Proper surgery groups and Wall-Novikov groups

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The lifting of a surgery problem of closed manifolds to a covering leads usually to a proper surgery problem on open locally compact manifolds, and this proceedure gives by the present work new informations about the original problem. This is the motivation of proper surgery. In [8] , proper surgery groups are constructed formally as in [9, §9] and our goal has been to "compute" these groups in terms of Wall-Novikov groups (both [8] and the present work have been done sinultaneously and ignoring each other). I am indebted to W.Browder, J.Wagoner, R.Lee, A.Ranicki for useful and friendly conversations, and to L.Taylor who pointed out a gap.

1. Notations and conventions

We consider exclusively locally compact manifolds M and CW-complexes X of finite dimension, and proper maps f between them (i.e. f^{-1}(compact) is compact).

If X is connected, we can choose a fondamental sequence of ngbd of ∞: $X_1 \supset X_2 \supset X_3 \supset \ldots$ formed by subcomplexes X_n with only non compact components (in finite number). We denote by $\overline{X-X_n}$ any finite subcomplex of X such that $\overline{X-X_n} \cup X_n = X$, and let $\dot{X}_n = \overline{X-X_n} \cap X_n$ which is a finite subcomplex containing the frontier of X_n in X. For any pointed connected CW-complex A, with associated universal covering $\tilde{A}$, one usually denotes by C(A) the chain complex of cellular chains on $\tilde{A}$ with integer coefficients (and if $B \subset A$, then C(A) mod C(A,B) is denoted by C(A,B)) We denote by $C(X_n)$ the family $C(X_n^i)$ obtained by choosing implicitely one base point in each connected component X^i of X_n.

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Similarly, we denote by $Z\pi_1X_n$ the family of rings $Z\pi_1X_n^i$ and by a $Z\pi_1X_n$-module M, we mean a family of $Z\pi_1X_n^i$ - modules M^i. The homology of $C(X_n)$ is denoted by $H_k(X_n)$, while the homology of its dual $C^*(X_n)$, the family of $\mathrm{Hom}_{\pi_1X_n^i}(C(X_n^i), Z\pi_1X_n^i)$, is denoted by $H^k(X_n)$, where $C(X_n^i)$ is given the right structure via the anti-automorphism $\alpha \to w(\alpha)\alpha^{-1}$ of $Z\pi_1X_n^i$, w being given by some fixed homomorphism $\pi_1X \to \pm 1$.
The U-groups of [4] or [5] will be denoted by $L_m^p(G)$ while $L_m(G)$ denotes the ordinary Wall groups (or V-groups). As an inner automorphism of G induces $\pm$ identity on $L_m^p(G)$, $L_m^p(\pi_1X_n^i)$ is well defined and we write $L_m^p(\pi_1X_n) = \bigoplus_i L_m^p(\pi_1X_n^i)$. Similarly $X_{n+1} \to X_n$ induces a unique homomorphism on L_m^p. In particular, $\varprojlim_n L_m^p(\pi_1X_n)$ only depends on X, as well as $\varprojlim_n{}^1 L_m^p(\pi_1X_n)$. As for the latter it maybe useful to recall Milnor's definition of $\varprojlim{}^1$ of an inverse system of abelian groups $A_1 \xleftarrow{\#} A_2 \xleftarrow{\#} A_3 \xleftarrow{\#} \ldots$: this is the Coker of $\prod_{n>1} A_n \xrightarrow{1-S} \prod_{n>1} A_n$, where $(1-S)(a_1,a_2,a_3,\ldots) = (a_1-a_2^{\#},a_2-a_3^{\#},\ldots)$. Observe that a subsequence of $\{A_n\}$ gives the same result : e.g. $\varprojlim{}^1 A_{2n+1} \simeq \varprojlim{}^1 A_n$ by mapping $(a_1,a_2,a_3,\ldots)$ to $(a_1+a_2^{\#},a_3+a_4^{\#}\ldots)$ in the range of 1-S.

2. Homology and cohomology inverse systems

Having implicitely choosen one base point for each connected component of X_n, we join the base points of X_{n+1} to those of X_n by paths in X_n (in this usy, a tree grows in each connected component of X_1). The latter determine maps $\tilde{X}_{n+1}^j \to \tilde{X}_n^i$ and so pseudo-linear homomorphisms $C(X_{n+1}^j) \to C(X_n^i)$.

This gives rise to an inverse system $\{C(X_n)\}$. Note that

$$\bigoplus_j \left(Z\pi_1 X_n^i \underset{Z\pi_1 X^j_{n+1}}{\otimes} C(X^j_{n+1}) \right) \text{ (one summand for each } j \text{ such that } X^j_{n+1} \subset X^i_n)$$

is isomorphic to the subcomplex of $C(X_n^i)$ determined by $\tilde{X}_n^i | X_{n+1}$

Two choices of base points and paths give two inverse systems related by a diagram of subsequences

$$\begin{tikzcd}
C(X_{r_1}) \arrow[d] & C(X_{r_2}) \arrow[l] \arrow[d] & C(X_{r_3}) \arrow[l] \arrow[d] & \arrow[l] \arrow[dl, phantom] \\
C(X_{s_1}) & C(X_{s_2}) \arrow[l] \arrow[ul] & C(X_{s_3}) \arrow[l] \arrow[ul] & \arrow[l] \arrow[ul]
\end{tikzcd}$$

which commutes up to the action of $\pi_1 X_n^i$ on itself by inner automorphisms. Such a diagram is called a conjugate equivalence.

Similarly, the families of cochain complexes $C_c^*(X_n, \dot{X}_n) \overset{\text{def}}{=}$ $= \varinjlim_r C^*(X_n, \dot{X}_n \cup X_r)$ form an inverse system by excision and is also well defined up to conjugate equivalence. Now, any element $[X] \in \varprojlim_r H_m(X, X_r; Z)$ (homology with coefficients extended by w : $Z\pi_1 X \to Z$) gives by cap products (see [1]) a commutative diagram

$$\begin{tikzcd}
C_c^*(X_1, \dot{X}_1) \arrow[d, "\cap [X]"] & C_c^*(X_2, \dot{X}_2) \arrow[l] \arrow[d, "\cap [X]"] & \arrow[l] \\
C(X_1) & C(X_2) \arrow[l] & \arrow[l]
\end{tikzcd}$$

i.e. by definition a morphism of inverse systems. The latter is called an equivalence if there is an "inverse" morphism $\{C(X_n)\} \xrightarrow{\Psi} \{C_c^*(X_n, \dot{X}_n)\}$, i.e. a commutative diagram of subsequences

$$\begin{tikzcd}
C_c^*(X_{r_1}, \dot{X}_{r_1}) \arrow[d, "\cap [X]"] & & C_c^*(X_{r_2}, \dot{X}_{r_2}) \arrow[ll] \arrow[d, "\cap [X]"] & & \arrow[l] \\
C(X_{r_1}) & C(X_{s_1}) \arrow[l] \arrow[ul, "\Psi_1"'] & C(X_{r_2}) \arrow[l] & C(X_{s_2}) \arrow[l] \arrow[ul, "\Psi_2"'] & \arrow[l]
\end{tikzcd}$$

where Ψ_n is pseudo-linear, and $r_1 < s_1 < r_2 < s_2 < \ldots$

Observe that in this case, we can assume $r_n = s_{n-1} = n$ without loss of generality. When $\cap[X]$ is an equivalence, we say that $[X]$ is a m-fundamental class at ∞, and that X is properly Poincaré at ∞. This turns out to be an invariant of the proper homotopy type of X. Now, a proper map $f\colon M \to X$ of properly Poincaré complexes is said of degree 1 if $f_*[M] = [X]$. By confusing X with the mapping cylinder of f, and denoting the k+1-homology of $C(X_n, M_n)$, resp. $C_c^*(X_n, \dot{X}_n \cup M_n)$ by $K_k(M_n)$, resp. $K_c^k(M_n, \dot{M}_n)$, where $M_n = X_n \cap M$, $\dot{M}_n = \dot{X}_n \cap M$ we get again inverse systems $\{K_k(M_n)\}$ and $\{K_c^k(M_n, \dot{M}_n)\}$ well defined up to conjugate equivalence. If $\dot{M}_n = \partial M_n$, then the composition :

$$\Psi : K_{m-k}(M_n) \xrightarrow{\partial} H_{m-k}(M_n) \overset{\text{Poincaré}}{\simeq} H_c^k(M_n, \partial M_n) \to K_c^k(M_n, \dot{M}_n)$$ turns out

to be a canonical equivalence of inverse systems with an inverse shifting n by 4 (and so shifting n by 1 on a subsequence). Of course in the above, $H_*(M_n)$ and $H_c^*(M_n, \partial M_n)$ are with $\pi_1 X_n$-coefficients.

3. Homology and cohomology direct systems

For $r<n<s$, let $C^*(X_n, X_s)_r$ be the family $\bigoplus_j \mathrm{Hom}_{Z\pi_1 X_r^i}\left(Z\pi_1 X_r^i \underset{Z\pi_1 X_n^j}{\otimes} {}_j C(X_n^j, X_s), Z\pi_1 X_r^i\right)$ and let $C_c^*(X_n)_r$ be the family

$\varinjlim_s C^*(X_n, X_s)_r$. For r fixed, the restriction maps $C_c^*(X_n)_r \to C_c^*(X_{n+1})_r$ determine a direct system $\{C_c^*(X_n)_r\}$.

Similarly, if $C(X_n, \dot{X}_n)_r$ denotes the family $\bigoplus_j \left(Z\pi_1 X_r^i \underset{Z\pi_1 X_n^j}{\otimes} {}_j C(X_n^j, \dot{X}_n^j)\right)$

(chains of $\tilde{X}_r^i | X_n$ mod $\tilde{X}_r^i | X_r - X_n$), then the quotient maps $C(X_n, \dot{X}_n)_r \to C(X_{n+1}, \dot{X}_{n+1})_r$ form a direct system, for r fixed and $n > r$. Now, given a proper map $f\colon M \to X$ of degree 1, if we write $K_*(M_n, \dot{M}_n)_r$, resp $K_c^*(M_n)_r$, for the homology of $C(X_n, \dot{X}_n \cup M_n)_r$, resp $C_c^*(X_n, \dot{M}_n)_r$, we find again an equivalence ψ :

$$\{K_{m-k}(M_n, \dot{M}_n)_r\} \to \{K_c^k(M_n)_r\}.$$

Of course, these direct systems are well defined only up to conjugate equivalence, the latter notion being the same as for inverse systems.

4. End homology and cohomology

The dual of $C_c^*(X_n,\dot{X}_n)$ is canonically isomorphic to $C'(X_n,\dot{X}_n) \overset{\text{notat.}}{=} = \varprojlim_s C(X_n,\dot{X}_n \cup X_s)$, which is nothing but the chain complex of locally finite chain on X_n mod $\dot{X}_n$, with $Z\pi_1X_n$-coefficients. The quotient complex $C'(X_n,\dot{X}_n)/C(X_n,\dot{X}_n)$ yields the end homology $H_*^e(X_n)$ by definition. As usually, the cochain complex $\varinjlim_s C^*(X_s)_n$ yields the end cohomology $H_e^*(X_n)$ by definition. Now, one can prove [see 3] that, if [X] is a m-fundamental class at ∞ coming from $C_m'(X;Z)$, then $\cap[X]$ gives rise to an isomorphism $H_e^k(X_n) \simeq H_{m-k}^e(X_n)$. All this applies to a proper map $f : M \to X$ of degree 1, to yield an isomorphism $K_e^k(M_n) \simeq K_{m-k}^e(M_n)$. Our end homology can be viewed as an ε-construction (see [8] or [2]) with π_1X_n-coefficients as follows : consider the diagram of families of pointed subcomplexes

$$\begin{array}{cccc}
(X_n,x_n) & (X_n,x_{n+1}) & (X_n,x_{n+2}) & \cdots \\
 & \cup & \cup & \\
 & (X_{n+1},x_{n+1}) & (X_{n+1},x_{n+2}) & \cdots \\
 & & \cup & \\
 & & (X_{n+2},x_{n+2}) & \cdots \\
 & & & \cdots
\end{array}$$

Then let $\mu C(X_s)_n$ be the quotient complex $\prod_{r>s} C(X_s,x_r)_n / \bigoplus_{r>s} C(X_s,x_r)$ and $\varepsilon C(X_n)=\varprojlim_{s>n} \mu C(X_s)_n$. An isomorphism $C^e(X_n) \simeq \varepsilon C(X_n)$ arises by decomposing $z \in C'(X_n)$ into $z_n \oplus z'_{n+1} \in C(X_n,X_{n+1}) \oplus C'(X_{n+1})_n$, then z'_{n+1} into $z_{n+1} \oplus z'_{n+2}$, and so forth.

5. The category of (inverse or direct) systems

If one considers systems of families of modules $\{A_n\}$ over $\{Z\pi_1 X_n\}$ as abstract objects and takes their equivalence classes by the relation of (conjugate) equivalence, and if one does the same thing for the morphisms $\{A_n\} \to \{B_n\}$, then it is routine to verify that one gets an abelian category (see [3], compare [7]). A more specific result is the following.

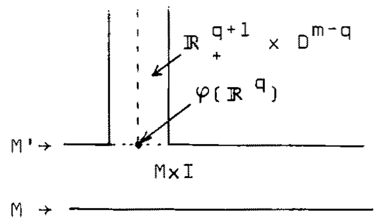
Proposition (see [3]) let $\{C(n)\}$ be a system of chain complexes. each of the form $0 \to C_\ell(n) \xrightarrow{\partial} \dots \to C_1(n) \to C_0(n) \to 0$ where $\ell > 0$ is fixed independant of n and $C_k(n)$ is free of countable rank. Suppose that the associated homology systems $\{H_k(n)\}$ are equivalent to 0 for all $k < \ell$. Then there is an equivalence $\{H_\ell(n)\} \to \{P_n\}$, where each P_n is a projective countably generated module and each homomorphism $H_\ell(n) \to P_n$ is injective. Moreover, in the system $\{P_n\}$, one can assume that the image of $P_{n+1} \to P_n$ a direct summand, in particular also projective.

These two results essentially allow us to elaborate an algebraic Whitehead torsion for proper homotopy equivalence (compare [8]).

6. Proper surgery

It is well known that any surgery rel, boundary on a compact m-submanifold of M^m extends to M, and similarly for a closed bicollared submanifold V^{n-1}. By definition, a proper surgery on M is the result of a diverging sequence of disjoint such surgeries. We distinguish the following particular case of carving out $\mathbb{R}^q \underset{\text{proper}}{\subset} M$. Let $f: M \to X$ be a proper normal map (relative to ξ on X), $\varphi: \mathbb{R}^q \to M$ be a proper embedding, $\Psi: \mathbb{R}^{q+1}_+ \to X$ a proper map such that $\Psi \mid \mathbb{R}^q = f_0 \varphi$ $(\mathbb{R}^q = \partial \mathbb{R}^{q+1}_+)$. Now the normal bundle of φ is trivial (because $\mathbb{R}^q$ is contractible) and we form W^{m+1} by gluing $M \times I$ and $\mathbb{R}^{q+1}_+ \times D^{m-q}$ along $\mathbb{R}^q \times D^{m-q} \subset M \times 1$.

As $(M\times I)\cup\mathbb{R}^{q+1}_{+}$ is a proper deformation retract of W, we can extend f to $F : W\to X\times I$ by using ψ . The stable trivialisation of $\tau_M \oplus f^*\xi$ on M extends to a stable trivialisation of $\tau_W \oplus F^*\xi$ on W because W retracts by deformation on M x 0. Now, W is a cobordism between M and $M' = M - \varphi(\mathbb{R}^q)$. The inclusions $M \subset W \supset M' \cup D^{mq}$ are homotopy equivalences



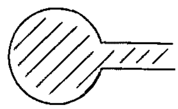
One can observe that M also results form M' by first a (m-q)-surgery and then carving out $\mathbb{R}^{m-q}$. To each cocompact submanifold $M_n \subset M$ corresponds a cocompact submanifold $M'_n \subset M'$ of the following shape : $M'_n = (M_n \cup \text{q-handle}) - \mathbb{R}^q$

7. Preliminary surgeries

Let M be an open m-manifold, X a proper Poincaré complex at ∞ and $f : M\to X$ a proper normal map of degree 1. We assume that X is connected and so we can choose cocompact subcomplexes X_n in X which have only non compact connected components. We can assume that each $\dot{X}_n$ is bicollared, and that f is transversal on each of them (see [1]). Then $f^{-1}(X_n)$ is a cocompact submanifolds $M_n \subset M$, such that $\partial M_n = f^{-1}(\dot{X}_n)$ and $\overline{M_n - M_{n+1}} = f^{-1}(\overline{X_n - X_{n+1}})$. Clearly, if $m=2q$, resp. $2q+1$, $q \geq 3$, we can assume that each map

$\overline{M_n - M_{n+1}} \xrightarrow{f} \overline{X_n - X_{n+1}}$ is q-connected, while $\partial M_n \xrightarrow{f} \dot{X}_n$ is q-1,resp. q-connected. In particular, f is bijective on ends spaces.

When $m=2q+1$, we can improve still the connectivity of f as follows. Each module of the family $K_q(M_n,M_{n+1}) \overset{\text{def}}{=} H_{q+1}(X_n,M_n \cup X_{n+1})$ is finitely generated, and each generator can be represented by an embedded q-sphere S^q in $\overline{M_n-M_{n+1}}$, provided with a nulhomotopy D^{q+1} in $\overline{X_n-X_{n+1}}$. We pipe S^q to ∞, getting $\mathbb{R}^q \underset{\text{proper}}{\subset} M_n$ and extend D^q into $\mathbb{R}^{q+1}_+ \xrightarrow{\text{proper}} X_n$:



Then the process of carving out $\mathbb{R}^q \subset M_n$ allows to kill each $K_q(M_n,M_{n+1})$. An immediate consequence is $K_c^q(M_n)=0$, hence the direct system $\{K_{q+1}(M_n,\partial M_n)_r\}$ is equivalent to 0 by duality. A more involved argument (see [3]) shows that $K'_q(M_n) \overset{\text{def}}{=} H'_{q+1}(X_n,M_n)$ also vanishes hence the inverse system $\{K^{q+1}(M_n,\partial M_n)\}$ is equivalent to 0 by duality. Moreover, the inverse system $\{K_q(M_n)\}$ and the direct system $\{K_q(M_n,\partial M_n)_r\}$ are both equivalent to systems of projective countably generated modules (*ibid*).

8. The case m=2q+1, M open

Assuming the prelimininary surgery already done the starting situation is described by a commutative square

$$\begin{array}{ccc} K_c^{q+1}(M_n,\partial M_n)_r & \longrightarrow & K_c^{q+1}(M_n)_r \\ \uparrow \Psi & & \uparrow \bar{\Psi} \\ K_q(M_n)_r & \longrightarrow & K_q(M_n,\partial M_n)_r \end{array} \qquad r < n.$$

where Ψ, resp. $\bar{\Psi}$, are equivalences of inverse, resp. direct, systems (r being fixed, n variable $> r$), with inverse equivalences shifting n by $+1$.

<u>The fundamental duality property</u> of this square is the following $\pm$ commutative diagrams of exact sequences

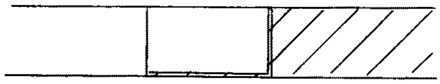
$$\begin{array}{ccccccc}
0 \to K^q(M_n,\partial M_n)_r & \to & K_e^q(M_n)_r & \to & K_c^{q+1}(M_n,\partial M_n)_r & & \\
\uparrow \bar{\Psi}^* & & \uparrow \Psi^e & & \uparrow \Psi & & \\
K'_{q+1}(M_n)_r & \to & K^e_{q+1}(M_n)_r & \to & K_q(M_n)_r & \to & 0
\end{array}$$

$$\begin{array}{ccccccc}
0 \to K^q(M_n)_r & \to & K_e^q(M_n)_r & \to & K_c^{q+1}(M_n)_r & & \\
\uparrow \Psi^* & & \uparrow \Psi^e & & \uparrow \bar{\Psi} & & \\
K_{q+1}(M_n,\partial M_n)_r & \to & K^e_{q+1}(M_n)_r & \to & K_q(M_n,\partial M_n)_r & \to & 0
\end{array}$$

where Ψ^* is the composition $K'_{q+1}(M_n,\partial M_n)_r \overset{\text{can.}}{\to}$ dual K_c^{q+1} $(M_n,\partial M_n)_r \overset{\text{dual }\Psi}{\to}$ dual $K_q(M_n)_r \simeq K^q(M_n)_r$, and similarly for Ψ^*, and $\Psi^e = \varinjlim_n \Psi^*$ is actually an isomorphism (see [3]). One sees that both Ψ and $\bar{\Psi}$ are induced by Ψ^e . Our aim is to improve the initial arbitrary choice of $X_n, \dot{X}_n$ in the mapping cylinder X of $M \overset{f}{\to} X$ so as to get Ψ bijective. One cannot do this for X itself but one can replace X by any complex simply homotopy equivalent to X rel. M. The first step is the following.

<u>Lemma</u> : Ker Ψ and Ker Ψ are finitely generated. Proof (sketched): using the results of §5, one finds an equivalence $\{K_c^{q+1}(M_n,\partial M_n)_r\} \overset{\text{inj.}}{\to} \{P_n\}$, Where each P_n is projective, the image of $P_{n+2} \to P_n$ being a direct summand P'_n . By composition with Ψ we get an equivalence $\alpha:\{K_q(M_n)\} \to \{P_{n-1}\}$ such that ker α =ker Ψ and im $\alpha = P'_n$, which is projective. Hence ker α is a direct summand. But ker Ψ is contained in the kernel of $K_q(M_{n+1})_r \to K_q(M_n)_r$, which is finitely generated, hence so is ker Ψ , as direct summand. The same argument applies to $\bar{\Psi}$. This shows actually that, for a <u>subsequence</u>, the kernel of $K_q(M_n)_{n-1} \overset{\Psi}{\to} K_c^{q+1}(M_n,\partial M_n)_{n-1}$ is finitely generated, and similarly for $\bar{\Psi}$.

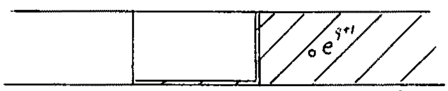
The first improvement is to replace X_n by $X_{n+1} \cup M_n$ and $\dot{X}_n$ by $X_{n+1} \cup \dot{M}_n$ where $\dot{M}_n = \overline{M_n - M_{n+1}}$



Then, in the square

$$\begin{array}{ccc} K_c^{q+1}(M_n, \dot{M}_n) & \longrightarrow & K_c^{q+1}(M_n) \\ \psi \uparrow & & \uparrow \bar{\psi} \\ K_q(M_n) & \longrightarrow & K_q(M_n, \dot{M}_n) \end{array}$$

ker ψ and ker $\bar{\psi}$ are finitely generated. The second improvement is to enlarge X_n inside $\overline{X_n - X_{n+1}}$ with $\overline{M_n - M_{n+2}} \cup e^{q+1}$, to kill ker $\bar{\psi}$:



By taking the quotient map, we find $K_q(M_n, \dot{M}_n) \to K_c^{q+1}(M_n)$ injective, and by the fundamental duality property we can restablish ψ and the initial square (see [3]). Assuming ψ injective, we can enlarge both X_n and $\dot{X}_n$ inside $\overline{X_{n-1} - X_n}$ with $\overline{M_{n-1} - M_{n+2}} \cup e^{q+2}$ to kill ker ψ. By taking the quotient map, we find $K_q(M_n) \xrightarrow{\psi} K_c^{q+1}(M_n, \dot{M}_n)$ injective, and we restablish $\bar{\psi}$ and the square by the fundamental duality property again. By using the proof of the above lemma, both $K_q(M_n)$ and $K_q(M_n, \dot{M}_n)$ are seen to be projective (ibid). Then one can still kill the kernel of the map $K_q(\dot{M}_n)^{\#} \to K_q(M_n)$ where $\#$ means with $\pi_1 X_n$-coefficients, and this will make ψ bijective (ibid). Then the fundamental duality property implies that ψ is injective. Now, the commutative diagram of exact sequence

$$\begin{array}{ccccccccc} 0 \to & K^q(\dot{M}_n)^{\#} & \to & K_c^{q+1}(M_n, \dot{M}_n) & \to & K_c^{q+1}(M_n) & & \\ & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & K_q(\dot{M}_n)^{\#} & \to & K_q(M_n) & \to & K_q(M_n, \dot{M}_n) & \to & 0 \end{array}$$

shows that ψ induces an isomorphism $K_q(\dot{M}_n)^{\#} \simeq K^q(\dot{M}_n)^{\#}$, i.e. a non degenerated quadratic projective finitely generated $Z\pi_1 X_n$ module $< K_q(\dot{M}_n) >$

Proposition : the quadratic form on $< K_q(M_n) >$ so obtained satisfies the following properties :

i) it is induced by the (degenerated) intersection form on $K_q(\partial M_r)^{\#}$ for some $r > n$, hence determine an element of $L^p_{2q}(\pi_1 X_n)$.

ii) it is defined stably, and the operation of carving out a trivial proper embedded $\mathbb{R}^q \subset M$ (bounding $\mathbb{R}^{q+1}_+ \underset{\text{proper}}{\subset} M$) adds a trivial free hyperbolic module

iii) there is a canonical equivalence between the quadratic $Z\pi_1 X_n$-modules $< K_q(\dot{M}_n)>$ and the $Z\pi_1 X_n$-extension of $< K_q(\dot{M}_{n+1})>$. In other words the sequence $< K_q(\dot{M}_n)>$ is an element of $\varprojlim_n L^p_{2q}(\pi_1 X_n)$

iv) the latter is well defined by the normal map $f: M \to X$, and is a cobordism invariant. For the proof of this proposition, we refer to [3] . As a result, we get a homomorphism σ : $L_m(eX) \to \varprojlim_n L^p_{m-1}(\pi_1 X_n)$ for m odd. Here, $L_m(eX)$ is the group of proper "surgery data over X at ∞", (same definition as in [8] , but use only proper h.e. at ∞ in defining 0) and satisfies actually an exact sequence

$L^p_m(\pi_1 X) \xrightarrow{\tau} L_m(X) \to L_m(eX) \to 0$, where $L_m(X)$ is the proper surgery group (see [8] for its construction).

Proposition : ker σ is isomorphic to $\varprojlim^1 L_{2q+1}(\pi_1 x_n)$.

The idea of the proof is to construct a map $\varprojlim^1 L_{2q+1}(\pi_1 x_n) \xrightarrow{\tau}$ Ker σ and an injective left inverse (see [3]).

Theorem (partial exact sequence) : for m odd, one has an exact sequence $\Pi_m \xrightarrow{1-s} L_m(\pi_1 X) \oplus \Pi \xrightarrow{\tau} L_m(X) \xrightarrow{\sigma} \Pi^p_{m-1} \xrightarrow{1-s} L^p_{m-1}(\pi_1 X) \oplus \Pi^p_{m-1}$

where Π_m is the product $\prod_{n>1} L_m(\pi_1 X_n)$, and S is the <u>shifting map</u>.

More precisely, $(1-S)(a_1, a_2, a_3, \ldots) = (-a_1^{\#}, a_1 - a_2^{\#}, a_2 - a_3^{\#}, \ldots)$

for $a_n \in L_m(\pi_1 X_n)$, $\#$ denoting the homomorphisms

$L_m(\pi_1 X) \leftarrow L_m(\pi_1 X_1) \leftarrow L_m(\pi_1 X_2) \leftarrow \dots$

Proof : observe that ker (1-S) is the subgroup of $\lim\limits_{\overleftarrow{n}} L^p_{m-1}(\pi_1 X_n)$ vanishing in $L^p_{m-1}(\pi_1 X)$. The range of σ is in ker (1-S) by the proof of iii in prop.above, replacing $\langle K_q(\dot{M}_n)\rangle$ by ϕ and $\langle K_q(\dot{M}_{n+1})\rangle$ by the $\pi_1 X$-extension of $\langle K_q(\dot{M}_n)\rangle$. The exactness Im$\sigma$=ker (1-S) is seen by constructing a cobordism between $N \xrightarrow{1} N$ and a proper h.e. $N' \to N$, where N is an open 2q-manifold provided with a 1-equivalence $N \to X$. The various map τ are also constructed by cobordiam on a 2q-manifold, and $\tau_o(1-S)$ vanishes. Hence we get induced maps τ satisfying the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\lim\limits_{\overleftarrow{n}} L_m(\pi_1 X_n) & \to & L_m(\pi_1 X) & \to & L_m(X) & \to & L_m(eX) & \to & 0 \\
 & & & & \uparrow\bar{\tau} & & \uparrow\sigma\ \uparrow\bar{\tau} & & \\
\lim\limits_{\overleftarrow{n}} L_m(\pi_1 X_n) & \to & L_m(\pi_1 X) & \to & \mathrm{Coker}(1-S) & \to & \lim\limits_{\leftarrow} L_m(\pi_1 X_n) & \to & 0
\end{array}$$

By the latter proposition, the right $\bar{\tau}$ is injective, hence so is the middle one. This proves the exactness Ker τ = Im (1-S). We also know that $\sigma_o\tau=0$. The exactness Ker σ= Im τ is a result of the above diagram

9. The case m=2q+2, M open

Assuming the preliminary surgery already done, we are left (as in) the case m odd) with only one inverse system $\{K_{q+1}(M)_n)_r\}$and one direct system $\{K_{q+1}(M_n,\partial M_n)_r\}$ not equivalent to 0. Following Wall's idea for the compact case, we want to consider the surgery data $M \xrightarrow{f} X$ as the union of two surgery cobordisms $M^o \cup V \to X^o \cup H$ along their common boundary $U \to \partial H$.

Lemma (see [8 chap.II th.3]) : X has the simple homotopy type of a CW-complex $X^o_{\partial H} \cup H$, where H is a locally finite m-handlebody of 0 and 1-handles. Actually, H is a regular ngbd of a tree in R^m, with 1-handles attached.

Proposition : assuming X of the above form, one can find a codimension 0-submanifold V of M such that, if $M^o = \overline{M-V}$, $f(M^o) \subset X^o$ and $f(V) \subset H$ up to a proper homotopy of f. Actually, V is a locally finite handlebody of 1, q and q+1-handles, formed by a regular ngbd of the union of immersed spheres $S^{q+1} \to M$ piped to ∞.
The proof relies on the same geometrical arguments than [6].
We refer to this as a Mayer-Vietoris decomposition of $M \xrightarrow{f} X$.
Actually, the ngbd of ∞ in ∂H, resp ∂V, can be chosen such that their frontier $\partial \dot{H}_n$, resp $\partial \dot{V}_n$, is S^{2q}, resp $S^q \times S^q$, and $f(\partial \dot{V}_n) \subset \partial \dot{H}_n$. This implies that $K_q(\partial \dot{V}_n)$ is a free hyperbolic module (with the intersection form). Then we can modify the choices of the ngbd of ∞: X^o_n in X^o, and the choice of $\dot{X}^o_n$, as in the proof of iv in the first prop. of §8 to get $K_q(\dot{M}^o_n)$ as a projective Lagrangian plane in $K_q(\partial \dot{V}_n)$. This determines an element of $L^p_{2q+1}(\pi_1 X_n)$ and we have results similar to those in §8, with m replaced by m+1.

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