PROCEEDINGS OF 'GROUPS - KOREA 1988'
PUSAN, August 1988

THE GROUPS OF AUTOMORPHISMS OF NON–ORIENTABLE HYPERELLIPTIC KLEIN SURFACES WITHOUT BOUNDARY

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1. Introduction.

Let X be a Klein surface (i.e. a compact surface equipped with a dianalytic structure [1]). If X is an orientable surface without boundary then this surface is a classical Riemann surface. A Klein surface X is said to be hyperelliptic if X admits an involution ϕ such that $X/\langle\phi\rangle$ has algebraic genus 0.

The study of automorphism groups of hyperelliptic Riemann surfaces is a classical topic from the 19th century, whilst the studies of groups of automorphisms of hyperelliptic Klein surfaces not being Riemann surfaces have been started to investigate in the recent decade. This paper concerns the problem of the groups of automorphisms of non–orientable hyperelliptic Klein surfaces without boundary. We here solve it completely when the topological genus g of the surface is odd. In the case of bordered hyperelliptic Klein surface, similar results, but concerning only the order of the group, have been established recently in [6].

By the well known correspondence between Klein surfaces and algebraic curves [1], our results can be expressed in terms of the automorphism groups of purely imaginary algebraic curves.

2. NEC groups and hyperelliptic surfaces.

A non–Euclidean crystallographic group (N.E.C. group) Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{D}$ with compact quotient $\mathcal{D}/\Gamma$. Each NEC group Γ has associated with a signature σ that has the form [11]

$$\sigma : (g; \pm; [m_1, \cdots, m_r], \{C_1, \cdots, C_k\}) \tag{2.1}$$

where $C_i = (n_{i1}, \cdots, n_{is_i})$; C_i are called cycle-periods, n_{ij} periods of cycle-periods and m_i proper periods.

The numbers in σ are non–negative integers; m_i and n_{ij} are greater than or equal to 2, and the number g is the topological genus of the surface $\mathcal{D}/\Gamma$ (the algebraic genus $p = \alpha g + k - 1$, where $\alpha = 2$ or $\alpha = 1$ according to the sign '+' or '–' in σ). This surface is orientable or not, according as the sign in σ is '+' or '–' respectively.

If $r = 0$ or $k = 0$, we write in σ [–] or {–} respectively. If the number s_i is zero for some i, we denote C_i by (–).

Partially supported by *CAICYT, **Ministerio de Educación y Ciencia de España.

The signature σ determines a canonical presentation of the group Γ, which as shown in [11] and [14], is given by the generators

$$\begin{array}{lll} x_i & i=1,\cdots,r & \\ e_i & i=1,\cdots,k & \\ c_{ij} & i=1,\cdots,k, & j=0,\cdots,s_i \\ a_i,\ b_i & i=1,\cdots,g & (\text{if sign `}+\text{'}) \\ d_i & i=1,\cdots,g & (\text{if sign `}-\text{'}) \end{array}$$

subject to the relations

$$\begin{array}{ll} x_i^{m_i}=1 & i=1,\cdots,r \\ c_{ij-1}^2=c_{ij}^2=(c_{ij-1}c_{ij})^{n_{ij}}=1 & i=1,\cdots,k;\quad j=1,\cdots,s_i \\ e_i^{-1}c_{i0}e_ic_{is_i}=1 & i=1,\cdots,k \\ \prod_{i=1}^{r}x_i\prod_{i=1}^{k}e_i\prod_{i=1}^{g}[*]=1 & \end{array}$$

where [*] is $a_ib_ia_i^{-1}b_i^{-1}$ or d_i^2 according to the sign in σ.

The area of Γ is

$$\mu(\Gamma)=2\pi[\alpha g+k-2+\sum_{i=1}^{r}(1-1/m_i)+1/2\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1-1/n_{ij})] \tag{2.2}$$

If Γ is a subgroup of Γ' of index N then the following relation between areas holds:

$$\mu(\Gamma)=N\mu(\Gamma'). \tag{2.3}$$

A non-orientable Klein surface without boundary X of algebraic genus $p\geq 2$ can be expressed as $X=\mathcal{D}/\Gamma$ where Γ is an NEC group with signature

$$(g;-;[-],\{-\}) \tag{2.4}$$

and if $G=\mathrm{Aut}(X)$ is the full group of automorphisms of X then G is isomorphic to $N_{\mathcal{G}}(\Gamma)/\Gamma$ where $\mathcal{G}$ is the group of all isometries of $\mathcal{D}$ (see [12]).

In [9] the non-orientable hyperelliptic Klein surfaces without boundary are characterized by means of NEC groups. We summarize the results in the following.

Theorem 2.1. *Let $X=\mathcal{D}/\Gamma$ be a non-orientable Klein surface without boundary of genus $g\geq$ 3.* Then

(a) *X is hyperelliptic if and only if there exists a unique NEC group Γ_1 containing Γ as a subgroup of index 2, Γ_1 having one of the following signatures*

$$\text{(i).}\ \ (0;\pm;[\overbrace{2,\cdots,2}^{g}],\{(-)\})\qquad \text{(ii).}\ \ (1;-;[\overbrace{2,\cdots,2}^{g}],\{-\})\ \ (g\ \text{even})$$

(b) *The automorphism of hyperellipticity ϕ where $\langle\phi\rangle=\Gamma_1/\Gamma$ is a central element in the full group of automorphisms of G of X.*

In our paper a hyperelliptic Klein surface (in short HKS) will mean a non–orientable hyperelliptic Klein surface without boundary.

3. Groups of automorphisms of non–orientable hyperelliptic Klein surfaces without boundary.

In this section, we will give a necessary and sufficient conditions for a finite group to be the group of automorphisms of a HKS when a group of hyperellipticity Γ_1 has signature (i) (see theorem 2.1). Thus our results will hold for all HKS for which a group of hyperellipticity has signature (i) and so in particular they are complete for HKS of odd topological genus g.

Lemma 3.1. (1) *Let $G/\mathbf{Z}_2 \cong \mathbf{Z}_N$ and $\mathbf{Z}_N \subseteq G$. Then*

(i) *if N is odd then $G \cong \mathbf{Z}_{2N}$ and*

(ii) *if N is even then $G \cong \mathbf{Z}_2 \times \mathbf{Z}_N$ or $G \cong \mathbf{Z}_{2N}$.*

(2) *Let $G/\mathbf{Z}_2 \cong \mathrm{D}_{N/2}$ and $\mathrm{D}_{N/2} \subseteq G$. Then*

(i) *if $N/2$ is odd then $G \cong \mathrm{D}_N$ and*

(ii) *if $N/2$ is even then $G \cong \mathrm{D}_N$ or $G \cong \mathbf{Z}_2 \times \mathrm{D}_{N/2}$, or $G = \mathrm{U}_{N/2}$, where $G = \mathrm{U}_{N/2}$ is the group with presentation $\langle x\ , y \mid x^N,\ y^2,\ yxyx^{N/2+1}\rangle$.*

Proof. Note first that $\mathbf{Z}_2$ is a central subgroup of G and the second cohomology $\mathrm{H}^2(\mathbf{Z}_N, \mathbf{Z}_2)$ is trivial (if N is odd) or is the cyclic group of order 2 (if N is even). Since $\mathrm{H}^2(G, A)$ classifies the extensions of G by A, the first case follows.

Now let $G/\mathbf{Z}_2 \cong \mathrm{D}_{N/2}$. If $N/2$ is odd then $\mathrm{H}^2(\mathrm{D}_{N/2}, \mathbf{Z}_2) = \mathbf{Z}_2$ and it turns out that there are two such groups: $G = \mathrm{D}_N$ and $G = G_{N/2}$, where $G_{N/2}$ is the group with the presentation $\langle x\ , y \mid x^N,\ x^{N/2}y^2,\ y^{-1}xyx\rangle$ (see [2]). Let H be a subgroup of index 2 in $G_{N/2}$. We will show that H is a cyclic group generated by x. In fact, $y2 \in H$ and so do $x^{N/2}$. Since $x^2 \in H$ and $N/2$ is odd we obtain that $x \in H$, and so H is a cyclic group of order N as required. Hereby $G_{N/2}$ does not contain $\mathrm{D}_{N/2}$

Now let $N/2$ be even. Although in this case $\mathrm{H}^2(\mathrm{D}_{N/2}, \mathbf{Z}_2)$ is a group of order 8, it turns out that there are only six groups G for which $G/\mathbf{Z}_2 \cong \mathrm{D}_{N/2}$ and the complete list of them can be found in [2]. Obviously D_N and $\mathbf{Z}_2 \times \mathrm{D}_{N/2}$ contain $\mathrm{D}_{N/2}$ as a subgroup. Moreover it is easy to check that (y, x^2) generate the dihedral subgroups of order N in $\mathrm{U}_{N/2}$. Now looking at the possible generators of quotients of G of order 2 and using the Reidemeister–Schreier algorithm for determining the presentation of a subgroup of a given group one can argue that there is no dihedral group $\mathrm{D}_{N/2}$ among subgroups of index 2 in the remaining three groups.

Lemma 3.2. *Let Γ_1 be an NEC group with signature $(0;+;[\overbrace{2,\cdots,2}^{g}],\{(-)\})$ and let Γ^* be an NEC group containing Γ_1 as a normal subgroup of index N. Then Γ^* has one of the following signatures:*

(i) $(0;+;[N,\overbrace{2,\cdots,2}^{k}],\{(-)\})$, *where $k = g/N$,*

(ii) $(0;+;[2N,\overbrace{2,\cdots,2}^{k}],\{(-)\})$, *where $k = (g-1)/N$,*

(iii) $(0;+;[\overbrace{2,\cdots,2}^{h}],\{(N/2,\overbrace{2,\cdots,2}^{l})\})$, *for some* h, *where* $l=2g/N-2h+2$,

(iv) $(0;+;[\overbrace{2,\cdots,2}^{h}],\{(N,\overbrace{2,\cdots,2}^{l})\})$, *for some* h, *where* $l=2(g-1)/N-2h+2$.

Proof. Let Γ^* has (2.1) and let p_i be the smallest integer for which $x_i^{p_i}\in\Gamma_1$. Since the only elements of Γ_1 of finite order are those of order 2, we have that $m_i/p_i=2$ or $m_i/p_i=1$. For notational convenience assume that $m_i/p_i=2$ for $i=1,\cdots,n$ and $m_i/p_i=1$ for $i=n+1,\cdots,r$. The elliptic elements in the second case do not produce proper periods in Γ_1, whilst in the first case they produce $\sum_{i=1}^{n}N/p_i$ proper periods, all of them being equal to 2 (see [3] and [4]).

Now consider the periods in Γ_1 provided by reflections of Γ^*. Let q_{ij} be the smallest integers for which $(c_{ij}c_{ij-1})^{q_{ij}}\in\Gamma_1$. The same argument, as in the first case, shows that $n_{ij}/q_{ij}=2$ or 1. Let $E=\{(i,j)\mid n_{ij}/q_{ij}=2\}$. Clearly the only proper periods in Γ_1 induced by reflections of Γ^* are those provided by the pairs of reflections corresponding to the elements of E, and each such pair produces $N/2q_{ij}$ periods, all of them being equal to 2 (see [3]). As a result

$$g=\sum_{i=1}^{n}N/p_i+\sum_{(i,j)\in E}N/2q_{ij} \tag{3.1}$$

Using (2.3) we obtain

$$\begin{aligned} g/2-1 &= N[\alpha g^*-2+k+\sum_{i=1}^{n}(1-1/m_i)+\sum_{i=n+1}^{r}(1-1/m_i)\\ &+ 1/2\sum_{(i,j)\in E}(1-1/n_{ij})+1/2\sum_{(i,j)\notin E}(1-1/n_{ij})] \end{aligned} \tag{3.2}$$

The number n has been chosen in such a way that $m_i=2p_i$ for $i=1,\cdots,n$ and $m_i=p_i$ for $i=n+1,\cdots,r$, whilst $n_{ij}=2q_{ij}$ for $(i,j)\in E$, and $n_{ij}=q_{ij}$ for $(i,j)\notin E$. So using (3.1) and (3.2) we obtain the following equation.

$$-1=\alpha Ng^*-2N+kN+N\sum_{i=1}^{n}(1-1/p_i)+N\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1-1/q_{ij}). \tag{3.3}$$

Clearly k cannot be bigger than 1, otherwise the right hand side of (3.3) would be ≥ 0. So let $k=1$. Then $g^*=0$, otherwise it again would be ≥ 0. Thus we see that Γ^* has signature

$$(0,+,[m_1,\cdots,m_r],\{(n_1,\cdots,n_s)\})$$

and so (3.3) becomes

$$N-1=N\sum_{i=1}^{r}(1-1/p_i)+N/2\sum_{j=1}^{s}(1-1/q_j). \tag{3.4}$$

Consider two cases:

Case 1. The cycle period of Γ^* is empty. Then $N-1=N\sum_{i=1}^{r}(1-1/p_i)$ and it is easy to check that the only solution of this equation is that all p_i but one, say p_1, are equal to 1 and $p_1=N$. Now for $i\neq 1$, $p_i=1$, and so $m_i=2$. If $m_1/p_1=1$ then $m_1=N$, whilst if $m_1/p_1=2$, $m_1=N/2$. So Γ^* has a signature (i) or (ii) (the number k can be found using (2.3)).

Case 2. The cycle period is non-empty. Then by [7], [8] two consecutive periods, say n_{s-1}, n_s are equal to 2. Moreover if c_{s-2}, c_{s-1}, c_s are the corresponding reflections, we can assume that $c_{s-1} \in \Gamma_1$, and c_{s-2}, $c_s \notin \Gamma_1$ (see [8]). Thus q_{s-1}, $q_s = 2$ and so the second summand of the right hand side of (3.4) is $\geq N/2$. Now all p_i are equal to 1, otherwise the first summand of the right hand side is also $\geq N/2$ and is $\geq N$, a contradiction. The relation (3.4) becomes

$$N/2 - 1 = N/2 \sum_{j=1}^{s-2} (1 - 1/q_j)$$

and the only solution of it is that all q_j but one, say q_1, are equal to 1 and $q_1 = N/2$. Now since all $p_i = 1$, $m_i = 2$ for $i = 1, \cdots, r$. For $q_j = 1$, the corresponding periods are equal to 2, and for $q_1 = N/2$ the corresponding period is $N/2$ or N according as $n_1/q_1 = 1$ or 2.So in this case Γ^* has a signature (iii) or (iv) (the number l can be found as in the first case using (2.3)).

Remark 3.3. From the proof of the previous lemma follows not only that Γ^* has one of the specified signatures but also an information how the group Γ_1 sits in Γ^*.

In the case of signatures (i), (ii) all elliptic elements x_i but x_1 belong to Γ_1 whilst $x_1^N \in \Gamma_1$ and N is the smallest integer with this property.

In the case of signatures (iii), (iv) all elliptic element x_i must belong to Γ_1, all c_i, but c_1, do not belong to Γ_1 and $c_i c_{i+1} \in \Gamma_1$ for $i = 1, \cdots, l-2$, whilst $(c_0 c_1)^{N/2} \in \Gamma_1$ and $N/2$ is the smallest integer with this property.

Lemma 3.4. *Let Γ_1 and Γ^* be NEC groups as in the previous lemma and assume that Γ is an NEC group with signature* (2.4) *being a subgroup of Γ_1 of index* 2 *and normal in Γ^*. Let $G = \Gamma^*/\Gamma$. Then*

(1) *If Γ^* has signature* (i) *then $G = \mathbf{Z}_{2N}$ or $G = \mathbf{Z}_2 \times \mathbf{Z}_N$.*

(2) *If Γ^* has signature* (ii) *then $G = \mathbf{Z}_{2N}$.*

(3) *If Γ^* has signature* (iii) *then $G = \mathrm{D}_N$ or $G = \mathbf{Z}_2 \times \mathrm{D}_{N/2}$ or $G = \mathrm{U}_{N/2}$, where*
$\mathrm{U}_{N/2} = \langle x,\ y \mid x^N,\ y^2, yxyx^{N/2+1} \rangle$.

(4) *If Γ^* has signature* (iv) *then $G = \mathrm{D}_N$.*

Proof. Let $G^* = \Gamma^*/\Gamma_1$ and $H = \Gamma_1/\Gamma \cong \mathbf{Z}_2$. Clearly $G/H \cong G^*$. Let x_i, e_i, c_i be the canonical generators of Γ^*. We will employ the notations introduced in the proof of the previous lemma.

(1) Let Γ^* have signature (i). We show that in this case $p_1 = N$ and so the image of x_1 in G is an element of order N. So $\mathbf{Z}_N \subseteq G$. Thus by Lemma 3.1 $G \cong \mathbf{Z}_{2N}$ or $G \cong \mathbf{Z}_2 \times \mathbf{Z}_N$.

(2) Let Γ^* have signature (ii). Then since Γ has no periods, x_1 induces an element of order $2N$ in G. Hereby $G \cong \mathbf{Z}_{2N}$.

(3) Now assume that Γ^* has signature (iii). We show that the pair of reflections c_0, c_1 corresponding to the period $N/2$ satisfy c_0, $c_1 \notin \Gamma_1$ and c_0, c_1 induces in G^* an element of order $N/2$. So $G^* \cong \mathrm{D}_{N/2}$. Moreover c_0, $c_1 \notin \Gamma$ and $c_0 c_1$ also induces an element of order $N/2$ in G. Thus the result follows from Lemma 3.1.

(4) Finally let Γ^* have a signature (iv) and let c_0, c_1 be the pair of reflections corresponding to

the period N as in the previous case. Then since Γ has no proper periods and no period cycles c_0, $c_1 \notin \Gamma_1$ and their product induces an element of order N in Γ. Therefore $G \cong \mathrm{D}_N$.

Theorem 3.5. *Let X be a HKS of odd topological genus g. Then $\mathrm{Aut}(X)$ may be one of the following groups $\mathbf{Z}_{2N}$, $\mathbf{Z}_2 \times \mathbf{Z}_N$, D_N or $\mathbf{Z}_2 \times \mathrm{D}_{N/2}$. Furthermore*

(i) *There exists a HKS of genus g having $\mathbf{Z}_{2N}$ as the group of automorphisms if and only if $N|g-1$ and $N \neq g-1$ or $N|g$, $N \neq g$, and N is odd.*

(ii) *There exists a HKS of genus g having $\mathbf{Z}_2 \times \mathbf{Z}_N$ $(N \neq 2)$ as the group of automorphisms if and only if $N|g$, $N \neq g$ and N is even.*

(iii) *There exists a HKS of genus g having D_N as the group of automorphisms if and only if $N|2(g-1)$ and N is even or $N|2g$, $4\nmid N$.*

(iv) *There exists a HKS of genus g having $\mathbf{Z}_2 \times \mathrm{D}_{N/2}$ as the group of automorphisms if and only if $N|2g$ and $4|N$.*

Proof. Let X be a HKS of odd topological genus $g \geq 3$. Then $X = \mathcal{D}/\Gamma$, where Γ is an NEC group with signature (2.4). Since X is a HKS there exists an involution $\phi \in \mathrm{Aut}(X)$ such that $X/\langle\phi\rangle$ is a surface of genus 0. By the theorem 2.1 ϕ is a central element in $\mathrm{Aut}(X)$. Let $\langle\phi\rangle = \Gamma_1/\Gamma$ and let G be a group of automorphisms of X containing ϕ. Then $G = \Gamma^*/\Gamma$ for some NEC group Γ^*, containing Γ_1 as a normal subgroup. By Theorem 2.1 Γ_1 has signature $(0; \pm; [\overbrace{2,\cdots,2}^{g}], \{(-)\})$. The Lemma 3.2 describes the possible signatures for Γ^* and Lemma 3.4 gives us necessary conditions for a group G to be represented in such case as a quotient Γ^*/Γ. Given a homomorphism $\theta: \Gamma^* \longrightarrow G$ and a central element ϕ of G of order 2 let π be the canonical projection $G \longrightarrow G/\langle\phi\rangle$ and $\theta^* = \phi \circ \theta$. We have to investigate homomorphisms θ from Γ^* onto groups specified in Lemma 3 such that $\mathrm{Ker}\theta = \Gamma$ and $\mathrm{Ker}\theta^* = \Gamma_1$, where Γ is a group with signature (2.4) and Γ_1 is a group with signature $(0; \pm; [\overbrace{2,\cdots,2}^{g}], \{(-)\})$.

1. Let Γ^* have signature (i). By Lemma 3.4 $G = \Gamma^*/\Gamma$ may be only $\mathbf{Z}_{2N}$ or $\mathbf{Z}_2 \times \mathbf{Z}_N$. Assume that $G = \mathbf{Z}_{2N}$. We have to see whether or not a homomorphism $\theta: \Gamma^* \longrightarrow \mathbf{Z}_{2N} = \langle x \mid x^{2N}\rangle$ satisfying the conditions in question exists. Since $\mathrm{Ker}\theta$ is a surface group, θ must preserve the orders of the canonical generators of Γ^*. So θ is forced to be defined as follows $\theta(x_1) = x^2$, $\theta(x_2) = \cdots = \theta(x_{k+1}) = x^N$, $\theta(c) = x^N$ and $\theta(e) = x^{-(kN+2)}$. Clearly $w = x_2 c$ is a non-orientable element in $\mathrm{Ker}\theta$ and so by [10] $\mathrm{Ker}\theta$ is a non-orientable surface group. But clearly this homomorphism is an epimorphism if and only if N is odd. Let this will be the case. Then $\theta^*: \Gamma^* \longrightarrow \mathbf{Z}_{2N}/\langle x^N\rangle \cong \mathbf{Z}_N = \langle \bar{x} \mid \bar{x}^N\rangle$ is given as follows $\theta^*(x_1) = (\bar{x})^2$, $\theta^*(x_2) = \cdots = \theta^*(x_{k+1}) = 1$, $\theta^*(c) = 1$ and $\theta^*(e) = (\bar{x})^{-2}$ and so $\mathrm{Ker}\theta^* = \Gamma_1$, by the proof of Lemma 3.2 (see Remark 3.3). Moreover the canonical Fuchsian subgroup $(\Gamma^*)^+$ of Γ^* has the signature $(0; \pm; [N, N, \overbrace{2,\cdots,2}^{2k}], \{-\})$ (see [13]) and so $k > 1$ (i.e. $N \neq g$) is a maximal signature (see [12]). Hence the signature of Γ^* is also maximal and so Γ^* can be chosen to be a maximal NEC group and so for the surface $X = \mathcal{D}/\Gamma$ just considered, $\mathbf{Z}_{2N}$ can be assumed to be the full group of automorphisms. The signature $(0, +, [N, 2], \{-\})$ corresponding to the case $g = N$ is not maximal, and so for $g = N$ the group $\mathbf{Z}_{2N}$ cannot be the full group of automorphisms

of a HKS of genus g (see Proposition 2.4 [5]).

Now let $G = \mathbf{Z}_2 \times \mathbf{Z}_N$ and let N be even. Consider the homomorphism $\theta : \Gamma^* \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_N = \langle x,\ y \mid x^2, y^N,\ [x,y]\rangle$ defined by $\theta(x_1) = xy$, $\theta(x_2) = \cdots = \theta(x_{k+1}) = x$, $\theta(c) = x$ and $\theta(e) = y^{-1}x^{(k+1)}$. Since θ preserves the orders of the canonical generators of Γ^* and $w = x_2c$ is a non-orientable element in Kerθ, Ker$\theta = \Gamma$. Moreover for the homomorphism $\theta^* : \Gamma^* \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_N/\langle x\rangle = \langle y \mid y^N\rangle$ we have $\theta^*(x_1) = y$, $\theta^*(x_2) = \cdots = \theta^*(x_{k+1}) = 1$, $\theta^*(c) = 1$ and $\theta^*(e) = y^{-1}$. So by the proof of Lemma 3.2 Ker$\theta^* = \Gamma_1$.

As in the case of $\mathbf{Z}_{2N}$ we can argue that for $N \neq g$ a dianalytic structure on X can be so chosen that $\mathbf{Z}_{2N}$ is the full group of its automorphisms whilst this is not the case for $N = g$.

2. Let Γ^* have signature (ii). By Lemma 3.4 $G = \Gamma^*/\Gamma$ may be only the cyclic group of order $2N$. We will show that this is so. Let $\theta : \Gamma^* \longrightarrow \mathbf{Z}_{2N} = \langle x \mid x^{2N}\rangle$ be the homomorphism defined by $\theta(x_1) = x$, $\theta(x_2) = \cdots = \theta(x_{k+1}) = x^N$, $\theta(c) = x^N$ and $\theta(e) = x^{-(kN+1)}$. Clearly Im$(\theta) = \mathbf{Z}_{2N}$. Moreover x_2c is a non-orientable element in Kerθ and so Ker$\theta = \Gamma$. Now for the homomorphism $\theta^* : \Gamma^* \longrightarrow \mathbf{Z}_{2N}/\langle x^N\rangle \cong \mathbf{Z}_N = \langle \bar{x} \mid \bar{x}^N\rangle$ we have $\theta^*(x_1) = \bar{x}$, $\theta^*(x_2) = \cdots = \theta^*(x_{k+1}) = 1$, $\theta^*(c) = 1$ and $\theta^*(e) = (\bar{x})^{-1}$. So by Remark 3.3 Ker$\theta^* = \Gamma_1$. As in the previous case we argue that for $N \neq g-1$, $\mathbf{Z}_{2N}$ can be assumed to be the full group of automorphisms of the surface just constructed, whereas this is not the case for $N = g-1$.

3. Let Γ^* be the group with signature (iii). By Lemma 3.4 $G = \Gamma^*/\Gamma$ may be only one of the groups D_N, $\mathbf{Z}_2 \times \mathrm{D}_{N/2}$ or $\mathrm{U}_{N/2}$. Clearly N must be even in this case, otherwise D_N would have the trivial center whilst the remaining two groups are not defined. We will show first that the group D_N is a group of automorphisms of a surface in question if and only if $N/2$ is odd. Let $\theta : \Gamma^* \longrightarrow \mathrm{D}_N = \langle x,\ y \mid x^2,\ y^2,\ (xy)^N\rangle$ be the homomorphism we are looking for. Let z be the central element of order 2 in D_N for which $\mathrm{D}_N/\langle z\rangle \cong \mathrm{D}_{N/2}$. Then by Remark 3.3 $\theta(x_1), \cdots, \theta(x_h)$ must be equal to z, $\theta(c_0) = \theta(c_{l+1} = az^{\varepsilon_0}$ ($\varepsilon_0 = 0$ or 1), $\theta(c_i) = bz^{\varepsilon_i}$ ($\varepsilon_i = 0$ or 1) for $i = 1, \cdots, l-1$, $\theta(c_l) = z$ for some elements a, b of order 2 whose product has order $N/2$. So in particular $\mathrm{D}_N \cong \mathbf{Z}_2 \times \mathrm{D}_{N/2}$. But the last is the case if and only if $N/2$ is odd, i.e. $4 \nmid N$.

Conversely let $N/2$ be odd. Let $h = 0$ and let $\theta : \Gamma^* \longrightarrow \mathrm{D}_N$ be the homomorphism defined by $\theta(c_0) = \theta(c_{l+1}) = y$ $\theta(c_i) = xyx$ for $1 \leq i \leq l-1$ and i odd, $\theta(c_i) = xyx(xy)^{N/2}$ for $1 \leq i \leq l-1$ and i even, $\theta(c_l) = (xy)^{N/2}$, and $\theta(e) = 1$. It is easy to check that Im$(\theta) = \mathrm{D}_N$ and Ker$\theta = \Gamma$ whilst Ker$\theta^* = \Gamma_1$. The same argument as used before shows that the dianalytic structure on X just considered can be so chosen that Aut$(X) = \mathrm{D}_N$.

Now let $N/2$ be even and let for $h = 0$ $\theta : \Gamma^* \longrightarrow \mathbf{Z}_2 \times \mathrm{D}_{N/2} = \langle z \mid z^2\rangle \otimes \langle x,\ y \mid x^2,\ y^2,\ (xy)^{N/2}\rangle$ be the homomorphism defined by $\theta(c_0) = \theta(c_{l+1}) = x$, $\theta(c_i) = y$ for $1 \leq i \leq l-1$ and i odd, $\theta(c_i) = yz$ for $1 \leq i \leq l-1$ and i even, $\theta(c_l) = z$, and $\theta(e) = 1$. As in the previous case we show that Ker$\theta = \Gamma$ and for the homomorphism $\theta^* : \Gamma^* \longrightarrow \mathbf{Z}_2 \times \mathrm{D}_{N/2}/\langle z\rangle = \mathrm{D}_{N/2}$, Ker$\theta^* = \Gamma_1$. Again the same argument as used before shows that the dianalytic structure on X just considered can be so chosen that Aut$(X) = \mathbf{Z}_2 \times \mathrm{D}_{N/2}$.

So it remains to show that the group $\mathrm{U}_{N/2}$ cannot stand for a group of automorphisms of a surface in question. If this were so, then arguing as in the case of the dihedral group, one can show that the corresponding homomorphism were considered as the one defined there. Consequently, $\mathrm{U}_{N/2}$ must be isomorphic to the direct product $\mathbf{Z}_2 \times \mathrm{D}_{N/2}$.

4. Now consider the group Γ^* with signature (iv). By the Lemma 3.4 $G = \Gamma^*/\Gamma$ may be only the dihedral group D_N of order $2N$. Clearly N is even since D_N has trivial center for N odd. Let for $h = 0$ $\theta : \Gamma^* \longrightarrow D_N = \langle x,\, y \mid x^2,\, y^2,\, (xy)^N \rangle$ be the homomorphism defined by $\theta(c_0) = \theta(c_{l+1}) = y$, $\theta(c_1) = x$, $\theta(c_i) = (xy)^{N/2}x$ for $2 \le i \le l-1$ and i even, $\theta(c_i) = x$ for $2 \le i \le l-1$ and i odd, $\theta(c_l) = (xy)^{N/2}$, and $\theta(e) = 1$. Clearly $\mathrm{Im}\theta = D_N$ and $(c_0c_1)^{N/2}c_1$ is a non–orientable element in $\mathrm{Ker}\theta$. So $\mathrm{Ker}\theta = \Gamma$. Moreover for the homomorphism $\theta^* : \Gamma^* \longrightarrow D_N/\langle (xy)^{N/2} \rangle \cong D_{N/2} = \langle \bar{x},\, \bar{y} \mid \bar{x}^2,\, \bar{y}^2,\, (\bar{x}\bar{y})^{N/2} \rangle$ we have $\theta^*(c_0) = \theta^*(c_{l+1}) = \bar{y}$, $\theta^*(c_1) = \cdots = \theta^*(c_{l-1}) = \bar{x}$, $\theta^*(c_l) = 1$, $\theta^*(e) = 1$. By the proof of the Lemma 3.2 (see Remark 3.3) $\mathrm{Ker}\theta^* = \Gamma_1$. As before we argue that the dianalytic structure on the surface just considered can be chosen in such a way that D_N is the full group of its automorphisms.

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Additional Refernces added in proof.

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