x},abcd)$. By infiniteness of $\underline{A}^n$, we may choose a hyperplane H in $\underline{P}^n$ not containing any of a,b,c,d,e,$\bar{x}$; in the ordered case we also require H to lie outside the convex hull in $\underline{A}^n$ of a,b,c,d,$\bar{x}$. Removing H from $\underline{P}^n$, we obtain $\underline{B}^n \cong \underline{A}^n$ in which ab intersects cd in e, but still $\phi(\bar{x},abcd)$, because a,b,c,d,$\bar{x}$ satisfy the same C_{n-1} and B relations as in $\underline{A}^n$.

□

3.3. Any $\{C_{n-1},B\}$-structure is also a $\{C_{n-1},S\}$-structure, by the definition: Sxyuv iff x,y,u,v are pairwise distinct, collinear, and

$$Bxuy \leftrightarrow (Bxyz \vee Bzxy).$$

An <u>embedding</u> between an ordered affine and projective space (of the same fixed dimension n) will be understood to be a $\{C_{n-1},S\}$ embedding between the affine $\{C_{n-1},S\}$-structure determined in this way and the projective structure. With these understandings in the ordered case, we have

<u>Corollary</u>. Let $n \geqslant 2$ and k an infinite (possibly ordered or skew) field. For any embedding $\mathbb{A}^n(k) \to \mathbb{P}^n(k)$ there is an embedding $\mathbb{P}^n(k) \to \mathbb{A}^n(K)$, for some K elementarily extending k, such that the composition map j: $\mathbb{A}^n(k) \to \mathbb{A}^n(K)$ is a $\{C_{n-1},B\}$-embedding (in the ordered case); no two parallels have parallel images under j.

Proof: The hyperplane removal argument of (3.2) shows that the diagram of $\mathbb{P}^n(k)$, together with the B-diagram on the image of $\mathbb{A}^n(k)$ is consistent with Th $(\mathbb{A}^n(k))$, which by (2.4.4) is $\mathbb{A}^n(\mathrm{Th}(k))$.

□

The argument in fact justifies taking K an ultrapower of k.
But no ring embedding $k \to K$ induces j via the Δ interpretation $\mathbb{A}^n$(2.1.1-3); for then parallelism would be preserved. Further examination of the argument of (3.2) would show how to preserve parallelism in n-i directions and destroy it in the i

remaining (independent) ones. But a logic-free algebraic approach (3.10) gives this, together with tighter control of $k \to K$. Direct algebraic constructions have been given by Szczerba (cf. [Sch] §6, 59-61) and (for the commutative, ordered, plane case) by Diller [D].

3.4. model companions, linear case.

A theory T is model consistent with a theory T^* if every model of T may be embedded in a model of T^*. T^* is the model companion of T if T^* is model complete and mutually model consistent with T.

Theorem. Let $n \geq 2$, and T, T^* theories of infinite skew fields. Then

(a) $\mathbb{P}^n(T)$ is the model companion of $\mathbb{A}^n(T)$ iff T is model complete.

(b) $\mathbb{P}^n(T^*)$ is the model companion of $\mathbb{A}^n(T)$ (and $\mathbb{P}^n(T)$) iff T^* is the model companion of T.

Proof: equivalence of model completeness is (3.1.i), so we need only the equivalences between model consistency statements. These are easily worked out using the constructions of § 2 ; that an embedding $\mathbb{A}^n(k) \to \mathbb{P}^n(K)$ induces an embedding $k \to K$ uses general affine coordinatisation (2.6), or (3.8).

□

3.5. In the ordered case, the claims of theorem (3.4) may only be made by stretching the accepted definition of model consistency: the projective and affine order primitives are different, so we do not really have embeddings. Nor can we force the issue by extending the definition of betweenness to projective closures of affine spaces (putting the point at infinity at an extreme point of each line); for while this can be done consistently with the universal theory of $\{C_{n-1}, B\}$ affine n-space -this is what implicitly happens in the embedding of (3.3)- the resulting structures simply are not model complete.

Still, (3.3) gives more than just mutual model consistency of $\mathbb{P}^n$ and $\mathbb{A}^n$ as $\{C_{n-1}, S\}$-structures: it allows the basic alternating chain arguments derived from mutual model consistency (see [M], §3.3) to be made between projective and affine structures, with the extension chain on the affine side consisting of $\{C_{n-1}, B\}$-embeddings, and B,S consistently related as in (3.3). One alternative in this situation is to name this type of mutual model consistency, intermediate between mutual model consistency in $\{C_{n-1}, S\}$ and in $\{C_{n-1}, B\}$, weak mutual model consistency; and correspondingly to speak of weak model companions and completions. By the familiar alternating chain argument, weak model companions and completions (in language $\{C_{n-1}, S\}$, w.r.t. the definition of S in (3.3)) are unique. We do this for the rest of this section. In particular, the argument of (3.4) gives

Theorem. Let $n \geq 2$, and T, T^* theories of ordered (skew) fields. Then

(a) $\mathbb{P}^n(T)$ is the weak model companion of $\mathbb{A}^n(T)$ iff T is model complete.

(b) $\mathbb{P}^n(T^*)$ is the weak model companion of $\mathbb{A}^n(T)$ iff T^* is the model companion of T.

On the other hand, we see that if there is a model companion of $\mathbb{A}^n$ in the $\{C_{n-1},B\}$ language, it too is weakly mutually model consistent with the model complete $\mathbb{P}^n$'s, and by the alternating chain companion structures would have to be model complete <u>projective</u> spaces. As model companion of $\mathbb{A}^n$'s, they would have to satisfy the extension axiom $\forall xy\,\exists z\neq y\ Bxyz$. Such <u>projective betweenness spaces</u> satisfying the extension axiom were discovered by Szczerba [Sz2] in another context. We study them further in §4.

3.6. Weak general affine geometry. [Sz1]

For any theory T of <u>ordered</u> (skew) fields, let

$$\begin{aligned} WA^n(T) &= Th(\{\underline{A}\models WGA_n;\ \underline{A}\text{ has coordinate ring }k\models T\}) \\ &= WGA_n \cup I^*(T) \end{aligned}$$

where I is the weak general affine coordinate ring construction (2.6); taking parameters into account, $I^*(T)$ abbreviates

$$\{\forall\bar{y}[\psi_p(\bar{y})\rightarrow I^*(\phi)]:\ \phi\in T\}.$$

The arguments of (3.2-5) may be extended to this more general class of theories. First, projective closure may be replaced by general projective closure (2.6): any model $\underline{A}\models WA^n(T)$, say with coordinate ring k, is embedded as a $\{C_{n-1},S\}$-structure in $\mathbb{P}^n(k)\models\mathbb{P}^n(T)$. Second, the supplement for the ordered case in (3.3) may be recovered by a consideration about finitely generated convex subsets of $\mathbb{P}^n$. (We avoid defining these, as we use only very simple properties, and anyhow only need to consider images of finitely generated convex subsets of $\underline{A}$: smallest sets containing the generators and containing any point between two contained points.)

<u>Lemma</u>. Any finitely generated convex subset of $\mathbb{P}^n$ misses a hyperplane of $\mathbb{P}^n$.

Proof: induction on n, trivial for $n=1$. For $n>1$, take any point x not in the subset; (a hyperplane through one of the generating points must contain such x, for if no hyperplane through generator g contains a point not in the set, the generator may be deleted); projecting to $\mathbb{P}^{n-1}$ through x we obtain a convex subset of $\mathbb{P}^{n-1}$ with the same generators. This misses a hyperplane of $\mathbb{P}^{n-1}$, with inverse image a hyperplane of $\mathbb{P}^n$ outside the set. □

<u>Theorem</u>. Let $n\geqslant 2$, and T,T^* be theories of ordered (skew) fields.

(a) $\mathbb{P}^n(T^*)$ is the weak model companion of $WA^n(T^*)$ iff T^* is model complete.

(b) $\mathbb{P}^n(T^*)$ is the weak model companion of $WA^n(T)$ iff T^* is the model companion of T.

3.7. Hyperbolic geometry ([ST], Example 6.3).

For ordered (skew) fields k, $n \geqslant 2$, let $H^n(k)$ be the (result of) the Δ interpretation which gives the restriction of $\mathbb{A}^n(k)$ to the interior of the unit hypersphere; and for theories T of ordered (skew) fields, let $H^n(T)$ be the associated theory of restricted affine planes, defined as in (3.1). These are just the affine reducts of Klein models of hyperbolic geometry over ordered (skew) fields k, and their theories. They satisfy the axioms of WGA_n. Thus Theorem (3.6) extends to $H^n(T), H^n(T^*)$ once we show that $\mathbb{P}^n(T)$ is model consistent with $H^n(T)$ for any theory T of ordered skew fields. But this is a trivial refinement of the ordered case of (3.3): once we have removed the hyperplane from $\mathbb{P}^n(k)$ avoiding the convex hull of the finitely many given points, we may find a hypersphere in $\mathbb{A}^n(k)$ including the given points; so the image of $\mathbb{P}^n(k)$ in $\mathbb{A}^n(K)$ is within a hypersphere of $\mathbb{A}^n(K)$.

In particular, <u>real projective space is the weak model companion of real affine hyperbolic space</u>. But this is a red herring, as the linear (C_{n-1}) structure of $\mathbb{P}^n$ is Euclidean. The proper conclusion is that while the affine structure fully determines the structure of hyperbolic space in terms of first-order definability or invariance under automorphisms, the congruence hyperbolic structure is not robust under affine morphisms, and should be studied with an additional primitive.

3.8. Let an <u>affine space</u> (for implicitly fixed dimension n) be (a) in the linear case, $\mathbb{A}^n(k)$ for some (skew) field k, or (b) in the ordered case, a model of WGA_n, hence necessarily with coordinate ring an ordered (skew) field; in language C_{n-1} resp. $\{C_{n-1}, B\}$.

<u>Theorem</u>. Let $j: \underline{A} \to \underline{A}'$ be an embedding of affine spaces with all lines infinite. Then j extends uniquely to an embedding $\hat{j}: \hat{A} \to \hat{A}'$ of the (weak) projective closures (up to an automorphism of $\underline{\hat{A}}'$ over $\underline{\hat{A}}$).

We originally obtained a self-contained synthetic proof of this result- even for $j: \mathbb{A}^n(k) \to \mathbb{A}^n(k')$ it is not trivial, for if j does not preserve parallelism, affine coordinatisation of $\mathbb{A}^n$ does not commute with j. The theorem has also been shown by Carter and Vogt [CV]. for the linear plane case.

The arguments for $n = 2$ and $n > 2$ are genuinely different, at least in our proof, as $n = 2$ uses converse Desargues and $n > 2$ a projection technique, both of which fail in the other case.

For brevity, we use another approach here, unfortunately not self-contained: The general projective closure (2.6) is a Δ interpretation. Therefore, j simply induces the embedding of general projective closures ! (1.1) (Ordinary projective closure is not Δ; it implicitly determines the location of the hyperplane at infinity.)

3.9. A prime model of a theory is one which is embeddable in all other models of the theory; a prime model of T over a structure $\underline{M}$ is a prime model of $T \cup \text{Diagram}(\underline{M})$. Here all primitives on $\underline{M}$ should be among those of T; if $\underline{M}$ is ordered affine and T is ordered projective, we make the convention that Diagram($\underline{M}$) be taken in the $\{C_{n-1}, S\}$-language following (3.3).

Corollary. Let $n \geqslant 2$; T, T^* theories of infinite (possibly ordered or skew) fields.

(a) $\mathbb{P}^n(k)$ is the prime model of the theory of Desarguean projective spaces over any affine space with (general affine) coordinate ring k.

(b) If there is a prime model of T^* over any model k of T, then (and only then) is there a prime model of $\mathbb{P}^n(T^*)$ over any affine space with coordinate ring a model of T.

Proof: general affine coordinatisation on the affine space $\underline{A}$ agrees with projective coordinatisation on the overlying $\mathbb{P}^n(K)$ and induces the factorisation of any given embedding into $\underline{A} \to \mathbb{P}^n(k) \to \mathbb{P}^n(K)$, where in (b), k is the prime model of T over the coordinate ring of $\underline{A}$. Conversely in (b), $k \to K$ induces $\mathbb{A}^n(k) \to \mathbb{P}^n(K)$, and using the prime projective space $\mathbb{P}^n(K')$ over $\mathbb{A}^n(k)$ in $\mathbb{P}^n(T^*)$ this factors and induces $K' \to K$, so K' is prime over k.

□

The model completion T^* of T is a model companion such that $T^* \cup \text{Diagram}(\underline{M})$ is complete for any $\underline{M} \models T$. In our ordered case, we speak of weak model completion, continuing to take Diagram($\underline{M}$) in language $\{C_{n-1}, S\}$. Thus a (weak) model companion of T with prime models over models of T is the (weak) model completion.

Corollary. n, T, T^* as above.

(a) $\mathbb{P}^n(T)$ is the model completion of $\mathbb{A}^n(T)$ (the weak model completion of $\mathbb{A}^n(T)$ and $W\mathbb{A}^n(T)$) iff T is model complete.

(b) $\mathbb{P}^n(T^*)$ is the model completion of $\mathbb{A}^n(T)$ (the weak completion of $\mathbb{A}^n(T)$ and $W\mathbb{A}^n(T)$) if T^* is the model completion of T and T^* has a prime model over any model of T.

With more detailed analysis (partially avoidable in the linear case) we will eliminate the bothersome restriction in (b). But all examples of model completions of commutative field theories appear to be covered by the present statement. In particular, <u>the theory of $\mathbb{P}^n(\mathbb{C})$ is the model completion of the theory of Pappian affine n-spaces</u> (= $\mathbb{A}^n$ (commutative fields)); in the ordered case, <u>$\mathbb{P}^n(\mathbb{R})$ is the weak model completion of Pappian ordered affine n-spaces</u>, and of Pappian weak general affine n-spaces, also without extension axiom. This last result is quite strong; the topological primitive B allows a local result -the models of $WA^n(\mathbb{R})$ are convex open n-dimensional neighborhoods of $\mathbb{A}^n_n(\mathbb{R})$;- this suggests looking for conditions under which projective closure commutes with glueing in real manifolds, so as to consider model completion phenomena among such topological structures.

3.10. We now study embeddings $\mathbb{A}^n(k) \to \mathbb{A}^n(K)$, $\mathbb{P}^n(k) \to \mathbb{A}^n(K)$ algebraically. Assume first we have $\mathbb{P}^n(K)$ coordinatised, with parameters $0 = (1,\bar{0})$, $e_i = (1:\bar{0}:1_i:\bar{0})$, $u_i = (0:\bar{0}:1_i:\bar{0})$, $i = 1\ldots n$. Suppose we want to fix $(0,\bar{e})$, but move u_i to u_i^*, where $u_i^* = u_i$ or $u_i^* = (1:\bar{0}:\alpha_i:\bar{0})$, $\alpha_i \neq 0,1$. This is accomplished by the map $\eta(\theta_1,\ldots,\theta_n)$:

$$\begin{cases} (\pi_0:\ldots\pi_n) \mapsto (\pi_0 + \Sigma\pi_i\cdot(\theta_i - 1): \ \ldots : \pi_i\theta_i : \ldots), \text{ all } \pi_i \varepsilon k, \\ \theta_i = 1, \text{ if } u_i^* = u_i, \\ \theta_i = \alpha_i(\alpha_i - 1)^{-1} = 1 + (\alpha_i - 1)^{-1}, \text{ if } u_i^* = (1:\bar{0}:\alpha_i:\bar{0}). \end{cases}$$

Here $1,\theta_1,\ldots,\theta_n$ are the eigenvalues associated with eigenvectors $0,e_1\ldots e_n$ of a K-linear transformation of $\mathbb{A}^{n+1}(K)$ which induces η, which is therefore evidently an automorphism of $\mathbb{P}^n(K)$. Conversely, for any $\theta_1\ldots\theta_n \neq 0$ in K one solves for x_i's such that $\eta(\theta_1\ldots\theta_n)$ produces a transformation of the type described initially.

Also, $\eta(\theta_1,\ldots,\theta_n)^{-1} = \eta(\theta_1^{-1},\ldots,\theta_n^{-1})$, and $\theta_i^{-1} - 1 = -\alpha_i^{-1}$.

Now let $j: \underline{A} \to \underline{A}^*$ be an arbitrary embedding of affine n-spaces. Choosing parameters 0, $\bar{e}$ in $\underline{A}$ and forming projective closures $\underline{P}$, $\underline{P}^*$, we obtain projective parameters $0,\bar{e},\bar{u}$, u_i the ideal point on $0e_i$ in $\underline{P}$ (resp. u_i^* in $\underline{P}^*$). We have a diagram

$$\begin{tikzcd}
(\underline{A},0,\bar{e}) \arrow[r, "(2.5)"] \arrow[d, "j"'] & (\underline{P},0,\bar{e},\bar{u}) \arrow[r, dash, "(2.4)"] \arrow[d, "\hat{j}", "(3.8)"'] & (k,0,e_1,+,\cdot) \arrow[r] \arrow[d, "\hat{\hat{j}}", "(2.4{,}1.1)"'] & (\mathbb{P}^n(k),0,\bar{e},\bar{u}) \arrow[d, "\hat{j}", "(2.4.3{,}1.1)"'] \\
(\underline{A}^*,0,\bar{e}) \arrow[r, "(2.5)"] & (\underline{P}^*,0,\bar{e},\hat{j}(\bar{u})) \arrow[r, dash, "(2.4.)"] & (k,0,e_1,+,\cdot) \arrow[r] & (\mathbb{P}^n(k),0,\bar{e},\hat{j}(\bar{u}))
\end{tikzcd}$$

In general, $\hat{j}(\bar{u}) \neq \bar{u}^*(3.3)$. We may describe j as obtained from $(\underline{A},0,\bar{e})$ and an embedding $\hat{\hat{j}}$: $k \to K$, k its (projective) coordinate ring, by removal of a suitable hyperplane, spanned by points $\bar{u}^*$, say with coordinates $u_i^* = (1:\bar{0}:\alpha_i:\bar{0})$ w.r.t. $0,\bar{e},\hat{j}(\bar{u})$ if $u_i^* \neq \hat{j}(u_i)$. If we had an embedding $\underline{P} \to \underline{A}^*$ of a projective n-space $\underline{P}$, the same analysis would hold after picking a full parameter set $0,\bar{e},\bar{u}$ in $\underline{P}$.

<u>Theorem</u>. Let $n \geqslant 2$, $\underline{A} = \mathbb{A}^n(k)$, $\underline{A}^* = \mathbb{A}^n(K)$; k,K infinite (possibly ordered or skew) fields; notation further as above.

1. Relative to a choice $(0,\bar{e})$ of affine parameters, the embeddings $\underline{A} \to \underline{A}^*$ are exactly the maps obtained from embeddings $k \to K$ by removal of the hyperplane $\eta(\theta_1 \ldots \theta_n)[\bar{u}] = \bar{u}^*$ from $\mathbb{P}^n(K)$, for $\alpha_1 \ldots \alpha_n \in K$ such that (writing $k^* = k \setminus 0$)
   (i) linear case: $\sum_{i=1}^{n} p_i(\alpha_i^{-1}) \notin k^*$, all $p_1, \ldots, p_n \in k$;
   (ii) ordered case: the θ_i all belong to the k-monad of 1 in K (i.e., no θ_i is bounded away from 1 in K by an element of k),
   up to automorphisms induced by those of K over the given embedding.
2. An embedding $\underline{A} \to \underline{A}^*$ extends to $\mathbb{P}^n(k) \to \underline{A}^*$ iff also
   (iii) $1, \alpha_1^{-1}, \ldots \alpha_n^{-1}$ are left k-linearly independent.

Proof: (i) given $\underline{A} \to \mathbb{P}^n(k) \to \mathbb{P}^n(K)$, the hyperplane $H(\bar{u}^*)$ spanned by $\bar{u}^* = \eta[\bar{u}]$ misses $\underline{A}$ iff $\eta^{-1}[\underline{A}]$ misses $H(\bar{u})$ iff for all $p_1 \ldots p_n \varepsilon k, p_o \varepsilon k^*$, $p_o - \sum_{i=1}^{n} p_i \alpha_i^{-1} \neq o$ (because $\theta_i^{-1} - 1 = \alpha_i^{-1}$). (ii) In the ordered case, we obtain a B-embedding iff no points of $\underline{A}$ are separated by $H(\bar{u})$ and $H(\bar{u}^*)$ iff each line of $\underline{A}$ intersects $H(\bar{u}^*)$ at a k-infinite parameter value $\lambda \varepsilon K$ (in the formula for $\lambda - 1$ in 4.2, with $p_o \neq o$) iff all α_i^- are k-infinite iff all θ_i are in the monad of 1 in K. (iii) For $(\underline{P}, 0, \bar{e}, \bar{u}) \to \underline{A}^*$, $H(\bar{u}) \cap \underline{P}$ should also have η^{-1}-image missing $H(\bar{u})$. η^{-1} maps $(0: \ldots p_i \ldots)$ to $H(\bar{u})$ iff $\sum_{i=1}^{n} p_i \alpha_i^{-1} = o$; given (i), this fails iff $\sum p_i \alpha_i^{-1} \notin k$ iff (iii) holds. $\square$

For example, we have $\mathbb{P}^i(\mathbb{R}) \to \mathbb{A}^i(\mathbb{C})$, $i = 1,2$, but $\mathbb{P}^3(k) \to \mathbb{A}^3(K)$ requires an extension of degree at least 3. For the unordered plane case, a complete analysis of these embeddings including the exceptional cases for small finite planes is given by Carter and Vogt [CV].

3.11. A theory T has <u>amalgamation</u> if any two extensions of a model of T to models of T have a common extension to a model of T such that the embeddings commute. Suppose we are given an $\exists$-definably inverse pair of interpretations (1.3) on T_1, <u>both</u> of which are Δ interpretations, in T_1 resp. $T_2 = Th(I_{12}(Mod(T_1)))$. Then T_1 has amalgamation iff T_2 has amalgamation; for we may convert the given extensions in (say) T_1 to extensions in T_2; amalgamate there, and the amalgamating extensions induce T_1 extensions, which by $\exists$-definably inverseness serve to solve the original amalgamation problem. This gives all but $(1,2 \Rightarrow 3)$ of

<u>Theorem</u>. For a theory T of (ordered; skew) fields, $n \geq 2$, the following are equivalent:

1. T has amalgamation

2. $\mathbb{P}^n(T)$ has amalgamation

3. $\mathbb{A}^n(T)$ has amalgamation.

Proof: (1) $\Rightarrow$ (3). Given embeddings of affine spaces, say

$$\mathbb{A}^n(k) \to \mathbb{A}^n(k_1),\ \mathbb{A}^n(k) \to \mathbb{A}^n(k_2),$$

we may amalgamate the field extensions induced by taking the projective closures (3.8) of the given affine embeddings, and a system of coordinatisation parameters $0, e_1, \ldots e_n$ from $\mathbb{A}^n(k)$, $u_1 \ldots, u_n$ from $\mathbb{P}^n(k)$, the points at infinity on $0e_1, \ldots, 0e_n$. By (3.10), the given affine embeddings arise from the canonical ones over the field extensions by† automorphisms determined by $\Theta^1 = (\Theta^1_1, \ldots \Theta^1_n)$ resp. Θ^2, satisfying (3.10.i) or (3.10.ii), w.r.t. $k \to k_1$ resp. $k \to k_2$. Making sure that the amalgamating field K contains an element t

(i) in neither k_1 nor k_2, and

(ii) in the ordered case, in the k_1 and the k_2-monad of 1 in K,

both of which may certainly be accomplished by moving to an elementary extension of the amalgamating field if necessary, and embedding $\mathbb{A}^n(k_1) \to \mathbb{A}^n(K)$ resp. $\mathbb{A}^n(k_2) \to \mathbb{A}^n(K)$ by the canonical embedding followed by† the automorphism determined by $((\Theta^1_1)^{-1}t, \ldots, (\Theta^1_n)^{-1}t)$, resp. $((\Theta^2_1)^{-1}t, \ldots, (\Theta^2_n)^{-1}t)$, the amalgamation is accomplished by (3.10. i,ii). □

†suppression of the hyperplane determined by

3.12. If T* is the model companion of T, then T* is the model completion of T iff T has amalgamation, as one verifies easily from mutual model consistency; and <u>weak</u> mutual model consistency (3.5) in our ordered case gives exactly what is needed for the "weak" analog of this argument (with T in language $\{C_{n-1}, B\}$ for amalgamation). Thus (3.9) may be completed:

<u>Corollary</u>. n,T,T* as in (3.9).

<u>$\mathbb{P}^n(T^*)$ is the (weak) model completion of $\mathbb{A}^n(T)$ iff T* is the model completion of T.</u>

Wheeler [W] has shown that for a <u>universal</u> theory of (ordered or unordered) commutative fields with amalgamation, existential closure coincides with real resp. algebraic closure; therefore these cases are already covered by (3.9). It would be interesting to have an example of a model completion of some theory of non commutative skew fields; especially one where the model completion does not have prime models over the models of the original theory. (The first type probably arises by the Δ interpretations associated with finite dimensional algebras over commutative fields; the second type could require a genuinely new example.) While these results on model completions may apply in relatively few cases, certainly the discussion of model companions (3.4) is widely applicable; Macintyre, McKenna and van den Dries [MMD] quote a result from McKenna's thesis, that there are very many model complete theories of fields.

Van den Dries has asked whether, say in the result that projective planes over algebraically closed commutative fields are the model completion of affine planes over commutative fields, the latter theory could be weakened. The most attractive candidate would of course be the universal part of that theory. But a model completion of a universal theory has elimination of quantifiers ([M], p. 155), and $\mathbb{P}^n$ (alg. closed) does not, in this language. In the ordered case, one does a bit better, getting most results for weak general affine geometry.

3.13. We show that in any infinite Desarguean affine or projective n-space, the relation C_{i+1}, $i+1<n$, has no nonempty subset universally definable in $C_i,\ldots C_1$, so that we must indeed take C_{n-1} as a primitive if model completeness is to be preserved. The argument, for $i=1$, was used in the affine case by Kordos [K] to show non $\forall\exists$-axiomatisability of the lower dimension axiom for $n>2$.

First consider $\mathbb{P}^n(k)$, and suppose $\sim C_{i+1}\, a_0 \ldots a_{i+2}$, and these satisfy $\exists\bar{x}\,\phi(\bar{x}\bar{y})$, ϕ quantifierfree, and $\mathbb{P}^n \models \forall\bar{y}[\exists\bar{x}\,\phi(\bar{x}\bar{y}) \to \sim C_{i+1}\,\bar{y}]$, ϕ in $C_i,\ldots C_1$, or in the ordered case, $C_i,\ldots C_1$ and S. By infiniteness, we may choose a point $p \in \mathbb{P}^n$ which lies neither (i) on a line through any two of $a_0,\ldots a_{i+2}$, $x_1\ldots,x_t$; nor (ii) on an (n-1) hyperplane spanned by n of these points. Then projection through p gives a $\{C_{n-2},S\}$-homomorphism $\mathbb{P}^n(k) \to \mathbb{P}^{n-1}(k)$ which is injective on the given points $a_0,\ldots,x_t$. Proceeding inductively, we repeat this until we obtain a $\{C_i,S\}$-homomorphism on $\mathbb{P}^{i+1}(k)$, then embed as a subspace in $\mathbb{P}^n(k)$. For the images $\hat{\bar{x}},\hat{\bar{a}}$, we have

$$\phi(\hat{\bar{x}},\hat{\bar{a}}) \wedge C_{i+1}\,\hat{\bar{a}}.$$

a contradiction.

Next, for $\mathbb{A}^n(k)$, we embed in $\mathbb{P}^n(k)$ and proceed as above, removing a hyperplane avoiding the given points in $\mathbb{P}^{i+1}(k)$ to get the image in $\mathbb{A}^{i+1}(k)$. In the ordered case, we must make all our choices (p, and the hyperplane to be removed) so as to avoid the image of the convex hull in $\mathbb{A}^n(k)$ of the given points; this is conveniently done using the fact that some hyperplane in projective space always avoids a finitely generated convex set (3.6); p can be chosen in such a hyperplane, and in the end it may be removed.

3.14. Note that (3.11) applies to skew fields, for they amalgamate. [C]

The alternating chain intended in (3.5), referred to in §4: all $k_i \models T^*$.

$$\begin{tikzcd} \mathbb{A}^n(k_1) \arrow[r, "{C_{n-1},S}"] \arrow[rr, bend right, "{C_{n-1},B}"'] & \mathbb{P}^n(k_1) \arrow[r, "{C_{n-1},S}"] & \mathbb{A}^n(k_2) \arrow[r, "{C_{n-1},S}"] \arrow[rr, bend right, "{C_{n-1},B}"'] & \mathbb{P}^n(k_2) \arrow[r] & \ldots \end{tikzcd}$$

4. PROJECTIVE BETWEENNESS SPACES.

The alternating chains argument suggested in (3.5) would show that the model companions of affine ordered spaces $\mathbb{A}^n(T)$ in the language $\{C_{n-1},B\}$ would have to contain at least, for some model complete theory of ordered skew fields T^*,

(i) $\mathbb{P}^n(T^*)$, in language $\{C_{n-1},S\}$,

(ii) The $\forall\exists$-part of $\mathbb{A}^n(T^*)$, in language $\{C_{n-1},B\}$,

and the definition of S in terms of B; by that argument, all this is mutually model consistent with $\mathbb{A}^n(T)$.

4.1. In order to investigate model completeness of this theory, we develop the representation theory for <u>projective betweenness n-spaces</u>, defined by

4.1.1. Desarguean projective n-space axioms (C_{n-1} only),

4.1.2. $\forall xyz\ C_1xyz \leftrightarrow Bxyz \vee Byzx \vee Bzxy$,

4.1.3. order axioms for B:

$\forall xy\ Bxyx \to x=y$, $\forall xyzu\ Bxyz \wedge Byzu \wedge y\neq z \to Bxyu$,

$\forall xyzu\ Bxyz \wedge Bxyu \wedge x\neq y \to Byzu \vee Byuz$,

4.1.4. Pasch, which by 4.1.1. may be written

$\forall xyzuvw\ Bxyz \wedge Buvy \wedge C_1wvz \wedge C_1xwu \to Bwvz \wedge Bxwu$,

4.1.5. $\forall xy\exists z\ Bxyz \wedge z\neq y$ (extension axiom).

C_1 is an abbreviation if $n>2$. 4.1.1-4 together with the definition of S from B (3.3) entail the theory of Desarguean projective n-space in $\{C_{n-1},S\}$. Verifying the ordinary $\forall\exists$-form of Pasch ([Sz1]A5), one sees that 4.1.1-5 entail the axioms of WGA_n and its Euclidean strengthening WEA_n (op. cit.).

4.2. If $\underline{A}$ is an affine space (3.8), $\underline{A} \models WGA_n$ (even without extension axiom), with coordinate ring k, then there is a $\{C_{n-1},B\}$-embedding $\underline{A} \to \mathbb{A}^n(K)$ for some K elementarily extending k by (3.6). Thus if $\underline{A} = \underline{P}^n_B$ is a projective betweenness space, we may regard $\underline{A}$ as $\mathbb{P}^n(k)$ endowed with a B-relation by the $\{C_{n-1},-S\}$-embedding $\mathbb{P}^n(k) \to \mathbb{P}^n(K)$ followed by removal of a hyperplane H, as in (3.10).

Choosing a set of parameters $(0,\bar{e},\bar{u})$ in $\underline{A}$, the lines $0e_iu_i$ intersect H in points $(1:\bar{0}:\alpha_i:\bar{0})$, $\alpha_i \in K\setminus k$. Defining Θ_i from α_i as in (3.10), we have (3.10.iii), that $1,\alpha_1^{-1},\ldots\alpha_n^{-1}$ must be left k-linearly independent.

However, the extension axiom for $\underline{P}^n_B$ requires now that no point in $\underline{P}^n_B$ be the last point in $\underline{P}^n_B$ on any line in $\underline{P}^n_B$. Choose a pair of points $p,r \in \mathbb{P}^n(k)$, $p \notin k.r$ i.e., distinct, say

$p = (p_0:\ldots p_i\ldots)$, $p_0 = \sum_{i>0} p_i$ (so $p \in H(\bar{e})$, hyperplane through $\bar{e}$),

$r = (0 : \ldots r_i \ldots)$ so $r \in H(\bar{u})$,

and some canonical normalisation fixing the ratio $p_i : r_i$. Then we parametrize the line pr of $\underline{P}_B^n$ in $\mathbb{P}^n(K)$ as

$$x = p + (\lambda-1)r \quad , \lambda-1 \in \mathbb{P}^1(K)$$

with H in particular given by

$$x_i = e_i + (\alpha_i - 1)u_i \qquad i = 1 \ldots n,$$

and the line pr intersects H for

$$\lambda-1 = (p_0 - \Sigma p_i \alpha_i^{-1}) \cdot (\Sigma r_i \alpha_i^{-1})^{-1},$$

which is defined and not in k by k-linear independence. The extension axiom is satisfied on pr iff $\mathbb{P}'(k)$ contains no point closest to $\lambda-1$.

This holds for all pr iff

(i) α_i is <u>archimedean</u>, $i = 1 \ldots n$ $(\exists \gamma_1 \gamma_2 \in k: 0 < \gamma_1 < |\alpha_i| < \gamma_2)$, and

(ii) for any $q = (q_0 : \ldots : q_n) \in \mathbb{P}^n(k)$,

$q_0 + \Sigma q_i \alpha_i^{-1}$ is bounded away from 0 by an element of k.

We abbreviate the latter condition as: $1, \alpha_1 \ldots \alpha_n$ are <u>strongly left k-linearly</u> independent. Violation of (i) violates the extension axiom on $e_i u_i$; given (i), violation of (ii) for $q = p$ or $q = r$ violates the axiom on pr; conversely, if (i,ii) hold, and $\ell \in \mathbb{P}'(k)$ is closest to $\lambda-1$, then $\ell \neq 0, \infty$, and $\Sigma r_i \alpha_i^{-1}$ is archimedean, $\ell \sim \lambda-1$ $(\forall \varepsilon > 0$ in $k \, |\ell - (\lambda-1)| < \varepsilon)$, so

$$\ell \cdot \Sigma r_i \alpha_i^{-1} \sim p_0 - \Sigma p_i \alpha_i^{-1}$$

$$0 \sim p_0 - \Sigma(\ell r_i + p_i)\alpha_i^{-1}, \text{ contradicting (ii).}$$

4.3. Conversely, let an extension $k \to K$ be given, with $\alpha_1 \ldots \alpha_n \in K$ satisfying (4.2. i,ii); define $\mathbb{P}_B^n(k, \alpha_1 \ldots \alpha_n)$ as $\mathbb{P}^n(k)$ endowed with the B-relation induced by the maps $\mathbb{P}^n(k) \to \mathbb{A}^n(K) \to \mathbb{P}^n(K)$ obtained from the canonical embedding $\mathbb{P}^n(k) \to \mathbb{P}^n(K)$ by removal of the hyperplane H determined by $\alpha_1 \ldots \alpha_n$ as in (4.2). First (4.2. ii) guarantees that H misses $\mathbb{P}^n(k)$; then this does make $\mathbb{P}^n(k)$ into a $\{C_{n-1}, B\}$-structure. As a $\{C_{n-1}, B\}$-substructure of $\mathbb{A}^n(K)$, it satisfies the universal axioms (4.1.2-4). As argued in (4.2), conditions (i,ii) now guarantee the extension axiom (4.1.5).

The models of WGA_n (resp. WGA_n without extension axiom) are the n-dimensional open B-convex (resp. n-dimensional B-convex) $\{C_{n-1}, B\}$-substructures of suitable $\underline{P}_B^n$, by the initial argument of 4.2, valid in this generality, together with the corresponding result of [Sz1] for $\{C_{n-1}, S\}$-embeddings in $\mathbb{P}^n(k)$. So the same analysis applies to these theories, replacing (4.2i,ii) in the case without extension axiom by the weaker (3.10. iii)

(iii) $1, \alpha_1^{-1} \ldots \alpha_n^{-1}$ are left-k-linearly independent.

Theorem. Let $n \geqslant 2$; k,K ordered skew fields.

(1) The projective betweenness n-spaces are exactly the structures $\mathbb{P}^n_B(k,\alpha_1\ldots\alpha_n)$ obtained as above from $k \to K, \alpha_1\ldots\alpha_n$ satisfying (4.2. i,ii).

(2) The models of WGA_n are exactly the n-dimensional B-convex open substructures of these.

(3) The models of WGA_n without extension axiom are exactly the n-dimensional B-convex substructures of $\mathbb{P}^n_B(k,\alpha_1\ldots\alpha_n)$ obtained as above from $k \to K, \alpha_1\ldots\alpha_n$ satisfying (iii).

Parts (1),(2) follow closely the representation theorems [Sz2, Thm. 3.1], [Sz1, Thm. 8]; however, the statements and proofs of these results are incorrect in that the author substitutes the weaker (iii) for (4.2. ii), inferring the extension axiom from (iii) by the lemma [Sz2. 1.1] that k-rational combinations of archimedean elements are archimedean. (This fails for differences.)

4.4. The representation theorem does not give the type of connection between skew fields and spaces which allows transfer of model completeness following §1. Nor could there be such a connection, for the coordinate rings of projective betweenness spaces do not form a first-order (EC_Δ) class: $\mathbb{R}$ cannot be the coordinate ring of a projective betweenness space (4.5), but by an alternating chain

$$\mathbb{A}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R}) \to \mathbb{A}^n(K_1) \to \mathbb{P}^n(K_1) \to \mathbb{A}^n(K_2)\ldots$$

we obtain a projective betweenness space with coordinate ring an elementary extension of $\mathbb{R}$.

The point is that projective betweenness spaces are correlated with ordered skew fields with extra structure; for example, after coordinatisation, the betweenness relation itself corresponds to some 3n-ary relation $\underline{R}_B$ on the coordinate ring. This gives a Δ construction of ordered skew fields-with-R_B from projective betweenness spaces such that the $\psi_{\simeq}$ of projective coordinatisation makes an $\exists$-definably inverse pair. (The construction of projective betweenness space from ordered skew field with R_B is evident.) Hence the model complete projective betweenness spaces correspond to the model complete skew fields-with-R_B satisfying the relevant conditions: the R_B-translations of (4.1.2-4.1.5).

4.5. This seems too complicated for useful analysis, and so we extract an additional axiom from our original goal of characterising model companions of theories $\mathbb{A}^n(T)$. In this case, we may replace R_B by something simpler, the cuts $\kappa_1\ldots\kappa_n$ in k of $\alpha_1\ldots\alpha_n$ from (4.2).

Given $\underline{P}^n_B$, we may choose the parameters $0,\bar{e},\bar{n}$ such that $B0e_iu_i$. Then the Δ coordinate ring construction may be extended to give $(k,\kappa_1\ldots\kappa_n)$, by the Δ

definitions

$$a<\kappa_i \quad \text{iff} \quad B0u_i(1:\bar{0}:a:\bar{0}), \qquad (*)$$

where the coordinate term is eliminable in favor of a Δ description in terms of the Δ formula $\psi_{\simeq}$. However, in general the structure $(K,\kappa_1 \ldots \kappa_n)$ does <u>not</u> determine the B relation of $\underline{P}_B^n$.

From this definition of κ_i we see that by the extension axiom (4.1.5) the κ_i are <u>k-finite</u>, <u>open</u> cuts: They partition k into two nonempty intervals, both of which are open $(-\infty,\kappa_i),(\kappa_i,\infty)$. Szczerba points out by this argument that no such structure has coordinate ring $k=\mathbb{R}$. If $\underline{P}_B^n$ satisfies the model companion of an $\mathbb{A}^n(T)$, the cuts κ_i are also <u>Dedekind</u>: their breadth is no greater than (hence, equals) that of the cut at 0 in k, i.e. $\{y-x\colon x<\kappa_i<y\}$ is downward cofinal in (0,1] of k. (Perhaps any model complete projective betweenness space has all such B-definable cuts Dedekind ?) To see this, consider the alternating chain from mutual model consistency with $\mathbb{A}^n(T)$:

$$\ldots \to (\underline{P}_B^n(k_i),0,\bar{e},\bar{u}) \to \mathbb{A}^n(k_i') \to \underline{P}_B^n(k_{i+1}) \to \ldots$$

The images of $(0,\bar{e},\bar{u})$ in $\mathbb{A}^n(k_i')$ give rise to cuts $\kappa_1 \ldots \kappa_n$ in k_i' by $(*)$; but these cuts simply indicate the position w.r.t. $0,\bar{e},\bar{u}$ of the infinite hyperplane in $\mathbb{P}^n(k_i')$, and hence are not open in k_i'. But then they must be Dedekind. Taking unions of chains,

$$\cup\,\underline{P}_B^n(k_i) \simeq \cup\,\mathbb{A}^n(k_i') \models \text{all } \kappa_i \text{ are Dedekind w.r.t. } 0,\bar{e},\bar{u},$$

as this statement is $\forall\exists$ and true at each i. By model completeness, the statement holds of $\underline{P}_B^n(k_0)$, which could have been chosen arbitrarily.

But if the cuts $\bar{\kappa}=\kappa_1 \ldots \kappa_n$ in k are Dedekind, there is at most one way to make $\mathbb{P}^n(k)$ into a projective betweenness space with the κ_i given by $(*)$. For if there were two distinct such B relations on $\mathbb{P}^n(k)$, say

$$B \text{ given by } \alpha_1' \ldots \alpha_n' \text{ in } K',\ k\to K',\ \alpha_i' \text{ in } \kappa_i\ \forall i$$

$$B' \text{ given by } \alpha_1'' \ldots \alpha_n'' \text{ in } K'',\ k\to K'',\ \alpha_i'' \text{ in } \kappa_i\ \forall i$$

then without loss of generality we may assume that $k\to K'$ is an elementary extension, and therefore we may obtain a common extension $k\to K$ containing the α_i' and the α_i''. But now each $\alpha_i'-\alpha_i''\in K$ is k-infinitesimal because κ_i is Dedekind. Consider any line pr of $\mathbb{P}^n(k)$ as in (4.2). As both $\Sigma r_i(\alpha_i')^{-1}$ and $\Sigma r_i(\alpha_i'')^{-1}$ are k-finite nonzero, the parameter difference of the intersection points of H' resp. H'' with pr

$$(\lambda'-1)-(\lambda''-1) = (p_0-\Sigma p_i(\alpha_i')^{-1})(\Sigma r_i(\alpha_i')^{-1})^{-1}) - (p_0-\Sigma p_i(\alpha_i'')^{-1})\cdot(\Sigma r_i(\alpha_i'')^{-1})^{-1}$$

is k-infinitesimal; but both $\lambda'-1$ and $\lambda''-1$ differ from any value in k by a k-finite amount by the extension axiom, and so give the same B. Note that this last argument also shows that if the cuts $\kappa_1 \ldots \kappa_n$ determined in a projective betweenness space w.r.t. one set $0, \bar{e}, \bar{u}$ of parameters are Dedekind, this will hold w.r.t. any other set $0, \bar{e}', \bar{u}'$ as well; thus we can speak of Dedekind projective betweenness spaces. This condition is expressable as an $\forall\exists$ statement in $\{C_{n-1}, B\}$.

This argument for uniqueness of projective betweenness spaces over k with given Dedekind cuts $\bar{\kappa}$ also shows that for Dedekind projective betweenness spaces with or without the extension axiom, strong left k-linear independence of $1, \alpha_1 \ldots \alpha_n$ in $\kappa_1 \ldots \kappa_n$ in K extending k is a property of $\kappa_1 \ldots \kappa_n$ in k; all extensions $k \to K$ and $\alpha_i \in K$ equally give dependence or independence in this sense. We say that $1, \kappa_1 \ldots \kappa_n$ are k-linearly independent cuts iff for any $(q_0 : \ldots : q_n) \in \mathbb{P}^n(k)$ the linear form $q_0 + \Sigma q_i x_i^{-1}$ is uniformly bounded away from 0 in some k-finite neighborhood of $(\kappa_1, \ldots, \kappa_n)$ in k^n; more formally iff k satisfies

$$\forall q_0 \ldots q_n \neq (0 \ldots 0)\ \exists\delta\ \exists x_i^- < \kappa_i < x_i^+ < 0,\ i = 1 \ldots n,$$

$$\forall i \forall y_i \in \{x_i^-, x_i^+\}\ q_0 + \Sigma q_i y_i^{-1} \text{ is bounded away from zero by } \delta.$$

This condition clearly entails the extension axiom in any structure $\underline{P}_B^n$ giving $(k, \bar{\kappa})$ which satisfies (4.1.1-4.1.4); if the κ_i are all Dedekind it is also necessary for the extension axiom, for if it were to fail for $\bar{\kappa}, q_0 \ldots q_n$, then some extension $k \to K$ has α_i in κ_i, $i = 1 \ldots n$ such that $q_0 + \Sigma q_i \alpha_i^{-1} = 0$, and then this linear form must be k-infinitesimal or zero for any $\alpha_1 \ldots \alpha_n$ in $\bar{\kappa}$ in any K extending k.

Augment the language of ordered rings by unary predicates for the cuts $\kappa_1 \ldots \kappa_n$; we continue to write these as '$x < \kappa_i$'. The theory of Dedekind $\bar{\kappa}$-independently ordered (skew) fields is the theory of ordered (skew) fields together with the statements that the κ_i are k-finite open k-linearly independent Dedekind cuts. By (4.3), projective betweenness spaces are obtained from such $(K, \bar{\kappa})$ by expansion by a suitable relation R_B. We now see that there is exactly one such expansion of a Dedekind $\bar{\kappa}$-independently ordered skew field, and so by Beth's theorem R_B is explicitly first-order definable in this theory. The diagram

$$(k, \bar{\kappa}) \to (k', \bar{\kappa}') \to (K, \alpha_1 \ldots \alpha_n);\ \alpha_i \text{ in } \kappa_i' \text{ so in } \kappa_i,$$

where $(K, \bar{\kappa}) \to (K', \bar{\kappa}')$ is an extension of Dedekind $\bar{\kappa}$-independently ordered skew fields, then shows that this definition has both an $\exists$ and a $\forall$ equivalent in this theory. (The definition could be worked out explicitly from the formula for the parameter value λ of $\mathrm{pr} \cap H$ in (4.2), but we won't need it.) Augmenting the Δ interpretation $\mathbb{P}^n(k)$ by this definition, we have a Δ interpretation $\mathbb{P}_B^n(k, \bar{\kappa})$ constructing Dedekind projective betweenness spaces from Dedekind $\bar{\kappa}$-independently ordered skew fields. This and the Δ interpretation $(k,\bar{\kappa})$ of projective coordinate ring construction augmented by (*) form an $\exists$-definably inverse pair in either order w.r.t. the appropriate projective $\psi_{\simeq}$ (2.4.2-3).

For any theory T of ordered (skew) fields, let $T_D = T +$ "Dedekind $\bar{\kappa}$-independently ordered". For any theory T' of Dedekind $\bar{\kappa}$-independently ordered skew fields, let $\mathbb{P}^n_B(T') = Th(\{\mathbb{P}^n_B(k,\bar{\kappa}) : (k,\bar{\kappa}) \models T'\})$. By alternating chains between $\mathbb{P}^n(T)$ and $\mathbb{A}^n(T)$ with limits giving projective betweenness spaces over the limit fields, $\mathbb{A}^n(T)$ is mutually model consistent with $\mathbb{P}^n_B(T_D)$, and for T_1, T_2 skew field theories, $(T_1)_D$ is model consistent with $(T_2)_D$ iff T_1 is model consistent with T_2. Together with the conclusions underlined above, this gives

Theorem.

1. The Dedekind projective betweenness spaces are exactly those obtained as $\mathbb{P}^n_B(k,\bar{\kappa})$ from Dedekind $\bar{\kappa}$-independently ordered skew fields $(k,\bar{\kappa})$.
2. $\mathbb{P}^n_B(k,\bar{\kappa})$ is model complete iff $(k,\bar{\kappa})$ is, for any Dedekind $\bar{\kappa}$-independently ordered skew fields $(k,\bar{\kappa})$.
3. $\mathbb{P}^n_B(T')$ is the model companion of $\mathbb{A}^n(T)$ iff T' is the model companion of T_D.
4. If $T \mapsto T^*$ is the Kaiser Hull operator, and T a theory of ordered skew fields, $(T^*)_D \subseteq (T_D)^*$.

Perhaps equality holds in (4.). In particular, one would conjecture that if T is the theory of real-closed fields, T_D is model complete. One should also be able to tell which real-closed fields can be expanded to models of T_D. By alternating chains between $\mathbb{P}^n(T)$ and $\mathbb{A}^n(T)$, this holds at least for any real-closed field $\underline{M} = \cup \underline{M}_i$ which can be obtained as a union of an increasing chain of extensions such that $\underline{M}_{i+1}$ contains $\underline{M}_i$-infinitesimals, i.e. $\underline{M}$ has an infinite decreasing chain of Archimedean classes. Finally, does one obtain prime models–over given affine spaces $\mathbb{A}^n(k)$–of the model companion of $\mathbb{A}^n(T)$ (assuming that it exists)?

5. DISCUSSION: THE ROLE OF PROJECTIVE CLOSURE.

Justifying our geometrical primitives leads to philosophical questions. Our remarks below, exceedingly exploratory in character, are intended to set the stage not for mathematical advance, but for philosophical catching up: The traditional semantic theory implicit in classical languages and Tarski semantics, which fit so closely with pure algebra (as in Robinson's work) appears unsuited for an "intrinsic" presentation of algebraic geometry (except via the artifice of first-order set theory).

An investigation of classical geometries which is so very sensitive to choices of primitive notions in a formal description as the preceding may seem surprising, or even somewhat suspect. I surmise that such intuitions rest on the notion, which goes back to the influence of Klein's "Erlanger Programm", that geometrical properties are intrinsic and in no way depend on choices of primitives or description.

Of course such views, while illuminating, are not unassailable; limitations of Klein's automorphism group based classification program have long been understood; foundations of geometry include a host of different conceptions, each illuminating a distinctive realm of geometrical investigation. From the logician's point of view, one would want to add: (1) definability classification gives positive information, e.g. on constructibility from given primitives, which it is in the very nature of Klein's method to ignore; (2) experience in model theory, as in algebra, shows the importance of morphisms between structures. And morphisms bring out definability distinctions, as was recognized in model theory from the very beginning, and comes out above most starkly in (3.7): automorphisms of hyperbolic planes may be characterisable as collinearity automorphisms, but collinearity <u>embeddings</u> lose the metric structure. Just so for affine parallelism structure.

Our particular choice of morphisms -hence of primitive notions- was made in order to obtain a formal counterpart to the commonplace intuition of the naturalness of projective geometry and projective closure - the desirability of considering projective versions of geometrical objects obtained in other contexts as a first step in classification and understanding of the original objects. To give just two random examples: (i) The headings of the motivational sections of [Cℓ]: "§2. Real Projective Space - the Unifier. §3. Complex Projective Space - the Great Unifier." (ii) From the introduction to [Seg]: "I shall show how these results can be completed and given a simple form, when the Galois spaces are considered from the projective point of view instead of the affine one." (p. 129). A similar phenomenon has long been recognised in the movement to the algebraic closure (real closure) of fields in (real) algebraic geometry; Robinson developed the notion of model completion as a formal model for these relationships. In the body of the paper, we have seen how to formally assimilate projective closure to these other cases. Now we ask, in a more philosophical and speculative mood, to what extent these formal features of the situation may be taken to account for the mathematical usefulness of moving to a projective setting.

5.1. The <u>homogenisation</u> of equations arising in projective closure -both $X^2+Y^2 = 1$ and $X^2-Y^2 = 1$ become $U^2+V^2 = W^2$- is a process of existential closure (adjoining solutions) which leads to unification - hyperbolas are circles intersecting the line at infinity in two points. The further unification in the complex case alluded to in the headings of [Cℓ] is similarly due to adjoining solutions in the algebraic closure. So this fits rather nicely, but in fact points out an inadequacy in the concept of model completion: Homogenisation gives unification regardless of whether the ground field is model complete. So we would want a concept of <u>relative model completion</u>: a closure operation which eliminates additional quantifier complexity of definable sets contributed by a construction [say $\mathbb{A}^n(\cdot)$] regardless of the base field. This is what our arguments show of projective closure.

5.2. What appears to go far deeper is that projective closure, at least to $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{P}^n(\mathbb{R})$, is a <u>compactification</u>. Compactness is a crucial property in the study of geometric structures by analytic methods. Consideration of examples quickly shows that the power of the method of compactification in geometric problems does not lie in compactification per se —e.g., in the point-set topological sense— but rather in the choice of a compactification intimately related to the geometric structures at hand. Indeed, one would expect something like this from a naive point of view, as study of a compactification (that is, a new structure) appears unlikely to give information about the original structure unless some mechanism for transfer of information back to that structure is available. It is therefore an interesting <u>logical</u> problem to give a formal description of such compactifications which would illuminate how they function in geometrical reasoning. Here we consider the correlation between such compactifications and existential closure. Such connections are especially intriguing because existential closure is a first-order phenomenon whereas compactness is restricted to topological fields such as $\mathbb{R}$ and $\mathbb{C}$.

In classical geometry, as considered in the body of the paper, such compactifications typically have to do with extending the action of a group which becomes the automorphism group of the resulting structure. This is brought out very clearly in [Bℓa] §1-3, first for projective transformations by extension of continuous bijective collinearity-preserving maps between affine open regions, and then for circle geometries of Möbius, Laguerre and Lie. (This last, largest group requires a two-sheeted cover of the euclidean plane circles, ramified in the circles of radius 0 and one ideal point.) Another such "classical" example is the conformal compactification of Minkowski space-time (e.g., [We], p. 38). Immensely more subtle are the compactification problems in modern algebraic geometry, such as those for moduli spaces. The question of what objects (singular curves) to add in order to obtain a parametric definition of all nonsingular curves of a given type as a projective variety now becomes very intricate, see [Mu 1]; [Mu 2], p. 182. Again a group action plays a crucial, though somewhat different role in the construction, in which compactification is eventually achieved as closure to a projective variety.

The "logical" point is, that in each case, one is extending an algebraic action of an algebraic group, which from our point of view might be read as: a group of maps whose graphs are explicitly definable on the geometrical space by a formula with parameters. This is at least clear in the case of $\mathbb{A}^n \to \mathbb{P}^n$, where we can coordinatise explicitly by $\psi_{\underset{\sim}{}}$, and obtain a Δ definition with parameters giving the graph of all projective transformations. It follows that points witout image or preimage satisfy a universal formula, and adding ideal points to function as such images or preimages is an obvious case of existential closure. So far, it remains mysterious, (a) how to deal with multiply-sheeted coverings instead of simple adjunctions of points, (b) what to say about definability on nontrivial algebraic varieties (e.g. moduli spaces) where no intrinsic primitives are in evidence. We return to the last point in (6.4); but one is at least tempted to speculate at this point that something akin to existential closure is involved in these compactifications, and that some kind of definability control from below, as in construction of a prime model over a given model, plays a role in the transferability of information about the compactification to the original structure.

5.3. The algebraic geometers in fact have an algebraic (as opposed to topological) analysis of the effect of "projective closure" of varieties. Just as the Hausdorff property of varieties over $\mathbb{C}$ is analogous to the condition that the diagonal map

$$(id_V, id_V) \,:\, V \to V \times V$$

be closed in the Zariski topology (where closed sets are solution sets of polynomial systems, at least locally), compactness has an algebraic analogue: V is complete iff any projection map

$$V \times W \to W$$

is closed (w.r.t. the Zariski topology). Now compactification may be compared with suitable conservative embedding in a complete variety; as all projective varieties are complete, projective closure will constitute such an embedding. Catch: once we have an embedding as a suitably definable (quasi projective) subset of some $\mathbb{P}^n$.

From a logical point of view, it is not clear how to evaluate this. The algebraic character of the notion of completeness leads one to suspect it is close to first-order. Indeed, the proof of completeness of projective varieties recently given by Van den Dries [VdD] makes explicit that a logical property is involved, namely a positive quantifier elimination - but in a language based of field primitives rather than geometric primitives. Also, the Lie geometry example (or Mumford's observation, that well-behaved moduli spaces can only be obtained for "polarised" abelian varieties, [Mu 2] p. 97) suggests that a general model theoretic understanding of projective closure requires a more structured relationship than simple embedding, e.g., capturing a notion of many-sheeted coverings extending a given definable group action.

.4. This brings us back to the original conceptual problems of this discussion -the choice of geometrical primitive notions. In the context of algebraic geometry, which is certainly an interesting and most important one, this problem appears quite intractable in traditional model theoretic terms, if one is serious about geometrical notions. In part, the difficulty seems parallel to one encountered in philosophy of physics. For on the one hand, the geometers appear convinced that they are dealing and must deal with intrinsically and inherently geometrical notions- just as the physicist must be taken to be studying an intrinsically physical world ultimately independent of mathematical objects, mathematical quantities and functions. On the other hand, just as the physicist has no other formulation of his theory except in terms of such mathematical objects, the algebraic geometer actually studies objects defined in terms of polynomial rings over fields and entities derived from these. To avoid making assertions which depend on these geometrically non-intrinsic objects, or specific embeddings of varieties in an ambient space such as $\mathbb{A}^n(k)$, $\mathbb{P}^n(k)$ one studies equivalence classes of varieties under some notion of isomorphism (birational equivalence, proper birational equivalence) and attempts to discover structural invariants w.r.t. the equivalence relation. This contrasts sharply with the simple cases of classical geometries dealt with in earlier sections, where the intrinsic geometrical structure is spelled out in advance by the explicit choice of primitives.

This raises the critical question -just as for physics- whether the intuition that one is studying intrinsic geometric objects which are independent of representation used to study them can be fully justified. A simple-minded (but ambitious) way to tackle this is to try to spell out intrinsic geometrical primitives on specific varieties just as for $\mathbb{A}^n$ and $\mathbb{P}^n$ (which after all are just the most trivial algebraic varieties.) This is not to be confused with defining a given variety V in $\mathbb{P}^n$ in the language of $\mathbb{P}^n$ -this is simple, we just translate the algebraic definition of V back via the coordinate ring interpretation; but gives us V as embedded in $\mathbb{P}^n$, which fixes quantities such as the degree of homogeneous defining polynomials, which are not intrinsic geometric structure (birationally invariant) of V.

Given a variety V of dimension $n \geqslant 2$, let us say a surface, taken as a point set in some $\mathbb{P}^n(k)$, we now look for some primitive geometrical relations on V. Experience in algebraic geometry seems to suggest that we should look at curves on V; perhaps select one or more algebraic families and look at their intersection behavior. Thus one would start with primitives such as

$$R^i x_0 \ldots x_{n_i} : x_0 \text{ lies on the curve determined by its points } x_1 \ldots x_{n_i}$$
$$\text{(among the i-th family of curves)}$$

This gives us one or more incidence relations. We would like to recover the base field k from this (provided that V contains enough points over k), and some sort of coordinatisation isomorphism. It is not clear how one should accomplish these things, as the intersection behavior of our families of curves could be quite complicated. Presumably, one might need additional primitives ? Nor is it clear how the families of curves should be selected in the first place, and to what extent this could be done uniformly for different V. In higher dimensions, one would expect to look at families of subvarieties of codimension 1; rather more complicated behavior is to be anticipated.

Regardless of the outcome of such attempts -they do not seem altogether without hope- one may have one's doubts about the semantic analysis of the objects of algebraic geometry provided thereby. The modern language of schemes reflects, via the mechanism of base change, a view of geometrical objects as <u>functors</u> rather than solution sets of equations over fields, much as one might view projective geometry as the construction $\mathbb{P}^n$ of (2.3) rather than $\mathbb{P}^n(\mathbb{C})$ or even all $\mathbb{P}^n(k)$ together with their embeddings. The difference hardly comes out as long as one is interested in a single fixed rich base field such as $\mathbb{R}$ or $\mathbb{C}$, as in classical algebraic geometry.

One difference between the conceptions becomes visible when one considers our embeddings $\mathbb{P}^n(k) \to \mathbb{A}^n(K)$: this makes perfect sense to a classical model theorist, but looks very strange to an algebraic geometer. If varieties such as $\mathbb{P}^n$ and $\mathbb{A}^n$ are functors, and embeddings relations between functors, then embeddings will mean: of varieties over the same base field, uniformly depending on that base field. As Jan Denef pointed out, there are "intrinsic geometric reasons" why one could not have an embedding $\mathbb{P}^n \to \mathbb{A}^n$: Over $\mathbb{A}^n$, there is a large ring of regular functions, whereas over $\mathbb{P}^n$ there are very few. If we had an embedding -functorial and always over the same base field; but this is automatically understood- the regular functions would restrict. Contradiction.

5.5. In summary, we seem to have at least the following difficulties. (i) While the formalism of modern algebraic geometry is perfectly clear mathematically, we do not see how to design a well-adapted "logic" (language, with accompanying semantic theory) which would give a descriptively close fit with the mathematical concepts in question--unless, again, indirectly via formalisation in some general framework such as set theory. And if we are really forced to modify our semantic conceptions, the philosopher (!) wants to understand the nature of the change, and what has forced it. Why must even innovative attempts to use Tarski semantics, say with unobvious but geometrically intrinsic primitives, break down in describing modern algebraic geometry? Here the philosopher seeks his own, critical, understanding of what the geometer has grasped. (ii) To what extent does the formal notion of model completion capture what is important in general about moving to projective settings?

As I hope the above will have brought out, these questions have significant mathematical components (though the necessary definitions are unavailable). Their motivation is largely non-mathematical.

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ADDENDA

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It is consistent with the description in Zentralblatt that Sperner's early work [SP] already contains results on general affine coordinatisation, for which we have referred to [ST]. We have not been able to consult [SP].