Some remarkable properties of the Wiener algebra W$^+$

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An algebra which is of interest both in harmonic analysis and in function theory is the Wiener algebra W^+ (also denoted by A^+ (see [7]) consisting of functions f analytic in the open unit disc $\mathbf{D}$ whose Taylor series converges absolutely; i.e.,

$$W^+ := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n| < \infty \right\}.$$

W^+ is a commutative complex Banach algebra with identity element under the usual pointwise algebraic operations and the norm $\|f\| = \sum_{n=0}^{\infty} |a_n|$. Let $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbf{D}\}$. Since $\|f\|_\infty \leq \|f\|$, convergence in the norm implies uniform convergence; therefore each function $f \in W^+$ may be extended continuously to the closed unit disc, i.e., W^+ is a subalgebra (in the algebraic sense) of the disc algebra $A(\mathbf{D})$. The inclusion is proper. This was already proved by Hardy (see [10], §14), a much nicer proof using conformal mapping was given by Gaier (see [10], p. 135 f.).

It is known that there are functions in W^+ which behave quite erratic, e.g., Salem and Zygmund have shown (see [10], p. 136) that there exists a function $f \in W^+$ such that the image of the unit circle $T = \partial\mathbf{D}$ under the mapping f covers a domain, i.e., $f(T)$ is a Peano curve. For further examples see Landau/Gaier ([10], p. 136 f.).

For algebraic properties of the algebra W^+, the zero sets of elements f of W^+ are of great importance. Unfortunately, the zero set

$$Z(f) := \{z \in \overline{\mathbf{D}} : f(z) = 0\}$$

or the boundary zero set $Z(f) \cap T$ of a function $f \in W^+$ may be very complicated. In case of the disc algebra $A(\mathbf{D})$ the boundary zero sets are just the closed sets of Lebesgue measure zero on T, by a theorem of Fatou. No description is known in the case W^+. Sufficient conditions on a subset E of T to be a zero set of a function in W^+ were given by Carleson (see [7], p. 146).

Now we turn to the whole algebra W^+ as a Banach algebra. In the following we present some unusual properties of W^+ which complicate the investigation of the algebraic structure. To compare the results, we introduce the Banach algebras

$$A^n := \{f \in A(\mathbf{D}) : f, f', \ldots, f^{(n)} \text{ extends continuously to } \overline{\mathbf{D}}\}$$

with the norms

$$\|f\|_n = \sum_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_\infty,$$

where $n \in \mathbf{N}_0$ and $f^{(k)}$ is the k-th derivative of f. For $n = 0$ we obtain the disc algebra, i.e. $A^0 = A(\mathbf{D})$. Applying Hardy's inequality ([7], p. 141 f.) to the derivative f' of f we get the following inclusion

$$A^1 \subset W^+ \subset A^0.$$

Since the algebras W^+ and A^n are subsets of the Hardy space H^1, each function $f \not\equiv 0$ has the following canonical factorization

$$f = B \cdot S_\mu \cdot F,$$

where

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \bar{a}_n z}$$

is the Blaschke product associated with the zeros a_n of f,

$$S_\mu(z) = \exp\left\{-\int_T \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right\}$$

is a singular inner function with positive singular measure μ on the boundary T of $\mathbf{D}$ and

$$F(z) = \lambda \exp\left\{\frac{1}{2\pi}\int_T \frac{e^{it} + z}{e^{it} - z} \log|f(e^{it})|\, dt\right\}$$

is the outer function associated with f, where λ is a constant of modulus 1.

A subset S of the Hardy space H^1 is said to have the F-property (factorization property) if for any function $f \in S$ and for any inner function φ, $f/\varphi \in H^1$ implies $f/\varphi \in S$. Almost all important subspaces of H^1 have the F-property, the best known examples are the Hardy spaces H^p for $1 \leq p \leq \infty$, the disc algebra $A(\mathbf{D})$ and the algebras A^n. In order to describe the closed ideals in Banach algebras of analytic functions, the F-property is an essential tool (see [6] and [9]). Gurarii ([6]) now has shown that W^+ has not the F-property. This is one of the reasons why we know very little about the closed ideals in W^+. Nevertheless, we know several characterizations of the closed primary ideals in W^+ (Feldman [5], Guararii [6], Kahane [8], Atzmon [1]). Probably the most far going result is Faivyschewski's theorem ([4]) which says that a closed ideal I in W^+ whose boundary zero set is at most countable has the form

$$I = I(K, \varphi) = \{f \in W^+ : f/\varphi \in H^1, f_{|K} = 0\},$$

where φ is an inner function, $K = Z(I) \cap T$ and

$$Z(I) = \bigcap \{Z(f) : f \in I\}$$

is the zero set of the ideal I. This is analogue to the result in the disc algebra. We now present three results: The first and the last theorem generalize results of our former paper [12].

To this end we have the following lemma.

Lemma *Let $I = (f_1, f_2, \ldots)$ be a countably generated closed ideal in a commutative Banach algebra A with identity element. Assume that the norms of the generators are uniformly bounded. Then there exists a constant $C > 0$ such that every function $f \in I$ has a representation of the form*

$$f = \sum_{i=1}^{n} g_i f_i,$$

where $n = n(f)$ depends on f; and $g_i \in A$ fulfill $\|g_i\| \leq C\|f\|$ for all $i = 1, \ldots, n$.

This lemma is well known and in the case of a finite number of generators an easy consequence of the open mapping theorem. A proof of the general case may be found in ([3], p. 72) and depends on the Banach-Dieudonné-Schwartz theorem.

Theorem 1 *A closed ideal $I \neq (0)$ in the algebra W^+ is countably generated if and only if it is a principal ideal generated by a finite Blaschke product.*

Proof. If I is an ideal in W^+ generated by a finite Blaschke product, then I is closed. This follows from the fact that W^+-convergence implies uniform convergence and that $f \in W^+$ implies $[f(z) - f(a)]/(z-a) \in W^+$ for any $a \in \mathbf{D}$.

To prove the converse, let $I = (f_1, f_2, \ldots)$ be a countably generated ideal in W^+. Without loss of generality we may assume that we have $\|f_i\| \leq 4^{-i}$ for all $i = 1, 2, \ldots$. We show that the boundary zero set of the ideal I is empty, i.e. $Z(I) \cap T = \emptyset$. Suppose not, then there exists a point on the boundary of the unit disc, take without loss of generality $z = 1$, such that all functions in I vanish in $z = 1$. Then I is contained in the maximal ideal $M := \{f \in W^+ : f(1) = 0\}$. Since M has an approximate identity (see [4], p. 1069) we may apply P.J. Cohen's factorization theorem to the Banach algebra M. Then any generator f_i (we take for our purpose $f_i/\|f_i\|$) has a factorization of the following form

$$\frac{f_i}{\|f_i\|} = g_i h_i \quad (i = 1, 2, \ldots),$$

where g_i, h_i are functions in M. Moreover, the functions h_i may be chosen in the closure of the ideal generated by f_i such that

$$\left\| \frac{f_i}{\|f_i\|} - h_i \right\| \leq 1 \text{ and } \|g_i\| \leq K$$

for some constant K (see [2], p. 76). We put

$$G_i = \sqrt{\|f_i\|}\, g_i \text{ and } H_i = \sqrt{\|f_i\|}\, h_i \quad (i = 1, 2, \ldots).$$

Then we have $\|G_i\| \leq K \cdot 2^{-i}$, $\|H_i\| \leq 2^{1-i}$, $H_i \in I$ and $f_i = G_i H_i$. By the lemma above every function $f \in I$ can be represented in the form $f = \sum_{i=1}^{n} q_i f_i$, where $n = n(f)$, $q_i \in W^+$, $\|q_i\| \leq C\|f\|$ for some constant $C > 0$. Since $H_i \in I$ and $\|q\|_\infty \leq \|q\|$ for every $q \in W^+$ we have

$$\sum_{i=1}^{\infty} |H_i| \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C \|H_i\| |f_j| = C \sum_{i=1}^{\infty} \|H_i\| \sum_{j=1}^{\infty} |f_j|$$

$$\leq 2C \sum_{j=1}^{\infty} |f_j| \leq C < \infty \text{ in } \overline{\mathbf{D}}.$$

The Cauchy-Schwarz inequality now yields

$$\sum_{i=1}^{\infty} |H_i|^2 \leq \left(\sum_{i=1}^{\infty} |H_i| \right)^2 \leq 4C^2 \left(\sum_{j=1}^{\infty} |f_j| \right)^2$$

$$= 4C^2 \left(\sum_{j=1}^{\infty} |H_j G_j| \right)^2 \leq 4C^2 \sum_{j=1}^{\infty} |H_j|^2 \sum_{j=1}^{\infty} |G_j|^2 \text{ in } \overline{\mathbf{D}}.$$

In order to divide let

$$Z := \left\{ z \in \overline{\mathbf{D}} : \left(\sum_{j=1}^{\infty} |H_j|^2 \right)(z) = 0 \right\}.$$

Then Z is nowhere dense in $\overline{\mathbf{D}}$. Hence, by continuity, we get

$$\frac{1}{4C^2} \leq \sum_{j=1}^{\infty} |G_j|^2 \text{ on } \overline{\mathbf{D}} \quad !$$

This contradicts the fact that every function G_j belongs to M, i.e. vanishes in the point $z = 1$. Thus we have established that $Z(I) \cap T = \emptyset$.

By the identity theorem for analytic functions we conclude that $Z(I)$ is a finite subset of the open unit disc. Let B be the Blaschke product formed with the common zeros of the functions in I including multiplicities. Then B is a finite Blaschke product and therefore belongs to W^+. Since any generator f_i vanishes on the zeros of B with the same or higher multiplicity and since B is finite, we have $f_i/B \in W^+$ for all $i = 1, 2, \ldots$. Hence $I \subset (B)$. On the other hand, the set

$$I/B := \{f/B : f \in I\}$$

is an ideal in W^+ satisfying $Z(I/B) = \emptyset$. Therefore I/B is contained in no maximal ideal of W^+, i.e. $1 \in I/B$ or $B \in I$. Then $(B) \subset I$, and we conclude $I = (B)$.

So far we have considered closed ideals in W^+ which are countably generated. Now we turn to closed ideals which are prime. The next result gives a complete characterization of the closed prime ideals.

Theorem 2 *$P \neq (0)$ is a closed prime ideal in the algebra W^+ if and only if P is a maximal ideal.*

Proof. Since one direction is obvious, let us assume that $P \neq (0)$ is a closed prime ideal in W^+. By a result of Waelbroeck ([13, p. 291) it follows that the zero set $Z(P)$ of any prime ideal P in W^+ is a connected set, hence the F. & M. Riesz uniqueness theorem for bounded analytic functions implies that $Z(P)$ is a singleton. Therefore every prime ideal in W^+ is primary, i.e. contained in exactly one maximal ideal. Every closed primary ideal I in W^+ has one of the following three forms (see [1], [8]):

1. $I = M(z_0) = \{f \in W^+ : f(z_0) = 0\}, \quad z_0 \in T$.
2. $I = ((z - z_0)^n)$ for an $n \in \mathbf{N}, \quad z_0 \in \mathbf{D}$.
3. $I = I_\alpha(z_0) := \left\{f \in W^+ : \exp\left\{\alpha \frac{z_0+z}{z_0-z}\right\} f \in H^\infty\right\}$ for some $z_0 \in T$ and real number $\alpha > 0$.

Noting that

$$\left[(z - z_0)^3 \exp\left(-\frac{\alpha}{2} \cdot \frac{z_0 + z}{z_0 - z}\right)\right]^2 \in I_\alpha(z_0),$$

we see that an ideal of the form 3 cannot be prime. The same is true for ideals of the form 2 if $n \geq 2$, therefore it must be of the form 2 with $n = 1$ or of the form 1, i.e., I is a maximal ideal.

Our third theorem connects Theorems 1 and 2 in so far as we consider countably generated prime ideals, but without the assumption of closedness. We show that many of those ideals are automatically closed. Indeed, they are even maximal. Here is the result. Recall that H^∞ denotes the algebra of all bounded analytic functions in the open unit disc.

Theorem 3 *Let $P = (f_1, f_2, \ldots)$ be a countably generated prime ideal in the algebra W^+ ($f_i \neq 0$, $i = 1, 2, \ldots$). If the generators $f_1, f_2, \ldots$ have, with respect to the algebra H^∞, a greatest common divisor, then P is a maximal ideal $M(z_0)$ corresponding to a point z_0 of the open unit disc.*

Proof. As in the proof of Theorem 2 we may assume that $\|f_i\| \leq 4^{-i}$ $(i = 1, 2, \ldots)$. First we show that $Z(P) \cap T = \emptyset$. Assuming the contrary, there exists a point $z_0 \in T$ with $P \subset M(z_0)$. Since $z_0 \in T$, $M(z_0)$ has an approximate identity ([4]). Applying Cohen's factorization theorem as in the proof of Theorem 2, we have $f_i = g_i h_i$ with functions $g_i, h_i \in M(z_0)$ satisfying $\|h_i\| \leq C2^{-i}$, $\|g_i\| \leq C2^{-i}$ $(i = 1, 2, \ldots)$ for some constant C. Since P is a prime ideal, g_i or h_i belongs to P.

If we change our notation we may denote those functions which belong to P by g_i. Then we have

$$P = (f_1, f_2, \ldots) = (g_1, g_2, \ldots).$$

Let $F \in H^\infty$ be the greatest common divisor of the functions $f_1, f_2, \ldots$. Then we conclude that F is a common divisor (with respect to the algebra H^∞) of all functions in P. In particular, F is a common divisor of the functions g_i.

Let

$$H(z) = \exp\left\{ \int_T \frac{e^{it} + z}{e^{it} - z} \log \sum_{j=1}^{\infty} |h_j(e^{it})|\, dt \right\}.$$

Then H is an outer function in H^∞ which divides all the functions h_i. Since $f_i = g_i h_i$ $(i = 1, 2, \ldots)$ and F divides g_i $(i = 1, 2, \ldots)$, we see that FH divides every function f_i $(i = 1, 2, \ldots)$. Because F is a greatest common divisor of the f_i, FH must divide F. Hence H is invertible in the algebra H^∞. This is, however, a contradiction to the fact that the continuous extension of $|H|$ to $\overline{\mathbf{D}}$ has value zero at z_0. Therefore we have $Z(P) \cap T = \emptyset$ and $P \subset M(z_0)$, $z_0 \in \mathbf{D}$. Since P is a prime ideal it is easy to see that we actually have $P = M(z_0)$.

Remark. 1. There actually exist subalgebras of the disc algebra $A(\mathbf{D})$ with the property that the functions in every non-maximal prime ideal have (with respect to the algebra H^∞) a proper common divisor. As an example, we may take the algebra A^n. This is an unpublished result of H.-M. Lingenberg.

2. Finally we note that in the algebra H^∞ there do exist non-maximal, countably generated prime ideals ([11], p. 59).

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