G E O M E T R Y A N D L O O P S

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§ 0 Introduction

This article contains some samples from a bigger project which A. Barlotti and I want to realize. Its aim is to show that classical principles of projective geometry and of the foundations of geometry can be applied successfully for the study of loops.

In the second paragraph natural analogues of the classical fundamental theorem of projective geometry are proved for loops and the abelian groups are characterized within the wide class of loops with the help of the group of projectivities.

In the third paragraph it is shown that the transitivity of the collineation group on the points of the 3-net which is associated to a loop Q is equivalent to the fact that every element of Q is a companion of a right and of a left pseudoautomorphism. The stabilizers of the collineation group on the horizontal line 1_h, on the vertical line 1_v and on the point $(1,1)$ is determined and the algebraic consequences for loops with transitive automorphism groups are discussed.

In the last paragraph we present a classification for loops analogous to the Lenz-Barlotti-classification for projective planes and show that this principle which has been propagated by H. Lenz with great success in the foundations of geometry can also be applied for other classes of mathematical structures.

§ 1. Quasigroups, nets and projectivities

Definition (1.1): A k-net (≥ 3) is a structure consisting of a set P of points and a set of lines which is partitioned into k disjoint families $L_i (i = 1, \dots k)$ for which the following conditions hold:

i) every point is incident with exactly one line of every $L_i (i = 1, \dots k)$;

ii) two lines of different families have exactly one point in common;

iii) there exist 3 lines belonging to 3 different L_i and which are not incident with the same point.

Lines of the same [different] families are said to have the same [different] directions.

It is well known that to every quasigroup Q (see e.g. [8], p. 16) we can associate a 3-net (see e.g. [8], p. 251) such that the three families of parallel lines consist of the following sets of points:

$g_h = \{(x, g) \mid g \text{ constant}, x \in Q\}$, horizontal lines;

$g_v = \{(g, x) \mid g \text{ constant}, x \in Q\}$, vertical lines;

$g_t = \{(x, y) \mid x \cdot y = g ; x, y \in Q, g \text{ constant}\}$, transversal lines.

We shall denote by $\mathfrak{H}$, $\mathfrak{V}$ and $\mathfrak{T}$ the families of horizontal, vertical and transversal lines respectively.

Conversely every 3-net leads to a class of isotopic quasigroups (see e.g. [6], p. 20).

Let N be a k-net, L a line in it, and $\mathfrak{X}$ one of the k families of parallel lines such that $L \notin \mathfrak{X}$. A perspectivity $\alpha = [L, \mathfrak{X}]$ assigns to a point $x \in L$ the line X of $\mathfrak{X}$ through x. The perspectivity $\alpha^{-1} = [\mathfrak{X}, L]$ assigns to $X \in \mathfrak{X}$ the point $x = X \cap L$.

A projectivity γ of a line onto a line is given by a set of consecutive perspectivities α_i, or in other words γ is the product of these α_i:

$$\gamma = \prod_{i=1}^{n} \alpha_i .$$

The projectivities of a line L onto itself in a k-net N form a group Π_L with respect to the composition of mappings. If H is any other line in N and β any projectivity from L onto H, then we have $\Pi_H = \beta^{-1}\Pi_L\beta$. Therefore all groups of projectivities of a line onto itself in a k-net are isomorphic as permutation groups, and we can speak of the group of projectivities of N .

If Q is a quasigroup then we define the group Π of projectivities of Q as the group of projectivities in a 3-net N which naturally arises from Q . Clearly all the members of the isotopy class of Q - as quasigroups corresponding to N - have Π as group of projectivities. The same holds even for all quasigroups isostrophic to Q ([3], p. 13).

Therefore Π can be seen as a group of projectivities of an isotrophy class of quasigroups. In any isotopy class of quasigroups there are loops: if (Q, $\cdot$) is a quasigroup and a, b are fixed elements of Q , then it is well known that (Q,$*$), with $(x\cdot a)*(b\cdot y) = x\cdot y$, is a loop, with the identity $b\cdot a$, and is isotopic to (Q,$\cdot$). Therefore it is enough to study the group of projectivities for loops.

Every projectivity α of a line L onto itself in a k-net N can be represented as

$$\bar{\alpha} = \prod_{i=1}^{n} [L_{i-1}, \mathfrak{X}_i] [\mathfrak{X}_i, L_i]$$

with $L_o = L_n = L$.

We say that the representation $\bar{\alpha}$ is irreducible (of length n) if $L_i \neq L_{i+1}$ and $\mathfrak{X}_i \neq \mathfrak{X}_{i+1}$. To a representation $\bar{\alpha}$ and to the set of points $c = \{ a_i^o \mid i = s,\dots, m; a_i^o \in L \}$ we associate the configuration $\Omega(\bar{\alpha} , c)$ consisting of all lines L_i (the "generators" of Ω), of the points

$$a_i^{(k)} = (a_i^o)\bar{\alpha}^k = (a_i^o) \prod_{i=1}^{k} [L_{i-1}, \mathfrak{X}_i] [\mathfrak{X}_i, L_i]$$

and of the lines (the non-trivial "projection lines") joining the different pairs of points $a_i^{(k-1)}$ and $a_i^{(k)}$.

The general problem of determining the group Π of projectivities of a given loop Q seems to be difficult. If however the loop Q satisfies some additional algebraic properties then we can determine the group Π of projectivities explicitly. Now we will compute Π for loops Q having the inverse property. We remember that a loop Q has the <u>inverse property</u> if and only if for every x there exist a and b in Q such that

$$a(xy) = y \quad \text{and} \quad (yx)b = y \quad \text{for all } y \in Q$$

[6], p. 111.

If G is a loop with the inverse property, then we will call P(G) the group which is generated by the mappings

$$\{x \to (ax)b \ \text{ and } \ x \to a(xb) \ ; \ G \to G\}.$$

Let L respectively R be the set of all left translations $x \to ax$, respectively of all right translations $x \to xa$. Let $L(G)$ respectively $R(G)$ be the group generated by L respectively R . If G is a group then P(G) is a product of $L(G)$ and $R(G)$; moreover $L(G)$ and $R(G)$ are then normal subgroups of G .

In any case $L \cap R$ consists of translations $x \to ax$ such that a is contained in the centre of G ; that is $ax = xa$ for all x . This assertion follows from the fact that if a left multiplication $x \to ax$ belongs to R , then there exists an element b such that for every x the equation $ax = xb$ holds; therefore $(ax)b^{-1} = x$, since a loop with the inverse property has a unique inverse element (see [6], p. 111). For $x = 1$ it follows $ab^{-1} = 1$ and so $b=a$. Then we have $ax = xa$ for all $x \in G$.

If in particular G is a group then, moreover $R(G) \cap L(G)$ consists of all translations with elements out of the centre of G . Moreover in P(G) every element of $L(G)$ commutes with every element of $R(G)$; since $(g_1x)g_2 = g_1(xg_2)$ holds for all $x \in G$ we have for the left translation λ_{g_1} and the right translation ρ_{g_2} the equation

$$\lambda_{g_1} \rho_{g_2} = \rho_{g_2} \lambda_{g_1} .$$

Theorem (1.1). The group Π *of projectivities of a loop* G *with the inverse property is isomorphic as permutation group to the group generated by the group* $P(G)$ *and the mapping* $\tau = (x \to x^{-1} : G \to G)$. *The group* $P(G)$ *is a normal subgroup of* Π .

The mapping τ *operates on* $L \cup R \subseteq P(G)$ *in the following way. For* $\rho_a = (x \to xa)$ and $\lambda_a = (x \to ax)$ *holds* $\tau\rho_a\tau = \lambda_a{-1}$ and $\tau\lambda_a\tau = \rho_a{-1}$.

If $\tau \notin P(G)$ *- and this for instance is the case when* G *is a group - then* Π *is the semidirect product of* $P(G)$ *and* $< \tau >$.

Proof. If $N(G)$ is the 3-net associated to G then we can describe the action of the different types of perspectivities within $N(G)$ as follows.

Perspectivity	Preimage	Image
$[g_h, \mathfrak{v}]$	the point (x, g)	the line x_v
$[g_h, \mathfrak{v}]^{-1} = [\mathfrak{v}, g_h]$	the line x_v	the point (x, g)
$[g_h, \mathfrak{t}]$	the point (x, g)	the line $(xg)_t$
$[g_h, \mathfrak{t}]^{-1} = [\mathfrak{t}, g_h]$	the lines x_t	the point (xg^{-1}, g)
$[g_v, \mathfrak{h}]$	the point (g, x)	the line x_h
$[g_v, \mathfrak{h}]^{-1} = [\mathfrak{h}, g_v]$	the line x_h	the point (g, x)
$[g_v, \mathfrak{t}]$	the point (g, x)	the line $(gx)_t$
$[g_v, \mathfrak{t}]^{-1} = [\mathfrak{t}, g_v]$	the line x_t	the point $(g, g^{-1} x)$
$[g_t, \mathfrak{h}]$	the point (x, y) (with $xy = g$)	the line y_h
$[g_t, \mathfrak{h}]^{-1} = [\mathfrak{h}, g_t]$	the line x_h	the point (gx^{-1}, x)
$[g_t, \mathfrak{v}]$	the point (x, y) (with $xy = g$)	the line x_v
$[g_t, \mathfrak{v}]^{-1} = [\mathfrak{v}, g_t]$	the line x_v	the point $(x, x^{-1}g)$

We assume now that α is a projectivity of a line g_h onto itself. Then α can be decomposed in projectivities γ_i of smallest length such that the preimage line and the image line of γ_i are always in the set $\mathfrak{h} \cup \mathfrak{v}$. We discuss now the possibilities for γ_i and describe the action on the points explicitly.

$$
\left.\begin{array}{lll}
\text{If } \gamma_i = [\, g_h , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_h'\,] & \text{then} & (x, g)\gamma_i \to (x, g')\,; \\
\text{if } \gamma_i = [\, g_h , \mathfrak{z}\,]\,[\,\mathfrak{z} , g_h'\,] & \text{then} & (x, g)\gamma_i \to [(xg)g'^{-1} , g']\,; \\
\text{if } \gamma_i = [\, g_v , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_v'\,] & \text{then} & (g, x)\gamma_i \to (g', x)\,; \\
\text{if } \gamma_i = [\, g_v , \mathfrak{z}\,]\,[\,\mathfrak{z} , g_v'\,] & \text{then} & (g, x)\gamma_i \to [g', g'^{-1}(g\,x)]\,; \\
\text{if } \gamma_i = [\, g_h , \mathfrak{z}\,]\,[\,\mathfrak{z} , g_v'\,] & \text{then} & (x, g)\gamma_i \to [g', g'^{-1}(xg)]\,; \\
\text{if } \gamma_i = [\, g_v , \mathfrak{z}\,]\,[\,\mathfrak{z} , g_h'\,] & \text{then} & (g, x)\gamma_i \to [(gx)g'^{-1}, g']\,; \\
\text{if } \gamma_i = [\, g_h , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_t'\,]\,[\, g_t' , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_v''\,] & & \\
 & \text{then} & (x, g)\gamma_i \to (g'' , x^{-1}g')\,; \\
\text{if } \gamma_i = [g_v , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_t'\,]\,[g_t' , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_h''\,] & & \\
 & \text{then} & (g, x)\gamma_i \to (g'\,x^{-1}, g'').
\end{array}\right\}\quad (1)
$$

If one piece of a projectivity is of the form:

$$[\, g_h , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_t^{(1)}\,]\,[\, g_t^{(1)} , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_t^{(2)}\,] \cdots\cdots$$

or $[\, g_v , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_t^{(1)}\,]\,[g_t^{(1)} , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_t^{(2)}\,] \cdots\cdot$

then this can be written in (reducible) form as

$$[\, g_h , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_t^{(1)}\,]\,[g_t^{(1)} , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_v^{*}\,]\,[\, g_v^{*} , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_t^{(2)}\,] \cdots \quad (2)$$

or respectively

$$[g_v , \mathfrak{h}\,]\,[\,\mathfrak{h} , g_t^{(1)}\,]\,[g_t^{(1)} , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_h^{*}\,]\,[g_h^{*} , \mathfrak{v}\,]\,[\,\mathfrak{v} , g_t^{(2)}\,] \cdots \quad (3)$$

Since (2) and (3) can be expressed as products of the last two projectivities given in (1) it is clear that the projectivity α can be decomposed in projectivities γ_i such that every γ_i occurs in (1). Every γ_i acts only on the variable coordinate x. The image of x arises under each γ_i by a suitable composition of the following mappings:

$$x \to a\,x, \qquad x \to x\,b, \qquad x \to x^{-1}$$

Therefore α has the same property and the theorem follows since for every two elemente c , d we have $(cd)^{-1} = d^{-1}c^{-1}$ [6], p. 111, (1.8)). □

Let N be a 3-net which is embedded in an affine plane A . It should be noticed that not every collineation of A leaving N invariant induces a projectivity of N . In the classical cases also if A is desarguesian and N is the additive 3-net (cf. [21], p. 61) besides the translations of A the only collineations which induce projectivities are the reflections on a point.

At the end of this section we give examples of loops G with the inverse property such that the map $\tau = (x \to x^{-1} ; G \to G)$ is contained in $P(G)$.

Let Q_m be the free loop over a set of generators with the cardinality m . Let us denote by x_l^{-1} , respectively x_r^{-1} the elements of Q_m defined by $x_l^{-1} x = 1 = xx_r^{-1}$. Let N be the normal subloop belonging to the relations

$$y_l^{-1} [(yx)x_r^{-1}] = 1 , [x_l^{-1} (xy)] y_r^{-1} = 1 .$$

The factor loop $\Psi_m = Q_m/N$ is the free loop with the inverse property over a set of generators with the cardinality m . In Ψ_m holds $x_l^{-1} = x_r^{-1} = x^{-1}$ ([6], p. 111). Let now M be the normal subloop of Ψ_m belonging to the relation $[y^{-1}(xy)]x=1$. It is clear that in the factor loop $\Phi_m = \Psi_m/M$ the map τ is contained in $P(\Phi_m)$. Since the loop Φ_m is not power-associative the stabilizer of $\Pi(\Phi_m)$ on the point $(1,1)$ of the line 1_h contains - besides τ - many other elements different from the identity, for instance the maps $\tau_a = [x \to (a^{-1} x)a : \Phi_m \to \Phi_m]$ which are all different from τ .

§ 2. The Staudt's Theorems for loops

In this section we give geometric characterizations for the abelian groups in the wide class of loops.

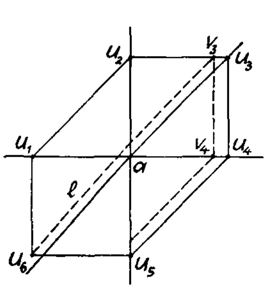
Contrary to the case of projective planes there is no chance of characterizing the whole class of abelian groups by the condition that the stabilizer $\Pi_{x_1,\dots,x_n}$ of the group of projectivities Π fixing every element of an arbitrary n-tuple consists only of the identity. In fact let G be an abelian group which is not an elementary 2-group and which has s involutions. Then $x \to x^{-1}$ is a projectivity $\neq 1$ having $s + 1$ fixed points. However for the class of abelian groups without involutions we have a direct analogue of the classical Staudt theorem for planes.

Theorem (2.1). A loop G is an abelian group without involutions if and only if the pointwise stabilizer of Π on every two distinct points consists only of the identity.

Before we give the proof of theorem (2.1) we notice the following

Proposition (2.2). If an a 3-net N all those Thomsen configurations close for which the three diagonals do not intersect at the same point then the hexagonal condition holds (i.e. the Thomsen condition holds without restrictions).

Proof.



If the hexagonal condition with respect to the point a does not hold then the point u_6 in figure 1 does not belong to the line au_3 . Then the line ℓ through u_6 which belongs to the same family $\mathfrak{Z}$ as au_3 intersects the two lines through a of the other two families in two different points, and meets the line u_2u_3 in a point $v_3(\neq u_3, u_2)$. Let v_4 be the intersection of u_1u_4 with the line through v_3 belonging to the same family as au_2. The points u_6, u_1, u_2, v_3, v_4 and the lines u_1v_4, u_2a

and ℓ lead to a Thomsen configuration which satisfies our assumption but which does not close. □

Proof of theorem (2.1). Let us consider the assumptions of the Thomsen condition. Let V_1, V_2, V_3 be three different vertical lines and H_1, H_2, H_3 be three different horizontal lines and assume that the points $V_1 \cap H_1$ and $V_3 \cap H_3$ are on a transversal line T_1 and that the points $H_1 \cap V_2$ and $H_2 \cap V_3$ are on a transversal line T_2. Under these hypotheses we want to prove that the points $H_2 \cap V_1$ and $H_3 \cap V_2$ are also on a transversal T_3. We can(assume because of proposition (2.2)) that the two points $a = H_2 \cap V_2$ and $b = H_2 \cap T_1$ are different. Consider then the following projectivity:

$$\delta = [H_2, \mathfrak{w}][\mathfrak{w}, T_1][T_1, \mathfrak{h}][\mathfrak{h}, V_2][V_2, \mathfrak{z}][\mathfrak{z}, H_2]$$

The projectivity δ^2 fixes the points a and b; therefore $\delta^2 = 1$. Let be $x = H_2 \cap V_1$ then $x^{\delta^2} = x$ and therefore the transversal line T_3 through x carries the point $V_2 \cap H_3$. Hence the Thomsen condition holds and Q is an abelian group.

Viceversa if Q is an abelian group then the stabilizer of Π on one point has order at most two (cf. § 1) and the assertion follows. □

In order to characterize the whole class of abelian groups by the group of projectivities we take a condition on the order of a stabilizer of Π on a point.

Theorem (2.3). A loop G is an abelian group if and only if the stabilizer Π_a of a point a in the group Π of projectivities of G has order at most two.

Proof. If G is an abelian group then one part of the theorem follows from theorem (1.1). Let us consider the other direction. In the isotopy class of G there exists a loop Q such that the multiplication of Q is given by the natural multiplication in $N(G)$ with respect to the point $0 = (1,1)$(which is chosen in an arbitrary way in $N(G)$ but which gives the 1 of the natural multiplication). Let $N(Q)$ be the not corresponding to Q, and let l_v, l_h, l_t be the vertical, horizontal and transversal lines passing through the point $(1,1)$. The hexagonal condition for the point $(1,1)$ is equivalent with the fact that every element in Q has exactly one inverse ([21], p. 54). Let $\Pi(Q) = \Pi$ be the group of projectivities of Q and let $\Pi_{(1,1)}$ be the stabilizer of Π on $(1,1)$. The maps $(x, 1) \rightarrow (x_r^{-1}, 1)$ and $(x, 1) \rightarrow (x_l^{-1}, 1)$ (where x_r^{-1} and x_l^{-1} are respectively the right and the left inverse of x) are given by the projectivities

$$[1_h, \mathfrak{V}]\,[\mathfrak{V}, 1_t][1_t, \mathfrak{H}][\mathfrak{H}, 1_v][1_v, \mathfrak{T}][\mathfrak{T}, 1_h]$$

and respectively

$$[1_h, \mathfrak{T}]\,[\mathfrak{T}, 1_v][1_v, \mathfrak{H}][\mathfrak{H}, 1_t][1_t, \mathfrak{V}][\mathfrak{V}, 1_h] \,.$$

Since $|\Pi_{(1,1)}| \leq 2$ holds, it follows that in N the hexagonal condition holds for the point $O = (1,1)$.

Since $|\Pi_x| \leq 2$ for every point $x \in N(G)$ and every point x can be chosen as origin for a loop belonging to the isotopy class of G, the hexagonality condition holds in general and the isotopy class contains power associative loops only ([1], thm. 3.5, p. 406).

If $|\Pi_x| = 1$ for one point x then Π is sharply transitive , i.e. $|\Pi_x| = 1$ for every x and in the net the condition of "parallel diagonals" holds and ([21] , p. 60) the loop Q is a group such that every element different from 1 is an involution. Therefore Q is an elementary abelian 2-group and the theorem holds.

We assume now that for every x is $|\Pi_x| = 2$, and that Q is a power associative loop. Let G_1, G_2, G_3 be three lines belonging to the same class $\mathfrak{X}$, let a be a point on G_1 and $\mathfrak{Y}$, $\mathfrak{Z}$ the two remaining classes of lines. Let us consider the projectivity:

$$\varepsilon = [G_1, \mathfrak{Y}][\mathfrak{Y}, G_2][G_2, \mathfrak{Z}][\mathfrak{Z}, G_3][G_3, \mathfrak{Y}] ;$$

the two lines $(a)\varepsilon$ and $(a)[G_1,\mathfrak{Z}]$ have exactly one point s in common. Let G_4 be the line of $\mathfrak{X}$ through s .

The projectivity

$$\delta = \varepsilon [\mathfrak{Y}, G_4][G_4, \mathfrak{Z}][\mathfrak{Z}, G_1]$$

is contained in the stabilizer Π_a of the point a .

Let us first choose $a = (1,1)$, $G_i \in \mathfrak{h}$,
$G_1 = 1_h$, $G_3 = g''_h$, $G_2 = G_4 = g'_h$ and $G_1 \neq G_2 \neq G_3 \neq G_4 \neq G_1$
and $\mathfrak{Y} = \mathfrak{v}$, $\mathfrak{Z} = \mathfrak{z}$. Then $\delta \in \Pi_{(1,1)}$ is equivalent to $g' = g'^{-1}g''$.
Since Q is power associative it follows $g'' = g'^2$. For any point $(x, 1) \in G_1$ we have then

$$(x, 1)^{\delta} = (\omega, 1)$$

with $xg' = \eta g'^2$ and $\eta g' = \omega$. Since $|\Pi_{(1,1)}| = 2$ then for every x is either $\omega=x$ or $\omega=x^{-1}$. If it is $\omega=x$ for every x then in Q holds one Bol condition. Assume that there exists some $x \in Q$ with $x^{\delta} = x^{-1} \neq x$. Now if $g'^{-1} \neq g'$ we can choose $x = g'^{-1}$ and then it would follow $\eta = g'^{-2}$ and $g' = \omega = g'^{-1}$, therefore a contradiction. If g' is an involution then we have $xg' = \eta$ and $\eta g' = x^{-1}$ and get for every involution g' and every x the following rule of computation

$$(xg')\, g' = x^{-1} . \tag{1}$$

Let us choose now on the other hand $a = (1,1)$, $G_i \in \mathfrak{v}$,
$G_1 = 1_v$, $G_3 = g''_v$, $G_2 = G_4 = g'_v$ and $G_1 \neq G_2 \neq G_3 \neq G_1$ and

$\mathfrak{Y} = \mathfrak{h}$, $\mathfrak{Z} = \mathfrak{z}$, then $\delta \in \Pi_{(1,1)}$ is equivalent with $g'' = g'^2$.

For any point $(1, x) \in G_1$ we have then $(1, x)^{\delta} = (1, \omega)$ with $g'x = g'^2\eta$ and $g'\eta = \omega$. An analogous computation as before shows us that we have either a further Bol condition or that $\omega = x^{-1}$ holds for every x and that there exists some $x \in Q$ such that $x^{\delta} \neq x^{-1} = x$. In the second case we obtain for every involution g' and every x the following relation

$$g'(g'x) = x^{-1} . \tag{2}$$

We choose now $a = (1,1)$, $G_i \in \mathfrak{h}$, $G_1 = 1_h$, $G_2 = g'_h$, $G_3 = g''_h$, $G_2 = g'''_h$ in such a way that all the G_i are different and it holds $\mathfrak{H} = \mathfrak{V}$, $\mathfrak{Z} = \mathfrak{z}$. In order to have $\delta \in \Pi_{(1,1)}$ we must have $g' = \zeta\, g''$, $\zeta\, g''' = 1$ and therefore $g' = g'''^{-1} g''$.

If $(x, 1) \in G_1$ is any point, then we have $(x, 1)^{\delta} = (\omega, 1)$ with $x(g'''^{-1} g'') = \eta g''$ and $\eta g''' = \omega$. As before, because of $(x \to x^{-1}) \in \Pi_{(1,1)}$, it follows that $\omega = x^{\pm 1}$. Moreover if we choose g''' involutory we obtain that for $x = g'''$ holds $\omega = g'''$ for $x = g'''$.

Then we have

$$g''' (g''' \; g'') = g'' \tag{3}$$

for every involutory element g''' and every element g''. But (3) contradicts (2).

Finally we choose $a = (1,1)$, $G_i \in \mathfrak{V}$, $G_1 = 1_v$, $G_2 = g'_v$, $G_3 = g''_v$, $G_4 = g'''_v$ in such a way that all the G_i are different and it holds $\mathfrak{V} = \mathfrak{h}$ and $\mathfrak{Z} = \mathfrak{z}$. In order that δ fixes the point $(1,1)$ it is necessary that $g' = g''\, g'''^{-1}$.

For $(1,g''')$, where g''' is any involution, we obtain by a computation analogous to the above the relation

$$(g''\, g''')g''' = g'' \tag{4}$$

for every element g''. But (4) contradicts (1).

Therefore in the loop Q two Bol conditions are fulfilled and so all three Bol conditions. Then the loop Q is a Moufang loop ([21], p. 57-58). Now it follows (see for instance [1], p. 416) that every loop of the isotopy class of Q is a Moufang loop. As a Moufang loop Q is di-associative (see [6], p. 117).

We choose now $a = (1,1)$, $G \in \mathfrak{h}$, $G_1 = 1_h$, $G_2 = g'_h$, $G_3 = g''_h$, $G_4 = g'''_h$ in such a way that all the G_i are different and $\mathfrak{h} = \mathfrak{V}$, $\mathfrak{Z} = \mathfrak{z}$. In order that δ fixes the point $(1,1)$ we must have $g' = g'''^{-1} g''$. If in Q the Reidemeister condition is not fulfilled then three follows $(x,1)^{\delta} = (x^{-1}, 1)$ for every x and there exist x with $x \neq x^{-1}$.

Now we have $x(g''^{\prime -1} g'') = \eta g''$ and $\eta g''' = x^{-1}$.

With $x = g''$ it follows from these that $g''(g'''^{-1} g'') = \eta g''$ and then $[(g'' g'''^{-1})g'']g''^{-1} = (\eta g'')g''^{-1}$ and therefore $\eta = g'' g'''^{-1}$ and then $(g'' g'''^{-1})g''' = g''^{-1}$ and so $g'' = g''^{-1}$. This gives us a contradiction since we have assumed that there is an element in Q different from its inverse.

Hence in Q the Reidemeister condition holds and Q is a group. From the theorem (1.1) we know that the stabilizer $\Pi_{(1,1)}$ of the group of projectivities consists of the mappings $x \to a^{-1} x a$ for every $a \in Q$ and of the mapping $x \to x^{-1}$ which in our case is different from the identity. If G would not be abelian then the only inner automorphism different from the identity should be the inversion $x \to x^{-1}$, since $\Pi_{(1,1)}$ has the order two; but a group for which $x \to x^{-1}$ is an automorphism is abelian. □

A further very easy characterization of the abelian groups is given by the following

Corollary (2.4). A loop Q is an abelian group if and only if the group Π of projectivities of Q contains a sharply point transitive subgroup of index at most two.

Proof. The stabilizer of a point has order at most two. Hence the result follows from theorem (2.3).

At the end of this section we want to exhibit a theorem which is analogous to the theorems which characterize the pappian planes as planes in which every projectivity can be represented by a chain of length less or equal 4 (cf. [21], p. 139).

Theorem (2.5) A loop Q *is an abelian group if and only if the group* Π *of projectivities contains a subgroup* N *of index* ≤ 2 *in which every projectivity can be represented by a chain with the length 0 or 4 .*

Proof. If Q is an abelian group then the group of projectivities is known (see § 1). For N we can take the group of all maps $x \to ax$.

Let Q be a loop satisfying our conditions. The map $\alpha_\ell : x \to x_\ell^{-1}$ where $1 = x_\ell^{-1}x$ is a projectivity such that (if $\alpha_\ell \neq 1$) its shortest representation has length six. If α_ℓ is the identity then every element of $Q \setminus \{1\}$ is an involution and the net $N(Q)$ satisfies the condition of the parallel diagonals. From [21] p. 60 follows that Q is an elementary abelian 2-group. Hence we may assume that $\alpha \neq 1$ holds. Every projectivity of the line 1_h of lenght 4 is either

$$\rho_a : x \to xa \quad , \quad a \neq 1$$

or ρ_a^{-1}. Therefore N operates fixed point free on 1_h, i.e. only the identity of N has fixed points. If N would not operate transitively on 1_h then N would have (since Π is transitive) two distinct domains D_1 and D_2 of transitivity. Assume for instance $(1,1) \in D_1$. One has $\Pi = N \langle \alpha_\ell \rangle$. Now $(1,1)^{\alpha_\ell} = (1,1)$ and hence $D_1^{\alpha_\ell} = D_1 = D_1^{\Pi}$ which is a contradiction to the transitivity of Π. Therefore N is a sharply transitive normal subgroup of Π. Now corollary (2.4) gives the result. $\square$

§ 3. The collineation group of a loop.

The collineation group Σ of a quasigroup Q is the (full) collineation group of the 3-net $N(Q)$ belonging to Q (*). Therefore Σ is the same for every quasigroup out of the same isotopy class, and we can assume that Q is a loop. The group Σ has a normal subgroup Γ of index ≤ 6 in Σ which maps into itself every class of parallel lines; this subgroup will sometimes be called the group of collineations of $N(Q)$ which preserves the directions. If the group Γ contains a subgroup Ψ which leaves every line out of one given class invariant and operates transitively on the line as a point set, then Q is a group and vice-versa ([4] p. 189). For non-associative loops then Σ cannot contain such a transitive "glide group" and the determination of Σ is difficult (cf. [3], chap. V).

On the other hand the determination of the stabilizers Γ_{1_v} , Γ_{1_h} and $\Gamma_{(1,1)}$ on the lines 1_v, 1_h and the point $(1,1)$ is easy.

A permutation α of a loop Q is called a <u>right</u> respectively <u>left pseudoautomorphism</u> if there exists at least one element c of Q , called a <u>companion</u> of α such that for every x, y

$$(x^{\alpha})(y^{\alpha}c) = (xy)^{\alpha}c \qquad \text{respectively}$$

$$(cx^{\alpha})(y^{\alpha}) = c(xy)^{\alpha} \qquad \text{holds.}$$

If α is such a one-sided pseudo-automorphism then we have $1^{\alpha} = 1$. If it is clear which class of one-sided pseudo-automorphism is considered or if it does not matter whether a pseudo-automorphism is one-sided or two-sided, we sometimes only use the term pseudo-automorphism and its companions . If Q is a loop with the inverse property then every one-sided pseudo-automorphism is two-sided ([6], p. 113); the same holds naturally for commutative loops.

(*) A collineation is defined to be a permutation of this points of $N(Q)$ such that a line is always mapped on a line.

We have the following

<u>*Theorem (3.1).*</u> *Let* Q *be a loop and* $N(Q)$ *the net belonging to* Q *. Let us denote by* Γ *the group of all collineations of* $N(Q)$ *which leave the set of horizontal lines and the set of vertical lines invariant. Then the stabilizer* Γ_{1_v} *on the line* 1_v *consists exactly of the set of mappings* $(x, y) \to (x^{\alpha_v}, y^{\alpha_h})$ *where* α_v *is a right pseudo-automorphism of* Q *and* $\alpha_h = \alpha_v \tau_c$ *, denoting by* τ_c *the right translation by a companion* c *of* α_v *. Likewise the stabilizer* Γ_{1_h} *consists of the set of mappings* $(x, y) \to (x^{\alpha_v}, y^{\alpha_h})$ *where* α_h *is a left pseudo-automorphism of* Q *and* $\alpha_v = \alpha_h \lambda_c$ *, denoting by* λ_c *the left translation by a companion* c *of* α_h *.*

The stabilizer Γ_{1_v} *(respectively* Γ_{1_h} *) is exactly then transitive on* 1_v *(resp.* 1_h *) if every element* $c \neq 1$ *is a companion of a right (respectively left) pseudo-automorphism of* Q *.*

If Q *has the inverse property then the stabilizer* Γ_{1_v} *(respectively* Γ_{1_h} *) operates exactly then transitively on* 1_v *(respectively* 1_h *) if* Q *is a Moufang loop.*

If Q *is a group then* Γ *consists of the maps* $(x, y) \to (c\, x^{\alpha}, y^{\alpha} d)$ *where* $c, d \in Q$ *and* α *is an automorphism of* Q *.*

<u>Proof.</u> Let α be a collineation out of Γ_{1_v} and $(x, y)^{\alpha} = (x^{\alpha_v}, y^{\alpha_h})$. The point (x, y) lies on the line $(xy)_t$; this also contains the point $(1, xy)$. Since α leaves invariant also the set of the transversal lines it follows

$$x^{\alpha_v} y^{\alpha_h} = (1^{\alpha_v})(xy)^{\alpha_h} = (xy)^{\alpha_h}$$

since $\alpha \in \Gamma_{1_v}$. With $x = g$ and $y = 1$ we have $g^{\alpha_v}(1^{\alpha_h}) = g^{\alpha_h}$.

Therefore if we put $1^{\alpha_h} = c$ we have

$$(x^{\alpha_v})(y^{\alpha_v} c) = (xy)^{\alpha_v} c$$

and α_v is a right pseudo-automorphism with companion c .

Conversely let α_v be a right pseudo-automorphism of Q with companion c . We shall prove that the mapping

$$(x, y) \to (x^{\alpha_v} , y^{\alpha_v} c) \qquad (*)$$

is a collineation of N(Q) . It is clear that the image of a vertical (respectively horizontal) line is a vertical (respectively horizontal) line. Let now (x, y) be such that $xy = d$ where d is a fixed element. Then $x^{\alpha_v} (y^{\alpha_v} c) = (xy)^{\alpha_v} c = d^{\alpha_v} c$ and so the image of a transversal line is a transversal line and $(*)$ is a collineation belonging to Γ_{1_v} since α_v is a right pseudo-automorphism.

The second statement of the theorem is a trivial consequence of the first since Γ_{1_v} is exactly then transitive to the points of 1_v if (1,1) can be mapped by Γ_{1_v} on every point (1, x) with $x \in Q$.

If Q has the inverse property then the group Γ_{1_v} is transitive on 1_v exactly then when every element is a companion of a pseudo-automorphism. Then the loop Q is isomorphic to the isotopic loop defined by $x * (b^{-1} y) = x \cdot y$ since there is a collineation moving (1,1) to any point of the line 1_v (cf. [21], p. 50) . From [6], p. 115 , thm. 2.3 , follows now that b is a Moufang element. Then every element is a Moufang element and Q is then a Moufang loop ([6], p. 113, lemma 2.2). □

It is interesting to notice that the pseudo-automorphisms and their companions have a deep geometrical meaning and appear in such a natural way in the study of the collineation group of a loop as the above theorem shows. Also the automorphism group A(Q) of a loop Q has a natural geometric interpretation. It is clear that we have a natural injection Φ from A(Q) into the stabilizer $\Gamma_{(1,1)}$ of the direction preserving group Γ , namely

$$\varphi : \alpha \to (\hat{\alpha} = [(x,y) \to (x^{\alpha} , y^{\alpha})]) .$$

The following theorem shows that φ is even a bijection.

Theorem (3.2) . *In every loop* Q *the injection* φ *gives an isomorphism between the automorphisms group* A(Q) *and the stabilizer* $\Gamma_{(1,1)}$ *of the direction preserving collineation group* Γ *on the point* (1,1).

Proof. One has $\Gamma_{(1,1)} \subseteq \Gamma_{1_v}$

and

$$\Gamma_{1_v} = \{(x,y) \to (x^\alpha , y^\alpha c)\}$$

where α is a right pseudo-automorphism of Q and c is a companion of α . If $\lambda \in \Gamma_{(1,1)}$ then we have

$$(1,1)^\lambda = (1^\alpha, 1^\alpha c) = (1, c) = (1,1)$$

i.e., c = 1 and α is an automorphism. □

The full stabilizer of the collineation group Σ on the point (1,1) can be computed for more special classes of loops, namely for loops having the inverse property or - expressing the same property geometrically - if in the corresponding net of Q both Bol condition hold for the point (1,1).

Theorem (3.3). *Let* Q *be a loop having the inverse property. Then the stabilizer* $\Sigma_{(1,1)}$ *of the full collineation group* Σ *of* Q *is the direct product of* $\Gamma_{(1,1)} \cong$ Aut Q *with a group* θ *isomorphic to the symmetric group* S_3, *of order* 6 ; *the group* θ *can be generated by the two following involutory collineations:*

$$\mu = [(x,y) \to (xy,y^{-1})] \quad , \quad \nu = [(x,y) \to (x^{-1}, xy)] .$$

Proof. The mappings μ and ν centralize every element out of $\Gamma_{(1,1)}$ and are collineations since the inverse property holds.

Since in Q holds $(xy)^{-1} = y^{-1} x^{-1}$ ([6], p. 111) one has $\mu\nu \neq 1$ and $(\mu\nu)^3 = 1$; therefore θ has order 6 and acts on the three sets of horizontal, vertical and transversal lines as the symmetric group S_3. □

We are going now to study under which circumstances the direction preserving group Γ of collineations of a loop Q has some transitivity properties. The next theorem shows that the class of loops for which the collineation group Γ is transitive is very large.

<u>*Theorem (3.4).*</u> *If Q is a loop such that every element is a companion of a right and of a left pseudo-automorphism then the group Γ of collineations of Q which preserves the directions is point transitive, and viceversa .*

<u>Proof.</u> From the theorem (3.1) we know

$$\Gamma_{1_v} = \{(x,y) \to (x^\alpha, y^\alpha a)\}$$

$$\Gamma_{1_h} = \{(x,y) \to (b x^\beta, y^\beta)\}$$

where α is a right and β is a left pseudo-automorphism and a and b are companions of α and respectively of β .

The complex $\Phi = \Gamma_{1_v} \cdot \Gamma_{1_h} \subseteq \Gamma$ consists of the mappings $(x, y) \to (b\cdot x^{\alpha\beta}, (y^\alpha \cdot a)^\beta)$.

Therefore we have $(1,1)^\Phi = \{(b, a^\beta)\}$ where b and a are freely chosen in Q , and so the result follows. □

<u>*Theorem (3.5).*</u> *For a loop Q the following three conditions are equivalent:*

1) The direction preserving collineation group is point transitive on the net $N(Q)$.

2) Every element of Q is a companion of a right and of a left pseudo-automorphism of Q .

3) Every loop which is isotopic to Q is isomorphic to Q .

<u>Proof.</u> The equivalence of 1) and 2) is given by theorem (3.4). If Q' (with the identity $1'$) is isotopic to Q then Q' belongs to $N(Q)$ (but possibly $(1,1) \neq (1', 1')$). If there is a collineation β , mapping $(1,1)$ into $(1', 1')$ then Q is isomorphic to Q' and viceversa every isomorphism between Q and Q' induces a collineation

in $N(Q)$ (cf. [21], p. 50). □

The theorem (3.5) is a solution of the problem presented in [6], p. 57 as an unsolved problem: Find necessary and sufficient conditions for the loop Q in order that every loop isotopic to Q be isomorphic to Q. An algebraic expression for the required condition is that every element of Q is a companion of a right and of a left pseudo-automorphism.

Remark (3.6): Let Q be a loop such that every element of Q is a companion of a right (respectively left) pseudo-automorphism. If the collineation group of Q contains an element which interchanges the set of the horizontal lines and the set of the vertical lines then every element of Q is a companion also of a left (respectively right) pseudo-automorphism and the collineation group Γ of Q which preserves the directions is point transitive.

The geometrical theorems (3.1) and (3.4) can be applied to obtain algebraic results on loops with transitive automorphism group (cf. for definition [6], p. 88).

Theorem (3.7). If Q is a loop with a transitive automorphism group then exactly one of the three following properties occur:

1) *Only the identity is a companion of a pseudo-automorphism, i.e. every pseudo-automorphism is an automorphism,*
2) *Every element is a companion of a right (respectively left) pseudo-automorphism, but no element $\neq 1$ is a companion of a left (respectively right) pseudo-automorphism.*
3) *Every element of Q is a companion of a right and of a left pseudo-automorphism.*

Proof. Since every element different from 1 can be mapped by an automorphism on every other element different from 1 the stabilizer Γ_{1_v} respectively Γ_{1_h} is exactly then transitive on 1_v respectively 1_h^v when there exists an element different from 1 which is a companion. But if Γ_{1_v} respectively Γ_{1_h} is transitive then every element of Q is a companion. □

<u>Theorem (3.8).</u> *If Q is a loop with a transitive automorphism group then either the right nucleus or the left nucleus consists only of the identity or every element of Q is a companion of a right and of a left pseudo-automorphism.*

<u>Proof.</u> The assertion follows from the previous theorem if we remember that for every element c contained in the right nucleus respectively left nucleus the equation $(x\cdot y)c = x(y\cdot c)$ respectively $c(x\cdot y) = (c.x)y$ holds. □

<u>Theorem (3.9).</u> *If Q is a proper commutative Moufang loop then the full collineation group of Q is not point transitive.*

<u>Proof.</u> We consider the stabilizer Γ_{1_v} of the group Γ of all collineations which leave the set of vertical and the set of horizontal lines invariant. Since every pseudo-automorphism of Q is an automorphism of Q ([6], p. 115, thm 2.2) Γ_{1_v} cannot operate transitively on 1_v since the companions of automorphisms lie in the nucleus of Q which is a proper subgroup of Q ([6], p. 114, thm. 2.1). □

From (3.9) and (3.5) follows

<u>Proposition (3.10)</u> (cf. [1], [2] and [6] , p. 58).
A necessary and sufficient condition that every loop isotopic to a Moufang loop Q be commutative is that Q be an abelian group.

<u>Theorem (3.11).</u> *Let Q be a loop which possesses the inverse property and has a transitive automorphism group. Then either the left nucleus, the right nucleus and the middle nucleus consist only of the identity which is the only Moufang element of Q or Q is a proper non commutative Moufang loop in which every element is a companion of a pseudo-automorphism or Q is a group. If the nucleus of Q consists only of the identity then the collineation group of Q which preserves the directions consists exactly of the maps:*

$$(x, y) \to (x^{\alpha}, y^{\alpha})$$

where α is an automorphism of Q ; also the full collineation group of Q leaves the point $(1,1)$ invariant.

Proof. Since Q has the inverse property, the left, middle, and the right nucleus coincide (cf. [6], thm. 2.1, p. 114). Now if the right nucleus is not equal 1 then the stabilizer Γ_{1_v} operates point transitively and every element of Q is a companion of a pseudo-automorphism of Q . Therefore Q is a Moufang loop ([6], lemma 2.2, p. 113). If Q is not a group then the first part of the assertions follows from theorems (3.8) and (3.9).

If β is a collineation of $N(Q)$ which maps $(1,1)$ on (m, n) then Q is isomorphic to the isotopic loop $(Q, *)$ where the composition $*$ is given by

$$(x \cdot m) * (n \cdot y) = x \cdot y$$

with the identity $n \cdot m$ ([21], p. 48). Therefore the isotopic loop has also the inverse property and it follows that m and n are Moufang elements ([6], thm. 2.3, p. 115). But the only Moufang element of Q is the identity and the last assertion follows with (3.3). □

Corollary (3.12). Let Q be a connected topological loop which possesses the inverse property and which is realized on a 1-dimensional manifold. If Q has a transitive automorphism group then either the left nucleus, the right nucleus and the middle nucleus consist only of the identity which is the only Moufang element of Q or Q is a group (which is isomorphic to $(\mathbb{R}, +)$ or to SO_2) .

Proof. From theorem (3.9) we have to exclude that Q is a proper non-commutative Moufang loop. This follows from [10], thm. (6.4) c) since Q is power associative. □

With the previous results we can also obtain some properties of division neorings (for the definition of this structure see [13],p. 507).

Theorem (3.13). If $(R, +, .)$ *is a finite planar division neoring such that its multiplicative loop is power associative then the additive loop* $(R, +)$ *is either an abelian group, or the left, the right and the middle nucleus of the additive loop* $(R, +)$ *which must have the inverse pro-*

perty consist only of the neutral element: in particular (R, +) *has no Moufang elements* $s \neq 0$.

Proof. From Th(II, 8) in [13] we know that the loop (R, +) is commutative and has the inverse property. Now the assertion follows from thm. (3.9) and lemma 2.2 of [6], p. 113. □

From (3.13) we can deduce by help of (3.9) the

Remark (3.14) . If (R, +) is a proper additive loop of a finite planar division neoring (R, +, .) whose multiplicative loop is power associative then the group Γ which preserves the directions leaves the point (1,1) fixed and operates transitively on the other points of a line through (1,1). Also the full group of collineations of (R, +) leaves (1,1) fixed.

Any division neoring F which has an element $a \neq 0$ contained in some additive subgroup possesses a prime subfield K lying in the center of F. The characteristic of K is called the characteristic of F (cf. [12] pp. 38-40).

A division neoring (R, +, .) is called a topological division neoring if the operations "+" and "." and all the binary operations which arise from solving the equations in (R, +) and (R, .) are simultaneously continuous in both variables.

Theorem (3. 15).Let (R, +, .) *be a division neoring with associative multiplication such that the additive loop* (R, +) *possesses the inverse property.*

(A). If the characteristic of R *is different from* 3 *then the neoring is either a skewfield or the left, the right and the middle nucleus of the additive loop consist only of the identity which is the only Moufang element of* (R, +).

(B). If (R, +, .) *is a connected, locally compact, finite dimensional, topological neoring then either* (R, +, .) *is one of the three classical fields (real numbers, complex numbers, quaternions) or the second alternative of (A) holds.*

Proof. From [12] , (1.18), p. 41 (cf. also [21] § 3.4 and [12] (4.13)) follows that (R, +) cannot be a proper Moufang loop. The rest follows from thm. (3.11) since (R, +) has a transitive automorphism group. □

We notice that there exist planar associative division neorings which possess an additive loop with the inverse property and which are homeomorphic to the real line $\mathbb{R}$ (cf. [22], pp. 459-461, § 13). The whole collineation group of such a loop leaves in the net N(Q) the point (1,1) fixed.

Paige gave necessary and sufficient conditions that the additive loop of an associative neoring is a commutative Moufang loop ([20]thm. II,11). From (3.11 (cf. also [5], 70 corollary 2)) follows that this condition can be satisfied only if the additive loop is an abelian group.

We study now the collineation group of a Moufang loop.

Theorem (3.16). Let Q *be a Moufang loop and* N(Q) *the net belonging to* Q *. Let us denote by* Σ *the full collineation group of* **Q** *and by* Γ *the subgroup of index* 6 *in* Σ *which leaves the set of horizontal lines and the set of vertical lines invariant.* Σ *(respectively* Γ *) operates transitively on the flags (respectively points) of* N(Q) *if and only if every element* $c \neq 1$ *of* Q *is a companion of a pseudo-automorphism of* Q *. On the set of lines* Σ *operates always transitively.*

Proof. The mappings

$$\gamma_a = [(x, y) \to (ax, ya);\ a \in Q] \qquad (o)$$

are collineations of N(Q) . For it is clear that γ_a maps the vertical (respectively horizontal) lines on vertical (respectively horizontal) lines. Let us consider now all points (x, y) with $xy = d$ where d is a fixed element. Then we have ([6], p. 115, lemma 3.1)

$$(ax)(ya) = [a (xy)] a = ada$$

and so the image of a transversal line is a transversal line.

The set of the collineations (o) operates transitively on the set of the vertical (respectively horizontal) lines. The group Σ contains (see thm. (3.3)) an involution μ which maps the set of the transversal lines onto the set of vertical lines such that $(1_t)^{\mu} = 1_v$. Let now W be any transversal line; then there exist a γ_a with

$$(1_v)^{\gamma_a} = W^{\mu} = (1_t)^{\mu\gamma_a}$$

and we have $(1_t)^{\mu\gamma_a\mu} = W^{\mu\mu} = W$. Since Γ is a normal subgroup of Σ one has $\mu\gamma_a\mu \in \Gamma$ and Γ operates transitively also on the set of transversals.

Let us denote by Φ the subgroup of Γ generated by (o) . Since $< \Phi\, \Gamma_{1_v} > \subset \Gamma$ we can map every point (x, y) on $(1,1)$ exactly then when every element of Q is a companion of a pseudo-automorphism of Q ; using a suitable γ_a we obtain $(x, y)^{\gamma_a} = (1, t)$ and the rest of the statement follows from the theorem (3.4). $\square$

§ 4. The Lenz classification for loops and 3-nets.

Let N be a k-net ($3 \leq k$). A translation α is a collineation of N which preserves the directions and leaves invariant every line of a direction $\mathfrak{X}$. If $\alpha \neq 1$ we shall call the direction $\mathfrak{X}$ the axis of α.

Remark (4.1). If $\alpha \neq 1$ is a translation of a k-net N, then α has no fixed points.

Proof. If s is a fixed point of α then every line incident with s consists only of fixed points. □

A collineation β of a k-net N ($3 \leq k$) which preserves the directions will be called a homology if all elements different from 1 of the group $< \beta >$ generated by β have exactly one fixed point p, the centre of β.

The translationswith the same axis $\mathfrak{X}$ and all homologies with the same centre p form according to the case a subgroup $T(\mathfrak{X})$ and $S(p)$ of the group Γ of all collineations which preserve the directions. We shall call the group $T(\mathfrak{X})$ and then also the axis $\mathfrak{X}$ transitive if the direction $\mathfrak{X}$ contains a line G such that $T(\mathfrak{X})$ is transitive on the points of G; in this case $T(\mathfrak{X})$ operates sharply point transitively on every line belonging to $\mathfrak{X}$. In an analogous way we shall call the group $S(p)$ of homologies and then also the centre p transitive if $S(p)$ is transitive on the points, different from p, of a line G incident with p; then the group $S(p)$ operates sharply transitively on the points different from p of every line incident with p.

Theorem 4.2. If N is a 3-net then N belongs to exactly one of the following seven Lenz classes:

I.1. - In N there does exist neither a transitive axis nor a transitive centre.

I.2. - In N there is no transitive axis, but there exists exactly one transitive centre.

I.3. - In N *there exists no transitive axis, but on every line there exists exactly one transitive centre.*

I.4. - In N *there exists no transitive axis, but the transitive centres of* **N** *are exactly the points of one line of* N .

I.5. - In N *there exists no transitive axis, but every point is a transitive centre.*

II.1 - In N *every direction is a transitive axis, but there is no transitive centre.*

II.2 - In N *every direction is a transitive axis and every point is a transitive centre.*

If Q is a loop then we will say that Q is of <u>Lenz type</u> A.a, where $A \in \{I,II\}$ and $a \in \{1,2,3,4,5\}$ if the net belonging to Q has the Lenz class A.a.

Let Q be a loop with respect to the multiplication " • " . The operations $(a, b) \to a\backslash b : Q \to Q$ respectively $(a, b) \to a/b : Q \to Q$ which are defined by $a\cdot(a\backslash b) = b$ respectively $(a/b)\, a = b$ give us on Q two further loop structures; one can assign in such a natural way to $(Q,\cdot)$ the <u>right</u> and the <u>left</u> <u>reversed</u> <u>loop</u>.

Under the cardinality of the isotopy class I(Q) of a loop Q we understand the number of different isomorphy classes of loops within I(Q).

<u>Corollary (4.3).</u> Every loop Q belongs to exactly one of the seven Lenz classes I.1 till II.2.

A loop Q is of type I.1 if and only if Q is not a group and no loop out of the isotopy class I(Q) admits a sharply transitive group of automorphisms.

A loop Q is of Lenz type I.2 if and only if the cardinality of the isotopy class I(Q) is at least five and I(Q) contains a loop admitting a sharply transitive group of automorphisms.

A loop Q is of Lenz type I.3 if and only if the cardinality of the isotopy class I(Q) is exactly two, I(Q) contains a loop Q^* admitting a sharply transitive group of automorphisms and no element $\neq 1$ of Q^* and its reversed loops is a companion of a pseudo-automorphism.

A loop Q is of Lenz type I.4 if and only if the cardinality of the isotopy class I(Q) is exactly two, I(Q) contains a loop Q^* admitting a sharply transitive group of automorphisms and every element of Q^* or of one of its reversed loops is a companion of a suitable right (respectively left) pseudo-automorphism.

A loop Q is of Lenz type I.5 if and only if Q admits a sharply transitive group of automorphisms, every loop isotopic to Q is isomorphic to Q and Q is not a group.

A loop Q is of Lenz type II.1 if and only if Q is a group which cannot be seen as the additive group of a vector space (over a field).

Q is of Lenz type II.2 if and only if Q is the additive group of a vector space (over a field).

Remark (4.4). If Q is a loop of Lenz type I.2 or I.3 which admits a sharply transitive group of automorphisms, then every pseudo-automorphism of Q is an automorphism.

This remark follows immediately from (4.3); if Q would admit a proper pseudo-automorphism α then every companion of α would be different from 1 . Then, however, the stabilizer Γ_{1_h} or Γ_{1_v} of the group Γ of all collineations which preserve the directions would be transitive on the line 1_h or on the line 1_v (cf. 3.1). □

For the proof of (4.2) and (4.3) one uses the following

Lemma (4.5). Let Q be a loop and N(Q) the net belonging to Q; assume that in N(Q) there exists a transitive axis $\mathfrak{X}$. Then N(Q) belongs either to the Lenz class II.1 or to the Lenz class II.2. The net N (Q) belongs to the class II.1 exactly then if Q is a group which cannot be seen as the additive group of a vector space; $N(Q)$ belongs exactly then to the class II.2 if Q is the additive group of a vector space (over a field).

Proof. From our hypothesis follows that every loop belonging to $N(Q)$ is a group isomorphic to Q and that every direction is a transitive axis ([4], p. 189). If $N(Q)$ does not belong to the class II.1, then every point p of $N(Q)$ is a transitive centre since the collineation group of $N(Q)$ is point transitive. Thus $N(Q)$ belongs to II.2. In this case every group Q which belongs to $N(Q)$ admits a sharply transitive group A of automorphisms.

The semidirect product $\theta = QA$ can be seen as a collineation group of $N(Q)$ which is contained in the stabilizer Γ_{1_h} of the group Γ of all collineations which preserve the directions, and which operates on 1_h sharply transitively. The nearfield F associated to θ has as additive group just Q and therefore Q is abelian ([16] (8.2)) . As a commutative group with a transitive automorphism group Q is the additive group of a vector space over a field ([6] , thm. 8.1). □

Proof of (4.2) and (4.3). Let N be a 3-net such that no direction is a transitive axis, but such that there exist two different transitive centres p_1 and p_2 .

We assume first that p_1 and p_2 are incident with a line L of the net N . Then the stabilizer Γ_L of the group Γ of all collineations which preserve the directions is point transitive on L . The line L can be seen as the line 1_h respectively 1_v of a loop Q such that either Q or one of the reversed loops of Q belongs to N .
Since every element of Q is a companion of a left pseudo-automorphism respectively of a right pseudo-automorphism of Q , the group Γ operates point transitively on 1_v or 1_h (3.4). Thus N belongs either to the class I.4 or to the class I.5 according to the case whether there exists an element $\neq 1$ in Q which is companion of a left and of a right pseudo-automorphism or not.

If in N there is no line incident with the transitive centres p_1 and p_2, then the collineation group Γ is transitive on each one of the three sets of the horizontal lines, of the vertical lines and of the transversal lines. If N does not belong to the class I.5 then on every line of N there exists exactly one transitive centre. In this case N belongs to the class I.3.

If a loop Q is of Lenz type I.1 no loop in the isotopy class $I(Q)$ admits a sharply point transitive group of automorphisms (3.2).

If a loop Q is of Lenz type I.2 then the collineation group Γ of the net $N(Q)$ which preserves the directions, has at least five different orbits on the set of points. From [21] p. 50 (cf. also (3.5)) follows that the isotopy class $I(Q)$ has at least five different isomorphy classes of loops. From (3.2) is clear that the isotopy class $I(Q)$ contains a loop admitting a sharply transitive group of automorphisms.

If a loop Q is of Lenz type I.3 then the collineation group Γ of the net $N(Q)$, which preserves the directions, operates transitively on the points which are transitive centres. If we take a transitive centre as the point $(1,1)$ for a loop Q^* which belongs also to $N(Q)$ then Q^* admits a sharply transitive group of automorphisms (3.2). Therefore Γ is transitive on those points of N which are not transitive centres. Thus Γ has on N exactly two point orbits and the cardinality of the isotopy class $I(Q)$ is exactly two.

The rest of the assertions in (4.2) and (4.3) follows from (3.2), (4.5) and (3.5). □

<u>Remark (4.6).</u> Let Q be a loop which admits a sharply transitive group of collineations. Then the cardinality of the isotopy class $I(Q)$ is different from three and four.

Another characterization of loops of Lenz type I.4 is

<u>Remark (4.7).</u> A loop Q is of Lenz type I.4 if and only if the isotopy class $I(Q)$ contains a loop Q^* admitting a sharply transitive

group of automorphisms and in Q^* or in one of its reversed loops there are elements $\neq 1$ which are companions of right (respectively left) pseudo-automorphisms but no elements $\neq 1$ which are companions of left (respectively right) pseudo-automorphisms.

The Lenz class I.1 contains not only loops satisfying only few algebraic rules (e.g. the free loops) but also for instance all proper Moufang loops admitting no transitive group of automorphisms. This follows e.g. from (4.4) and from the fact that in a Moufang loop there are always elements different from 1 which are companions of pseudo-automorphisms [6], p. 113 lemma 2.2 and [5], p. 70. Therefore every connected Lie Moufang loop is of Lenz type I.1 (cf. [17]). Also every finite,or every commutative, proper Moufang loop M is of Lenz type I.1; otherwise M would admit a sharply transitive group of automorphisms. Then M would be a simple loop such that no element has order 3 ([5], p. 70, cor. 2). If M is commutative we have a contradiction to (3.9) because of [6] p. 113 lemma 2.2 or p. 161 thm. 11.4. If M is finite then every element of M would be an involution ([9], p. 387) and this emplies again that M is commutative.

Also the Lenz class I.2 contains many examples of loops. For instance let $(R, +, \cdot)$ be a division neoring with associative multiplication such that the additive loop $(R, +)$ possesses the inverse property but is not a group. If either the characteristic of R is different from 3 , or if R is a finite planar division neoring, or if R is a connected,locally compact, finite dimensional, topological neoring, then the loop $(R, +)$ is of Lenz type I.2 (cf. (3.14) till (3.16), [12],[13], [22], pp. 459-461, [11] § 17, p. 229).

Definition (4.8). A k-net N is called strongly planar if it is embeddable in an affine plane E in such a way that the set of points of N and E is the same; moreover every translation of N can be extended to a collineation of E and every homology of N can be extended to a homology of E . If a k-net N is embedded in this way in an affine plane E , we say that N is strongly embedded in E .

There are many examples of loops of Lenz type I.1, I.2, II.1 and II.2. For instance strongly planar examples of groups of type II.1 can be constructed in the following way. Let G be an infinite group which does not admit a sharply transitive group of automorphisms (e.g. let G be an infinite group in which there are elements of different order). From [14] and [25] follows that there exists a projective plane P with the following properties: In P there exists a point p on a line L such that the group Λ of elations with the centre p and the axis L is transitive and Λ is isomorphic to G. Consider now the affine plane P_L which arises from P by omitting the line L and all its points, and let $N(p)$ be a 3-net consisting of 3 pencils of parallel lines of P_L, one of which is the pencil whose lines have the direction of the improper point p. A group which belongs to $N(p)$ is strongly planar and of Lenz type II.1.

In contrast to the existence of many examples of the types mentioned above we have the following

Remark (4.9). There are no strongly planar 3-nets of Lenz types I.3, I.4, and I.5 .

Proof. If N would be a 3-net of Lenz type I.3 or I.4 or I.5 which is strongly embedded in an affine plane A then it follows from [21] p. 67-70 that the collineation group of A would contain all translations of A. Then A would be desarguesian and N could be considered as a 3-net belonging to the additive group of a skew-field. But then N would belong to the Lenz class II.2. □

In general we have not been able to decide whether there exist examples of loops of Lenz type I.3, I.4 and I.5. If such examples exist, their order is at least 7 (cf. [8], § 4.2).

The loops of Lenz type I.4 are most peculiar. Since this class does contain neither commutative loops nor loops with the inverse property one cannot expect that the search for examples will be in the next time positive. Also our attempts to obtain examples of loops of Lenz type I.3 and I.5 in the class of additive loops of neofields ([20] and [15] were not successful. On the other hand it is not known whether there exists a proper infinite simple Moufang loop admitting a sharply transitive group of automorphism; such a loop would be of the Lenz type I.5.

Another class of loops which may be considered in order to obtain examples is the class of totally symmetric loops. A totally symmetric loop is a commutative loop in which the following identity is satisfied: $x(xy) = y$. The totally symmetric loops correspond in a one-to-one way to the Steiner triple systems(cf. [8] p. 75); therefore there are totally symmetric loops which are not groups and which admit a sharply transitive group of automorphisms ([7],[18],[19],[23]) . Since the class of totally symmetric loops is not too difficult to handle we obtained the following

Remark (4.10). Let Q be a totally symmetric loop which admits a sharply transitive group of automorphisms. If Q possesses a pseudo-automorphism which is not an automorphism then Q is of Lenz type I.5.

Proof. Let N be the net belonging to the totally symmetric loop Q . The points of N are the pairs (x, y) with $x, y \in Q$ and the transversal lines c_t can be described by the equations $y = xc$. Let γ be a collineation of N which preserves the directions. Then γ can be described as a mapping of the type $(x, y) \to (x^\alpha , y^\beta)$ where α and β are permutations of Q such that for every $c \in Q$ there exists a suitable c' satisfying for all x the equation

$$x^\alpha (xc)^\beta = c' \tag{1}$$

For $x = 1$ from (1) follows $1^\alpha c^\beta = c'$ and we have

$$x^\alpha (xc)^\beta = 1^\alpha c^\beta \tag{2}$$

This equation for $x = c$ leads to $c^\beta = 1^\alpha(c^\alpha 1^\beta)$ and we have

$$x^\alpha (1^\alpha [(xc)^\alpha 1^\beta]) = c^\alpha 1^\beta \tag{3}$$

For $c = 1$ we have from (3):

$$x^\alpha (1^\alpha [x^\alpha 1^\beta]) = 1^\alpha 1^\beta = x^\alpha [x^\alpha (1^\alpha 1^\beta)]$$

and

$$1^\alpha (x^\alpha 1^\beta) = x^\alpha (1^\alpha 1^\beta) .$$

Now (3) is equivalent to

$$x^{\alpha}\,[(x\,c)^{\alpha}\,(\,1^{\alpha}\,1^{\beta}\,)\,] = c^{\alpha}\,1^{\beta} = x^{\alpha}\,[\,x^{\alpha}\,(c^{\alpha}\,1^{\beta}\,)\,] \tag{4}$$

This emplies

$$(x\,c)^{\alpha}\quad(1^{\alpha}\,1^{\beta}\,) \;=\; x^{\alpha}\,(c^{\alpha}\,1^{\beta}\,) \tag{5}$$

If we take for α a proper pseudo-automorphism and for 1^{β} a companion of α then α leads to a collineation γ which does not leave the point (1,1) fixed. With our assumptions follows now that Q is of Lenz type I.5. □

In general we have been unable to decide whether or not there exist loops Q of Lenz type I.4 or I.5 admitting a group of collineations which in the corresponding net $N(Q)$ preserves the directions, leaves a line L invariant and operates on the points of L sharply 2-transitively. The non-existence of such loops would follow from the non-existence of near-domains which are not near-fields.

Thus for instance there are no finite loops of such kind (cf. [16], p. 31) or no locally compact, connected loops with the above property (cf. [24]).

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