FACTORIZATION OF SECOND ORDER DIFFERENCE EQUATIONS AND ITS APPLICATION TO ORTHOGONAL POLYNOMIALS

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ABSTRACT. It is shown that a second order recurrence expression with coefficients having bounded variation, written as a second degree polynomial of the forward shift operator, can be factored as the product of two first order expressions. This result is used to obtain asymptotics over the complex plane for a class of polynomials orthonormal over the real line.

CONTENTS

1. Introduction

Consider the second order recurrence equation

$$f(n+2) + \alpha(n)f(n+1) + \beta(n)f(n) = h(n) \quad (-\infty < n < \infty) \tag{1.1}$$

with f as the unknown function, such that the coefficients α and β are of bounded variation (i.e., (2.4) below holds). Writing A and B for the limits of the sequences $\alpha(n)$ and $\beta(n)$, respectively, the equation

$$t^2 + At + B = 0 \tag{1.2}$$

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is called the characteristic equation of (1.1). Provided the roots of (2.1) have different absolute values, we will be able to express the left-hand side of (1.1) as an operator product. That is, introducing the forward shift operator E, the left-hand side of (1.1) can be written as

$$((E^2 + \alpha E + \beta)f)(n)$$

(E will be described in some detail in the next section.) This will be factored as

$$((E - \varsigma_2)(E - \varsigma_1)f)(n)$$

where ς_1 and ς_2 are functions on integers (it is shown in (2.9) below how to multiply out this product). Thus (1.1) can be written as

$$(E - \varsigma_2)(E - \varsigma_1)f = h,$$

which can be solved explicitly as

$$f = (E - \varsigma_1)^{-1}(E - \varsigma_2)^{-1}h;$$

the inverses on the right can be written as infinite sums, and they will be convergent if h is well-behaved.

Now, polynomials orthogonal on the real line satisfy a recurrence equation analogous to (1.1):

$$a_{n+2}p_{n+2}(x) + (b_{n+1} - x)p_{n+1} + a_{n+1}p_n(x) = 0 \quad (n \geq -1) \tag{1.3}$$

(see (4.3) below). If the coefficient sequences here are of bounded variation, then we can use the above method to obtain an asymptotic expression for p_n outside the real line (see Theorems 4.1, 5.1 and 5.2 below). The results we obtain go partly beyond Theorem 1 of Máté-Nevai-Totik [5, p. 232] in that our asymptotic estimate holds on unbounded sets as well.

While equation (1.3) is a homogeneous equation, it is only valid for $n \geq -1$. Before the method outlined above can be applied, it has to be extended to an inhomogeneous equation valid for $-\infty < n < \infty$.

PART I. RECURRENCE EQUATIONS
2. The Factorization Theorem

In this section $\alpha, \beta, \ldots, f, g, \ldots$ will denote functions on integers and E will denote the forward shift operator, that is

$$E^k f(n) = (E^k f)(n) = f(n + k) \qquad (-\infty < k, n < \infty).$$

We will write terms involving products of several functions, indicating the argument only once on the right. That is, e.g.,

$$fg(n) = (fg)(n) = f(n) \cdot g(n).$$

Such terms may be interlaced with integral powers of E. An occurrence of E in the term will affect all functions to the right in the same term. That is, e.g.,

$$fEg(n) = f(n)g(n+1),$$
$$f_1E^2f_2E^{-3}f_3f_4E^4f_5(n) = f_1(n)f_2(n+2)f_3(n-1)f_4(n-1)f_5(n+3)$$

(e.g. to obtain $f_4(n-1)$, observe that f_4 is affected by powers of E to the left of it, i.e., by $E^2E^{-3} = E^{-1}$), and

$$\begin{aligned}(E\alpha)^3(n) &= E\alpha E\alpha E\alpha(n) \\ &= \alpha(n+1)\alpha(n+2)\alpha(n+3).\end{aligned} \tag{2.1}$$

Using this notation, we can now state the Factorization Theorem:

THEOREM 2.1. *Consider the recurrence polynomial*

$$(E^2 + \alpha E + \beta)f(n) \qquad (-\infty < n < \infty) \tag{2.2}$$

with f as the indeterminate function, where

$$(i)\ \lim_{n\to\infty} \alpha(n) = A, \qquad (ii)\ \lim_{n\to\infty} \beta(n) = 1, \tag{2.3}$$

and

$$\sum_{n=0}^{\infty}(|\alpha(n+1) - \alpha(n)| + |\beta(n+1) - \beta(n)|) < \infty. \tag{2.4}$$

Assume here A is a complex number such that

$$A \notin [-2,2]. \tag{2.5}$$

Then (2.2) can be factored as

$$(E - \varsigma_2)(E - \varsigma_1)f(n) \tag{2.6}$$

for large enough n, say $n \geq n_0$, where ς_1 and ς_2 are functions on integers. Moreover, writing t_{1n} and t_{2n} with $|t_{1n}| \geq |t_{2n}|$ for the roots of the equation

$$t^2 + \alpha(n)t + \beta(n) = 0, \tag{2.7}$$

we can choose ς_1 and ς_2 such that

$$\sum_{n=n_0}^{\infty} | t_{1n} - \varsigma_1(n) | < \infty. \tag{2.8}$$

It may be possible to gain some insight into the meaning of the factorization in (2.6) by multiplying it out. We obtain

$$\begin{aligned}(E - \varsigma_2)(E - \varsigma_1)f(n) &= E^2 f(n) - \varsigma_2 E f(n) - E\varsigma_1 f(n) + \varsigma_2\varsigma_1 f(n) \\ &= f(n+2) - (\varsigma_2(n) + \varsigma_1(n+1))f(n+1) \\ &+ \varsigma_2(n)\varsigma_1(n)f(n).\end{aligned} \tag{2.9}$$

The equation

$$t^2 + At + 1 = 0 \tag{2.10}$$

is called the characteristic equation of the difference polynomial in (2.2). Condition (2.5) ensures that the roots of this polynomial have different absolute values, and this will be crucial for our method. Condition (2.3)(ii) is only technical, and it can be replaced with the more general condition

$$\lim_{n\to\infty} \beta(n) = B \neq 0, \tag{2.11}$$

but then (2.5) will have to be modified appropriately. The characteristic equation in this case is

$$t^2 + At + B = 0,$$

but this can be transformed back to the form in (2.10) by using the substitution $t = \sqrt{B}t'$. This substitution corresponds to the substitution

$$f(n) = B^{n/2} f'(n)$$

in (2.2), and so condition (2.11) can be reduced to condition (2.3)(ii).

As $A \neq \pm 2$ according to (2.5), the roots of equation (2.10) are distinct. Hence (2.4) implies

$$\sum_{n=0}^{\infty} (| t_{1,n+1} - t_{1n} | + | t_{2,n+1} - t_{2n} |) < \infty \tag{2.12}$$

in virtue of the case $m = 2$ of the following simple

LEMMA 2.2. *Assume that the roots $w_1, \ldots, w_m$ of the equation*

$$\sum_{j=0}^{m} u_j w^j = 0 \qquad (u_m = 1)$$

are pairwise distinct. Then the Jacobian

$$\det[\frac{\partial u_j}{\partial w_k}]_{0\le j\le m-1,\ 1\le k\le m}$$

is different from zero.

PROOF. Write

$$P(w) = \sum_{j=0}^{m} u_j w^j = \prod_{k=1}^{m} (w - w_k).$$

As the z_k 's are distinct, we have

$$\frac{\partial P(w)}{\partial w_k} = -\frac{P(w)}{w - w_k} = \sum_{j=0}^{m-1} w^j \frac{\partial u_j}{\partial w_k}.$$

Now the conclusion of the lemma follows from the observation that the vectors $[\partial u_j/\partial w_k]_{0\le j\le m-1}$ $(1 \le k \le m)$ are linearly independent. Namely, the equation

$$0 = \sum_{k=1}^{m} c_k \sum_{j=0}^{m-1} w^j \frac{\partial u_j}{\partial w_k} = -\sum_{k=1}^{m} c_k \frac{P(w)}{w - w_k}$$

implies $c_k = 0$ for each k; to see this, make $w \to w_k$. The proof is complete.

As we will apply Theorem 2.1 to obtain uniform asymptotics for orthogonal polynomials, we will need a more precise version of this theorem in that we will need an estimate for the remainder of the sum in (2.8).

THEOREM 2.3. *Assume the hypotheses of Theorem 2.1, and let* ρ *be a real with* $0 < \rho < 1$. *Let* n_0 *be such that*

$$|\beta(n)/t_{1n}| \le \rho \tag{2.13}$$

and

$$|t_{1n}| \ge 1 + \sum_{n=n_0}^{\infty} |t_{1,n+1} - t_{1n}| \tag{2.14}$$

hold for $n \ge n_0$. *Then* ς_1 *and* ς_2 *in (2.6) can be chosen such that*

$$\sum_{\nu=n}^{\infty} |t_{1\nu} - \varsigma_1(\nu)| \le \sum_{\nu=n_0}^{n-1} |t_{1,\nu+1} - t_{1\nu}| \rho^{n-1-\nu}/(1-\rho) + \sum_{\nu=n}^{\infty} |t_{1,\nu+1} - t_{1\nu}|/(1-\rho) \tag{2.15}$$

holds for every $n \geq n_0$.

Observe that an integer n_0 satisfying (2.13) and (2.14) does exist. Indeed, writing t_1 and t_2 with $|t_1| \geq |t_2|$ for the roots of the characteristic equation (2.10), we have $|t_1| > |t_2|$ in view of (2.5), and so we have $|t_1| > 1$, as $t_1 t_2 = 1$. Now, since $\beta(n) \to 1$ and $t_{1n} \to t_1$, (2.13) will hold for large enough n. Moreover, writing $\eta = (t_1 - 1)/2$, we will have $|t_{1n}| \geq 1 + \eta$ for large enough n, and so (2.14) will hold for large enough n_0 in view of (2.12).

3. The Proof of the Factorization Theorem

The reason the Factorization Theorem is useful is that it reduces the solution of certain second orders difference equation to the successive solution of two first order difference equation. Under certain conditions, first order difference equations can be solved explicitly, as shown by the following simple

LEMMA 3.1. *Given the functions* f, g, *and* h *on integers, suppose that we have*

$$(E - g)f(n) = h(n) \qquad (-\infty < n < \infty). \tag{3.1}$$

Then

$$f(n) = \sum_{\ell=0}^{\infty} (E^{-1}g)^{\ell} E^{-1} h(n) \tag{3.2}$$

holds for each n *provided that*

$$\lim_{\ell \to \infty} (E^{-1}g)^{\ell} f(n) = 0 \tag{3.3}$$

holds for each n.

Observe that e.g.,

$$(E^{-1}g(n))^2 E^{-1} h(n) = g(n-1)g(n-2)h(n-3)$$

(cf. example (2.1) above). Condition (3.3) will be fulfilled in our applications, as we will have $f(n) = 0$ when n is a large enough negative number.

PROOF. Replacing $h(n)$ with the left-hand side of (3.1), the right-hand side of (3.2) becomes

$$\begin{aligned}\sum_{\ell=0}^{\infty}(E^{-1}g)^{\ell}E^{-1}(E-g)f(n) &= \sum_{\ell=0}^{\infty}((E^{-1}g)^{\ell} - (E^{-1}g)^{\ell+1})f(n)\\ &= \lim_{N\to\infty}(I - (E^{-1}g)^{N+1})f(n) = f(n),\end{aligned}$$

where I is the identity operator; the last qeuality follows from (3.3). This shows that (3.2) is indeed valid, completing the proof of the lemma.

Next we turn to the proof of the Factorization Theorem. As Theorem 2.1 follows from Theorem 2.3, it will be sufficient to present only the

PROOF OF THEOREM 2.3. As the expressions in (2.2) and on the right-hand side of (2.9) (which is the multiplied-out version of (2.6)) must agree for any choice of the function f, the respective coefficients must agree, i.e., we must have

$$-\alpha(n) = \varsigma_1(n+1) + \varsigma_2(n)$$

and

$$\beta(n) = \varsigma_2(n)\varsigma_1(n). \tag{3.4}$$

That is

$$\varsigma_1(n+1) = -\alpha(n) - \beta(n)/\varsigma_1(n). \tag{3.5}$$

Now, dividing equation (2.7) by t and replacing t with t_{1n} (which is a root of this equation), we obtain

$$t_{1n} = -\alpha(n) - \beta(n)/t_{1n}.$$

Subtracting (3.5) from this, we obtain

$$t_{1n} - \varsigma_1(n+1) = \frac{\beta(n)}{t_{1n}\varsigma_1(n)}(t_{1n} - \varsigma_1(n)),$$

that is

$$| t_{1,n+1} - \varsigma_1(n+1) | \leq | t_{1,n+1} - t_{1n} | + | \frac{\beta(n)}{t_{1n}\varsigma_1(n)} || t_{1n} - \varsigma_1(n) | . \tag{3.6}$$

Now choose

$$\varsigma_1(n_0) = t_{1n_0} \tag{3.7}$$

for the n_0 described in connection of (2.13) and (2.14), and define $\varsigma_1(n)$ for $n \geq n_0$ with the aid of (3.5). We will prove by induction on n that

$$| \varsigma_1(n) | \geq 1 \tag{3.8}$$

holds for $n \geq n_0$. This is true for $n = n_0$, since $\varsigma_1(n_0) = t_{1n_0} \geq 1$ holds in view of (2.14). Let $n \geq n_0$ and assume that (3.8) holds with ν such that $n_0 \leq \nu \leq n$ replacing n. Then

$$| \frac{\beta_1(\nu)}{t_{1\nu}\varsigma_1(\nu)} | \leq \rho \qquad (n_0 \leq \nu \leq n)$$

holds in view of (2.13), and so (3.6) becomes

$$| t_{1,\nu+1} - \varsigma_1(\nu + 1) | \leq | t_{1,\nu+1} - t_{1\nu} | + \rho | t_{1\nu} - \varsigma_1(\nu) | .$$

Using this for $\nu = n,\ n-1, \ldots, n_0$ repeatedly and noting that $t_{1n_0} - \varsigma_1(n_0) = 0$ by (3.7), we obtain

$$| t_{1,n+1} - \varsigma_1(n + 1) | \leq \sum_{\nu=n_0}^{n} | t_{1,\nu+1} - t_{1\nu} | \rho^{n-\nu} . \tag{3.9}$$

As $\rho < 1$, this implies

$$| \varsigma_1(n + 1) | \geq | t_{1,n+1} | - \sum_{\nu=n_0}^{n} | t_{1,\nu+1} - t_{1\nu} | \geq 1 , \tag{3.10}$$

where the last inequality holds in view of (2.14). Thus (3.8) holds with $n + 1$ replacing n. This completes the inductive argument, showing that (3.8) holds for all $n \geq n_0$. As a by-product, we also obtain that (3.9) holds for all $n \geq n_0$; actually it vacuously holds for $n = n_0 - 1$ as well in view of (3.7). In fact, (3.9) will be our key result, and (3.8) was needed only in order to establish it.

Conclusion (2.15) of the theorem to be proved now readily follows. In fact, using (3.9), for $n \geq n_0$ we obtain

$$\begin{aligned}\sum_{\nu=n}^{\infty} | t_{1\nu} - \varsigma_1(\nu) | &\leq \sum_{\nu=n}^{\infty} \sum_{\ell=n_0}^{\nu-1} | t_{1,\ell+1} - t_{1\ell} | \rho^{\nu-1-\ell} \\ &= \sum_{\ell=n_0}^{n-1} | t_{1,\ell+1} - t_{1\ell} | \sum_{\nu=n}^{\infty} \rho^{\nu-1-\ell} + \sum_{\ell=n}^{\infty} | t_{1,\ell+1} - t_{1\ell} | \sum_{\nu=\ell+1}^{\infty} \rho^{\nu-1-\ell} .\end{aligned}$$

By evaluating the inner sums we obtain (2.15). The proof of Theorem 2.3 is complete.

PART II. AN APPLICATION TO ORTHOGONAL POLYNOMIALS

4. The Main Asymptotic Result

In what follows, by measure we will mean a positive finite measure α on the real line R whose support supp(α) (the smallest closed set $S \subset R$ with $\alpha(R \backslash S) = 0$) is an infinite set, and all the moments of which are finite, that is, for every integer $n \geq 0$ the integral

$$\int_{-\infty}^{\infty} x^n d\alpha(x)$$

exists (i.e., it is absolutely convergent). Associated with the measure α there is a unique sequence of orthonormal polynomials

$$p_n(x) = p_n(d\alpha, x) = \gamma_n x^n + \cdots \quad (\gamma_n = \gamma_n(d\alpha) > 0,\ n \geq 0) \tag{4.1}$$

(it is traditional to use the differential notation $d\alpha$ instead of α in these formulas) satisfying

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)d\alpha(x) = 1 \quad or \quad 0 \tag{4.2}$$

according as $m = n$ or $m \neq n$ $(m, n \geq 0)$. These polynomials satisfy a recurrence relation

$$a_{n+2}p_{n+2}(z) + (b_{n+1} - z)p_{n+1}(z) + a_{n+1}p_n(z) = 0 \quad (n \geq -1), \tag{4.3}$$

where $p_{-1} = 0$, $p_0 = 0$, $a_0 = 0$, and $a_n > 0$ for $n \geq 0$ (cf. e.g., Freud [2, formula (I.2.4) on p. 17] or Szegö [7, formula (3.2.1) on p. 42]). If one wants to indicate the dependence of the coefficients a_n, b_n on the measure α, one may write $a_n(d\alpha)$ and $b_n(d\alpha)$ instead.

The connection between the behavior of the coefficients a_n, b_n and the properties of the measure α is frequently investigated. In the most studied cases, the limits

$$\lim_{n\to\infty} a_n = a\,(\neq 0) \quad and \quad \lim_{n\to\infty} b_n = b$$

exist (and are finite). By a linear change of variables, we may assume $a = \frac{1}{2}$ and $b = 0$ here. In this case, the support of α is [-1,1] plus countable many isolated atoms (singletons with positive measure); see Blumenthal's theorem in Chihara [1, Sections IV.3-4, pp. 113-124]. The set of measures for which $a = \frac{1}{2}$ and $b = 0$ is often called $M(0, 1)$, and it is studied in detail e.g., in Nevai [6] (from p. 10 on at several places).

In studying equation (4.3), one can make use of the corresponding algebraic equation:

$$a_{n+2}t^2 + (b_{n+1} - z)t + a_{n+1} = 0 \quad (n \geq 0) \tag{4.4}$$

(we will not consider the case $n = -1$, even though it is allowed in (4.3), because the fact that $a_0 = 0$ would cause complications). Define τ as a holomorphic function on $C\backslash[-1, 1]$, where C is the complex plane, by putting

$$\tau(z) = z + \sqrt{z^2 - 1}, \tag{4.5}$$

where that branch of τ is chosen for which

$$\lim_{z\to\infty} \tau(z) = \infty. \tag{4.6}$$

For $-1 \leq x \leq 1$ put e.g.,

$$\tau(x) = \lim_{y\to 0^+} \tau(x + iy).$$

Then the roots of equation (4.4) can be written as

$$t_{1n}(z) = \sqrt{\frac{a_{n+1}}{a_{n+2}}}\,\tau\left(\frac{z - b_{n+1}}{2\sqrt{a_{n+1}a_{n+2}}}\right) \quad (n \geq 0), \tag{4.7}$$

and

$$t_{2n}(z) = \frac{a_{n+1}}{a_{n+2}}(t_{1n}(z))^{-1} \quad (n \geq 0). \tag{4.8}$$

Using the Factorization Theorem of Section 2 (or, rather, its variant, Theorem 2.3), we will establish the following:

THEOREM 4.1. *Let α be a finite positive measure on the real line with finite moments such that* supp(α) *is an infinite set. Assume that, writing $a_n = a_n(d\alpha)$ and $b_n = b_n(d\alpha)$, we have*

$$(i)\ \lim_{n\to\infty} a_n = \frac{1}{2}, \quad (ii)\ \lim_{n\to\infty} b_n = 0 \tag{4.9}$$

and

$$\sum_{n=0}^{\infty}(|\, a_{n+1} - a_n \,| + |\, b_{n+1} - b_n \,|) < \infty. \tag{4.10}$$

Then, writing $p_n(z) = p_n(d\alpha, z)$, the limit

$$g(z) = \lim_{n\to\infty}(p_n(z)\prod_{\nu=0}^{n-1}(t_{1\nu}(z))^{-1}) \tag{4.11}$$

exists for every complex $z \notin [-1,1]$. Moreover, the convergence in (4.11) is uniform on every closed set $K \subset C\backslash[-1,1]$. The limit

$$L = \lim_{z\to\infty} g(z) \tag{4.12}$$

exists and $L \neq 0$.

When we say that a limit exists, we do, of course, require that it not be ∞. Given K as described, $t_{1\nu}^{-1}$ is holomorphic on K for large enough ν in view of (4.9). Hence the uniformness of the convergence in (4.11) implies that

$$g(z)\prod_{\nu=0}^{N} t_{1\nu}(z) \tag{4.13}$$

is holomorphic on K for large enough N. In fact, we do not quite need the uniformness of the convergence in (4.11) to reach this conclusion: clearly, it is sufficient to know that the convergence in (4.11) is uniform on compact subsets of K (this observation will be of some use in Section 6).

The above result is only partly new. It was setablished earlier as Theorem 1 in Máté-Nevai-Totik [5, formula (9) on p. 232] except that the uniformness of the convergence in (4.11) was established only for compact (and hence bounded) $K \subset C\backslash[-1,1]$, and the existence of the limit in (4.12) was not discussed. Condition (4.10) was first considered in Máté-Nevai [4] and (somewhat later) Dombrowski [2].

5. The Main Lemma

A major step toward establishing Theorem 4.1 is represented by

LEMMA 5.1. *Assume the hypotheses of Theorem 4.1, and let D be an open subset of the complex plain such that its closure $\overline{D}$ is disjoint from [-1,1]. Then for every large enough integer n_0 there are functions F_n and G_n satisfying*

$$p_n(z)\prod_{\nu=0}^{n-1}(t_{1\nu}(z))^{-1} = F_n(z)(p_{n_0}(z) + (p_{n_0+1}(z) - t_{1n_0}(z)p_{n_0}(z))G_n(z)) \tag{5.1}$$

for $n > n_0$ and $z \in D$ such that for certain function F, G, ψ_{1n}, and ψ_{2n} we have

$$F_n(z) = F(z)(1 + \psi_{1n}(z)) \tag{5.2}$$

and

$$G_n(z) = G(z) + \psi_{2n}(z) \tag{5.3}$$

for every $n > n_0$ and $z \in D$, and

$$\lim_{n\to\infty}\psi_{jn}(z) = 0 \quad (j = 1,2) \tag{5.4}$$

uniformly for $z \in D$. Moreover, the functions F and G are bounded on every compact subset of D.

It is clear from (5.2)-(5.4) that for any fixed $z \in D$ the limit of the right-and side of (5.1) exists as $n \to \infty$, and in fact this limit is

$$g(z) = F(z)(p_{n_0}(z) + (p_{n_0+1}(z) - t_{1n_0}(z)p_{n_0}(z))G(z)). \tag{5.5}$$

Moreover, if $K \subset D$ is a compact set, then this limit is uniform on K, since the functions F, G, p_{n_0}, p_{n_0+1}, and t_{1n_0} are bounded on K (for t_{1n_0} this is true in view of (4.7), since τ is bounded on compact sets – cf. (4.5)). Hence the pointwise existence and the uniformness on every compact set $K \subset C\backslash[-1,1]$ of the limit in (4.11) follows from the above lemma. To establish the uniformness of the convergence in (4.11) on unbounded K we need to study the behavior of the functions F, G, and G_n near infinity. This will be done in the next section.

PROOF. We are going to use Theorem 2.3 to factor the left-hand side of the recurrence equation

$$p_{n+2}(z) + \frac{b_{n+1} - z}{a_{n+2}} p_{n+1}(z) + \frac{a_{n+1}}{a_{n+2}} p_n(z) = 0; \tag{5.6}$$

this equation holds for $n \geq 0$ (actually for $n \geq -1$, but cf. the remark after (4.4)) according to (4.3). The roots of the corresponding algebraic equation (4.4) were given by (4.7) and (4.8).

Let D_1 be an open set such that $\overline{D} \subset D_1$ and $\overline{D}_1 \subset C\backslash[-1,1]$. As $1/z$ is bounded on D and the derivative of $\tau(z)$ (cf. (4.5)) is bounded on D_1, it is easy to conclude from (4.7), (4.9), and (4.10) that

$$\lim_{n\to\infty} \frac{1}{z} \sum_{\nu=n}^{\infty} | t_{1,\nu+1}(z) - t_{1\nu} | = 0 \tag{5.7}$$

uniformly for $z \in D$.

By (4.7) and (4.9) we have

$$\lim_{n\to\infty} \frac{t_{1n}(z)}{z} = \frac{\tau(z)}{z} \tag{5.8}$$

uniformly on D. As $|\tau(z)| > 1$ for $z \in C\backslash[-1,1]$, it follows that there are constants $\eta > 1$ and $C_1, C_2 > 0$ such that if n is sufficiently large, say $n \geq n_1$ for some n_1, then

$$| t_{1n}(z) | > \eta \tag{5.9}$$

and

$$C_1 | z | < | t_{1n}(z) | < C_2 | z | \tag{5.10}$$

hold for $z \in D$. The constants C_1 and C_2 (and $C_3, C_4, \ldots$ below) may of course depend on D and the measure α. (4.9)(i) and (5.9) imply that there are an integer $n_0 \geq n_1$ and a real ρ with $0 < \rho < 1$ such that

$$\left| \frac{a_{n+1}/a_{n+2}}{t_{1n}(z)} \right| \leq \rho \tag{5.11}$$

holds for $z \in D$ and $n \geq n_0$; e.g., one can take $\rho = 2/(1+\eta)$. Using (5.9) (for $|z|$ small) and (5.10) (for $|z|$ large) it is easy to conclude from (5.7) that

$$| t_{1n}(z) | \geq 1 + C_3 | z | + \sum_{\nu=n_0}^{\infty} | t_{1,\nu+1}(z) - t_{1\nu}(z) | \tag{5.12}$$

holds for $z \in D$ and $n \geq n_0$ with some positive constant C_3, provided n_0 is chosen large enough; hence the conditions analogous to (2.13) and (2.14) are satisfied by equation (4.4) replacing (2.7). Clearly, writing

$$\alpha(n) = \frac{b_{n+1} - z}{a_{n+2}} \quad \text{and} \quad \beta(n) = \frac{a_{n+2}}{a_{n+1}} \tag{5.13}$$

(these are the coefficients in equation (5.6)), conditions (2.3) and (2.4) are satisfied in view of (4.9) and (4.10). Finally, condition (2.5) corresponds to the relation $z \notin [-1, 1]$, which holds for $z \in D$.

Hence, according to Theorem 2.3, the left-hand side of (5.6) can be factored as

$$(E - \varsigma_{2_z})(E - \varsigma_{1_z})q_z(n) \quad (n \geq n_0), \tag{5.14}$$

where ς_{1_z} and ς_{2_z} are functions on integers (depending on z, as indicated by the second subscript) and

$$q_z(n) = p_n(z).$$

We wrote z as a subscript so as to retain the original use of the forward shift operator as acting on arguments.

If we choose n_0 above large enough, then $t_{1n}(z)$ will be holomorphic in D for $n \geq n_0$. Then, defining $\varsigma_{1_z}(n)$ and $\varsigma_{2_z}(n)$ as in the proof of Theorem 2.3, that is, by formulas (3.7), (3.5), and (3.4) (cf. (5.13) for $\alpha(n)$ and $\beta(n)$), these functions will also be holomorphic in D for $n \geq n_0$.

The analogue of (2.15) in Theorem 2.3 is satisfied, i.e.,

$$\begin{aligned}\sum_{\nu=n}^{\infty} | t_{1\nu}(z) - \varsigma_{1_z}(\nu) | \leq \sum_{\nu=n_0}^{n-1} | t_{1,\nu+1}(z) - t_{1\nu}(z) | \rho^{n-1-\nu}/(1 - \rho) \\ + \sum_{\nu=n}^{\infty} | t_{1,\nu+1}(z) - t_{1\nu}(z) | /(1 - \rho) \quad (z \in D,\ n \geq n_0).\end{aligned} \tag{5.15}$$

In virtue of the uniformness of the convergence in (5.7), this implies that

$$\begin{aligned}&\lim_{n\to\infty} \sum_{\nu=n}^{\infty} \left| \frac{t_{1\nu}(z)}{z} - \frac{\varsigma_{1_z}(\nu)}{z} \right| \\ &= \lim_{n\to\infty} \frac{1}{|z|} \sum_{\nu=n}^{\infty} | t_{1\nu}(z) - \varsigma_{1_z}(\nu) | = 0\end{aligned} \tag{5.16}$$

uniformly for $z \in D$.

In view of inequality (5.10), this implies that

$$\lim_{n\to\infty}\sum_{\nu=n}^{\infty} \left| \log\frac{t_{1\nu}(z)}{z} - \log\frac{\varsigma_{1_z}(\nu)}{z} \right| = 0$$

uniformly for $z \in D$, i.e.,

$$\lim_{n\to\infty}\prod_{\nu=n}^{\infty}(\varsigma_{1_z}(\nu)/t_{1\nu}(z)) = 1 \tag{5.17}$$

uniformly for $z \in D$.

Now (5.14) factors the left-hand side of (5.6) for $n \geq n_0$; however, we need a factoring valid for all n, $-\infty < n < \infty$. To this end, put

$$\varsigma_{1_z}(n) = \varsigma_{2_z}(n) = 0 \tag{5.18}$$

for $z \in D$ and $-\infty < n < n_0$ (for $n \geq n_0$, $\varsigma_{1_z}(n)$ and $\varsigma_{2_z}(n)$ have already been defined), and write

$$r_z(n) = \begin{cases} p_n(z)(= q_z(n)) & \text{if } n \geq n_0, \\ 0 & \text{if } n < n_0. \end{cases} \tag{5.19}$$

Define $h_z(n)$ by the equation

$$(E - \varsigma_{2_z})(E - \varsigma_{1_z})r_z(n) = h_z(n) \tag{5.20}$$

for $z \in D$ and $-\infty < n < \infty$. Then

$$h_z(n) = 0 \quad \text{unless} \quad n = n_0 - 1 \text{ or } n = n_0 - 2, \tag{5.21}$$

$$h_z(n_0 - 2) = p_{n_0}(z), \tag{5.22}$$

and

$$h_z(n_0 - 1) = p_{n_0+1}(z) - \varsigma_{1_z}(n_0)p_{n_0}(z)$$

for $z \in D$. As

$$\varsigma_{1_z}(n_0) = t_{1n_0}(z) \tag{5.23}$$

according to (3.7), the last equation becomes

$$h_z(n_0 - 1) = p_{n_0+1}(z) - t_{1n_0}(z)p_{n_0}(z). \tag{5.24}$$

Using Lemma 3.1 twice, we can solve equation (5.20) for $r_z(n)$. The analogue of condition (3.3) is satisfied, since $r_z(n) = 0$ for $n < n_0$ according to (5.19). We obtain

$$r(n) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}(E^{-1}\varsigma_1)^k E^{-1}(E^{-1}\varsigma_2)^\ell E^{-1}h(n). \tag{5.25}$$

To simplify our notation, everywhere in this formula we dropped the subscript z; that is, we wrote $r = r_z$, $\varsigma_1 = \varsigma_{1z}$, $\varsigma_2 = \varsigma_{2z}$, and $h = h_z$. Note that only finitely many terms are nonzero in the sums on the right-hand side in view of (5.21). In fact, according to (5.21), the only nonzero terms on the right-hand side are those for which $n - k - \ell - 2 = n_0 - 1$ or $n - k - \ell - 2 = n_0 - 2$. Moreover, by (5.18) we can see that the terms corresponding to the latter case are zero unless $\ell = 0$ (since otherwise this term contains $\varsigma_2(n_0 - 1)$ as a factor). Thus, assuming $n \geq n_0$, (5.25) becomes

$$r(n) = (E^{-1}\varsigma_1)^{n-n_0}E^{-2}h(n) + \sum_{k=0}^{n-n_0-1}(E^{-1}\varsigma_1)^k E^{-1}(E^{-1}\varsigma_2)^{n-n_0-1-k}E^{-1}h(n).$$

Eliminating the operator E as done in example (2.1), we obtain

$$\begin{aligned} r(n) &= h(n_0-2)\prod_{\lambda=1}^{n-n_0}\varsigma_1(n-\lambda) + h(n_0-1)\sum_{k=0}^{n-n_0-1}\left(\prod_{\mu=1}^{k}\varsigma_1(n-\mu)\right)\left(\prod_{\nu=k+2}^{n-n_0}\varsigma_2(n-\nu)\right) \\ &= \left(\prod_{\lambda=1}^{n-n_0}\varsigma_1(n-\lambda)\right)\left(h(n_0-2) + h(n_0-1)\sum_{k=0}^{n-n_0-1}\frac{1}{\varsigma_1(n-k-1)}\prod_{\nu=k+2}^{n-n_0}\frac{\varsigma_2(n-\nu)}{\varsigma_1(n-\nu)}\right). \end{aligned}$$

By introducing the new variables $j = n - \lambda$, $\ell = n - k - 1$, and $m = n - \nu$, this becomes

$$r(n) = \left(\prod_{j=n_0}^{n-1}\varsigma_1(j)\right)\left(h(n_0 - 2) + h(n_0 - 1)\sum_{\ell=n_0}^{n-1}\frac{1}{\varsigma_1(\ell)}\prod_{m=n_0}^{\ell-1}\frac{\varsigma_2(m)}{\varsigma_1(m)}\right). \tag{5.26}$$

Note that the denominators here are not zero in view of (3.8).

Write

$$F_n(z) = \left(\prod_{\nu=0}^{n_0-1}(t_{1\nu}(z))^{-1}\right)\prod_{j=n_0}^{n-1}\frac{\varsigma_{1z}(j)}{t_{1j}(z)} \tag{5.27}$$

and

$$G_n(z) = \sum_{\ell=n_0}^{n-1} \frac{1}{\varsigma_{1_z}(\ell)} \prod_{m=n_0}^{\ell-1} \frac{\varsigma_{2_z}(m)}{\varsigma_{1_z}(m)} \tag{5.28}$$

for $n \geq n_0$. Then (5.26) becomes

$$r_z(n) = \left(\prod_{\nu=0}^{n-1} t_{1\nu}(z) \right) F_n(z)(h_z(n_0 - 2) + h_z(n_0 - 1)G_n(z)).$$

Hence (5.1) follows by (5.19), (5.22), and (5.24).

Putting

$$F(z) = \lim_{n\to\infty} F_n(z) \qquad \text{and} \qquad G(z) = \lim_{n\to\infty} G_n(z), \tag{5.29}$$

we can see that

$$\lim_{n\to\infty} F_n(z)/F(z) = 1$$

uniformly for $z \in D$ according to (5.17). Thus (5.2) and (5.4) with $j = 1$ hold.

To show (5.3) and (5.4) with $j = 2$, i.e., that

$$\lim_{n\to\infty} G_n(z) = G(z) \tag{5.30}$$

uniformly for $z \in D$, note that if n is large enough, then

$$\varsigma_{1_z}(n) \geq (1 + \eta)/2 \tag{5.31}$$

for every $z \in D$ according to (5.9) (for $|z|$ small), (5.10) (for $|z|$ large), and (5.16); recall that $\eta > 1$. Moreover, as we have

$$\varsigma_{2_z}(n)\varsigma_{1_z}(n) = a_{n+1}/a_{n+2}$$

(cf. (3.4) and (5.13)) and the right-hand side here tends to 1 as $n \to \infty$ according to (4.9)(i), it follows from (5.31) that, if n is large enough, then

$$|\varsigma_{2_z}(n)| \leq 1 \tag{5.32}$$

for every $z \in D$. This and (5.31) imply that the convergence in (5.30) is uniform.

Finally, we have to show that, given a compact set $K \subset D$, F and G are bounded on K. In view of (5.2)-(5.4), it is sufficient to show for this that F_n and G_n are bounded on K for each $n > n_0$. For G_n this is so because G_n is holomorphic in D (cf. (3.8) and the paragraph preceding (5.15)). For F_n this is so because

$$H_n(z) = \prod_{j=n_0}^{n-1} \frac{\varsigma_{1j}(j)}{t_{1j}(z)} = F_n(z) \prod_{\nu=0}^{n_0-1} t_{1\nu}(z) \tag{5.33}$$

holomorphic in D (cf. the same paragraph), and $t_{1\nu}(z)$ is bounded away from zero on K for every $\nu \geq 0$ (since τ is bounded away from zero on the whole plane – cf. (4.5) and (4.7)). The proof Lemma 5.1 is complete.

6. Near Infinity

In order to complete the proof of Theorem 4.1, we are going to study the behavior of the functions F and G near infinity. We have

LEMMA 6.1. *Assume the hypotheses of Lemma 5.1 (and so, also those of Theorem 4.1), and suppose that the set D includes a deleted neighborhood of ∞. Let $K \subset D$ be closed. Then for the functions F, G, G_n, and the integer n_0 described in Lemma 5.1 we have*

$$C_4 \mid z \mid^{-n_0} < \mid F(z) \mid < C_5 \mid z \mid^{-n_0} \tag{6.1}$$

and

$$\mid G(z) - 1/t_{1n_0}(z) \mid \leq C_6 \mid z \mid^{-2} \tag{6.2}$$

for $z \in K$ with some positive constants C_4, C_5 and C_6, and

$$\mid G_n(z) - 1/t_{1n_0}(z) \mid \leq C_6 \mid z \mid^{-2} \tag{6.3}$$

for $z \in K$ and $n > n_0$.

PROOF. We are going to show (6.1) first. To this end, observe that there are positive constants C_7 and C_8 such that

$$C_7 \mid z \mid < \mid \varsigma_{1n}(z) \mid < C_8 \mid z \mid \tag{6.4}$$

holds for $n \geq n_0$ and $z \in D$. Indeed, the first inequality holds for $n > n_0$ with $C_7 = C_3$ in view of (3.10) and (5.12). For $n = n_0$, it holds in view of (5.10) and (5.23). As for the second inequality, it follows from (5.15) that

$$\sum_{\nu=n_0}^{\infty} \mid t_{1\nu}(z) - \varsigma_{1\nu}(z) \mid \leq \frac{1}{1-\rho} \sum_{\nu=n_0}^{\infty} \mid t_{1,\nu+1}(z) - t_{1\nu}(z) \mid .$$

The right-hand side here is less than

$$| t_{1n_0}(z) | /(1 - \rho)$$

(cf. (5.12)); thus the second inequality in (6.4) follows from (5.10).

We are now going to estimate $F(z)$ as given by (5.27) and (5.29). As

$$\lim_{z \to \infty} \tau(z)/z = 2$$

(cf. (4.5)), the limit

$$\lim_{z \to \infty} (z^{n_0} \prod_{\nu=0}^{n_0-1} (t_{1\nu}(z))^{-1})$$

exists and is different from 0 (cf. (4.7)). Now, according to (4.7), the expression after the limit is bounded, since $| \tau(z) | \geq 1$ for $z \in C$, and it is bounded away from zero on compact subsets of D since $\tau(z)$ is bounded on compact subsets of C and z is bounded away from zero on D. Therefore, we have

$$C_9 | z |^{-n_0} < | \prod_{\nu=0}^{n_0-1} (t_{1\nu}(z))^{-1} | < C_{10} | z |^{-n_0} \tag{6.5}$$

for $z \in D$ with some positive constants C_9 and C_{10}.

Moreover according to (5.17) there is an $N \geq n_0$ such that

$$\frac{1}{2} < | \prod_{j=N}^{\infty} \frac{\varsigma_{1z}(j)}{t_{1j}(z)} | < \frac{3}{2} \tag{6.6}$$

holds for $z \in D$. Finally we have

$$(\frac{C_7}{C_2})^{N-n_0} < | \prod_{j=n_0}^{N-1} \frac{\varsigma_{1z}(j)}{t_{1j}(z)} | < (\frac{C_8}{C_1})^{N-n_0} \tag{6.7}$$

for $z \in D$ according to (5.10) and (6.4). Now (6.1) follows from (5.27) and (5.29) with the aid of (6.5)-(6.7). Note that (6.1) actually holds for $z \in D$, and not only for $z \in K$.

As for (6.2), it is an obvious consequence of (6.3) and (5.29). To show (6.3), observe that in view of (5.23) we can write (5.28) as

$$G_n(z) = \frac{1}{t_{1n_0}(z)} + \sum_{\ell=n_0+1}^{n-1} \frac{1}{\varsigma_{1z}(\ell)} \prod_{m=n_0}^{\ell-1} \frac{\varsigma_{2z}(m)}{\varsigma_{1z}(m)}$$

for $n > n_0$ (and $z \in D$). From here (6.3) follows by virtue of (5.32) and (6.4) provided $| z |$ is large enough (so that $C_7 | z | > 1 + \epsilon$ for $z \in K$ in (6.4), where $\epsilon > 0$). For $z \in K$ not large (6.3) simply says that G_n is bounded uniformly in n on each compact subset of D; this is indeed so in view of the last sentence of Lemma 5.1 and the uniformness of the convergence in (5.30). The proof of Lemma 6.1 is complete.

We are now in the position to complete the

PROOF OF THEOREM 4.1. Let D be an open set with $K \subset D$ such that $\overline{D}$ is disjoint from $[-1, 1]$ and D includes a deleted neighborhood of infinity. We have to establish (4.12) and the uniformness of the convergence in (4.11); the existence of the limit in (4.11) was pointed out right after Lemma 5.1. In what follows we assume $z \in K$.

As for (4.12), observe that

$$p_n(z) = \gamma_n z^n + O(|z|^{n-1})$$

holds for fixed n as $z \to \infty$, where $\gamma_n \neq 0$ (cf. (4.1)). Thus (5.5), (5.10), and (6.2) imply that

$$g(z) = F(z)(\gamma_{n_0+1} z^{n_0+1}/t_{1n_0}(z) + O(|z|^{n_0-1})) \tag{6.8}$$

as $z \to \infty$. Hence g is bounded away from 0 and ∞ in a deleted neighborhood of infinity, according to (5.10) and (6.1). Therefore, the existence of the limit L in (4.12) and $L \neq 0$ follow if we can show that g is holomorphic in a deleted neighborhood of ∞. Now g is indeed holomorphic in a deleted neighborhood of infinity, since the function in (4.13) is so and $t_{1\nu}(z)$ $(0 \leq \nu \leq N)$ has no zeros if $|z|$ is large enough. Here the remark made after (4.13) is significant, since we do not yet know the uniformness of the convergence in (4.11) on D, but we know it on compact subsets of D (cf. the discussion after Lemma 5.1).

Next we turn to the question of the uniformness of the convergence in (4.11). Writing $g_n(z)$ for the left-hand side of (5.1) and using (6.3) instead of (6.2) (and (5.1) instead of (5.5)), we obtain

$$g_n(z) = F_n(z)(\gamma_{n_0+1} z^{n_0+1}/t_{1n_0}(z) + O(|z|^{n_0-1}))$$

for $n > n_0$ as $z \to \infty$, where the bound implicit in the symbol $O(\cdot)$ is independent of n (it depends only on the constants in (5.10) and (6.2), and the coefficients of p_{n_0} and p_{n_0+1}). Thus, by (5.2) and (6.8) we have

$$|g_n(z) - g(z)| \leq F(z)(|\psi_{1n}(z)\gamma_{n_0+1} z^{n_0+1}/t_{1n_0}(z)| + (1 + |\psi_{n1}(z)|)C_{11} |z|^{n_0-1})$$

for $n > n_0$ and $|z| > R$, with some positive constants C_{11} and R. According to (5.10) and (6.1), the right-hand side here is less than

$$C_{12} |\psi_{1n}(z)| + C_{13} |z|^{-1} \qquad (|z| > R),$$

with some positive constants C_{13} and C_{14}. Given $\epsilon > 0$, this will be less than ϵ provided $|z|$ and n are large enough, say $|z| > R_1$ and $n > n_2$. That is

$$|g_n(z) - g(z)| < \epsilon \tag{6.9}$$

whenever $|z| > R_1$ and $n > n_2$. Since we know that the convergence

$$\lim_{n \to \infty} g_n(z) = g(z)$$

is uniform on each compact subset of D (cf. the discussion after Lemma 5.1), (6.9) now implies that this convergence is uniform on each closed subset of D. Thus the uniformness of the convergence on K in (4.11) follows. The proof of Theorem 4.1 is complete.

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