9. Examples on $\mathcal{S}_{g,1}$

In this chapter we describe $\mathcal{P}_g$ (or $\mathcal{P}_{g,\eta}$) and $\mathcal{P}'_g$ for some low values of g and any characteristic p, and use the results to study the structure of the locus $\mathcal{S}_{g,1}$ of principally polarized abelian varieties of dimension g.

9.1. Example, $g = 1$.

When $g = 1$, the set

$$\mathcal{S}_{1,1}(\bar{\mathbb{F}}_p) \subset \mathcal{A}_{1,1} \otimes \bar{\mathbb{F}}_p \cong \mathbb{A}^1 \tag{9.1.1}$$

is the set of supersingular j-invariants. We write

$$h_p := \#(\mathcal{S}_{1,1}(\bar{\mathbb{F}}_p)). \tag{9.1.2}$$

This number equals the class number of $B = Q_{\infty,p}$ (see (1.2.5)), which is equal to

$$h_p = \frac{p-1}{12} + \{1 - (\frac{-3}{p})\}/3 + \{1 - (\frac{-4}{p})\}/4 \tag{9.1.3}$$

(cf. [9, p. 200] and [29, p. 312]), as was proved by Deuring (using a class number computation by Eichler), and later proved along different lines by Igusa, see [29, p. 312]. Explicitly: $h_2 = h_3 = 1$ and for $p \geq 5$,

$$h_p = [\frac{p-1}{12}] + \begin{cases} 0 & p \equiv 1 \pmod{12}, \\ 1 & p \equiv 5 \text{ or } 7 \pmod{12}, \\ 2 & p \equiv 11 \pmod{12}. \end{cases} \tag{9.1.4}$$

This can also be expressed by the mass formula:

$$\sum \frac{1}{\#(\mathrm{Aut}(C))} = \frac{p-1}{24}, \tag{9.1.5}$$

where the summation is over all isomorphism classes of supersingular elliptic curves C over $\bar{\mathbb{F}}_p$.

9.2. Example, $g = 2$.

For $g = 2$, an FTQ over k is of the form

$$\rho_1 : E^2 \otimes k \cong Y_1 \to Y_0, \quad \ker(\rho_1) \cong \alpha_p. \tag{9.2.1}$$

Such an FTQ is automatically rigid. For any η satisfying (3.6.1) (i.e. $\ker(\eta) = E^2[F] \otimes k$), (9.2.1) is automatically a PFTQ with respect to η, hence

$$\mathcal{P}_{2,\eta} \cong \mathcal{P}'_{2,\eta} \cong \mathbf{P}^1 \tag{9.2.2}$$

(see Example 3.8). The number of irreducible components of $\mathcal{S}_{2,1} \otimes k$ is equal to $H_2(1,p)$ (see [35, Theorem 5.7]). This number was explicitly calculated by Hashimoto and Ibukiyama (see [25, p.696]). It is equal to 1 when $p = 2, 3$ or 5, and when $p > 5$,

$$\begin{aligned} H_2(1,p) =&(p^2-1)/2880 + (p+1)\Big(1 - \Big(\frac{-1}{p}\Big)\Big)/64 \\ &+ 5(p-1)\Big(1 + \Big(\frac{-1}{p}\Big)\Big)/192 + (p+1)\Big(1 - \Big(\frac{-3}{p}\Big)\Big)/72 \\ &+ (p-1)\Big(1 + \Big(\frac{-3}{p}\Big)\Big)/36 \\ &+ \begin{cases} 2/5 & \text{if } p \equiv 2 \text{ or } 3 \pmod 5 \\ 0 & \text{if } p \equiv 1 \text{ or } 4 \pmod 5 \end{cases} \\ &+ \begin{cases} 1/4 & \text{if } p \equiv 3 \text{ or } 5 \pmod 8 \\ 0 & \text{if } p \equiv 1 \text{ or } 7 \pmod 8 \end{cases} \\ &+ \begin{cases} 1/6 & \text{if } p \equiv 5 \pmod{12} \\ 0 & \text{if } p \equiv 1,\ 7 \text{ or } 11 \pmod{12} \end{cases} \end{aligned} \tag{9.2.3}$$

where $\left(\frac{q}{p}\right)$ denotes the Legendre symbol.

Let η be a polarization of $E^2 \otimes k$ such that $\ker(\eta) = E^2[F] \otimes k$. Then $G_\eta = \mathrm{Aut}(E^2 \otimes k, \eta)/\{\pm 1\}$ is isomorphic to one of the following groups:

$$\{1\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ S_3,\ A_4,\ D_{12},\ S_4,\ A_5. \tag{9.2.4}$$

Let $W_\eta \subset \mathcal{S}_{2,1}$ be the irreducible component corresponding to η (i.e. the closure of $\Psi(\mathcal{P}'_{2,\eta})$, see (4.2.1)). Then the action of G_η on $\mathcal{P}_{2,\eta}$ is generically free, and we have

$$\mathbb{P}^1 \cong \mathcal{P}_{2,\eta} \to \mathcal{P}_{2,\eta}/G_\eta \cong \tilde{W}_\eta \to W_\eta, \tag{9.2.5}$$

where $\tilde{W}_\eta$ is the normalization of W_η (cf. [35, Section 7, 8.1]). By [32, Theorem 7.1] we see that those in (9.2.4) are exactly the groups which do appear in this way.

Conclusion. Let Λ be a set of representatives of equivalence classes of polarizations η of $E^2 \otimes \bar{\mathbb{F}}_p$ satisfying $\ker(\eta) = E^2[F] \otimes \bar{\mathbb{F}}_p$. Then there is a one to one correspondence ψ between Λ and the set of irreducible components of $\mathcal{S}_{2,1} \otimes \bar{\mathbb{F}}_p$. Denote by W_η the irreducible component corresponding to η under ψ. The normalization of W_η is isomorphic to $\mathcal{P}_{2,\eta}/G_\eta$, where $\mathcal{P}_{2,\eta} \cong \mathbb{P}^1$ and $G_\eta = \mathrm{Aut}(E^2 \otimes \bar{\mathbb{F}}_p, \eta)/\{\pm 1\}$. We have $\#(\Lambda) = H_2(1,p)$ and

$$\mathcal{S}_{2,1} \otimes \bar{\mathbb{F}}_p = \bigcup_{\eta \in \Lambda} W_\eta. \tag{9.2.6}$$

9.3. Calculation via the truncation morphisms.

When $g > 2$, we proceed as follows. Let $\mathcal{V}_m$ be the fine moduli scheme of the category $\mathfrak{V}_m$ of truncated PFTQs $\{S; Y_i, \eta_i(m \leq i < g); \rho_i(m < i < g)\}$. (This moduli scheme exists by the same argument as that in Lemma 3.7.) Then we can calculate $\mathcal{V}_m$'s inductively. First we note the following two facts:

i) $\mathcal{V}_{g-2}$ is easy to calculate: To give a Y_{g-2} from $Y_{g-1} = E^g \times S$ is equivalent to giving a flat subgroup scheme $G \subset \alpha_p^g \times S$ of α-rank $g-1$ such that condition ii) in Definition 3.9 holds. This is then equivalent to choosing a section $(x_1, ..., x_g)$ of the α-sheaf of $\alpha_p \times S$ such that the following $[(g-1)/2]$ equations are satisfied:

$$\sum_i x_i^{p^{g-2j}+1} = 0 \quad (0 < j < g/2) \tag{9.3.1}$$

when g is odd, and

$$\sum_{i \leq g/2} (x_i x_{g-i}^{p^{g-2j}} - x_{g-i} x_i^{p^{g-2j}}) = 0 \quad (0 < j < g/2) \tag{9.3.2}$$

when g is even.

ii) It is also easy to determine $\mathcal{V}_0 = \mathcal{P}_{g,\eta}$ from $\mathcal{V}_1$: Since $G = \ker(Y_1 \to Y_1^t)$ is a self-dual α-group of α-rank 2, every flat subgroup scheme of G of α-rank 1 is isotropic. Hence to give a Y_0 is equivalent to giving a flat quotient of rank 1 of the α-sheaf of G. Therefore $\mathcal{V}_0$ is a $\mathbb{P}^1$-bundle over $\mathcal{V}_1$.

Remark. From (9.3.1) and (9.3.2) we see that $\mathcal{V}_{g-2}$ is singular (at a point where all $x_i \in \mathbb{F}_{p^2}$) when $g \geq 5$. Hence there is in general no hope to prove the smoothness of $\mathcal{P}'_g$ over $\mathbb{F}_{p^2}$ using the factorization $\mathcal{V}_0 \to ... \to \mathcal{V}_{g-1}$. Therefore we will use another factorization to prove Proposition 4.3.i) (see 11.3 and 9.7).

By the proof of Lemma 7.11, the truncation morphism $\mathcal{P}'_g \to \mathcal{V}_{g-2}$ is an epimorphism. Hence we have:

Proposition. *The subscheme $T_g \subset \mathbb{P}^{g-1}$ defined by the homogeneous equations in (9.3.1) (when g is odd) or (9.3.2) (when g is even) is irreducible of dimension $[g/2]$. Furthermore, a geometric point $(a_1, ..., a_g) \in T_g$ is non-singular iff the $\mathbb{F}_{p^2}$-linear space generated by $a_1, ..., a_g$ has dimension $\geq [(g-1)/2]$ over $\mathbb{F}_{p^2}$.*

For the second statement, by taking differentials, it reduces to an application of Fact 5.8.

9.4. Example, $g = 3$.

Let

$$E^3 \otimes k = Y_2 \xrightarrow{\rho_2} Y_1 \to Y_0 \tag{9.4.1}$$

be a PFTQ with respect to η, where η satisfies

$$\ker(\eta : E^3 \otimes k \to (E^3 \otimes k)^t) = E^3[p] \otimes k. \tag{9.4.2}$$

Note that

$$(\alpha_p^2 \cong \ker(\rho_2) \subset E^3[F]) \in \mathrm{Grass}_{2,3} \cong \mathbb{P}^2 \tag{9.4.3}$$

and that $\mathfrak{V}_1$ is represented by the Fermat curve:

$$\rho_2 \in \mathcal{V}_1 = \mathcal{Z}(X^{p+1} + Y^{p+1} + Z^{p+1}) \subset \mathbb{P}^2 \tag{9.4.4}$$

(see (9.3.2)) and a flat subgroup scheme $H \subset \alpha_p^3 \times \mathcal{V}_1$. The α-sheaf of $H_1 = \alpha_p^3 \times \mathcal{V}_1/H$ is isomorphic to the subsheaf of $O_{\mathcal{V}_1}^{\oplus 3}$ consisting of sections (a, b, c) such that $(a : b : c) = (X : Y : Z)$, hence it is isomorphic to $O_{\mathcal{V}_1}(-1)$.

Let

$$G = \ker(Y_1 \to Y_1^t) = \ker(E^3 \times \mathcal{V}_1/H \to (E^3 \times \mathcal{V}_1/H)^t). \tag{9.4.5}$$

Then G is an α-group of α-rank 2. Note that η induces an isomorphism $G \cong G^t$. Hence we have $G/H_1 \cong H_1^t$, whose α-sheaf is therefore isomorphic to $O_{\mathcal{V}_1}(1)$.

Let $\mathcal{F}$ be the α-sheaf of G. Then $\mathcal{F}$ is an extension of $O_{\mathcal{V}_1}(-1)$ by $O_{\mathcal{V}_1}(1)$. Since the structure sheaf $\mathcal{E}$ of $\ker(\eta) \times \mathcal{V}_1$ is trivial, the α-sheaf of $\alpha_p^3 \times \mathcal{V}_1$ can be lifted to a subsheaf of $\mathcal{E}$. Hence the α-sheaf of H_1, identified as a subsheaf of the α-sheaf of $\alpha_p^3 \times \mathcal{V}_1$, can also be lifted to a subsheaf of $\mathcal{E}$. Since $\mathcal{F} \cong \omega_{G/\mathcal{V}_1}$, we see that $\mathcal{F} \to O_{\mathcal{V}_1}(-1)$ has a section and hence

$$\mathcal{F} \cong O_{\mathcal{V}_1}(-1) \oplus O_{\mathcal{V}_1}(1). \tag{9.4.6}$$

By 9.3.ii), $\mathcal{P}_{3,\eta}$ is isomorphic to

$$\mathbb{P}_{\mathcal{V}_1}(O_{\mathcal{V}_1}(-1) \oplus O_{\mathcal{V}_1}(1)) \cong \mathbb{P}_{\mathcal{V}_1}(O_{\mathcal{V}_1} \oplus O_{\mathcal{V}_1}(2)). \tag{9.4.7}$$

This is a non-singular surface. Thus we have a $\mathbb{P}^1$-fibration

$$\mathcal{P}_{3,\eta} \xrightarrow{\pi} \mathcal{V}_1. \tag{9.4.8}$$

As in [73, Proposition 2.3], we see that there is a section of π

$$\mathcal{P}_{3,\eta} \supset T \xleftarrow[t]{\sim} \mathcal{V}_1 \tag{9.4.9}$$

given by

$$t(\rho_2) = (E^3 \otimes k \xrightarrow{\rho_2} Y_1 \to (E^3/E^3[F]) \otimes k = Y_0). \tag{9.4.10}$$

We have

$$\mathcal{P}'_{3,\eta} = \mathcal{P}_{3,\eta} - T. \tag{9.4.11}$$

Furthermore, if $x \in \mathcal{P}_{3,\eta}$ represents $\{Y_2 \to Y_1 \to Y_0\}$, then

$$x \in T \implies a(Y_0) = 3, \tag{9.4.12}$$

$$\pi(x) \in \mathcal{V}_1(\mathbb{F}_{p^2}) \iff a(Y_0) \geq 2, \tag{9.4.13}$$

$$x \notin T, \pi(x) \notin \mathcal{V}_1(\mathbb{F}_{p^2}) \iff a(Y_0) = 1. \tag{9.4.14}$$

Remark. The statement (9.4.12) is correct, while in [73, Proposition 2.3] there is a misprint.

Under the morphism

$$\mathcal{P}_{3,\eta} \xrightarrow{\Psi} W_\eta \subset \mathcal{S}_{3,1} \otimes k \tag{9.4.15}$$

the curve $T \subset \mathcal{P}_{3,\eta}$ is contracted to the point

$$\Psi(T) = (E^3 \otimes k, \eta/p) \in \mathcal{S}_{3,1} \otimes k, \tag{9.4.16}$$

where η/p is the principal polarization of $(E^3/E^3[F]) \otimes k \cong E^3 \otimes k$ induced by η (as the polarization of Y_0 in (9.4.10)). Outside T the morphism Ψ is finite to one, and generically equals dividing out by the action of $G_\eta = \mathrm{Aut}(E^3 \otimes k, \eta)/\{\pm 1\}$ on $\mathcal{P}_{3,\eta}$. Note that $\Psi(T) \in W_\eta$ is a singular point of W_η. In fact, if $W_\eta^{(n)}$ is an irreducible component of $\mathcal{S}_{g,1,n} \otimes k$ and $x = (E^g \otimes k, \eta/p, \alpha) \in W_\eta^{(n)}$ (where α is a level n-structure), then the tangent space of $W_\eta^{(n)}$ at x has dimension 6 (cf. [73, Corollary 2.9]).

The intersection pattern of components of $\mathcal{S}_{3,1} \otimes k$ seems fairly complicated. For example, let $\rho_2 \in \mathcal{V}_1(\mathbb{F}_{p^2})$, and let $T' := \pi^{-1}(\rho_2) \subset \mathcal{P}_{3,\eta}$ be the fiber above ρ_2. Then

$$\#\{x \in T' | a(\Psi(x)) = 3\} \;=\; p^2 + 1, \tag{9.4.17}$$

and W_η is non-singular at every superspecial point $x \neq \Psi(T) \in T'$. However, such an x equals $(E^3 \otimes k, \mu)$ for some principal polarization μ and is therefore a singular point in the component $W_{\eta'}$ with $\eta' = p\mu$.

The number of irreducible components of $\mathcal{S}_{3,1} \otimes k$ was shown in [36, Theorem 6.7] to equal $H_3(p,1)$. This number was explicitly computed by Hashimoto in [24, Theorem 4]. Note that $H_3(2,1) = 1$, furthermore $H_3(p,1) > 1$ for $p > 2$, and $H_3(p,1) \approx p^6/(2^9 \cdot 3^4 \cdot 5 \cdot 7)$ for p large.

For the action of $\mathrm{Aut}(E^3 \otimes k, \eta)$ on $\mathcal{P}_{3,\eta}$, see Proposition 9.12 below.

Conclusion. Let Λ be a set of representatives of equivalence classes of polarizations η of $E^3 \otimes \bar{\mathbb{F}}_p$ satisfying $\ker(\eta) = E^3[p] \otimes \bar{\mathbb{F}}_p$. Then there is a one to one correspondence ψ between Λ and the set of irreducible components of $\mathcal{S}_{3,1} \otimes \bar{\mathbb{F}}_p$. Again denote by W_η the irreducible component corresponding to η under ψ. Then W_η is birationally equivalent to $\mathcal{P}_{3,\eta}/G_\eta$, where $\mathcal{P}_{3,\eta}$ is a $\mathbb{P}^1$-bundle over a Fermat curve and $G_\eta = \mathrm{Aut}(E^3 \otimes \bar{\mathbb{F}}_p, \eta)/\{\pm 1\}$. We have $\#(\Lambda) = H_3(p,1)$ and

$$\mathcal{S}_{3,1} \otimes \bar{\mathbb{F}}_p = \bigcup_{\eta \in \Lambda} W_\eta. \tag{9.4.18}$$

Note that W_η has a singular point corresponding to $(E^3 \otimes \bar{\mathbb{F}}_p, \eta/p)$ (see (9.4.16)), and the tangent space at this point to W_η has dimension 6 (see [73, Proposition 2.3]).

9.5. Some other methods for the calculation.

When $g > 3$, there are many global equations for $\mathcal{P}_{g,\eta}$ (i.e. more than the difference of the number of variables and the dimension), and one can hardly see the structure of $\mathcal{P}_{g,\eta}$ from these equations. So we will write down local equations in the sequel.

For convenience we will also use the language of Dieudonné modules (see 11.3 for an explanation).

9.6. Example, $g = 4$.

When $g = 4$, we first see that $\mathcal{V}_2$ is isomorphic to the non-singular surface $S \subset \mathbb{P}^3$ defined by (see 9.3.2))

$$a^{p^2} b - ab^{p^2} + c^{p^2} d - cd^{p^2} = 0. \tag{9.6.1}$$

Let x, y, z, u be the corresponding generators of the skeleton of $M_3 = A_{1,1}^{\oplus 4}$ (satisfying $\langle x, F^4 y\rangle = \langle z, F^4 u\rangle = 1$).

We consider an open neighborhood of a point $(a, b, c, d) \in S$, where a, b, c, d are linearly independent over $\mathbb{F}_{p^2}$. The corresponding Dieudonné module M_2 at (a, b, c, d) is generated by Fx, Fy, Fz, Fu and $v = \tilde{a}x + \tilde{b}y + \tilde{c}z + \tilde{d}u$, where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are liftings of a, b, c, d in $W = W(k)$ respectively. To give an M_1 is equivalent to giving a vector $w = \tilde{r}v + \tilde{s}Fx + \tilde{t}Fy$ ($\tilde{r}, \tilde{s}, \tilde{t} \in W$, not all in pW) such that

$$\langle w, Fw\rangle \in W \tag{9.6.2}$$

or explicitly

$$rt^p a - rs^p b + sr^p b^p - tr^p a^p = 0 \tag{9.6.3}$$

where r, s, t are the images of $\tilde{r}, \tilde{s}, \tilde{t}$ in $W/pW \cong k$ respectively. Therefore we get two irreducible components $\mathcal{V}_{11}$ and $\mathcal{V}_{12}$ of $\mathcal{V}_1$, where $\mathcal{V}_{11}$ is defined by

$$t^p a - s^p b + sr^{p-1} b^p - tr^{p-1} a^p = 0, \tag{9.6.4}$$

hence $\mathcal{V}_{11} \to \mathcal{V}_2$ has fiber dimension 1, and $\mathcal{V}_{12}$ is defined by $r = 0$, hence it is a $\mathbb{P}^1$-bundle over $\mathcal{V}_2$. Therefore $\dim(\mathcal{V}_{11}) = \dim(\mathcal{V}_{12}) = 3$. (One can compare this with Remark 6.4. Here $r = 0$ means $\ker(F_{Y_3/S}) \subset \ker(Y_3 \to Y_1)$, in this case condition iii) in 6.2 automatically holds for $i = 1$.)

Since $\mathcal{V}_0 = \mathcal{P}_{4,\eta}$ is a $\mathbb{P}^1$-bundle over $\mathcal{V}_1$, we see that $\mathcal{V}_0$ also has two irreducible components $\mathcal{V}_{01}$ and $\mathcal{V}_{02}$ (both of dimension 4), where $\mathcal{V}_{02}$ is a $\mathbb{P}^1$-bundle over $\mathcal{V}_{12}$ and does not meet $\mathcal{P}'_{4,\eta}$. It is easy to check that the fiber of Y_0 over the generic point of $\mathcal{V}_{02}$ has a-number 2.

In general, if the fiber of Y_0 over the generic point of an irreducible component $\mathcal{V} \subset \mathcal{P}_{g,\eta}$ is not supergeneral, then we call $\mathcal{V}$ a "*garbage component*" of $\mathcal{P}_{g,\eta}$. Note that $\mathcal{V}$ is a garbage component iff it does not map surjectively to a component of $\mathcal{S}_{g,1}$. Note also that the generic point of a garbage component is not in $\mathcal{P}'_{g,\eta}$.

Thus $\mathcal{V}_{02}$ is a garbage component of $\mathcal{P}_{4,\eta}$.

On the other hand, when $p > 2$, we see that $\mathcal{V}_{11}$ is singular at a point with $r = 0$. Hence $\mathcal{V}_{01}$ is also singular.

By more calculation one can see that $\mathcal{P}_{4,\eta}$ is reduced.

9.7. A proof of Proposition 4.3.i) for $g = 4$.

We now show that $\mathcal{P}'_4$ is smooth over $\mathbb{F}_{p^2}$. This is simply an illustration of 11.3 for $g = 4$. By 3.9, it is enough to show that $\mathcal{P}'_{4,\eta}$ is non-singular for a special choice of η over $k = \bar{\mathbb{F}}_p$. We choose η such that for some decomposition $E^4 \otimes k \cong E_1 \times E_2 \times E_3 \times E_4$, we have $\eta = p(\eta'' \times \eta')$, where η' (resp. η'') is a polarization of $E_2 \times E_3$ (resp. $E_1 \times E_4$) such that $\ker(\eta') = (E_2 \times E_3)[F]$ (resp. $\ker(\eta'') = (E_1 \times E_4)[F]$).

Let $\{X_3 \to ... \to X_0\}$ be the universal PFTQ over $\mathcal{P}'_{4,\eta}$. Let $U_i \subset \mathcal{P}'_{4,\eta}$ be the largest open subscheme such that $E_i \times U_i \to X_0 \times_{\mathcal{P}'_{4,\eta}} U_i$ is a closed immersion $(1 \leq i \leq 4)$. Then $\mathcal{P}'_{4,\eta} = \bigcup_i U_i$. By symmetry it is enough to show U_1 is non-singular. For convenience we denote $X_0 \times_{\mathcal{P}'_{4,\eta}} U_1$ simply by X_0.

Since $E_1 \times U_1 \to X_0$ is a closed immersion, its dual

$$X_0 \cong X_0^t \to E_1^t \times U_1 \cong (E_4/E_4[F^3]) \times U_1 \tag{9.7.1}$$

is smooth. Therefore the projections $X_i \to (E_4/E_4[F^{3-i}]) \times U_1$ $(0 \leq i \leq 3)$ are all smooth. Let $X''_i = H_1(C^{i+1}_\cdot)$ $(i = 0, 1)$, where $C^i_\cdot$ is the complex

$$C^i_\cdot : \quad E_1 \times U_1 \to X_i \to (E_4/E_4[F^{3-i}]) \times U_1. \tag{9.7.2}$$

Then one sees that $\{X''_1 \to X''_0\}$ is a PFTQ with respect to η'. This induces a morphism

$$\psi : U_1 \to \mathcal{P}_{2,\eta'} \cong \mathbb{P}^1. \tag{9.7.3}$$

It is enough to show ψ is smooth.

We first decompose ψ. Let $\{X'_1 \to X'_0\}$ be the universal PFTQ over $\mathcal{P}_{2,\eta'}$ and $G' = \ker(X'_1 \to X'_0)$. Let $\mathfrak{U}_m$ $(0 \leq m \leq 3)$ be the category of sequences of isogenies $\{Y_3 \to ... \to Y_m\}$ of polarized abelian schemes (Y_i, η_i) over some $\mathcal{P}_{2,\eta'}$-scheme S such that

i) $Y_3 = E^4 \times S$, with $\eta_3 = \eta \times \mathrm{id}_S$;
ii) $\ker(Y_i \to Y_{i-1})$ is a flat α-group of α-rank i $(m \leq i \leq 3)$;
iii) $\ker(Y_3 \to Y_i) = \ker(Y_3 \to Y_m) \cap Y_3[F^{3-i}]$ $(m < i \leq 3)$;
iv) $\ker(\eta_i) \subset X_i[F^i]$ $(m \leq i \leq 3)$;
v) $E_1 \times S \to Y_m$ is a closed immersion, and there are induced isomorphisms $\phi_i : H_1(C^{i+1}_\cdot) \cong X'_i \times_{\mathcal{P}_{2,\eta'}} S$ $(m - 1 \leq i \leq 1)$, where $C^i_\cdot$ is the complex

$$C^i_\cdot : \quad E_1 \times S \to X_i \to (E_4/E_4[F^{3-i}]) \times S; \tag{9.7.4}$$

vi) (for $m < 3$ only) letting $G \subset Y_2[F]$ be the inverse image of $G' \times_{\mathcal{P}_{2,\eta'}} S$ in $Y_2[F]$ under

$$G' \times_{\mathcal{P}_{2,\eta'}} S \subset X'_1[F] \times_{\mathcal{P}_{2,\eta'}} S \hookrightarrow Y_2[F]/E_1[F] \times S \tag{9.7.5}$$

induced by ϕ_1 in v), we have $G^{(p)} \subset Y_2^{(p)}[F] \cap \ker(V : Y_2^{(p)} \to Y_2)$.

Let $\mathcal{U}_m$ be the fine moduli scheme of $\mathfrak{U}_m$. Then clearly $\mathcal{U}_0 \cong U_1$ and $\mathcal{U}_3 \cong \mathcal{P}_{2,\eta'}$. Furthermore, the truncations induce morphisms $\psi_i : \mathcal{U}_i \to \mathcal{U}_{i+1}$ $(0 \leq i \leq 2)$, and $\psi = \psi_2 \circ \psi_1 \circ \psi_0$. Hence it is enough to show each ψ_i is smooth. By 9.3.ii), we see ψ_0

is a line bundle (it is not a $\mathbb{P}^1$-bundle because of the open condition v)). It remains to check the smoothness of ψ_1 and ψ_2.

First we consider ψ_2. For a given $\{S; Y_3\} \in \mathrm{Ob}(\mathfrak{U}_3)$, let $G_1 = \ker(Y_3 \to Y_3^t) = Y_3[F^3]$ and $G_2 = Y_3[F]$. Note that G_2 is an α-group, and we denote by $\mathcal{F}$ the α-sheaf of G_2.

To extend $\{S; Y_3\}$ to an object of $\mathfrak{U}_2$, we need to find an α-subgroup $G_3 \subset G_2$ of α-rank 3, or equivalently a nowhere zero section s of $\mathcal{F}$. Condition v) simply says the s_1-coordinate of s is non-zero. Hence we can assume

$$s = s_1 + x_1 s_2 + x_2 s_3 + x s_4. \tag{9.7.6}$$

Let $G_4 = E_1[F] \times S$. Then $G_4 \subset Y_2 = Y_3/G_3$, and the projection $Y_2 \to Y_3/Y_3[F]$ gives an exact sequence

$$0 \to G_4 \to Y_2[F] \to G_5 \to 0, \tag{9.7.7}$$

where $G_5 = (E_2 \times E_3 \times E_4)[F]^{(p)} \times S$.

We check condition vi). Let $G_6 = Y_2^{(p)}[F] \cap \ker(V : Y_2^{(p)} \to Y_2)$. Then $G_7 = G_6/G_4^{(p)}$ is a subgroup scheme of $G_5^{(p)}$ by (9.7.7). It is easy to see that the ideal sheaf of $G_7 \hookrightarrow G_5^{(p)}$ is generated by the section $F^* s^{(p)} - V^* s = s^{(p^2)} - s$ of the α-sheaf of $G_5^{(p)}$. On the other hand G' is defined by the section $y_1 s_2^{(p)} + y_2 s_3^{(p)}$ of the α-sheaf $\mathcal{F}'$ of $X_1'[F]$, where y_1, y_2 are the homogeneous coordinates of $\mathcal{P}_{2,\eta'} \cong \mathbb{P}^1$. Hence vi) is equivalent to that the restriction of $s^{(p^2)} - s$ to $\mathcal{F}' \otimes_{O_{\mathcal{U}_3}} O_S$ is proportional to $y_1 s_2^{(p)} + y_2 s_3^{(p)}$, or explicitly

$$(x_1^{p^2} - x_1) y_2 = (x_2^{p^2} - x_2) y_1. \tag{9.7.8}$$

Next we check condition iv). Since G_2^D is a quotient group scheme of $G_1^D \cong G_1$ and $\ker(G_1 \to G_2^D) = G_1[F^2]$, we have an induced isomorphism $f : G_2^D \to G_2^{(p^2)}$, which is equivalent to an O_S-linear map $\mathcal{F}^{(p^2)} \to \mathcal{F}^\vee$, or equivalently an O_S-bilinear form $\langle , \rangle : \mathcal{F} \otimes_{O_S} \mathcal{F}^{(p^2)} \to O_S$. Take a generator s_i of the α-sheaf of $E_i[F]$ for each i. Then s_1, s_2, s_3, s_4 can be viewed as a set of generators of $\mathcal{F}$. We can choose s_1, s_2, s_3, s_4 such that

$$\langle s_1, s_4^{(p^2)} \rangle = -\langle s_4, s_1^{(p^2)} \rangle = \langle s_2, s_3^{(p^2)} \rangle = \langle s_3, s_2^{(p^2)} \rangle = 1, \tag{9.7.9}$$

and we have

$$\begin{aligned} &\langle s_1, s_2^{(p^2)} \rangle = \langle s_1, s_3^{(p^2)} \rangle = \langle s_4, s_2^{(p^2)} \rangle = \langle s_4, s_2^{(p^2)} \rangle = 0, \\ &\langle s_i, s_i^{(p^2)} \rangle = 0 \quad (1 \le i \le 4). \end{aligned} \tag{9.7.10}$$

Let $G_8 = G_2/G_3$. Then G_8^D is a subgroup scheme of G_2^D. Let $\phi : G_8^D \to G_8^{(p^2)}$ be the composition of the inclusion $G_8^D \hookrightarrow G_2^D$, f and the projection $G_2^{(p^2)} \twoheadrightarrow G_8^{(p^2)}$. Then iv) is equivalent to $\phi = 0$, and this is then equivalent to $\langle s, s^{(p^2)} \rangle = 0$, or explicitly

$$x^{p^2} - x + x_1 x_2^{p^2} - x_2 x_1^{p^2} = 0. \tag{9.7.11}$$

Note that we also have $G_2^D \cong G_1/G_1[p]$, which induces another bilinear form $\langle,\rangle_1 : \mathcal{F} \otimes_{O_S} \mathcal{F} \to O_S$. We automatically have $\langle s, s\rangle_1 = 0$ since $\langle,\rangle_1$ is alternating. Therefore we have $\ker(Y_2 \to Y_2^t) \subset Y_2[p]$ for any choice of G_3.

We see that $\mathcal{U}_2 \to \mathcal{U}_3$ is defined by variables x_2, x_3, x with defining relations (9.7.8) and (9.7.11), hence ψ_2 is smooth.

Finally we consider ψ_1. Assume we are given an object $\{S; Y_3 \to Y_2\}$ of $\mathfrak{U}_2$. Let $G_9 = Y_2[F]$ and $G_{10} = \ker(Y_2 \to Y_2^t)$. Then condition vi) says that we have an α-group $G \subset G_9$ of α-rank 2. On the other hand, condition iv) (for $i = 2$) says $G_{10} \subset Y_2[F^2]$, and the above note says $G_{10} \subset Y_2[p]$, hence $\mathrm{coker}(G_9 \to G_{10})$ has Verschiebung 0. Therefore

$$\ker(G_{10} \cong G_{10}^D \to G_9^D) \subset G_{10}[F] = G_9. \tag{9.7.12}$$

Thus we have an induced homomorphism $\phi : G_9^D \to G_9^{(p)}$. It is easy to see that ϕ^D induces a homomorphism $\Phi_. : D'_. \to D_.$ of the following two complexes

$$D'_. : \quad E_1[F] \times S \hookrightarrow (G_9^D)^{(p)} \twoheadrightarrow E_4[F]^{(p)} \times S \tag{9.7.13}$$

and

$$D_. : \quad E_1[F] \times S \hookrightarrow G_9 \twoheadrightarrow E_4[F]^{(p)} \times S. \tag{9.7.14}$$

Note that Φ_0 and Φ_2 are isomorphisms and $H_1(\Phi_.) = 0$. Hence ϕ^D has a flat image $G_{11} \subset G_9$, which is an α-group of α-rank 2.

Note that $G \cap G_{11} = E_1[F] \times S$, hence G and G_{11} together generate an α-group $G_{12} \subset G_9$ of α-rank 3. Let $\mathcal{F}'$ be the α-sheaf of G_{12}. Locally we can lift s_1 to a section s'_1 of $\mathcal{F}'$. Locally we also take a section s' of $\mathcal{F}'$ which lifts a generator of the α-sheaf of $G' \times_{\mathcal{U}_3} S$. Thus $\mathcal{F}'$ is locally generated by $s'_1, s', s_4^{(p)}$.

To extend $\{S; Y_3 \to Y_2\}$ to an object of $\mathfrak{U}_1$, we need to find a subgroup scheme $G_{13} \subset G_9$ which is an α-group of α-rank 2 (and $Y_1 = Y_2/G_{13}$). We first show it is necessary that $G_{13} \subset G_{12}$. Indeed, since $G_{13} \cap E_1[F] \times S = 0$ by condition v), it is enough to show that the image G_{14} of G_{13} in $G_9/E_1[F] \times S$ is equal to $G_{12}/E_1[F] \times S$. Condition v) requires that $G' \times_{\mathcal{U}_3} S \cong G/E[F] \times S \subset G_{14}$. On the other hand $E_1[F] \times S \subset Y_1$ and the above note gives a subgroup scheme

$$E_1^t[F]^{(p)} \times S \cong E_4[F]^{(p)} \times S \hookrightarrow G_9/E_1[F] \times S \tag{9.7.15}$$

which maps to 0 in $Y_1/E_1 \times S$ by the dual of iv). Hence we have $E_4[F]^{(p)} \times S \cong G_{11}/E_1[F] \times S \subset G_{14}$.

It reduces to finding a section $s = s'_1 + x_1 s' + x s_4^{(p)}$ of $\mathcal{F}'$. It remains to check condition iv). As in the case of ψ_2, we have an induced homomorphism $G_{12}^D \to G_{12}^{(p)}$ which is equivalent to an O_S-bilinear form $\langle,\rangle_2 : \mathcal{F}' \otimes_{O_S} \otimes \mathcal{F}'^{(p)} \to O_S$, and iv) is equivalent to

$$\langle s, s^{(p)}\rangle_2 = 0. \tag{9.7.16}$$

Note that we have

$$\begin{aligned} &\langle s'_1, s_4^{(p^2)}\rangle_2 = -\langle s_4^{(p)}, s_1'^{(p)}\rangle_2 = 1, \\ &\langle s', s_4^{(p^2)}\rangle_2 = \langle s_4^{(p)}, s'^{(p)}\rangle_2 = \langle s_4^{(p)}, s_4^{(p^2)}\rangle_2 = \langle s', s'^{(p)}\rangle_2 = 0 \end{aligned} \tag{9.7.17}$$

But $c = \langle s_1', s'^{(p)} \rangle_2$, $c' = \langle s', s_1'^{(p)} \rangle_2$ and $d = \langle s_1', s_1'^{(p)} \rangle_2$ may not equal 0 in general. Thus we can write (9.7.16) explicitly

$$x^p - x + cx_1 + c'x_1^p + d = 0. \tag{9.7.18}$$

Therefore $\mathcal{U}_1 \to \mathcal{U}_2$ is locally given by variables x_1, x with the defining relation of the form (9.7.18). Hence ψ_1 is smooth.

Remark. Let $\mathcal{V}_m' \subset \mathcal{V}_m$ (see Example 9.6) be the open subscheme representing sequences $\{Y_3 \to \dots \to Y_m\}$ (over a k-scheme S) satisfying i-iv) above and $E_1 \times S \hookrightarrow Y_m$. Then we have $\mathcal{V}_0' \cong \mathcal{U}_0$ and $\mathcal{V}_1' \cong \mathcal{U}_1$. On the other hand, we have an induced morphism $\mathcal{U}_2 \to \mathcal{V}_2 \times \mathcal{P}_{2,\eta'}$ which is not an isomorphism because $\mathcal{V}_2 \times \mathcal{P}_{2,\eta'}$ represents isogenies $\{Y_3 \to Y_2\}$ satisfying i-v) but not vi). Condition vi) guarantees that an extension $\{Y_3 \to Y_2 \to Y_1\}$ of $\{Y_3 \to Y_2\}$ satisfies v) also.

9.8. Garbage components for large g.

Example. When $g = 5$, by the same way of calculation we see that $\mathcal{P}_{g,\eta}$ has a garbage component of dimension 6, which is equal to $\dim(\mathcal{S}_{5,1})$. When $g > 5$, we even have a garbage component of dimension $> [g^2/4]$.

9.9. The subsets defined by a-numbers.

For any $n > 0$, the points of $\mathcal{S}_{g,d}$ representing abelian varieties with a-number $\geq n$ form a Zariski closed subset, which will be denoted by $\mathcal{S}_{g,d}(a \geq n)$. For example, $\mathcal{S}_{g,d}(a \geq g)$ is the set of superspecial points, and $\mathcal{S}_{g,1}(a \geq 2)$ is a divisor, as will be shown in Corollary 10.3.

We now study $\mathcal{S}_{4,1}(a \geq n)$. There are two kinds of irreducible components in $\mathcal{S}_{4,1}(a \geq 2)$:

a) Let μ be a polarization of E^4 such that $\ker(\mu) = E^4[p]$. Consider sequences of isogenies of polarized abelian varieties

$$E^4 \to Y_1 \to Y_0 \quad (\ker(E^4 \to Y_1) \cong \alpha_p^3,\ \ker(Y_1 \to Y_0) \cong \alpha_p) \tag{9.9.1}$$

where the polarization of E^4 is μ. Such sequences admit a fine moduli scheme U_μ which is isomorphic to a $\mathbb{P}^1$-bundle of the hypersurface

$$X_1^{p+1} + X_2^{p+1} + X_3^{p+1} + X_4^{p+1} = 0 \tag{9.9.2}$$

in $\mathbb{P}^3$. The image of U_μ in $\mathcal{S}_{4,1}(a \geq 2)$ is an irreducible component of dimension 3, and there are $H_4(p,1)$ irreducible components of this kind.

b) Let μ be a polarization of E^4 such that $\ker(\mu) = E^4[F]$. Consider isogenies of polarized abelian varieties

$$E^4 \to Y_0 \quad (\ker(E^4 \to Y_1) \cong \alpha_p^2) \tag{9.9.3}$$

where the polarization of E^4 is μ. Such isogenies admit a fine moduli scheme $\mathcal{T}_\mu$ which is isomorphic to the subscheme of the Grassmannian $\text{Grass}_{4,2}$ consisting of points representing isotropic subspaces of $\{k^{\oplus 4}, \langle,\rangle\}$, where $\langle,\rangle$ is a non-degenerate alternating form. The image of $\mathcal{T}_\mu$ in $\mathcal{S}_{4,1}(a \geq 2)$ is an irreducible component of dimension 3, and there are $H_4(1,p)$ irreducible components of this kind.

9.10. Supersingular Dieudonné modules with a-number $g-1$.

Next we study $\mathcal{S}_{g,1}(a \geq g-1)$ (in particular $\mathcal{S}_{4,1}(a \geq 3)$). We make use of the following result.

Lemma. *Let M be a principally quasi-polarized supersingular Dieudonné module of genus g over $W(k)$ with $a(M) = g-1$. Then there is a decomposition $M = N \oplus N'$, where N' is a principally quasi-polarized superspecial Dieudonné module, and N is a principally quasi-polarized Dieudonné module of genus $2r$ ($r \leq g/2$) such that $S_0 N = FS^0 N$.*

Proof. By $a(M) = g-1$ we have $FS^0M \subset M$ (see [45, p.337]). Hence by Proposition 6.1 we have a decomposition $S^0M = N_0 \oplus N'$, where N' is a principally quasi-polarized superspecial Dieudonné module, and N_0 is a quasi-polarized superspecial Dieudonné module such that $N_0^t = FN_0$. Let $N = M \cap N_0$ and $r = \dim_k(N/N_0^t)$. Then $M = N \oplus N'$ and $N^t = N$. Finally, since $r = \dim_k(N_0/N^t) = \dim_k(N_0/N_0^t) - r$, we see that $g(N) = g(N_0) = 2r$. Q.E.D.

9.11. The structure of $\mathcal{S}_{g,1}(a \geq g-1)$.

Proposition. *Let $k = \bar{\mathbb{F}}_p$. For any $0 < r \leq [g/2]$ and any polarization μ of $E^g \otimes k$ such that $\ker(\mu) \cong \alpha_p^{2r}$, denote by $\mathcal{T}_\mu$ the fine moduli scheme of isogenies $\rho : E^g \otimes k \to Y$ of polarized abelian varieties satisfying*

$$\ker(\rho) \cong \alpha_p^r \subset \ker(\mu), \tag{9.11.1}$$

where the polarization of $E^g \otimes k$ is μ (hence Y is principally polarized). Denote by $T_\mu \subset \mathcal{T}_\mu$ the locally closed subset of points whose corresponding Y has $a(Y) = g-1$ (with reduced induced scheme structure).

- *i) The induced morphism $T_\mu \to \mathcal{S}_{g,1} \otimes k$ is generically finite to one, and T_μ is irreducible of dimension r.*
- *ii) The induced morphism*

$$\Psi_0 : \coprod_{\ker(\mu) \cong \alpha_p^{2[g/2]}} T_\mu \to \mathcal{S}_{g,1}(a \geq g-1) \otimes k \tag{9.11.2}$$

  is surjective and gives a one to one correspondence between the set of irreducible components of $\mathcal{S}_{g,1}(a \geq g-1) \otimes k$ and the set of equivalence classes of μ such that $\ker(\mu) \cong \alpha_p^{2[g/2]}$.
- *iii) Every irreducible component of $\mathcal{S}_{g,1}(a \geq g-1)$ has dimension $[g/2]$, and the number of irreducible components of $\mathcal{S}_{g,1}(a \geq g-1) \otimes k$ is equal to $H_g(1,p)$.*

Proof. i) To give an isogeny $\rho : E^g \otimes k \to Y$ satisfying (9.11.1) is equivalent to giving a totally isotropic subspace of dimension r of the α-sheaf $\mathcal{F}$ of $\ker(\mu)$, or an

$r \times r$ symmetric matrix $C = (c_{ij})$ over k under a choice of standard basis of $\mathcal{F}$. By an easy calculation of Dieudonné modules, one sees that $a(Y) = g - 1$ iff the corresponding C satisfies

$(*)$ $C - C^{(p^2)} = (c_{ij} - c_{ij}^{p^2})$ has rank 1.

By the symmetricity of C, $(*)$ is equivalent to $r(r-1)/2$ local equations on c_{ij} $(1 \le i, j \le r)$. Hence every irreducible component of T_μ has dimension $\ge r(r+1)/2 - r(r-1)/2 = r$.

Let $T'_{2r} \subset T_{2r}$ (in Proposition 9.3) be the set of points whose coordinates are linearly independent over $\mathbb{F}_{p^2}$. Then T'_{2r} is open dense in T_{2r} by Proposition 9.3, hence has dimension r. For any $(a_1, ..., a_{2r}) \in T'_{2r}$, under a choice of standard basis of $\mathcal{F}$, the subspace of $\mathcal{F}$ generated by

$$(a_1^{p^{2n}}, ..., a_{2r}^{p^{2n}}) \quad (0 \le n < r) \tag{9.11.3}$$

is totally isotropic of dimension r by (9.3.2), hence gives a minimal isogeny ρ as in (9.11.1). This gives a morphism $\phi_\mu : T'_{2r} \to T_\mu$ which is easily seen to be set-theoretically injective. Conversely, if the isogeny ρ in (9.11.1) is minimal, then ρ is represented by a point in $\mathrm{im}(\phi_\mu)$. Combining this with the fact that every irreducible component of T_μ has dimension $\ge r$ (as shown above), we see that ϕ_μ is generically surjective and T_μ is irreducible of dimension r.

Furthermore, if ρ is minimal, then μ is uniquely determined by the polarization of Y. Hence $T_\mu \to \mathcal{S}_{g,1} \otimes k$ is generically finite to one.

ii) By Lemma 9.10, the morphism Ψ_0 in (9.11.2) is surjective. We have also seen that $\Psi_0(T_\mu)$ determines the equivalence class of μ, hence Ψ_0 gives a one to one correspondence between the irreducible components of $\mathcal{S}_{g,1}(a \ge g-1) \otimes k$ and the equivalence classes of μ.

iii) By i) and ii) we see that every irreducible component of $\mathcal{S}_{g,1}(a \ge g-1) \otimes k$ has dimension $[g/2]$, and the number of irreducible components of $\mathcal{S}_{g,1}(a \ge g-1) \otimes k$ is equal to the number of equivalence classes of μ such that $\ker(\mu) \cong \alpha_p^{2[g/2]}$, which is equal to $H_g(1,p)$ by Corollary 4.8.iii). Q.E.D.

9.12. The action of the automorphism group of a polarization η on $\mathcal{P}_{g,\eta}$.

We study the action of $\mathrm{Aut}(E^g \otimes k, \eta)$ on $\mathcal{P}_{g,\eta}$ for any $g > 1$.

Proposition. *Let $g > 1$ and η be a polarization of $E^g \otimes k$ such that $\ker(\eta) = E^g[F^{g-1}] \otimes k$.*

i) If g is odd and $\eta = p^{(g-1)/2}\mu^g$ for some principal polarization μ of (a choice of) E, then the group $\mathrm{Aut}(E^g \otimes k, \eta)$ is isomorphic to the subgroup of $GL_g(\mathcal{O})$ consisting of matrices T such that each row of T has one entry in $\mathcal{O}^\times$ with the other entries$= 0$, hence we have

$$\mathrm{Aut}(E^g \otimes k, \eta) \cong (\mathcal{O}^\times)^g \rtimes S_g. \tag{9.12.1}$$

ii) If $g \ne 3$ or $p > 2$, then the action of $\mathrm{Aut}(E^g \otimes k, \eta)/\{\pm 1\}$ on $\mathcal{P}'_{g,\eta}$ is generically free.

iii) When $g = 3$ and $p = 2$, (9.12.1) holds since there is only one equivalence class of η. The stabilizer of the generic point of $\mathcal{P}_{3,\eta}$ under the action of $\mathrm{Aut}(E^3 \otimes k, \eta)$ is isomorphic to $\{\pm 1\}^3 \subset (\mathcal{O}^\times)^3$ under (9.12.1), and the degree of $\mathcal{P}_{3,\eta} \to \mathcal{S}_{3,1}$ is $2^7 3^4$.

Proof. From [31, Proposition 2.8] we see that for any choice of a principal polarization μ_0 of $E^g \otimes k$, there is an isomorphism

$$\mathrm{Aut}(E^g \otimes k, \eta) \cong \{T \in GL_g(\mathcal{O}) | \bar{T}^t A T = A\} \tag{9.12.2}$$

via $\mathrm{Aut}(E^g \otimes k, \eta) \subset \mathrm{Aut}(E^g \otimes k) \cong GL_g(\mathcal{O})$, where $A = \mu_0^{-1} \circ \eta$ and $\bar{T}$ is the Rosati involution of T with respect to μ_0. When g is odd, we can take $\mu_0 = \eta/p^{(g-1)/2}$, hence

$$\mathrm{Aut}(E^g \otimes k, \eta) \cong \{T \in GL_g(\mathcal{O}) | \bar{T}^t T = I_g\}. \tag{9.12.3}$$

Write $T = (a_{ij}) \in GL_g(\mathcal{O})$. When $\mu_0 = \mu^g$, we have $\bar{T}^t = (\bar{a}_{ji})$, where $\bar{a}_{ji}$ is the conjugate of $a_{ji} \in \mathcal{O}$ (i.e. the Rosati involution with respect to μ). Note that $\bar{a}_{ij} a_{ij}$ is a positive integer unless $a_{ij} = 0$. Hence $\bar{T}^t T = I_g$ is equivalent to that each row of T has one entry in $\mathcal{O}^\times$ with the other entries=0. Therefore we have an exact sequence

$$(\mathcal{O}^\times)^g \hookrightarrow \mathrm{Aut}(E^g \otimes k, \eta) \twoheadrightarrow S_g. \tag{9.12.4}$$

This proves i).

Next we prove ii). We have already seen the case $g = 2$ in 9.2, hence we assume $g > 2$ in the following. Note that $\mathrm{Aut}(E^g \otimes k, \eta)$ is a finite group.

Let $T_0 \subset \mathcal{P}'_{g,\eta}$ be the Zariski open subset of points representing PFTQs with supergeneral end. Let $x \in T_0$ represent a PFTQ $\{X_{g-1} \to \dots \to X_0\}$ with respect to η (with $a(X_0) = 1$). Then η induces a quasi-polarization $\langle , \rangle$ on $M_{g-1} = D(X_{g-1}) \cong A_{1,1}^{\oplus g}$ and we have $\langle M_0, M_0 \rangle \subset W$, where $M_0 = D(X_0)$. Since $a(M_0) = 1$, by Fact 5.6.ii) we have $M_0 = Av$ for some $v \in M_0$. Choose generators $x_1, \dots, x_g$ of the skeleton of M_{g-1} (see 5.7). Then we can write

$$v = (a_1 + b_1 F)x_1 + \dots + (a_g + b_g F)x_g \tag{9.12.5}$$

where $a_i, b_i \in W$ $(1 \le i \le g)$.

Let $\phi \in \mathrm{Aut}(E^g \otimes k, \eta)$. Then ϕ induces an automorphism $D(\phi)$ of M_{g-1} which preserves $\langle , \rangle$. Thus $D(\phi)$ can be expressed as an H-matrix $(\alpha_{ij} + \beta_{ij} F)$ $(\alpha_{ij}, \beta_{ij} \in W(\mathbb{F}_{p^2}))$ with respect to the generators $x_1, \dots, x_g$ (see (5.7.1) for the definition of H).

Suppose $\phi(x) = x$. Then $D(\phi)(M_0) = M_0$, i.e. $D(\phi)(v) \in Av$. Hence there exists $c \in k$ such that

$$\sum_i \bar{\alpha}_{ij} \bar{a}_i = c \bar{a}_j \quad (1 \le j \le g), \tag{9.12.6}$$

where $\bar{\alpha}_{ij}, \bar{a}_i$ are respectively the images of α_{ij}, a_i in k under the projection $W \to W/pW \cong k$. If v is general enough, we have $\bar{\alpha}_{ij} = c\delta_{ij}$. Since $D(\phi)$ preserves $\langle , \rangle$, we have $c^2 = 1$ (hence $c = \pm 1$) when g is even, and $c^{p+1} = 1$ when g is odd. Take a lifting $\tilde{c} \in W$ of c.

By $D(\phi)(v) \in Av$ we get

$$\sum_{i,j} (a_i + b_i F)(\tilde{c}\delta_{ij} + \beta_{ij} F)x_j \equiv \tilde{c}v + c_1 F v + c_2 V v \pmod{F^2 M_{g-1}} \tag{9.12.7}$$

for some $c_1, c_2 \in W$. By writing down the explicit equations one sees (9.12.7) is a non-trivial algebraic condition unless $c = \pm 1$ and $\beta_{ij} \in pW$ for all i, j. Therefore either

$$D(\phi) \equiv \pm \mathrm{id} \pmod{pH}, \tag{9.12.8}$$

or there is a non-empty Zariski open subset $U'_\phi \subset T_0$ such that $\phi(x) \neq x$ for any $x \in U'_\phi$.

Repeating the above argument inductively we can show that either

$$D(\phi) \equiv \pm \mathrm{id} \pmod{F^{g-1}H}, \tag{9.12.9}$$

or there is a non-empty Zariski open subset $U_\phi \subset T_0$ such that $\phi(x) \neq x$ for any $x \in U_\phi$.

Now we use [46, Lemma 2.5 and Remark 2.6], which in particular gives:

(∗) *Let* $\phi \in \mathrm{Aut}(E^g \otimes k, \eta)$. *If* $p > 2$ *and* $E^g[p] \otimes k \subset \ker(\phi - \mathrm{id})$, *or if* $p = 2$ *and* $E^g[F^3] \otimes k \subset \ker(\phi - \mathrm{id})$, *then* $\phi = \mathrm{id}$.

Note that (9.12.9) is equivalent to

$$E^g[F^{g-1}] \otimes k \subset \ker(\phi \mp \mathrm{id}), \tag{9.12.10}$$

hence implies $\phi = \pm\mathrm{id}$ if $g > 3$ or $p > 2$.

When $g > 3$ or $p > 2$, let $U = \bigcap_{\phi \neq \pm \mathrm{id}} U_\phi$. Then the stabilizer of any $x \in U$ in $\mathrm{Aut}(E^g \otimes k, \eta)$ is $\{\pm \mathrm{id}\}$. This proves ii).

Finally we consider the case $g = 3$, $p = 2$. In this case there is only one equivalence class of η because $H_3(2, 1) = 1$ (see [24, Theorem 4]). Hence we may assume $\eta = 2\mu^3$, where μ is a principal polarization of $E \otimes k$. Therefore $\mathrm{Aut}(E^3 \otimes k, \eta)$ is isomorphic to the group of 3×3-matrices each row of which has an entry in $\mathcal{O}^\times$ with other entries $= 0$. Note that $\mathcal{O}^\times$ is a non-commutative group of order 24 (isomorphic to a semi-direct product of the quaternion group with $\mathbb{Z}/3\mathbb{Z}$). Since $\mathcal{P}_{3,\eta} \cong \mathcal{P}_3 \otimes k$ which only depends on $E^3[2] \otimes k$ (see 3.9), if ϕ acts trivially on $E^3[2] \otimes k$, then it acts trivially on $\mathcal{P}_{3,\eta}$. The converse also holds by the argument of ii).

If ϕ acts trivially on $E^3[2] \otimes k$, then we can write $\phi = \mathrm{id} + 2\psi$, hence ϕ corresponds to a diagonal matrix $\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) \in GL_g(\mathcal{O})$ and we can write $\alpha_i = 1 + 2\beta_i$ $(1 \leq i \leq 3)$. Since the order of α_i divides 12, it is easy to check that $\alpha_i = \pm 1$. Hence ϕ acts trivially on $\mathcal{P}_{3,\eta}$ iff it corresponds to a diagonal matrix with ± 1 as its diagonal entries. This shows the first assertion of iii), and hence the degree of $\mathcal{P}_{3,\eta} \to \mathcal{S}_{3,1}$ is $2^7 3^4$. Q.E.D.

9.13. Different automorphism groups of polarizations.

Remark. When g is even, the structure of $\mathrm{Aut}(E^g \otimes k, \eta)$ (as a group) depends on η, as we have seen for $g = 2$ in 9.2 above. This is also the case when g is odd. For example, when $p = 17$, there are two supersingular elliptic curves E_1, E_2 over k with $j(E_1) = 8$ and $j(E_2) = 0$. We have $\mathrm{Aut}(E_1) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(E_2) \cong \mathbb{Z}/4\mathbb{Z}$. Take principal polarizations μ_1 of E_1 and μ_2 of E_2 respectively, and let $\eta_1 = p\mu_1^3$, $\eta_2 = p\mu_2^3$ which are polarizations of $E_1^3 \cong E_2^3$. Then by Proposition 9.12.i) we have $\mathrm{Aut}(E_1^3, \eta_1) \not\cong \mathrm{Aut}(E_2^3, \eta_2)$ (hence $\eta_1 \not\sim \eta_2$).