ON THE DEFINITIONS OF ATTRACTORS

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ABSTRACT. We introduce the notion of attractor and present its historical evolution. Then we show that previous definitions are too stringent. We present two equivalent definitions of attractors, show that in this case strange attractors are indeed attractors and give some properties.

Various authors have worked on the definition of the concept of attraction which is of basic importance for a deterministic dynamical system. Smale [16], Thom [18], and Bathia and Szegö [2] proposed several definitions in which the attractor was such that:

- a neighbourhood of the attractor is contained in the domain of attraction
- it satisfies a condition of minimality
- it is in general closed and invariant.

Recently, many numerical experiments on strange attractors have shown that the preceding characteristics are too stringent. A strange attractor is an attractor which is strange: different from a fixed point, a cycle, a closed curve... However it is a stranger fact that such a strange attractor is not in general an attractor, as defined classically. For such an example we refer to Thibault's paper [17] in these proceedings.

Using Bowen's notion of pseudo orbit [3], Conley [4] and Ruelle [14] proposed a more general approach of the concept of attractor.

In this paper, we introduce the notion of attractor and present its historical evolution. Then we show that previous definitions are too stringent and try to analyse what we would like to call an attractor. We deduce that an attractor must be invariant under the double dual action of taking its basin of attraction and taking the limit set of this basin.

Lastly we present two equivalent definitions of attractors, show that in this case strange attractors are indeed attractors and give some properties.

1. PREVIOUS DEFINITIONS

Let f be a continuous selfmap of a compact metric space (E,d). We call L(x) the set of limit points of $f^n(x)$:

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$$L(x) = \{ y \in E / \exists n_i \to +\infty ; \lim_{i \to +\infty} d(f^{n_i}(x),y) = 0 \} .$$

We recall three different definitions of attractor among others:

Definition 1 (Thom [18]). A is an attractor if

1. almost all trajectories of A are dense in A
2. $\exists \{V_i\}_{i \in I}$ fundamental system of neighbourhood of A such that

   2.1 $\forall\ M \subset V_i,\ \lim_{n\to\infty} d(f^n(M),A) = 0$

   2.2 $\forall\ M \subset V_i,\ \lim_{n\to\infty} f^{-n}(M) \cap A \neq 0 \Rightarrow M \subset A.$

Definition 2 (Guckenheimer and Holmes [11]). A is an attractor if

1. there exists a trajectory of A dense in A
2. $\exists\ V(A),\ \forall\ n \geq 0,\ f^n(V) \subset V$ and $\forall\ x \in V,\ L(x) \subset A.$

Definition 3 (Ruelle [14]). A is an attractor (included in an attracting set) if

1. $\exists\ V(A),\ A = \bigcap_{n\geq 0} f^n(V)$
2. A is an equivalence class of the Ruelle-Bowen equivalence relation.

These three definitions are constructed using the same model: existence of a neighbourhood contained in the domain of attraction, and a condition of minimality. It is not yet known if these definitions are equivalent.

However the first requirement is too stringent: the neighbourhoods of an attractor can contain trajectories which remain in these neighbourhoods but do not converge to the attractor: an example is constructed in part 3.

The condition of minimality is obtained through the use of dense trajectories or Ruelle-Bowen relation. In the following we shall use this relation which is more general than the density of trajectories.

2. RUELLE-BOWEN EQUIVALENCE RELATION

Let x and y belong to E. We say that there exists a pseudo trajectory from x to y and write $x \to y$ if

$$\forall\ \varepsilon > 0,\ \ \exists\ z_1,\dots,\ z_{j(\varepsilon)}\ \text{ and }\ \exists\ n_1,\dots,n_{j(\varepsilon)} \geq 1$$

such that

$$z_1 = x,\ z_j = y,\ d(f^{n_i}(z_i),z_{i+1}) \leq \varepsilon \quad i = 1,\dots,j-1$$

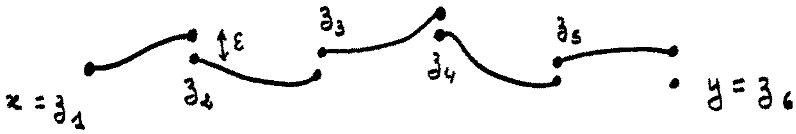


Fig. 1

We call heart of E and write C(E) the set of x such that there exists a pseudo trajectory from x to x:

$$C(E) = \{x \in E / x \rightarrow x\}.$$

Consider the following relation R on C(E):

$$\forall \; x, y \in C(E), \; x \, R \, y \Leftrightarrow x \rightarrow y \text{ and } y \rightarrow x \quad .$$

It is an equivalence relation, called Ruelle-Bowen equivalence relation.

3. A CANTOR ATTRACTOR

The Feigenbaum functional equation [9] is the following: find a continuous unimodal selfmap f of [-1,+1] such that

$$f(x) = (-1/\alpha) f^2(-\alpha x); \; \alpha = -f(1) > 0 \; .$$

This equation is close to that studied by Dubuc [8] in these proceedings. In Cosnard [5], an algorithm for constructing solutions of this equation is proposed and the iterative behaviour of such solutions is studied.

These functions (an example is presented in Fig. 2) admit a cycle of order 2^i for all integer i and a Cantor set which can attract almost all trajectories in the sense of Lebesgue measure. However given a neighbourhood of this Cantor set, there exists i such that the cycles of order greater than 2^i are contained in this neighbourhood.

This is an example of an attractor whose basin of attraction is of Lebesgue measure one but does not contain any neighbourhood of the attractor.

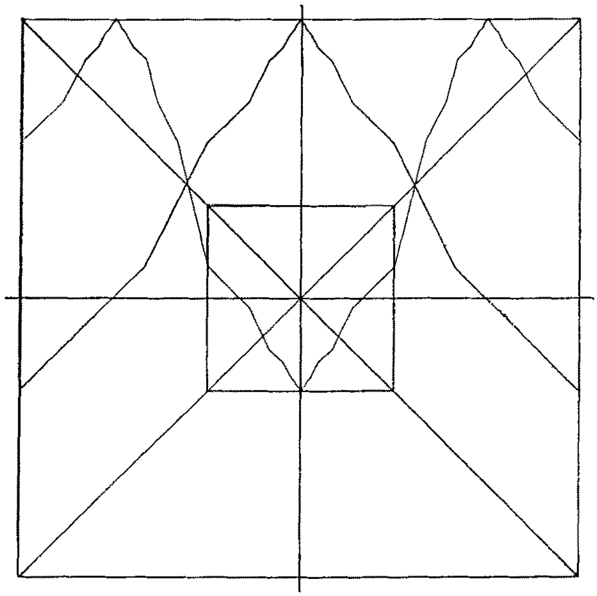


Fig. 2

4. INTUITIVE DEFINITION AND EXAMPLES

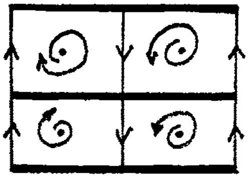
We would like that an attractor remains invariant under the double dual operations of:

- taking its basin of attraction
- taking the limit set of this basin.

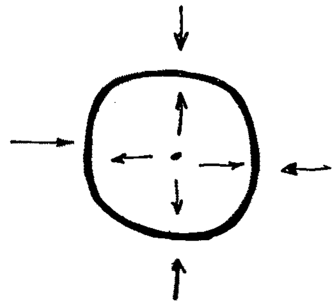
Moreover it must satisfy a condition of maximality with respect to the connexity under the iteration and a condition of minimality in order that the union of two attractors is not an attractor.

We would like to stress that in all general definitions there is some kind of personal touch. For examples:

- in Figure 3.1, is the attractor the whole grid or does there exist four attractors?
- in Figures 3.2 and 3.3 is the attractor the whole circle or each fixed point?
- in Figure 3.4 the whole grid with or without the bottom line or each horizontal line?



3.1



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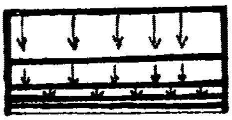

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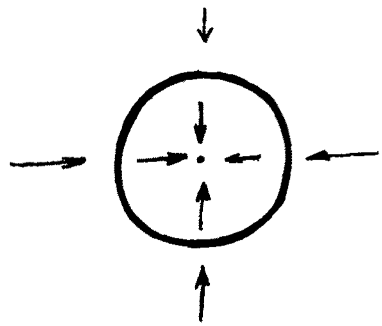

3.3

Fig.3. The thick lines correspond to fixed points

Moreover example 3.3 sets up the case of semi stable attractors and example 3.4 the case of non compact attractors.

We decided to choose arbitrarily the whole circle and the whole grid in each case by the use of a maximality condition.

5. FIRST DEFINITION

If $A \subset E$, we define $L(A) = \bigcup_{x \in A} L(x)$. The basin of attraction of A is defined to be

$$B(A) = \{x \in E \setminus A / L(x) \subset A\}$$

x of E is called super non recurrent if

$$\nexists\, y \in E; \quad x \in L(y) \ .$$

Let S be the set of super non recurrent points. We write $B_S(A) = S \cap B(A)$. L, B, and B_S are considered as operators on $\mathcal{P}(E)$.

Properties

1. $f(S) = S$; $S = E - L(E)$.
2. $(L \circ B)^2 = L \circ B$; $(L \circ B_S)^2 = L \circ B_S$.
3. $\forall\, x \in E$, $\forall\, y$, $z \in L(x)$; $y \to z$. We say that $L(x)$ is chain recurrent.
4. $L(E) \subset C(E)$.

Let C_i, $i \in I$, be the equivalence classes of $C(E)/R$. Hence $L(x)$ is chain recurrent if and only if

$$\exists\, i \in I; \quad L(x) \subset C_i \quad .$$

Definition I. $A \neq \emptyset \subset E$ is a weak attractor if:

1. $L \circ B_S(A) = A$.
2. $\nexists\, A'$, chain recurrent such that A' contains strictly a chain recurrent component of A and $L \circ B_S(A \cup A') = A \cup A'$.
3. $\nexists\, A''$ such that $A'' \subset A$ and A'' satisfies 1. and 2.

Lemma 1. If A is a weak attractor then

1. $f(A) = A$.
2. $L(A) \subset A$.
3. A is chain recurrent.

As an illustration, let us consider the dynamical system of Fig. 4:

$$S = \{\, (x,y) \in E / x \neq 0 \}; \quad A = \{\, (x,y) \in E / x = 0 \} \quad .$$

A is a weak attractor but (0,0) is not an attractor.

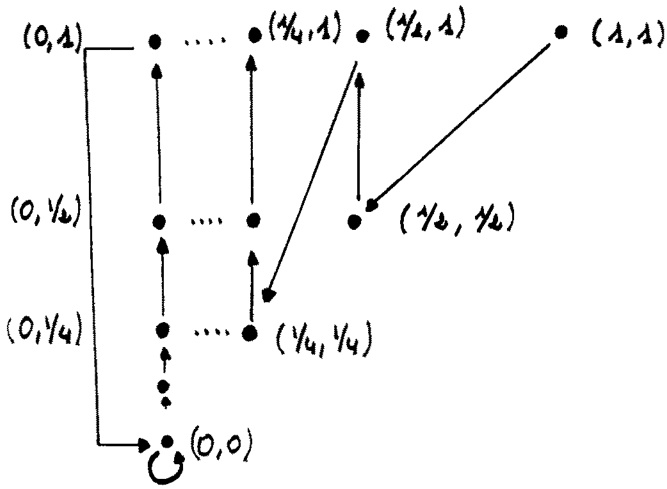

Fig. 4

6. SECOND DEFINITION

We call attracting set, the set A defined by

$$A = \bigcup_{A=L\,\circ\,B(A)} A \; .$$

We call super attracting set, the set A_S defined by

$$A_S = \bigcup_{A=L\,\circ\,B_S(A)} A \; .$$

Lemma 2.

1. $A = \bigcup_{A=L\,\circ\,B(A)} A = \bigcup_{A\in E} L \circ B(A) = L(N)$.

2. $A_S = \bigcup_{A=L\,\circ\,B_S} A = \bigcup_{A\in E} L \circ B_S(A) = L(S) = L(E-L(E))$.

3. $A_S \subset A \subset L(E)$.

4. $A = L_o \; B(A)$ and $A_S = L_o \; B_S(A_S)$.

A_S can be strictly included in A as is shown in an example of Cosnard and Demongeot [6].

Definition II. $A = \emptyset \subset E$ is a weak attractor if it is a chain recurrent component of A_S.

Equivalence theorem: Definition I and II are equivalent.

Definition III. $A \neq 0 \subset E$ is a strong attractor if A is a weak attractor such that $B(A) \neq 0$.

A is a super strong attractor if $A \cup B(A)$ contains a neighborhood of A.

7. PROPERTIES OF ATTRACTORS

P1.: A necessary and sufficient condition of existence of a weak attractor is that $S \neq 0$, i.e., $E \neq L(E)$.

P2.: If A is an attractor in the sense of definition 1, 2 or 3, then it is a super strong attractor.

P3: The CANTOR attractor described in section 3 is a strong attractor.

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