THE L_p MINIMALITY AND NEAR-MINIMALITY OF ORTHOGONAL POLYNOMIAL APPROXIMATION AND INTEGRATION METHODS

J.C. Mason

Computational Mathematics Group
Royal Military College of Science
Shrivenham, Swindon, Wiltshire, England

ABSTRACT

It is known that the Chebyshev polynomials of the first and second kinds are minimal in L_p on $[-1,1]$ with respect to appropriate weight functions, namely certain powers of $1-x^2$, for $1 \leq p \leq \infty$. These properties are here exploited in two applications. First, convergence and optimality properties are established for a "complete" Chebyshev polynomial expansion method for the determination of indefinite integrals. Second, conjectures are derived concerning the near-minimality of the Laguerre polynomials $L_n^{\mp 1/2}(2\beta x)$ for $\beta \approx 1$ with respect to appropriate exponentially weighted L_p norms on $[0,\infty)$.

1. INTRODUCTION

This paper discusses two distinct ways of exploiting minimal L_p properties of Chebyshev polynomials $T_k(x)$ and $U_k(x)$ of the first and second kinds, where k is the polynomial degree. Such minimal properties, together with a number of results concerning Chebyshev series, are discussed in full by Mason [1] and the two key properties are that, amongst all suitably normalised polynomials $Q_k(x)$ of degree k, for all $1 \leq p \leq \infty$

$$T_k(x) \quad \text{minimises} \quad \left[\int_{-1}^{1} (1-x^2)^{-1/2} |Q_k(x)|^p dx \right]^{1/p} \tag{1}$$

$$\text{and} \quad U_k(x) \quad \text{minimises} \quad \left[\int_{-1}^{1} (1-x^2)^{(p-1)/2} |Q_k(x)|^p dx \right]^{1/p} \tag{2}$$

In the first application, a function is integrated after first being expanded in a complete first and second kind Chebyshev series. Four minimal L_p properties (namely (1), (2) for $p = 1, \infty$) are then used to establish the optimality of the chosen method in certain canonical cases. We also establish L_∞ convergence for the integral and L_1 convergence for the integrand in the method. The present discussion extends and broadens the author's earlier treatment of integration methods in [1].

The second application is in the determination of orthogonal polynomial systems which have nearly minimal L_p norms on $[0,\infty)$, subject to weight functions closely related to e^{-x} . Two new conjectures are obtained, which extend to L_p norms some earlier results of the author in [2] for L_∞ . These conjectures have already been tested and found to be valid for polynomials up to degree 10 in L_∞ , and they are trivially valid for all polynomial degrees in L^2 .

2. INDEFINITE INTEGRATION

2.1. The Chebyshev Method

Suppose that we require the value of the indefinite integral

$$h(x) = \int_{-1}^{x} f(x)\,dx \qquad \text{in} \qquad -1 \leq x \leq 1 , \tag{3}$$

and that $f(x)$ takes the form

$$f(x) = f^A(x) + (1-x^2)^{-1/2} f^B(x) , \tag{4}$$

where f^A and f^B are given continuous functions. This means that we are integrating functions which have $x^{-1/2}$ singularities at end points and a complementary smooth behaviour. (The analysis is actually valid if f^A and f^B are at most L_2-integrable, although the methods can be of limited accuracy in such cases). Now let us approximate f^A and f^B by the partial sums f^A_{n-1} and f^B_n of their expansions in $\{U_k\}$ and $\{T_k\}$ respectively, namely

$$f^A(x) \simeq f^A_{n-1}(x) = \sum_{n-1}^{n} a_k\, U_{k-1}(x) \tag{5}$$

$$f^B(x) \simeq f^B_n(x) = \sum_{k=o}^{n} b_k\, T_k(x) \tag{6}$$

where $a_k = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{1/2} f^A(x)\, U_{k-1}(x)\,dx$

and $b_k = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{-1/2} f^B(x)\, T_k(x)\,dx$.

On integrating (4) between -1 and x and using the approximations (5), (6), we obtain an indefinite integral in the form

$$h(x) \simeq h^A_n(x) + H^B_{n-1}(x) , \tag{7}$$

where $$h_n^A(x) = \int_{-1}^{x} f_{n-1}^A(x)\,dx = \sum_{k=1}^{n} \frac{a_k}{k} \left[T_k(x) - T_k(-1)\right] \tag{8}$$

and

$$H_{n-1}^B(x) = \int_{-1}^{x} f_n^B(x) = \frac{1}{2} b_o(\pi - \cos^{-1}x) - \sum_{k=1}^{n} \frac{b_k}{k} (1-x^2)^{1/2} U_{k-1}(x) \tag{9}$$

The integration is here greatly simplified by the formulae

$$\frac{d}{dx}\left[T_k(x)\right] = k\, U_{k-1}(x) \ ,$$

$$\frac{d}{dx}\left[(1-x^2)^{1/2} U_{k-1}(x)\right] = -k(1-x^2)^{-1/2} T_k(x)$$

The above method is essentially a generalisation to complete T_k and U_k expansions of a method of Filippi [3], which was originally based on a U_k expansion. For practical implementation, however, the partial sums (5), (6) should normally be replaced by the (virtually indistinguishable) polynomials obtained by collocation at the respective Chebyshev zeros. This very much simplifies the calculation and, indeed, if the discrete orthogonality properties of T_k and U_k are exploited, then only $0(n)$ arithmetic operations are required in the method. (See [1]). However, it is more difficult to analyse the collocation method in the context of approximation theory, and that is why we have used the expansion method as our theoretical model here.

In the context of definite integration, the method can be viewed as a product integration rule with certain abscissae and weights, and then convergence can be studied from this viewpoint (See [4]).

2.2. The L_1 Convergence of the Integrand

Let us first analyse the error E_n in the approximation of the integrand using (5), (6), namely

$$E_n(x) = f(x) - f_{n-1}^A(x) - (1-x^2)^{1/2} f_n^B(x)$$

On setting $x = \cos\theta$ (for $0 \leq \theta \leq \pi$) and multiplying through by $\sin\theta$, we obtain

$$\sin\theta\, E_n(\cos\theta) = \sin\theta\, f(\cos\theta) - \sum_{k=o}^{n} a_k T_k(\cos\theta) -$$

$$- \sum_{k=1}^{n} b_k \; U_{k-1}(\cos\theta) =$$

$$= \sin\theta \; f(\cos\theta) - \sum_{k=o}^{n} (a_k \cos k\,\theta + b_k \sin k\theta)$$

The right hand side is the error in the partial sum of a Fourier series expansion of a continuous function, namely $\sin\theta\; f(\cos\theta)$, and so by classical theory it converges to zero in L_2 as $n \longrightarrow \infty$. Convergence to zero immediately follows in the weaker L_1 norm (in θ) . Hence

$$||E_n(x)||_1 = \int_{-1}^{1} |E_n(x)|dx = \int_{o}^{\pi} |E_n(\cos\theta)|\sin\theta \; d\theta$$

$$= ||\sin\theta \; E_n(\cos\theta)||_1 \longrightarrow 0 \qquad (10)$$

This establishes the L_1 convergence (in x) of the integrand to f(x) as $n \longrightarrow \infty$.

2.3. The Uniform Convergence of the Integral

Turning now to the indefinite integral (which is obtained from the approximate integrand), the error ε_n in this is given by

$$\varepsilon_n(x) = h(x) - h_n^A(x) - H_{n-1}^B(x) = \int_{-1}^{x} E_n(x)dx$$

Now $$|\varepsilon_n(x)| = |\int_{-1}^{x} E_n(x)dx| \leq \int_{-1}^{x} |E_n(x)|dx \; ,$$

and hence

$$||\varepsilon_n(x)||_\infty = \max_{-1\leq x\leq 1} |\varepsilon_n(x)| \leq \int_{-1}^{1} |E_n(x)|dx = ||E_n(x)||_1 \; . \qquad (11)$$

From (10) we immediately deduce the uniform convergence of ε_n to zero as $n \longrightarrow \infty$.

The bound (11) is extremely conservative, since the modulus of the integral of an oscillatory function has been bounded by the integral of the modulus. Nevertheless, we show in §2.4 below the remarkable fact that, in two canonical cases, the method optimises both $||\varepsilon_n||_\infty$ and $||E_n||_1$ simultaneously.

2.4. The Optimality of the Method

The tacit assumption was made above that the expansions (5) (6) were particularly appropriate ones to adopt, and indeed their use certainly ensured a very simple integration procedure. But it is not clear that it might not, for example, be better to adopt a $\{T_k\}$ expansion to $f^A(x)$ in (5), and indeed this is the approach used in the original Chebyshev integration method of Clenshaw and Curtis [5]. However, even though a wide variety of orthogonal polynomial expansion methods would probably give reasonably comparable results, the respective choices of U_{k-1} and T_k in (5) (6) are optimal, in the sense that $||\varepsilon_n||_\infty$ and $||E_n||_1$ are minimised in two canonical cases, provided an appropriate small error is introduced (through a constant of integration).

2.4.1. Polynomials of Degree n for $f(x)$

Consider first the function

$$f(x) = f^A(x) = x^n \qquad (\text{with } f^B(x) = 0) \quad , \tag{12}$$

which is representative of all n^{th} degree polynomials (of one degree higher than the approximation f^A_{n-1}). Now, in this case, the partial expansion of x^n up to degree n in $\{U_k\}$ is exact, and moreover the U_n term must have a unit coefficient of x^n. Hence

$$f^A_{n-1}(x) = x^n - 2^{-n}U_n(x) \tag{13}$$

Now integrating up to x,

$$\varepsilon_n(x) = \int^x E_n(x) + C = 2^{-n}(n+1)^{-1}\, T_{n+1}(x) + C$$

and, on setting $C = 0$,

$$\varepsilon_n(x) = 2^{-n}(n+1)^{-1}T_{n+1}(x) \tag{14}$$

By the minimality properties (1), (2) of $U_n(x)$, $T_{n+1}(x)$ for $p = 1, \infty$ applied to (13), (14), respectively, we deduce that both $||E_n||_1$ and $||\varepsilon_n||_\infty$ have been minimised (over all possible expansions of $f^A(x)$).

Note, however, that an error has been introduced at -1, namely

$$\varepsilon_n(-1) = 2^{-n}(n+1)^{-1}T_{n+1}(-1) = (-1)^{n+1}2^{-n}(n+1)^{-1} \tag{15}$$

However (15) tends to zero as $n \to \infty$, and hence $\varepsilon_n(x)$ still converges uniformly to zero.

2.4.2. Weighted Polynomials of Odd Degree $n+1$ for $f(x)$

Next consider the function

$$f(x) = (1-x^2)^{-1/2} f^B(x) = (1-x^2)^{-1/2} x^{n+1} \qquad (\text{with } f^A = 0) \tag{16}$$

where n is even, which is representative of all polynomials of degree $n+1$ (of one degree higher than f_n^B) weighted by $(1-x^2)^{-1/2}$. The oddness of f^B ensures that b_o vanishes in (9), so that the term in $\cos^{-1} x$ is not present. The partial expansion of x^{n+1} up to degree $n+1$ in $\{T_k\}$ is exact and the T_{n+1} term must have a unit coefficient of x^{n+1}. Hence

$$f_n^B(x) = x^{n+1} - 2^{-n} T_{n+1}(x) \tag{17}$$

Integrating from -1 to x,

$$\varepsilon_n(x) = \int_{-1}^{x} E_n(x)\,dx = -2^{-n}(n+1)^{-1}(1-x^2)^{-1/2} U_n(x) \tag{18}$$

By the minimal properties (1), (2) of $(1-x^2)^{1/2} T_{n+1}(x)$, $(1-x^2)^{-1/2} U_n(x)$ for $p=1,\infty$ applied to (17), (18), respectively, we deduce that both $||E_n||_1$ and $||\varepsilon_n||_\infty$ have been minimised (over all expansions of $f^B(x)$).

No error has been introduced at $x = -1$ in this case.

3. NEAR-MINIMALITY IN L_p WITH EXPONENTIAL WEIGHTS

Consider the following L_p norms of an appropriately normalised polynomial $P_k(x)$ of degree k, weighted by e^{-x} and $x^{1/2}e^{-x}$, respectively:

$$F_1(P_k) = \left[\int_o^\infty x^{-1/2} |e^{-x} P_k(x)|^p dx \right]^{1/p} \qquad (1 \leq p \leq \infty) \tag{19}$$

$$F_2(P_k) = \left[\int_o^\infty x^{-1/2} |x^{1/2} e^{-x} P_k(x)|^p dx \right]^{1/p} \qquad (1 \leq p \leq \infty) \tag{20}$$

Specific cases include

$$p = \infty: \quad F_1(P_k) = ||e^{-x} P_k||_\infty \quad , \quad F_2(P_k) = ||x^{1/2} e^{-x} P_k||_\infty \tag{21}$$

$$p = 2 : \quad F_1(P_k) = \left[\int_0^\infty x^{-1/2} e^{-2x} P_k^2 \, dx \right]^{1/2} , \qquad (22)$$

$$F_2(P_k) = \left[\int_0^\infty x^{1/2} e^{-2x} P_k^2 \, dx \right]^{1/2}$$

$$p = 1 : \quad F_1(P_k) = \int_0^\infty x^{-1/2} e^{-x} |P_k| dx , \quad F_2(P_k) = \int_0^\infty e^{-x} |P_k| dx \qquad (23)$$

Note that weights e^{-x} occur in F_1 for $p = \infty$ and F_2 for $p = 1$, $x^{-1/2}e^{-\alpha x}$ occur in F_1 for $p = 1,2$, and $x^{1/2}e^{-\alpha x}$ occur in F_2 for $p = 2,\infty$.

Now F_1, F_2 are functionals of P_k, which we desire to minimise. Explicit solutions are only known for $p = 2$, in which case

$$P_k = L_k^{-1/2}(2\beta x) \;, \quad L_k^{1/2}(2\beta x) \qquad (24)$$

where $\beta = 1$. For other values of p we aim instead to "nearly minimise" F_1, F_2. The appropriate specification for the term "nearly minimise" is a matter of personal taste. However, we suggest that in practice P_k be accepted for $k \leq 10$ if its functional is relatively within 20% of the minimum possible value.

This requirement can in fact be comfortably satisfied by (24) for $p = \infty$ and $\beta \simeq 1$ (see [2]). For example, for n=10, $\beta = .975$, F_1 and F_2 are within 10% of their minima. Let us now propose two conjectures which extend the above deductions for $p = 2,\infty$ and which have therefore already been confirmed in these 2 cases.

<u>Conjecture 3.1</u>. For any p $(1 \leq p \leq \infty)$, $F_1(P_k)$ is nearly minimised by $P_k = L_k^{-1/2}(2\beta_1 x)$ for some $\beta_1 = \beta_1(p)$ close to 1.

<u>Conjecture 3.2</u>. For any p $(1 \leq p \leq \infty)$, $F_2(P_k)$ is nearly minimised by $P_k = L_k^{1/2}(2\beta_2 x)$ for some $\beta_2 = \beta_2(p)$ close to 1.

Although we have no rigorous proofs, the following discussions give substantial support to the Conjectures, and are based on the application of bilinear transformations to (1), (2) which take $[-1,1]$ into $[0,\infty)$.

3.1. Discussion of Conjecture 3.1

$$\left[\int_{-1}^{1} (1-t^2)^{-1/2}|Q_k(t)|^p\, dt\right]^{1/p} \qquad (1\leq p\leq\infty) \tag{25}$$

is minimised over normalised polynomials Q_k of degree k (by (1)) when $Q_k = T_k$ and hence when

$$\int_{-1}^{1} (1-t^2)^{-1/2}\, Q_j(t)Q_k(t)\,dt = 0 \qquad (j < k) \tag{26}$$

Setting $t = \frac{Ax-1}{Ax+1}$, $A = r^{-1}$, $r = k+p^{-1}$, so that

$$1-t^2 = \frac{2Ax}{(Ax+1)^2} \quad \text{and} \quad dt = \frac{2\,dx}{(Ax+1)^2} ,$$

it follows from (25) (26) that

$$\left[\int_{0}^{\infty} x^{-1/2}\, |(1+x/r)^{-r}\, P_k(x)|^p\, dx\right]^{1/p} \tag{27}$$

is minimised over polynomials P_k of degree k when

$$\int_{0}^{\infty} x^{-1/2}(1+x/r)^{-(j+k+1)}P_j(x)P_k(x)\,dx = 0 \qquad (j<k) \tag{28}$$

Replacing $(1+x/r)^{-r}$ by the comparable weight e^{-x} in both (27) and (28) , $F_1(P_k)$ is nearly minimised when

$$\int_{0}^{\infty} x^{-1/2}\, e^{-2\beta_{jk}x}\, P_j(x)\; P_k(x)\,dx = 0 \qquad (j < k) \tag{29}$$

where $\beta_{jk} = 1/2\ (j+k+1)r^{-1}$.

Now $\beta_{jk} = k/(k+p^{-1}) \simeq 1$ for $j = k-1$ (the key value of j) , and hence on replacing β_{jk} by a constant $\beta_1 \simeq 1$ in (29) we obtain Conjecture 3.1.

3.2. Discussion of Conjecture 3.2

$$\left[\int_{-1}^{1} (1-t^2)^{(p-1)/2}|Q_k(t)|^p dt\right]^{1/p} \qquad (1\leq p\leq\infty) \tag{30}$$

is minimised over normalised polynomials Q_k of degree k (by (2)), when $Q_k = U_k$ and hence when

$$\int_{-1}^{1} (1-t^2)^{1/2}\, Q_j(t)\, Q_k(t)\, dt = 0 \qquad (j < k) \qquad (31)$$

Setting $t = \frac{Ax-1}{Ax+1}$, $A = r^{-1}$, $r = k+1+p^{-1}$,

it follows from (30), (31) that

$$\left[\int_0^\infty x^{(p-1)/2} \quad (1+Ax)^{-(p+1)} \left| (1+Ax)^{-k} P_k(x) \right| (1+Ax)^{-2} dx \right]^{1/p} =$$

$$= \left[\int_0^\infty x^{-1/2} \left| x^{1/2} (1+x/r)^{-r}\, P_k(x) \right|^p dx \right]^{1/p} \qquad (32)$$

is minimised over normalised polynomials P_k of degree k when

$$\int_0^\infty x^{1/2} (1+x/r)^{-(j+k+3)} P_j(x) P_k(x)\, dx = 0 \qquad (j < k) \qquad (33)$$

Replacing $(1+x/r)^{-r}$ by the comparable weight e^{-x} in both (32) and (33) , $F_2(P_k)$ is nearly minimised when

$$\int_0^\infty x^{1/2} e^{-2\beta_{jk} x} \quad P_j(x)\ P_k(x)\, dx = 0 \qquad (j < k) \qquad (34)$$

where $\beta_{jk} = \frac{1}{2}(j+k+3)\, r^{-1}$

Now $\beta_{jk} = (k+1)/(k+1+p^{-1}) \simeq 1$ for $j = k-1$ (the key value of j), and hence on replacing β_{jk} by a constant $\beta_2 \simeq 1$ in (34) we obtain Conjecture 3.2.

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