CHAPTER IV

COHEN EXTENSIONS OF ZF-MODELS

In this chapter we study Cohen's forcing technique for constructing extensions of ZF-models. This technique was introduced in 1963 by Paul J.Cohen. Using this method Cohen has solved the long outstanding problems of the independence of the Continuum-hypothesis from the axiom of choice and the independence of the axiom of choice from the ZF-axioms (including foundation):

[9] P.J.COHEN: The Independence of the axiom of choice; mimeographed notes(32 pages), Stanford University 1963.

[10] P.J.COHEN: The Independence of the Continuum Hypothesis; Proc. Nat.Acad.Sci.USA, part 1 in vol.50(1963)p.1143-1148, part 2 in vol.51(1964)p.105-110.

A sketch of the proofs is contained in:

[11] P.J.COHEN: Independence results in set theory; In: The Theory of Models-Symposium, North Holland Publ.Comp.Amst.1965, p.39-54.

In these papers the constructible closure is obtained by means of Gödel's F(α)-hierarchy (Gödel's monograph [25] of 1940). Dana Scott has remarked that the constructible closure can be obtained in a much more elegant way using Gödel's M_α-hierarchy (Gödel's paper [24] of 1939). The presentation of the independence proofs in Cohen's monograph is based on these improvements:

[12] P.J.COHEN: Set Theory and the Continuum Hypothesis; New York - Amsterdam 1966 (Benjamin, Inc.).

Since the publication of Cohen's papers [9], [10] and [11] the forcing technique has been modified in various ways by several authors. Using modified "Gödel-functions F" W.Felscher and H.Schwarz have studied systematically Cohen-generic models (see Tagungsberichte Oberwolfach April 1965 and the dissertation of H.Schwarz: Ueber generische Modelle und ihre Anwendungen; Freiburg i.Br.1966). A topological approach to forcing has been developed by C.Ryll-Nardzewsky and G.Takeuti:

[83] G.TAKEUTI: Topological Space and forcing; Abstract in the J.S.L. vol.32(1967)p.568-569.

A detailed exposition of this approach is contained in:

[66] A.MOSTOWSKI: Constructible Sets with applications; Amsterdam - Warszawa 1969(North Holland + PWN).

That forcing can be understood as a boolean valuation of sentences has been discovered by D.Scott, R.M.Solovay and P.Vopěnka -see the forthcoming paper by Scott-Solovay, or Scott's lecture notes of the UCLA set theory Institute (August 1967) and :

[72] J.B.ROSSER: Simplified Independence Proofs; Academic Press 1969.

[86] P.VOPĚNKA: General theory of ∇-models; Comment.Math.Univ. Carolinae (Prague) vol.8(1967)p.145-170.

For further litterature on ∇-models see the bibliography in [86]. Some of Vopěnka's papers have been reviewed by K.Kunen in the J.S.L. 34(1969)p.515-516. -We shall present here the forcing method in a way close to P.J.Cohen, using ideas which are due to D.Scott, R.M.Solovay and others. The following basic publications will be useful:

[39] R.B.JENSEN: Modelle der Mengenlehre; Springer-Lecture Notes, vol.37, 1967.

[40] R.B.JENSEN: Concrete Models of Set Theory; In Sets, Models and Recursion theory, Leicester Proceedings 1965, North Holland PublComp.Amsterdam 1967, p.44-74.

[80] J.SILVER: Forcing à la Solovay; unpublished lecture notes (28 pages).

[51] A.LÉVY: Definability in axiomatic Set Theory I; in: Logic, Methodology and Philosophy of Sci., Congress Jerusalem 1964, North Holland Publ.Comp.Amst.1965, p.127-151.

The main difficulties which arise when one wants to extend a given ZF-model $\mathfrak{M}$ by adjoining some new sets $a_0, a_1, \ldots$ to $\mathfrak{M}$ are that the sets a_i may contain undesired information encoded by the interior $\in$-structure of a_i. For instance, the interior $\in$-structure of a_i may give rise to mappings which destroy the replacement axiom in the extension. These "new" sets a_i which, when added to $\mathfrak{M}$, <u>generate</u> a ZF-model are called "<u>generic sets</u>". The forcing method is a technique to obtain generic sets. Herein the main idea is that every finite part of the interior $\in$-structure of a_i has to be in $\mathfrak{M}$, id est, a_i has to fulfill finite amounts of conditions which can be posed in $\mathfrak{M}$. Then a determination of the whole interior $\in$-structure of a_i is obtained in a way similar to Lindenbaum's completing process (see e.g. Mendelson [60] p.64) by choosing a "complete sequence of conditions".

In this chapter we shall not construct socalled "endextensions" The extensions we are dealing with are those which contain the same ordinals!

A) THE FORCING RELATION IN A GENERAL SETTING

The simplest general framework for constructing Cohen models of ZF is provided by considering partially ordered structures. This approach, a straightforward generalization of Cohen's original work, is due to R.M.Solovay. We shall present here a slight generalization of Solovay's approach.

Let $\mathfrak{M} \triangleq \langle M, \in_M \rangle$ be a standard model of ZF (see p.25 for the definition of "standard"). Let

$$\mathfrak{A} = \langle A; R_i \rangle_{i \in I}$$

be a first-order relational system in $\mathfrak{M}$ with domain A and some n_i-ary relations R_i ($i \in I$) defined on A. We assume that A is a set in the sense of $\mathfrak{M}$. We want to extend $\mathfrak{M}$ by adding to $\mathfrak{M}$ a generic copy of $\mathfrak{A}$. The properties which this copy has to fulfill in the extension $\mathfrak{N}$ of $\mathfrak{M}$ are expressed in a certain formal language $\mathcal{L}$. The language $\mathcal{L}$ describes $\mathfrak{N}$. Since $\mathcal{L}$ shall express in a very detailed way all that what "happens" in $\mathfrak{N}$, we construct $\mathcal{L}$ as a ramified language which has means to talk about every v.Neumann-Stufe V_α separately. Formally this is done by introducing limited comprehension terms E^α (intended interpretation of $E^\alpha x\Phi(x)$: set of sets x of rank less than α satisfying Φ; the E is taken from the french word "Ensemble") and limited quantifiers $\bigvee^\alpha$ (read $\bigvee_x^\alpha \Phi(x)$ as: "there exists an x of rank less than α such that $\Phi(x)$).

The Alphabeth of the ramified language $\mathcal{L}$

1) One sort of set-variables: $v_0, v_1, v_2, \ldots, v_n, \ldots$ ($n \in \omega$). $x, y, z, \ldots$ are used to stand for these variables.
2) Set-constants $\underline{x}$ for each set x of $\mathfrak{M}$.

3) Constants $\mathring{a}_j$ for each $j \in A$.
4) n_i-ary predicates π_i for each $i \in I$ and ε for membership.
5) logical symbols: $\neg$, $\vee$, $\bigvee$ (not, or, there exists).
6) limited comprehension operatirs E^α and limited quantifiers $\bigvee^\alpha$ for each ordinal α of $\mathfrak{M}$, and finally brackets.

It is possible to arrange that these symbols are sets of $\mathfrak{M}$ in the following way: $\neg = \langle 0,0 \rangle$, $\vee = \langle 0,1 \rangle$, $\bigvee = \langle 0,2 \rangle$, $\bigvee^\alpha = \langle 1,\alpha \rangle$, $\varepsilon = \langle 0,3 \rangle$, $v_i = \langle 0,4+i \rangle$, $E^\alpha = \langle 2,\alpha \rangle$, $\underline{x} = \langle 3,x \rangle$, $\mathring{a}_j = \langle 4,j \rangle$, $\pi_i = \langle 5,i \rangle$ and $(= \langle 6,0 \rangle$, $) = \langle 6,1 \rangle$.
The formulae of $\mathcal{L}$ are obtained from these symbols by concatenation as usual by recursion. It follows that the collection of all formulae constitutes a class of $\mathfrak{M}$.

Definition. The notions of a ranked (= limited) formulae and of a limited comprehension term of $\mathcal{L}$ are defined as follows:

(a) If $u_1, u_2, \ldots$ are limited comprehension terms, set-constants or constants $\mathring{a}_j$ or variables, then $u_1 \varepsilon u_2$ and $\pi_i(u_1, \ldots, u_{n_i})$ are limited formulae.

(b) If Φ and Ψ are limited formulas, then so are $\neg\Phi$, $\Phi \vee \Psi$ and $\bigvee_x^\alpha \Phi$ (for α in $\mathfrak{M}$).

(c) If Φ is a limited formula with no free variables other than x, and α is an ordinal of $\mathfrak{M}$ such that (i) Φ contains no occurrence of $\bigvee^\beta$ with $\beta > \alpha$, (ii) Φ contains no occurrence of E^β with $\beta \geqslant \alpha$, (iii) Φ contains no set-constant $\underline{x}$ for a set x of Mirimanoff-rank $\geqslant \alpha$, (iv) if $\alpha \leqslant \lambda$ then Φ contains no occurrence of $\mathring{a}_j$, then $E^\alpha x\Phi(x)$ is limited comprehension term.

The notion of a free variable is defined as usual; a limited formula without free variables is said to be a limited sentence. We shall refer to the set-constants, constants of the form $\mathring{a}_j$ ($j \in A$) and the limited comprehension terms as <u>constant terms</u>. Remark that the definition above of a limited comprehension term is given with respect to the parameter $\lambda \geq \omega$. In most applications we choose λ to be ω respectively $\omega+1$.

Definition. Let $\rho(x)$ be the Mirimanoff-rank of the set x in the sense of $\mathfrak{M}$ (see p.14). The degree $\delta(t)$ of a constant term t is given by:

(a) $\delta(\underline{x}) = \rho(x)$,
(b) $\delta(\mathring{a}_j) = \lambda$
(c) $\delta(E^\alpha x\Phi(x)) = \alpha$

Abbreviations. Let u and v be constant terms or variables; then $u = v$ stands for $\bigwedge_x (x \in u \leftrightarrow x \in v)$ where x is a variable distinct from u,v. For constant terms u and v, $u \simeq v$ will stand for $\bigwedge_x^\alpha (x \in u \leftrightarrow x \in v)$ where $\alpha = \mathrm{Max}\{\delta(u),\delta(v)\}$. $u \simeq v$ is thus a limited sentence.

Next we define in $\mathfrak{M}$ a well-founded, localizable partial-ordering between limited formulas Φ by assigning to Φ an ordinal Ord(Φ) of $\mathfrak{M}$. Read Ord(Φ) as "the order of Φ". This then allows to define in $\mathfrak{M}$ the the forcing relation $\Vdash$ between "conditions" and limited formulas by induction on Ord(Φ). Obviously instead of defining Ord(Φ) to be the ordinal $\omega^2.\alpha + \omega.e + m$ we could define Ord(Φ) to be $\langle \alpha,e,m \rangle$ and then taking the lexicographical ordering to these triples.

Definition. For a limited formula Φ define

$$\mathrm{Ord}(\Phi) = \omega^2.\alpha + \omega.e + m$$

where (i) α is the least ordinal such that Φ contains no quantifier $\bigvee^\beta$ with $\beta > \alpha$ and no constant term t of degree $\geq \alpha$,

(ii) e = 3 iff Φ contains at least one of the symbols π_i, e = 2 iff Φ does not contain any π_i but Φ contains at least one of the symbols $\mathring{a}_j$, e = 1 iff Φ contains no symbol π_i and no symbol $\mathring{a}_j$ but Φ contains a subformula $v \in u$ where v is either a constant term with $\delta(v) + 1 = \alpha$ or a variable which stands in the scope of a limited quantifier $\bigwedge^\alpha$ (for α defined in (i)), e = 0 in all other cases.

(iii) m is the length of Φ.

Let S be an infinite set in $\mathfrak{M}$ such that $x \in S \rightarrow \rho(x) < \lambda$ and $\lambda = \sup\{\rho(x); x \in S\}$ where $\lambda \geq \omega$. We want to find for each $j \in A$ (where $\mathfrak{A} = \langle A,R_i \rangle_{i\in I}$ is the given first order relational system) a generic subset a_j of S and generic relations B_i for $i \in I$ between these a_j's such that in the metatheory $\mathfrak{A}$ and $\langle \{a_j;j \in A\},B_i \rangle_{i\in I}$ are isomorphic. Id est: we want to find a generic copy of $\mathfrak{A}$.
The sets a_j have to fulfill certain properties, or in different words: they have to satisfy certain conditions p (like "$7 \in a_j$" for instance, or others) which can be posed in $\mathfrak{M}$.
Instead of defining the sets a_j directly we first give a list saying that the sets a_j and the relation B_i have in $\mathfrak{N}$ (the extension of $\mathfrak{M}$) only those properties which are "forced" by some finite amount of information.

<u>Definition</u>: A condition p is a finite partial function from S × A into 2 = {0,1}.

Let P be the set of $\mathfrak{M}$ of all conditions and let $\leq$ be the partial ordering in P defined by $p \leq q \leftrightarrow p \subseteq q$.
The definition of forcing is given first for limited sentences Φ by induction on Ord(Φ). Notice that p varies over the <u>set</u> P and that for a given ordinal β, all the ranked sentences Φ with Ord(Φ) $< \beta$ constitute a set in $\mathfrak{M}$. Therefore (by the recursion theorem) the definition of $p \Vdash \Phi$ by induction on Ord(Φ) is permissible.

<u>Definition</u> of the (strong) forcing relation $\Vdash$ for limited sentences Φ. The definition takes place in $\mathfrak{M}$. Let T be the $\mathfrak{M}$-class of constant terms and let u be a variable ranging over constant terms.

(1) $p \Vdash u \,\varepsilon\, \underline{x} \Leftrightarrow (\exists y \in x)(p \Vdash u \simeq \underline{y})$.

(2) $p \Vdash u \,\varepsilon\, E^{\alpha} x\Phi(x) \Leftrightarrow (\exists t \in T)(\delta(t) < \alpha \;\&\; p \Vdash u \simeq t \;\&\; p \Vdash \Phi(t))$.

(3) $p \Vdash u \,\varepsilon\, \dot{a}_j \Leftrightarrow (\exists x \in S)(p \Vdash \underline{x} \simeq u \;\&\; p(\langle x,j\rangle) = 1)$.

(4) $p \Vdash \neg\Phi \Leftrightarrow \sim(\exists q \geq p)(q \Vdash \Phi)$.

(5) $p \Vdash \Phi \vee \Psi \Leftrightarrow (p \Vdash \Phi \;\dot{\vee}\; p \Vdash \Psi)$.

(6) $p \Vdash \bigvee^{\alpha}_{x}\Phi(x) \Leftrightarrow (\exists u \in T)(\delta(u) < \alpha \;\&\; p \Vdash \Phi(u))$.

(7) $p \Vdash \pi_i(u_1,\dots,u_{n_i}) \Leftrightarrow (\exists j_1,\dots,j_{n_i} \in A)(\langle j_1,\dots,j_{n_i}\rangle \in R_i \;\& \;\& \; p \Vdash u_1 \simeq \dot{a}_{j_1} \;\&\; \dots \;\&\; p \Vdash u_{n_i} \simeq \dot{a}_{j_{n_i}})$.

To see that $p \Vdash \Phi$ is indeed defined by induction on Ord(Φ), notice that the formulae occurring on the right side of $\Leftrightarrow$ have order strictly smaller that the formulae occurring on the left side of $\Leftrightarrow$. Further remark that in the definition of $\mathrm{Ord}(u_1 \simeq u_2)$ we have $e = 1$.

The definition of $p \Vdash \Phi$ for arbitrary $\mathcal{L}$-sentences Φ will be given in the Metalanguage (and not in $\mathfrak{M}$) by induction on the (ordinary) length of Φ. This definition will be valid since p ranges over a set P and the collection of all formulae of $\mathcal{L}$ constitutes a set in the sense of the meta-theory (since $\mathfrak{M}$ is a set in the sense of the meta-theory). Again let u,v range over T and p,q range over P.

Definition of $p \Vdash \Phi$ for arbitrary (unlimited) $\mathcal{L}$-sentences Φ.

(8) $p \Vdash u \varepsilon v$ and $p \Vdash \pi_i(u_1,..,u_{n_i})$ are defined as above.

(9) $p \Vdash \neg \Phi \Leftrightarrow \sim(\exists q \geqslant p)(q \Vdash \Phi)$.

(10) $p \Vdash \Phi \vee \Psi \Leftrightarrow (p \Vdash \Phi \mathbin{\dot{\vee}} p \Vdash \Psi)$.

(11) $p \Vdash \bigvee^{\alpha}_{x}\Phi(x) \Leftrightarrow (\exists u \in T)(\delta(u) < \alpha \;\&\; p \Vdash \Phi(u))$.

(12) $p \Vdash \bigvee_{x}\Phi(x) \Leftrightarrow (\exists u \in T)(p \Vdash \Phi(u))$.

It is obvious that for limited sentences Φ of $\mathcal{L}$, $p \Vdash \Phi$ according to this definition iff $p \Vdash \Phi$ according to the former definition. The rest of this section is devoted to the study of the formal properties of the forcing relation $\Vdash$. In the following three lemmata let Φ be any $\mathcal{L}$-sentence.

Consistency-Lemma. For no $p \in P$ do we have both $p \Vdash \Phi$ and $p \Vdash \neg \Phi$.

Proof. If $p \Vdash \Phi$ and $p \Vdash \neg \Phi$ for some $p \in P$ and some $\mathcal{L}$-formula Φ, then by (9) $p \Vdash \neg \Phi \rightarrow \sim p \Vdash \Phi$ and we get a contradiction in the metalanguage, q.e.d.

First Extension Lemma. If $p \Vdash \Phi$ and $p \leqslant q$, then $q \Vdash \Phi$.

Proof by induction on the complexity of Φ (i.e. for limited sentences Φ by induction on Ord(Φ) and for unlimited Φ by induction on the length of Φ), see e.g. Jensen [39] p.94-95.

Second Extension Lemma. For every $p \in P$ there is a $q \in P$, $p \leqslant q$, such that either $q \Vdash \Phi$ or $q \Vdash \neg \Phi$.

Proof. Suppose that for no $q \geqslant p$ we do have $q \Vdash \Phi$. Then $p \Vdash \neg \Phi$ by (9). Suppose now that for no $q \geqslant p$ we do have $q \Vdash \neg \Phi$. Then by (9): $p \Vdash \neg(\neg \Phi)$. But applying (9) twice one gets

$$p \Vdash \neg\neg \Phi \Leftrightarrow \sim(\exists q \geqslant p)[\sim(\exists q' \geqslant q)(q' \Vdash \Phi)]$$
$$\Leftrightarrow (\forall q \geqslant p)(\exists q' \geqslant q)(q' \Vdash \Phi)$$

Thus there exists $q' \geqslant p$ such that $q' \Vdash \Phi$, q.e.d.

Remark that forcing does not obey some simple rules of the propositional calculus. Exempla gratia, p may force $\neg\neg\Phi$ but not Φ. Furthermore, the forcing relation $\Vdash$ has by definition (clauses (5), (10), (12)) a homomorphism property with respect to disjunction $(\vee,\dot{\vee})$ and existential quantification $(\bigvee, \exists)$. If we introduce conjunction $\wedge$ and universal quantification $\bigwedge$ as usual, then one notices that $\Vdash$ does not have the homomorphism property for conjunction $(\wedge, \&)$ or for universal quantification $(\bigwedge, \forall)$. For example only

$$p \Vdash \Phi \wedge \Psi \Rightarrow (\exists q_1 \geqslant p)(\exists q_2 \geqslant p)[q_1 \Vdash \Phi \;\&\; q_2 \Vdash \Psi]$$

holds. We shall introduce a relation $\Vdash^*$ (called weak forcing), which has the property that $p \Vdash^* \Phi \Leftrightarrow p \Vdash^* \neg\neg\Phi$ and the homomorphism property for conjunction and universal quantification. $\Vdash^*$ does not have the homomorphism property for disjunction and existential quantification and is, as we may say, dual to the strong forcing relation $\Vdash$.

Definition. $p \Vdash^* \Phi \Leftrightarrow p \Vdash \neg(\neg \Phi)$ "p weakly forces Φ"

$p \parallel \Phi \Leftrightarrow (p \Vdash \Phi \;\dot{\vee}\; p \Vdash \neg \Phi)$ "p decides Φ"

$p \parallel^* \Phi \Leftrightarrow (p \Vdash^* \Phi \;\dot{\vee}\; p \Vdash^* \neg \Phi)$ "p weakly decides Φ"

$\Vdash \Phi \Leftrightarrow (\forall p \in P)(p \Vdash \Phi)$.

p_1 and p_2 are compatible $\Leftrightarrow (\exists q \in P)(p_1 \leqslant q \;\&\; p_2 \leqslant q)$.

Lemma A: The weak forcing relation has the following properties (u,v are variables for terms and Φ,Ψ are any $\mathcal{L}$-formulae)

(i) $p \Vdash^* \Phi \Leftrightarrow \sim(\exists q)[p \leqslant q \;\&\; q \Vdash \neg \Phi]$,

(ii) $p \Vdash \Phi \Rightarrow p \Vdash^* \Phi$,

(iii) $p \Vdash^* \neg \Phi \Leftrightarrow p \Vdash \neg \Phi$,

(iv) If Φ is of the form $\Psi_1 \wedge \Psi_2, \Psi_1 \Leftrightarrow \Psi_2, \bigwedge_x \Psi, \bigwedge_x^\alpha \Psi$, $u = v$ or $u \approx v$, then $p \Vdash \Phi \Leftrightarrow p \Vdash^* \Phi$,

(v) $p \Vdash^* \Phi \wedge \Psi \Leftrightarrow [p \Vdash^* \Phi \;\&\; p \Vdash^* \Psi]$,

(vi) $p \Vdash^* \bigwedge_x \Phi \Leftrightarrow (\forall u \in T)[p \Vdash^* \Phi(u)]$,

(vii) $p \Vdash^* \bigwedge_x^\alpha \Phi \Leftrightarrow (\forall u \in T)[\delta(u) < \alpha \Rightarrow p \Vdash^* \Phi(u)]$,

(viii) $p \Vdash^* \Phi \leftrightarrow \Psi \Rightarrow [p \Vdash^* \Phi \Leftrightarrow p \Vdash^* \Psi]$,

(ix) $(\forall p' \geqslant p)(\exists q \geqslant p')[q \Vdash^* \Phi \Leftrightarrow q \Vdash^* \Psi] \Rightarrow p \Vdash^* \Phi \leftrightarrow \Psi$,

(x) $(\forall q \geqslant p)[q \Vdash^* \Phi \Leftrightarrow q \Vdash^* \Psi] \Rightarrow p \Vdash \Phi \leftrightarrow \Psi$.

Proof. Ad(i): Let Ψ be $\neg \Phi$. By (9) of the forcing definition $\sim(\exists q \geqslant p)[q \Vdash \Psi]$ is equivalent to $p \Vdash \neg \Psi$ which is $p \Vdash \neg \neg \Phi$; by definition of $\Vdash^*$ this is equivalent to $p \Vdash^* \Phi$.

Ad(ii): follows from the first extension lemma and (9).

Ad(iii): "$\Leftarrow$" follows from (ii). Now assume $p \Vdash^* \neg \Phi$ and suppose that $\sim p \Vdash \neg \Phi$. Then by (9): $q \Vdash \Phi$ for some $q \geqslant p$. Thus by (ii) $q \Vdash^* \Phi$. This is in contradiction with $p \Vdash^* \neg \Phi$ by the consistency lemma.

Ad(iv): Notice that all the forms of Φ listed are of the form $\neg \Gamma$, thus the claim follows from (iii). The symbols $\wedge, \leftrightarrow, \bigwedge, \bigwedge^\alpha$ are introduced by definition for longer expressions in terms of $\neg, \vee, \bigvee, \bigvee^\alpha$ only.

Ad(v): $p \Vdash^* \Phi \wedge \Psi$ is by (iv) equivalent with $p \Vdash \Phi \wedge \Psi$, which is by definition: $p \Vdash \neg(\neg \Phi \vee \neg \Psi)$. This is, using first (9) and then (10) of the forcing definition equivalent to

$$(\forall q)[p \leqslant q \Rightarrow \sim(q \Vdash \neg \Phi \dot{\vee} q \Vdash \neg \Psi)].$$

Using again (9) one gets equivalently $p \Vdash \neg \neg \Phi$ & $p \Vdash \neg \neg \Psi$.

Ad(vi): The proof is similar to the proof of (v)

Ad(vii): Again the proof similar to (v) or (vi).

Ad(viii): Assume $p \Vdash^* \Phi \leftrightarrow \Psi$ and $p \Vdash^* \Phi$ but $\sim p \Vdash^* \Psi$. By (i) there is an extension q of p such that $q \Vdash \neg \Psi$. Since $p \leqslant q$ the first extension lemma yields $q \Vdash \neg \neg \Phi$. Thus by (9) of the forcing definition

$$(\forall q')[q \leqslant q' \Rightarrow \sim(q' \Vdash \neg \Phi \dot{\vee} q' \Vdash \Psi)].$$

Using (10) and then again (9) of the forcing definition this gives us $q \Vdash \neg(\neg \Phi \vee \Psi)$. Thus: $q \Vdash \neg(\Phi \rightarrow \Psi)$ by definition of $\rightarrow$. Using (iii) and (v) one sees that this is in contradiction with $p \Vdash^* \Phi \leftrightarrow \Psi$.

Ad(ix): suppose that the conclusion does not hold and proceed using (i) and (9), (10) of the forcing definition and the second extension lemma. In this way one immediately gets a contradiction.

Ad(x): follows directly from (ix). This proves lemma A.

Lemma B. Let p and q be elements of the set P of conditions and let Φ and Ψ be $\mathcal{L}$-sentences.

(i) If p_1 and p_2 are compatible and $p_1 \Vdash \Phi$ and $p_2 \Vdash \Psi$, then $q \Vdash \Phi \wedge \Psi$ for every q greater than both, p_1 and p_2.

(ii) If $p \parallel \Phi$ and $p \parallel \Psi$ then $p \parallel \Phi \vee \Psi$, $p \parallel \Phi \rightarrow \Psi$, $p \parallel \Phi \wedge \Psi$ and $p \parallel \Phi \leftrightarrow \Psi$.

(iii) If $p \parallel \Phi$ and $p \parallel \Psi$, then $p \Vdash \Phi \wedge \Psi \Leftrightarrow (p \Vdash \Phi \;\&\; p \Vdash \Psi)$.

(iv) If $p \parallel \Phi$ and $p \parallel \Psi$, then $p \Vdash \Phi \leftrightarrow \Psi \Leftrightarrow (p \Vdash \Phi \Leftrightarrow p \Vdash \Psi)$.

Proof by direct computation.

Lemma C. Let p, Φ and Ψ be as in lemma B.

(i) If $p \parallel^* \Phi$ and $p \parallel^* \Psi$, then $p \parallel^* \neg \Phi$, $p \parallel^* \Phi \wedge \Psi$, $p \parallel^* \Phi \vee \Psi$, $p \parallel^* \Phi \rightarrow \Psi$ and $p \parallel^* \Phi \leftrightarrow \Psi$.

(ii) If $p \parallel^* \Phi$ and $p \parallel^* \Psi$, then $p \parallel^* \Phi \vee \Psi \Leftrightarrow (p \Vdash^* \Phi \;\dot{\vee}\; p \Vdash^* \Psi)$.

(iii) If $p \parallel^* \Phi$ and $p \parallel^* \Psi$, then $p \Vdash^* \Phi \leftrightarrow \Psi \Leftrightarrow (p \Vdash^* \Phi \Leftrightarrow p \Vdash^* \Psi)$.

Proof by direct computation (use lemma A, (viii) and (ix)).

Lemma D. If $p \parallel \Phi_i$ $(i = 1,\dots,n)$, C is an n-ary sentential connective id est: an operation which is an iteration of the primitive sentential connectives $\neg$ and $\vee$) and $\dot{C}$ is the corresponding sentential connective of the meta-language (id est: the analog of $\sim$ and $\dot{\vee}$), then:

(i) $p \parallel C(\Phi_1,\dots,\Phi_n)$, and

(ii) $p \Vdash C(\Phi_1,\dots,\Phi_n) \Leftrightarrow \dot{C}(p \Vdash \Phi_1,\dots,p \Vdash \Phi_n)$.

Proof by induction on the number of times $\neg$ and $\vee$ are used in C (use lemma B).

Lemma E. Let C be a n-ary sentential connective. If $C(\phi_1,\dots,\phi_n)$ is a tautology for all $\phi_1,\dots,\phi_n$, then for all p and for all $\mathcal{L}$-sentences $\Phi_1,\dots,\Phi_n$ it holds that

$$p \Vdash^* C(\Phi_1,\dots,\Phi_n).$$

Proof. Use lemma A and C (see e.g. A.Lévy [51] p.141).

Lemma F. Let u, v and w stand for constant terms; then for every p:

(i) $p \Vdash u = u$,

(ii) $p \Vdash u = v \Rightarrow p \Vdash v = u$,

(iii) $[p \Vdash u = v \;\&\; p \Vdash v = w] \Rightarrow p \Vdash u = w$,

Lemma G. Again let u, v and w be constant terms, then for every p:

(i) $p \Vdash u \simeq v \Leftrightarrow p \Vdash u = v$,

(ii) $[p \Vdash u \in w \;\&\; p \Vdash u = v] \Rightarrow p \Vdash v \in w$,

(iii) $[p \Vdash w \in u \;\&\; p \Vdash u = v] \Rightarrow p \Vdash^* w \in v$.

For a proof of lemmata F and G see Lévy [51] p.141 or Easton, Annals of Math. Logic, vol.1(1970):[14].

Corollary H: If $p \Vdash \pi_i(u_1,\ldots,u_{n_i})$ and $p \Vdash u_1 = v_1,..p \Vdash u_{n_i} = v_{n_i}$, then $p \Vdash \pi_i(v_1,\ldots,v_{n_i})$.

This follows easily from the definition of forcing, clause (7), and lemmata F and G.

Digression: The forcing definition $p \Vdash \Phi$ between elements p of the set of conditions (of $\mathfrak{M}$) and limited $\mathcal{L}$-formulae Φ was given in the "groundmodel" $\mathfrak{M}$ while the definition of $p \Vdash \Phi$ for unlimited Φ was given in the underlying meta-theory. We shall show in the sequel that for each specific $\mathcal{L}$-sentence Φ the forcing relation can be defined in $\mathfrak{M}$, because Φ is finite and the construction of the class K_Φ of p's forcing Φ can be done in finitely many steps. For each specific Φ the mechanism of constructing K_Φ can be implemented within $\mathfrak{M}$ but the mechanism is not universally applicable for all sentences Φ of $\mathcal{L}$, so that within $\mathfrak{M}$ we do not have the whole relation $\Vdash$. This is not too much surprising, since the definition of forcing resembles very much the definition of truth, and by the Epimenides-Tarski paradox we cannot define in ZF (or within the $\mathfrak{M}$-language) the notion of truth for $\mathcal{L}$-sentences [see Lévy [51] p.138, A.Tarski: Logic, Semantics, Metamathematics (Oxford 1956) p. 248, Fraenkel-BarHillel: Foundation of Set Theory (Amsterdam 1958) p.306 and Kleene: Introduction to Meta-Mathematics (Amsterdam-Groningen 1967) p.39,42, 501, see also Mendelson [60] p.151]. However we can define forcing for a single given sentence Φ or for some particular family of sentences within $\mathfrak{M}$.

Lemma I: Let $\Phi(x_1,\ldots,x_n)$ be an unlimited formula of $\mathcal{L}$. There is a class K_Φ of the model $\mathfrak{M}$ whose elements are the (n+1)-tuples $\langle p,u_1,\ldots,u_n\rangle$ such that $p \Vdash \Phi(u_1,\ldots,u_n)$, where the $u_i (1 \leqslant i \leqslant n)$ are constant terms.

According to our remark on page 79 the constant terms u_i are considered as certain special finite sequences of symbols which are in $\mathfrak{M}$ - for more details see Easton's thesis, Annals of math.Logic, vol 1(1970).

Proof by induction on the length of the formula Φ. Since the atomic formulae are all limited formulae, the lemma is true for atomic Φ. If $\Phi(x_1,\ldots,x_n)$ is $\psi_1(x_1,\ldots,x_n) \vee \psi_2(x_1,\ldots,x_n)$ and the classes K_{ψ_1} and K_{ψ_2} satisfy the lemma for ψ_1 and ψ_2 respectively, then $K_\Phi = K_{\psi_1} \cup K_{\psi_2}$ is the required class for Φ. If Φ is $\bigvee_y \Psi(y,x_1,\ldots,x_n)$ and if K_ψ satisfies the lemma for $\psi(x_0,x_1,\ldots x_n)$, then $\{\langle p,z\rangle ; \bigvee_y \langle p,y,z\rangle \in K_\psi\}$ is the required class. The case that Φ is $\bigvee_y^\alpha \Psi(y,x_1,\ldots,x_n)$ is similar to the previous one. If Φ is $\neg\, \Psi(x_1,\ldots,x_n)$ and if K_ψ satisfies the lemma for Ψ, then

$$\{\langle p,z\rangle ;\ p \in P \wedge \neg \bigvee_{q\in P}(p \leqslant q \wedge \langle q,z\rangle \in K_\psi)\}$$

is the required class K_Φ. This proves lemma I.

Definition: A set $\mathcal{K}$ of conditions is dense (cofinal) in $\langle P,\subseteq\rangle$, the set of all conditions, iff for every $p \in P$ there is a $q \in \mathcal{K}$ such that $p \leqslant q$.

Definition: A sequence $\mathfrak{R}$ of conditions is complete iff $\mathfrak{R}$ is well-ordered by $\subseteq$ and of ordertype ω, $\mathfrak{R} = \{p^{(0)}, p^{(1)}, \ldots, p^{(k)}, \ldots\}$, such that $\mathfrak{R} \cap \mathcal{K} \neq \emptyset$ for every dense set $\mathcal{K}$ of conditions.

Remark. Both definitions above are given in the meta-language (and not in $\mathfrak{M}$). The notion of a dense subset of a partially ordered set is due to F.Hausdorff who used the name "cofinal". The original definition of "completeness" for sets $\mathfrak{R}$ of conditions of P.J.Cohen was **a bit more** restrictive. The definition given above is due to W.B.Easton (Thesis, Princeton 1964, the main part appeared in the Annals of math.Logic, vol.1(1970)).

Lemma J: If $\mathfrak{R}$ is a complete sequence of conditions, then for every $\mathcal{L}$-sentence Φ there exists $p^{(k)} \in \mathfrak{R}$ such that $p^{(k)} \Vdash \Phi$.

Proof. Let Φ be given. By lemma I there is a set K in $\mathfrak{M}$ whose elements are just those conditions p for which $p \Vdash \Phi$ holds. By the second extension lemma K is a dense subset of P, thus $K \cap \mathfrak{R} \neq \emptyset$, and there are conditions $p \in \mathfrak{R}$ such that $p \in K$ and hence $p \Vdash \Phi$, q.e.d.

The following lemma is the only place where we need the countability of $\mathfrak{M}$. Notice that the weaker assumption, namely the $\mathfrak{M}$-set of $\mathfrak{M}$-subsets of P is countable, is already sufficient. This was used e.g. by R.Solovay in the construction of a model $\mathfrak{N}$ which contains a non-constructible Δ_3^1-set of reals.

Lemma K: There are complete sequences of conditions. Moreover, for every condition p there is a complete sequence $\mathfrak{R}$ in which p occurs as first element.

Proof. Since $\mathfrak{M}$ is countable (in the meta-theory), there is an enumeration of all sets of $\mathfrak{M}$ and in particular an enumeration of the set of all subsets of P which are in $\mathfrak{M}$. Let $\{s_n; n \in \omega\}$ be an enumeration of the powerset of P in the sense of $\mathfrak{M}$.
Take any condition p and define $p^{(0)} = p$. If $p^{(n)}$ is defined, let $p^{(n+1)}$ be any condition in s_n which extends $p^{(n)}$ if such an element exists, otherwise put $p^{(n+1)} = p^{(n)}$. We show that the so-defined sequence $\mathcal{R} = \{p^{(0)}, \dots, p^{(n)}, \dots\}$ intersects every dense set of conditions. If $\mathcal{K}$ is a dense set, then it has a number, say n, in the enumeration, thus $\mathcal{K} = s_n$. By definition $p^{(n+1)} \in s_n$ and $p^{(n+1)} \in \mathcal{R}$, q.e.d.

Definition: Let $\mathcal{K}$ be any collection (in the sense of the meta-language) of conditions and let Φ be an $\mathcal{L}$-sentence. We write $\mathcal{K} \Vdash \Phi$ for $(\exists\, p \in \mathcal{K})(p \Vdash \Phi)$ and similarly $\mathcal{K} \Vdash^* \Phi$ for $(\exists\, p \in \mathcal{K})(p \Vdash^* \Phi)$. Notice, that if $\mathcal{R}$ is a complete sequence of conditions, then $\mathcal{R} \Vdash \Phi$ and $\mathcal{R} \Vdash^* \Phi$ are equivalent.

Lemma L: Let $\mathcal{R}$ be a complete sequence of conditions and $\Phi(x_1, \dots, x_n)$ an $\mathcal{L}$-formula. If $\mathcal{R} \Vdash u_1 = v_1, \dots, \mathcal{R} \Vdash u_n = v_n$ for constant terms $u_1, \dots, u_n, v_1, \dots, v_n$, then
$\mathcal{R} \Vdash \Phi(u_1, \dots, u_n) \Leftrightarrow \mathcal{R} \Vdash \Phi(v_1, \dots, v_n)$.

This follows by induction on the length of Φ from lemmata F, G and corollary H.

So far we have investigated several useful properties of the forcing relation. In the next section we shall show that every complete sequence of conditions gives raise to a valuation of the predicates $\dot{a}_j$ so that the resulting sets are generic.

B) COHEN - GENERIC SETS

We shall use the terminology and formalism introduced in section A.

Definition: Let $\mathcal{R}$ be a complete sequence of conditions. Define the function $val_{\mathcal{R}}$ (valuation or interpretation with respect to $\mathcal{R}$) on the set T of all constant terms of the language $\mathcal{L}$ by induction on their degree as follows:

$$val_{\mathcal{R}}(u) = \{val_{\mathcal{R}}(v);\ v \in T \ \&\ \delta(v) < \delta(u) \ \&\ \mathcal{R} \Vdash v \in u\}$$

Finally define:

$\mathfrak{N}_{\mathfrak{R}} = \{val_{\mathfrak{R}}(u); u \in T\}$
(we shall usually omit the subscript $\mathfrak{R}$ from $val_{\mathfrak{R}}$ and $\mathfrak{N}_{\mathfrak{R}}$).

Lemma M: Let u and v be constant terms. If $p \Vdash u \varepsilon v$ then there is a constant term w such that $\delta(w) \leq \delta(u)$, $\delta(w) < \delta(v)$ and $p \Vdash u \simeq w$, $p \Vdash w \varepsilon v$.
(for a proof see e.g. A.Lévy [51] p.141).

Lemma N: $\mathfrak{R} \Vdash u = v \Leftrightarrow \mathfrak{R} \Vdash u \simeq v \Leftrightarrow val_{\mathfrak{R}}(u) \triangleq val_{\mathfrak{R}}(v)$.

Lemma O: $\mathfrak{N}_{\mathfrak{R}}$ is a transitive set. For each $x \in \mathfrak{M}$, $val_{\mathfrak{R}}(\underline{x}) = x$, hence $\mathfrak{M} \subseteq \mathfrak{N}_{\mathfrak{R}}$.

Proof: The transitivity of $\mathfrak{N}_{\mathfrak{R}}$ follows directly from the definitions of $val_{\mathfrak{R}}$ and $\mathfrak{N}_{\mathfrak{R}}$. $val(\underline{x}) = x$ follows easily by induction on $\delta(\underline{x})$ using the definition of $val(\underline{x})$, the forcing-definition and the lemma N. Thus the witnessing constants $\underline{x}$ ensure that $\mathfrak{M}$ is contained in $\mathfrak{N}$ as a transitive submodel.

The semantics of $\mathcal{L}$. For each $x \in \mathfrak{N}$ let $r(x)$ be the least $\delta(w)$ for which $val(w) = x$. Thus $x,y \in \mathfrak{N}$ & $x \in y \Rightarrow r(x) < r(y)$ by lemma N. Now the formulae of $\mathcal{L}$ can be interpreted in $\mathfrak{N}$ in the following way:

(i) A term u is interpreted in $\mathfrak{N}$ by val(u).

(ii) $u \varepsilon v$ holds in $\mathfrak{N}$ iff $val(u) \in val(v)$.

(iii) The sentential connectives $\neg, \vee$ and the existential quantifier $\bigvee$ are interpreted as usual by $\sim$, $\dot{\vee}$ and $\exists$.

(iv) $\bigvee_x^{\alpha}\Phi(x)$ holds in $\mathfrak{N}$ iff there exists $y \in \mathfrak{N}$ with $r(y) < \alpha$ such that $\Phi(y)$.

(v) $y_1,\ldots,y_{n_i}$ satisfy $\pi_i(x\ ,\ldots,x_{n_i})$ in $\mathfrak{N}$ iff there are $j_1,\ldots,j_{n_i} \in A$ such that $\langle j_1,\ldots,j_{n_i}\rangle \in R_i$ and $y_1 = val(\dot{a}_{j_1})$, $\ldots, y_{n_i} = val(\dot{a}_{j_{n_i}})$.

One of the most important steps in showing that $\mathfrak{N}$ is a model of ZF is by proving that $\mathfrak{N}$ can be described to a large extent within $\mathfrak{M}$. When one is dealing with inner models $\mathfrak{M}_1$ of some structure $\mathfrak{M}_2$ in the verification of the axioms in $\mathfrak{M}_1$ one usually uses the fact that $\mathfrak{M}_1$ can be described entirely within $\mathfrak{M}_2$, thus reducing the validity of some axioms in $\mathfrak{M}_1$ to the validity in $\mathfrak{M}_2$. Cohen-extensions $\mathfrak{N}$ of countable ZF-models $\mathfrak{M}$ have similar features. Though $\mathfrak{N}$ extends $\mathfrak{M}$, $\mathfrak{N}$ can be described to

a good extend in $\mathfrak{M}$, so that again questions about the validity of statements in $\mathfrak{N}$ can be reduced to questions which can be posed (and "answered") in $\mathfrak{M}$.
This is the content of the following lemma:

<u>Lemma P</u>: Let Φ be an $\mathcal{L}$-sentence. Then Φ holds in $\mathfrak{N}_{\mathcal{R}}$ iff $\mathcal{R} \Vdash \Phi$.

For a proof see e.g. A.Lévy [51] p.144 or Easton's thesis [14].

<u>Lemma Q</u>: p $\Vdash^{*} \Phi$ iff $\mathfrak{N}_{\mathcal{R}} \models \Phi$ for all complete sequences $\mathcal{R}$ containing p.

Proof (see Lévy or Easton, loc.cit.).

<u>Lemma R:</u> val($E^{\alpha}x\Phi(x)$) is the set of all members y of $\mathfrak{N}$ such that $r(y) < \alpha$ and y satisfies $\Phi(x)$ in $\mathfrak{N}$. val($\dot{a}_j$) = a_j is the set of all $y \in \mathfrak{N}$ such that $r(y) < \lambda$ and $y \varepsilon \dot{a}_j$ holds in $\mathfrak{N}$.

<u>Lemma S</u>: For every $j \in A$, val($\dot{a}_j$) $\subseteq$ S and val($\dot{a}_j$) $\notin \mathfrak{M}$, thus the sets val($\dot{a}_j$) = a_j are "new".

<u>Proof</u>. $a_j \subseteq S$ follows easily from clause (3) of the forcing definition and lemma R. We have to prove that a_j is "new". Suppose a_j is not new. Then $a_j \in \mathfrak{M}$ and since $a_j = x$ for some $x \in \mathfrak{M}$, val($\underline{x}$) = x by lemma O. Hence val($\underline{x}$) = x = a_j = val($\dot{a}_j$) and lemma N yields: $\mathcal{R} \Vdash \dot{a}_j \simeq \underline{x}$. Therefore p $\Vdash \dot{a}_j = \underline{x}$ for some $p \in \mathcal{R}$. Since p is finite there are sets $s \in S$ such that p is not defined for $\langle s,j \rangle$. Now define $q_0 = p \cup \{\langle\langle s,j\rangle,0\rangle\}$ and $q_1 = p \cup \{\langle\langle s,j\rangle,1\rangle\}$. q_0 and q_1 are extensions of p. Proceed by cases. If $s \not\varepsilon x$ then $s \not\varepsilon val_{\mathcal{R}}(\underline{x}) = x$ for every complete sequence $\mathcal{R}$. Also, if $q_1 \varepsilon \mathcal{R}$, then $\mathcal{R} \Vdash s \varepsilon \dot{a}_j$ and hence $s \in val_{\mathcal{R}}(a_j) = a_j$ by lemma R (since $s \varepsilon S \rightarrow \rho(s) < \lambda$ and $\delta(\dot{a}_j) = \lambda$). Thus $s \in \dot{a}_j$ holds in $\mathfrak{N}_{\mathcal{R}}$ for $\mathcal{R}$ containing q_1 while $s \varepsilon \underline{x}$ fails in $\mathfrak{N}_{\mathcal{R}}$. Hence $\mathcal{R} \Vdash^{*} \dot{a}_j \neq \underline{x}$ for every complete sequence $\mathcal{R}$ containing q_1 (by lemma Q). Lemma A (iii) yields: $q_1 \Vdash \dot{a}_j \neq \underline{x}$.
If $s \varepsilon x$ then proceed as above and obtain $q_0 \Vdash \dot{a}_j \neq \underline{x}$.
Thus we have shown that every condition p has an extension q such that $q \Vdash \dot{a}_j \neq \underline{x}$. By lemma A (i),(iii) this implies that every condition p forces $\dot{a}_j \neq \underline{x}$. It follows now from lemmata O and P that a_j = val($\dot{a}_j$) $\notin \mathfrak{M}$, q.e.d.

<u>Lemma T</u>: $\mathfrak{M}$ and $\mathfrak{N}$ have the same ordinals.

Proof. Since $\mathfrak{M} \subseteq \mathfrak{N}$ by lemma 0 every ordinal of $\mathfrak{M}$ is an ordinal of $\mathfrak{N}$ (notice that since $\mathfrak{M}$ and $\mathfrak{N}$ are transitive $\in$-models the notion of "being an ordinal" is absolut). Now let α be an ordinal of $\mathfrak{N}$. Since $\alpha = \rho(\alpha) \leq \delta(\alpha)$, $\delta(\alpha) \in \mathfrak{M}$, the transitivity of $\mathfrak{M}$ implies $\alpha \in \mathfrak{M}$. Here $\rho(\alpha) \leq \delta(\alpha)$ holds since $s,y \in \mathfrak{N}$ and $x \in y \Rightarrow r(x) < r(y)$, thus $x \in \mathfrak{N} \Rightarrow \rho(x) \leq r(x)$.
But by definition of $r(x)$ we have $r(x) \leq \delta(x)$, thus $\rho(x) \leq \delta(x)$ for all $x \in \mathfrak{N}$, q.e.d.

Having proved all these various lemmata we are able to prove the main-theorem of forcing manely that the structure $\mathfrak{N}$ is a model of ZF.

HAUPTSATZ of the forcing technique

Let $\mathfrak{M}$ be a countable transitive $\in$-model of ZF and let $\mathfrak{A} = \langle A, R_i \rangle_{i\in I}$ be a (1^{st}-order) relational system in $\mathfrak{M}$. If the forcing relation $\Vdash$ is defined as in section A, then for every complete sequence $\mathfrak{R}$ the structure $\mathfrak{N}_{\mathfrak{R}}$ is a countable, transitive $\in$-model of ZF which extends $\mathfrak{M}$.

Proof: That $\mathfrak{N}_{\mathfrak{R}}$ is transitive and includes $\mathfrak{M}$ has been shown in lemma 0.
Ad.axiom(0): Since val($\underline{\emptyset}$) = $\emptyset \in \mathfrak{N}$, the axiom of null-set holds in $\mathfrak{N}$.
Ad(I): Extensionality follows from the transitivity of $\mathfrak{N}$.
Ad(II): If a and b are sets of $\mathfrak{N}$, then $a = val(t_1)$, $b = val(t_2)$ for terms t_1, t_2 of $\mathcal{L}$. Let $\delta(t_1) = \alpha_1$, $\delta(t_2) = \alpha_2$ and $\alpha = \max\{\alpha_1, \alpha_2\} + 1$, then $t_3 = E^{\alpha}x(x \simeq t_1 \vee x \simeq t_2)$ is a term of degree α and val(t_3) is the unordered pair of a and b.
Ad(III): Similar to (II). Ad(IV): Since val($\underline{\omega}$) = ω by lemma 0, the axiom of infinity holds in $\mathfrak{N}$.
Ad(V): The power-set axiom holds in $\mathfrak{N}$: (see: COHEN [11] p.46-47)
Cohen's proof that the power-set axiom holds in the extension $\mathfrak{N}$ (see [10] part II) follows the proof of Gödel [25] that V = L implies the GCH. We follow, instead, an elegant proof due to R.M.Solovay which avoids Gödel's argument (see Easton's thesis [14]).

We shall show, that for any constant term t there exists an ordinal α such that if val(s) $\subseteq$ val(t) holds in $\mathfrak{N}$ (for some constant term s), then val(s) = val(s^*) for some constant term

s^* of degree less than α. Then the power-set of val(t) in the sense of $\mathfrak{M}$ is given by

$$\mathrm{val}(\mathrm{E}^{\alpha}x(x \subseteq t)).$$

Let t be a constant term. For each constant term s define (p is any condition):

$$\phi(s) = \{\langle p,w\rangle ; \delta(w) < \delta(t) \wedge p \Vdash w \in s\}$$

By lemma I each $\phi(s)$ is a set of $\mathfrak{M}$ and the correspondence $s \mapsto \phi(s)$ is a function of $\mathfrak{M}$. Let this function be represented by the classterm G. We claim that $\phi(s_1) = \phi(s_2)$ implies $\mathrm{val}(t) \cap \mathrm{val}(s_1) = \mathrm{val}(t) \cap \mathrm{val}(s_2)$.

Proof. Suppose $\phi(s_1) = \phi(s_2)$ and $\mathrm{val}(w) \in \mathrm{val}(t) \cap \mathrm{val}(s_1)$. Then by the definition of the valuation $\mathrm{val} \triangleq \mathrm{val}_{\mathcal{R}}$: $p^{(n)} \Vdash w \in t$ and $p^{(n)} \Vdash w \in s_1$ for some $p^{(n)}$ in the complete sequence $\mathcal{R}$. Further w may be taken so that $\delta(w) < \delta(t)$. Thus $\langle p^{(n)},w\rangle \in \Phi(s_1)$. Since $\phi(s_1) = \phi(s_2)$ we get $\langle p^{(n)},w\rangle \in \phi(s_2)$ and this entails $p^{(n)} \Vdash w \in s_2$. We conclude that $\mathrm{val}(w) \in \mathrm{val}(t) \cap \mathrm{val}(s_2)$ [more precisely : $p^{(n)} \Vdash w \in s_2$ implies by lemma M the existence of a term w^* such that $\delta(w^*) < \delta(s_2)$, $p^{(n)} \Vdash w \simeq w^*$ and $p^{(n)} \Vdash w^* \in s_2$. By lemma N: $\mathrm{val}(w) = \mathrm{val}(w^*)$, thus $\mathrm{val}(w^*) \in \mathrm{val}(s_2)$ implies $\mathrm{val}(w) \in \mathrm{val}(s_2)$. Further, since $\mathrm{val}(w) \in \mathrm{val}(t)$ we get $\mathrm{val}(w) \in \mathrm{val}(t) \cap \mathrm{val}(s_2)$ as stated above].

Thus we have shown that $\mathrm{val}(t) \cap \mathrm{val}(s_1) \subseteq \mathrm{val}(t) \cap \mathrm{val}(s_2)$. The inverse $\supseteq$ follows in the same way, and we have proved that $\phi(s_1) = \phi(s_2)$ implies that $\mathrm{val}(s_1)$ and $\mathrm{val}(s_2)$ are equal modulo val(t).

Let $T_{\delta(t)}$ be the set (in $\mathfrak{M}$) of terms of degree less than $\delta(t)$. Then $\phi(s) \subseteq P \times T_{\delta(t)}$, where P is the set of all conditions. For every $k \subseteq P \times T_{\delta(t)}$ let (T is the class of all constant terms)

$$\psi(k) = \{s;\ s \in T \wedge \phi(s) = k \wedge \bigwedge_{s_1}[s_1 \in T \wedge \phi(s_1) = k \rightarrow \delta(s) \leq \delta(s_1)]\}.$$

($\psi(k)$ may be empty but in any case $\psi(k)$ is a set of $\mathfrak{M}$). By the axioms of powerset and replacement in $\mathfrak{M}$, $\{\gamma;\ \bigvee_k \bigvee_s (k \subseteq P \times T_{\delta(t)} \wedge s \in \psi(k) \wedge \gamma = \delta(s))\}$ is a set of $\mathfrak{M}$. Let β_0 be the supremum of the ordinals of this set and define $\beta_1 = \beta_0 + 1$ and $\alpha = \beta_1 + 1$.

Now suppose that x and y are sets of $\mathfrak{M}$ such that $y = \mathrm{val}(t)$ and $x \subseteq y$ holds in $\mathfrak{M}$. Then $x = \mathrm{val}(u)$ for some constant term u and $\mathrm{val}(u) \subseteq \mathrm{val}(t)$. Thus $\phi(u) = k \subseteq P \times T_{\delta(t)}$ and $\psi(k)$ is not empty, since k is represented as a $\phi(u)$ for some constant term u and

$\psi(k)$ contains those terms of minimal degree. Hence let s_0 be some element of $\psi(k) = \psi(\phi(u))$. Then

$$val(u) = val(t) \cap val(u) = val(t) \cap val(s_0).$$

Define $s_1 = E^{\beta_1}x(x \in t \wedge x \in s_0)$. Since $\delta(t) \leq \beta_0 < \beta_1$ and $\delta(s_0) < \beta_1$, s_1 is a constant term of degree β_1 and $val(u) = val(s_1)$ is true in $\mathfrak{N}$ (by lemmata M, N and R). Moreover $\delta(s_1) = \beta_1 < \beta_1 + 1 = \alpha$. Thus we have obtained an ordinal α (of $\mathfrak{M}$) with the required properties, q.e.d.

Notice that the proof given above is similar to the proof that the powersetaxioms holds in L - see page 28. Also the proof that the replacementaxiom holds in $\mathfrak{N}$ will be inspired by the corresponding proof for L. We need two lemmata.

<u>Lemma U</u>: Let $\Phi(x_0, x_1, \ldots, x_n)$ be an unlimited formula of $\mathcal{L}$. Then for every ordinal α of $\mathfrak{M}$ there is an ordinal β of $\mathfrak{M}$ such that for every condition p and constant terms $t_1, \ldots, t_n$ of rank less that α,

$$p \Vdash \bigvee_{x_0} \Phi(x_0, t_1, \ldots, t_n) \Leftrightarrow p \Vdash \bigvee^{\beta}_{x_0} \Phi(x_0, t_1, \ldots, t_n).$$

<u>Proof</u>. Let Φ be given and suppose that Φ has no free variables other than $x_0, \ldots, x_n$. By lemma I (see section A) there exists in $\mathfrak{M}$ a class K whose elements are the n+2-tuples $\langle p, t_0, t_1, \ldots, t_n\rangle$ such that $p \Vdash \Phi(t_0, \ldots, t_n)$. Hence

$$C = \{\langle p, t_0, \ldots, t_n\rangle ; p \Vdash \Phi(t_0, \ldots, t_n) \wedge \delta(t_i) < \alpha \text{ for } 1 \leq i \leq n\}$$

is also a class of $\mathfrak{M}$. By the axiom of foundation in $\mathfrak{M}$ the following collection C^* is a <u>set</u> of $\mathfrak{M}$:

$$C^* = \{\langle p, t_0, \ldots, t_n\rangle ; \langle p, t_0, \ldots, t_n\rangle \in C \wedge [p \Vdash \bigvee_{x_0} \Phi(x_0, t_1, \ldots, t_n) \rightarrow \rightarrow \bigwedge_{\langle p, t_0^*, t_1, \ldots, t_n\rangle \in C} (\delta(t_0) \leq \delta(t_0^*))]\}$$

Thus C^* contains only those n+2-tuples $\langle p, t_0, \ldots, t_n\rangle$ from C for which t_0 has minimal degree whenever $\bigvee_{x_0} \Phi(x_0, t_1, \ldots, t_n)$ is forced by p. By the replacement axiom in $\mathfrak{M}$,

$$D = \{\gamma ; \bigvee_{\langle p, t_0, \ldots, t_n\rangle \in C^*} (\gamma = \delta(t_0))\}$$

is again a set of $\mathfrak{M}$, and using again the replacement axiom in $\mathfrak{M}$, there exists an ordinal β such that $\gamma \in D \rightarrow \gamma < \beta$. Then it is easily seen that the equivalence stated in the lemma holds for this β, q.e.d.

<u>Lemma V</u>: Let $\Phi(x_1, \ldots, s_n)$ be an unlimited formula of $\mathcal{L}$ and α be an ordinal of $\mathfrak{M}$. Then there exists a limited formula $\Phi^{\nabla}(x_1, \ldots, x_n)$ such that

$\bigwedge^{\alpha}_{x_1} \ldots \bigwedge^{\alpha}_{x_n} [\Phi(x\ ,\ldots,x_n) \leftrightarrow \Phi^{\nabla}(x_1,\ldots,x_n)]$
holds in $\mathfrak{N}$.

Proof by induction on the length of Φ.
Case 1. If Φ is atomic, then we can let Φ^{∇} be Φ.
Case 2. Φ is $\neg\Psi(x_1,\ldots,x_n)$. By the induction hypothesis there is a limited formula $\Psi^{\nabla}(x_1,\ldots,x_n)$ such that $\bigwedge^{\alpha}_{x_1} \ldots \bigwedge^{\alpha}_{x_n}[\Psi(x_1,\ldots,x_n) \leftrightarrow \leftrightarrow \Psi^{\nabla}(x_1,\ldots,x_n)]$ is true in $\mathfrak{N}$. Hence, we can let Φ^{∇} be $\neg(\Psi^{\nabla})$.
Case 3. Φ is $\bigvee^{\delta}_{y}\Psi(x_1,\ldots,x_n,y)$. Let $\gamma = \max\{\alpha,\delta\}$. By the induction hypothesis there is a limited formula $\Psi^{\nabla}(x_1,\ldots,x_n,y)$ corresponding to Ψ and γ such that:

$$\bigwedge^{\gamma}_{x_1} \ldots \bigwedge^{\gamma}_{x_n} \bigwedge^{\gamma}_{y}[\Psi(x_1,\ldots,x_n,y) \leftrightarrow \Psi^{\nabla}(x_1,\ldots,x_n,y)]$$

holds in $\mathfrak{N}$. It follows that

$$\bigwedge^{\gamma}_{x_1} \ldots \bigwedge^{\gamma}_{x_n} [\bigvee^{\delta}_{y}\Psi(x_1,\ldots,x_n,y) \leftrightarrow \bigvee^{\delta}_{y}\Psi^{\nabla}(x_1,\ldots,x_n,y)]$$

is also true in $\mathfrak{N}$. This shows that we can define Φ^{∇} to be $\bigvee^{\delta}_{y}\Psi^{\nabla}(x_1,\ldots,x_n,y)$.
Case 4. Φ is $\bigvee_{y}\Psi(x_1,\ldots,x_n,y)$. By lemma U there is for given Φ and α an ordinal β such that for each condition p and constant terms $t_1,\ldots,t_n$ of degree less than α,

$$p \Vdash \bigvee_{y}\Psi(t_1,\ldots,t_n,y) \Leftrightarrow p \Vdash \bigvee^{\beta}_{y}\Psi(t_1,\ldots,t_n,y).$$

This means by lemma A, (vii) of section A. that every condition p weakly forces

$$\bigwedge^{\alpha}_{x_1} \ldots \bigwedge^{\alpha}_{x_n} [\bigvee_{y}\Psi(x_1,\ldots,x_n,y) \leftrightarrow \bigvee^{\beta}_{y}\Psi(x_1,\ldots,x_n,y)]$$

Hence also every p in the complete sequence $\mathfrak{R}$ weakly forces this formula, and therefore $\mathfrak{R}$ also strongly forces the formula. By lemma P of this section, this formula holds in $\mathfrak{N}$. Let $\gamma = \max\{\alpha,\beta\}$ and proceed as in case 3 (using β rather than δ), q.e.d.

Using lemma V we are able to prove, that the replacementaxiom (VI) holds in the structure $\mathfrak{N}$. Notice that our proof ressembles very much the proof that (VI) holds in Gödel's model L.

Continuation of the proof for the Hauptsatz

Ad(VI): The replacement-schema holds in $\mathfrak{N}$ · Let t_1 be a constant term of degree α and let $\Phi(x,y)$ be a formula of $\mathcal{L}$ such that it holds in $\mathfrak{N}$ that for every $x \in t_1$ there is precisely one y such that $\Phi(x,y)$. By lemma U there is an ordinal β such that

$$\mathfrak{N} \models \bigwedge^{\alpha}_{x}[\bigvee_{y}\Phi(x,y) \leftrightarrow \bigvee^{\beta}_{y}\Phi(x,y)].$$

By lemma V, there is a limited formula $\Phi^{\nabla}(x,y)$ such that

(+) $\quad \mathfrak{N} \models \bigwedge_x^{\zeta} \bigwedge_y^{\zeta}[\Phi(x,y) \leftrightarrow \Phi^{\nabla}(x,y)]$,

where $\zeta = \max\{\alpha,\beta\}$. It follows from (+) that:

(0) $\quad \mathfrak{N} \models \bigwedge_x^{\alpha} \bigwedge_y^{\beta}[\Phi(x,y) \leftrightarrow \Phi^{\nabla}(x,y)]$.

Let $s = E^{\lambda}y(\bigvee_x^{\alpha} \bigwedge_z^{\beta} y = z \wedge x \varepsilon t \wedge \Phi^{\nabla}(x,y))$, then s is a constant term of $\mathcal{L}$, where $\lambda = \max\{\alpha,\beta\} + 1 = \zeta + 1$. It follows that val(s) is the image of val(t) under the function Φ in $\mathfrak{N}$, q.e.d.
Thus we have proved the Hauptsatz.

Digression. What have we done so far? Our main question was whether the independence results we have obtained by means of the Fraenkel-Mostowski-Specker-method for the system ZF^0 (without the axiom of foundation, but assuming the existence of reflexive sets $x = \{x\}$) are also true for ZF = ZF^0 + foundation. Obviously not all independence results carry over to independence results in ZF, since e.g. (AC) and (PW) - see p.62 - are equivalent in ZF while (AC) is independent from (PW) in ZF^0 alone. The general procedure in the construction of a permutation model $\mathfrak{M}$ of ZF^0 was to define some relations R_i $(i \in I)$ between a set A of atoms (i.e. reflexive sets) and then to construct the permutation model $\mathfrak{M}$ over the structure $\mathfrak{A} = \langle A,R_i\rangle_{i \in I}$.

In order to obtain independence results which apply to full ZF-set theory (including foundation) our general idea was to add to a given countable model $\mathfrak{M}$ of ZF a generic copy of a structure $\mathfrak{A} = \langle A,R_i\rangle_{i \in I}$. Obviously, we cannot construct within our meta-theory (which is ZF + (AC)) a model ab ovo, since this would give otherwise a contradiction to Gödel's theorem. But what we can do is to construct from some given model $\mathfrak{M}$ of ZF another model $\mathfrak{N}$ of ZF in which some interesting statements Φ are true while others Ψ fail, thus proving that $\Phi \rightarrow \Psi$ is not derivable from ZF. Again by Gödel's theorem, we have to use the fact that $\mathfrak{M}$ is a ZF-model when proving that the extension $\mathfrak{N}$ is a ZF-model. This we have done by reducing questions about $\mathfrak{N}$ by means of lemma P to questions which can be posed in $\mathfrak{M}$. This is the most astonishing fact, that the extension can be described in the groundmodel $\mathfrak{M}$ (see lemmata I and P). It was the aim of the forcing definition to determine the interior $\in$-structure of the "new" generic sets $a_j (j \in A)$ in such a way that in $\mathfrak{M}$ we have evough information what properties these sets a_j have. These finite amounts of informations were called "conditions". Notice that since we are dealing with finite conditions, the sets a_j generic over $\mathfrak{M}$ determined by these conditions, are called Cohen-generic over $\mathfrak{M}$. The name "Cohen-generic" was chosen in honor of the man who first invented forcing with finite conditions. Forcing with perfect-closed subsets of the real-line is usually called Sacks-forcing and the generic sets obtained by this way of forcing are called Sacks-generic. Forcing with Borel-sets is called Solovay-forcing and the corresponding generic sets Solovay-generic - see Silver's Seminar notes [80] and the articles of Sacks and Solovay.

We have developed Cohen-forcing in a general setting and have obtained for infinite sets S Cohen-generic subsets $a_j \subseteq S$. In many cases we shall take ω as S. The Cohen-generic subsets of ω will be called simply Cohen-generic reals, since every subset of ω determines a real number.

Instead of proving one independence result after the other we shall first collect some additional informations about the generic extensions $\mathfrak{N}$. We have shown (see lemma T) that $\mathfrak{M}$ and the extension $\mathfrak{N}$ have the same ordinals. We ask: do they have the same cardinals? id est: are the ordinals λ which are initial ordinals in the sense of $\mathfrak{N}$ just the initial ordinals of $\mathfrak{M}$? or better: under which conditions is this true? Another question: Under what conditions on $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$ is the axiom of choice (AC) true in the extension?
Further, what are the conditions $\mathfrak{A}$ has to fullfill in $\mathfrak{M}$ in order to ensure that the extension $\mathfrak{N}$ satisfies the ordering principle? In the following section we discuss these questions and give some solutions.

C) ORDERINGS AND WELLORDERINGS IN GENERIC EXTENSIONS

We start with the presentation of a theorem which says that, if $\mathfrak{M}$ is a countable standard model of ZF + (AC) and $\mathfrak{A}$ is finite, then the model $\mathfrak{N}$ obtained from $\mathfrak{M}$ by adding a generic copy of $\mathfrak{A}$ to $\mathfrak{M}$, satisfies ZF + Axiom of choice.

A necessary remark. In section B we have shown, that the model $\mathfrak{N}$ extends $\mathfrak{M}$ (see lemma O), but $\mathfrak{M}$ need not to be a $\mathfrak{N}$-definable subclass of $\mathfrak{N}$. This, however, can be attained by adding to the forcing language $\mathcal{L}$ a further unary predicate symbol $\dot{g}$. The intended interpretation of $\dot{g}(v)$ is "v is in the groundmodel $\mathfrak{M}$".

The forcing-definition has to be enriched by the clause:

(*) $\quad p \Vdash \dot{g}(t) \Leftrightarrow (\exists x \in \mathfrak{M})(t \hat{=} \underline{x})$.

for conditions p and constant terms t. The interpretation of $\mathcal{L}$ in $\mathfrak{N}$, id est the semantics for $\mathcal{L}$, then has to be enriched by:

"y satisfies $\dot{g}$ in $\mathfrak{N}$ iff there is a constant term t such that $y \hat{=} val(t)$ and $\mathfrak{R} \Vdash \dot{g}(t)$".

It follows that $\dot{g}$ defines $\mathfrak{M}$ in $\mathfrak{N}$, more precisely, $\{val(\underline{x});\ x \in \mathfrak{M}\}$ is $\mathfrak{N}$-definable by means of $\dot{g}$. Whenever we shall need the fact that $\mathfrak{M}$ is $\mathfrak{N}$-definable we shall assume that forcing was done in a way including clause (*). This assumption is made e.g. in the following theorem, the proof of which is close to Gödel's proof that (AC) holds in L.

<u>Theorem</u>. If $\mathfrak{M}$ is a countable standard model of ZF + (AC) and a is Cohen-generic over $\mathfrak{M}$, then the extension $\mathfrak{N} \hat{=} \mathfrak{M}[a]$ is a countable standard model of ZF + (AC).

<u>Proof</u>. (R.B.Jensen [40] p.69). Set up in $\mathfrak{N}$ a ramified language $\mathcal{L}$ with a name $\underline{a}$ for a, names $\underline{x}$ for x in $\mathfrak{M}$ (this is possible in $\mathfrak{N}$, since a and $\mathfrak{M}$ are $\mathfrak{N}$-definable), limited quantifiers $\bigvee^{\alpha}$, limited comprehension operators E^{α} for all ordinals α of $\mathfrak{N}$, so that $\{\langle \alpha, \bigvee^{\alpha}\rangle;\alpha \in On^{\mathfrak{N}}\}$ and $\{\langle \alpha, E^{\alpha}\rangle;\alpha \in On^{\mathfrak{N}}\}$ are classes of $\mathfrak{N}$. $\mathcal{L}$ has furthermore all the symbols of ZF. Obtain by recursion (as usual) the well-formed formulae, so that $\mathcal{L}$, the collection of all these wff's, is a class of $\mathfrak{N}$. Define an interpretation Ω for the constant terms of $\mathcal{L}$ by setting

$$\Omega(\underline{a}) = a,\ \Omega(\underline{x}) = x \quad (\text{for } x \in \mathfrak{M}),$$

and then extending to all constant terms of $\mathcal{L}$ by recursion on δ, the degree (defined here as on p.79). Since the correspondence $x \leftrightarrow \underline{x}$ was by definition $\mathfrak{N}$-definable (see e.g. the conventions on p.79), it follows from the recursion theorem, that the function $\Omega = \{\langle t,\Omega(t)\rangle;t \in T\}$ is $\mathfrak{N}$-definable (T is the class of all constant terms). Let T_{α} be the set (of $\mathfrak{N}$) of constant terms t of degree less than α. Define $N_{\alpha} = \{\Omega(t);t \in T_{\alpha}\}$. It follows that $\bigcup_{\alpha} N_{\alpha}$ is the $\mathfrak{N}$-class of all sets of $\mathfrak{N}$.

After these preparatory remarks. let us prove that in $\mathfrak{N}$ every set x can be well-ordered. Let x be any set of $\mathfrak{N}$; then there exists in $\mathfrak{N}$ an ordinal α such that $x \subseteq N_{\alpha}$. We claim that T_{α} can be wellordered in $\mathfrak{N}$. In fact

$K_\alpha = \{\underline{x}, x \in \mathfrak{M} \wedge \delta(\underline{x}) < \alpha\}$ is a set of $\mathfrak{N}$ and included in T_α. Since $M_\alpha = \{\Omega(t), t \in K_\alpha\}$ is a set of $\mathfrak{M}$, this set M_α can be well-ordered in $\mathfrak{M}$ and induces hence (via Ω^{-1}) a well-ordering of K_α. By definition (see p.79) constant terms t of degree less than α are constructed as finite sequences of symbols taken from $S = K_\alpha \cup \{E^\beta; \beta < \alpha\} \cup \{\bigvee^\beta; \beta \leq \alpha\} \cup$ {the ZF-symbols}.
The set of ZF-symbols is countable, hence wellorderable. Thus the set S (of $\mathfrak{N}$) is wellorderable, and the set T_α can be well-ordered, e.g. lexicographically. Let W_α be a well-ordering of T_α. For $y \in N_\alpha$ let t_y by the first $t \in T_\alpha$ (in the ordering W_α) such that $y = \Omega(t)$. The function $\{\langle y, t_y\rangle ; y \in x\}$ is $\mathfrak{N}$-definable and hence so is the well-ordering

$$\{\langle y_1, y_2\rangle ; y_1, y_2 \in x \wedge \langle t_{y_1}, t_{y_2}\rangle \in W_\alpha\}$$

of x. This proves the theorem.

Corollary 1. Let $\mathfrak{M}$ be a countable, standard model of the NBG-set theory Σ + (E) and let a be Cohen-generic over $\mathfrak{M}$. Then $\mathfrak{N} \doteq \mathfrak{M}[a]$ is a countable standard model of Σ + (E).

Here Σ is the set of axioms of groups A,B,C,D in Gödel's orange monograph [25] and (E) is the global version of the axiom of choice.

Corollary 2. Let $\mathfrak{M}$ be a countable standard model of ZF + (AC) and $a_1, \ldots, a_n$ be finitely many sets which are Cohen-generic over $\mathfrak{M}$. Then $\mathfrak{N} \doteq \mathfrak{M}[a_1, \ldots, a_n]$ is a countable standard model of ZF + (AC).

Corollary 2 was obtained by S.Feferman using ideas of Gödel and Cohen:

[16] S.FEFERMAN: Some applications of the notions of forcing and generic sets. Fund.Math. 56(1965)p.325-345. See also Feferman's article (with the same title) in the "Theory of Models"-Symposium volume, North Holland Publ.Comp. Amsterdam 1965.

Symmetry Properties of Generic Extensions. Let $\mathfrak{M}$ be a countable standard model of ZF and let $\mathfrak{A} = \langle A, R_i\rangle_{i \in I}$ be a relational system in $\mathfrak{M}$. Let $\mathfrak{G}$ be the group in $\mathfrak{M}$ of automorphisms of $\mathfrak{A}$. Let $\mathcal{L}$ be the ramified language having constants $\underline{x}$ for each $x \in \mathfrak{M}$, constants $\dot{a}_j$ for each $j \in A$ and n_i-ary predicate symbols π_i for each $i \in I$

(as defined in section A). Let $\Vdash$ be the forcing relation as defined in section A. For $\sigma \in \mathfrak{G}$ and $\mathcal{L}$-formulae Φ let $\sigma(\Phi)$ be the formula obtained from Φ by replacing every occurence of $\dot{a}_j$ in Φ by $\dot{a}_{\sigma(j)}$. For conditions p (id est: finite partial functions from S × A into 2 = {0,1}, see p.81) define $\sigma(p)$ by:

$$\langle\langle s,j\rangle,0\rangle \in p \Leftrightarrow \langle\langle s,\sigma(j)\rangle,0\rangle \in \sigma(p)$$
$$\langle\langle s,j\rangle,1\rangle \in p \Leftrightarrow \langle\langle s,\sigma(j)\rangle,1\rangle \in \sigma(p)$$

By definition a formula Φ of $\mathcal{L}$ may contain some particular terms like $\underline{x}$ or $E^{\alpha}x\Psi(x)$ but contains never variables for terms. Thus if $E^{\alpha}x\Psi(x)$ occurs in Φ, then $E^{\alpha}x\sigma(\Psi(x))$ occurs in $\sigma(\Phi)$. According to the forcing-relation defined on p.81-82 the following holds:

Symmetry-Lemma (P.J.Cohen). Let Φ be any $\mathcal{L}$-sentence and let p be any condition. Then for every $\sigma \in \mathfrak{G}$ we have $p \Vdash \Phi \Leftrightarrow \sigma(p) \Vdash \sigma(\Phi)$.

Proof by induction on Ord(Φ) for limited formulae Φ and then for unlimited Φ by induction on the length of Φ. Exempla gratia, suppose the lemma is true for limited formulae Φ of order $< \alpha$. If Φ has Order α, proceed by cases. If Φ is $u \varepsilon \underline{x}$, then
$p \Vdash u \varepsilon \underline{x} \Leftrightarrow (\exists y \in x)(p \Vdash u \approx \underline{y}) \Leftrightarrow (\exists y \in x)(\sigma(p) \Vdash \sigma(u) \approx \underline{y}))$ since $y \in x \Rightarrow \rho(y) < \rho(y)$, hence $\mathrm{Ord}(u \approx y) < \mathrm{Ord}(u \varepsilon \underline{x})$.
The latter is equivalent to $(\exists y \in x)(\sigma(p) \Vdash \sigma(u) \approx \underline{y})$ which in turn is by (1) of the forcing definition equivalent to $\sigma(p) \Vdash \sigma(u) \varepsilon \underline{x}$, id est $\sigma(p) \Vdash \sigma(u \varepsilon \underline{x})$. One proceeds similar in all the other cases.

The symmetry-lemma has the following consequence, if $p \Vdash \Phi$ and p is in the complete sequence $\mathfrak{R}$, which defines $\mathfrak{N}$, then by lemma P, Φ holds in $\mathfrak{N}$. If σ is an automorphism of $\mathfrak{A}$ and $\sigma(p) = p$, then $p \Vdash \sigma(\Phi)$, hence $\sigma(\Phi)$ holds in $\mathfrak{N}$ as well. Since $\mathfrak{G}$ is in $\mathfrak{M}$ we can handle symmetry-properties of $\mathfrak{N}$ in the ground model $\mathfrak{M}$.

We shall use the symmetry-lemma in order to prove that there are models $\mathfrak{N}$ of ZF in which choice fails, thus proving Cohen's theorem, that the axiom of choice (AC) is not deducible from ZF. However, we shall not present Cohen's original proof [9], [11]. In the proof given here a Cohen-extension $\mathfrak{N}$ of a countable standard model $\mathfrak{M}$ is constructed in which there exists an infinite, but Dedekind-finite set A. This construction is due to J.D.Halpern-A.Lévy:

[35] J.D.HALPERN-A.LÉVY: The Boolean Prime Ideal theorem does not imply the axiom of choice; Mimeographed Notes, 93 pages. To appear in the Proceedings of the 1967-Set Theory Symposium at UCLA (AMS-Publications).

(see also Jensen's lecture notes [39] p.164-167).

Definitions. A set x is called finite, iff x is equipotent to some member n of ω. A set x is infinite, iff it is not finite. A set x is called Dedekind-finite, iff there does not exist a function f mapping x one-to-one onto some proper subset of x (This definition of finiteness was used in 1888 by R.Dedekind in his famous monograph "Was sind und was sollen die Zahlen"). In ZF it holds obviously that every finite set is Dedekind-finite. In order to prove the converse one needs the axiom of choice; the following fragment of the axiom of choice turns out to be already sufficient:

(AC^{ω}): The countable axiom of choice: For every set x of non-empty sets such that x is countable, there exists a function f such that for all $y \in x$ it holds that $f(y) \in y$.

Lemma: ZF + (AC^{ω}) $\vdash$ Every infinite set x has a denumerable subset.

Proof. Let x be infinite. Define $S_n = \{y \subseteq x; \overline{\overline{y}} = n\}$ for $n \in \omega$. Then $T = \{S_n; 0 < n < \omega\}$ is denumerable. By (AC^{ω}) there exists a function f defined on T such that $f(S_n) \in S_n$ for all n, $1 \leq n < \omega$. Hence $f(S_n)$ contains n elements. Define $g(n) = f(S_n)$ for $1 \leq n < \omega$. The set $\{g(n); 1 \leq n \in \omega\} = G$ is countable.Thus using (AC^{ω}) one obtains a function h defined on G such that $h(g(n)) \in g(n)$. Define $h^*(n) = h(g(n))$, then $\{h^*(n); 1 \leq n < \omega\}$ is an infinite countable subset of x. This set is infinite since x is infinite and therefore every S_n for $1 \leq n < \omega$ non-empty, q.e.d.

Corollary. ZF + (AC^{ω}) $\vdash$ A set x is finite iff it is Dedekind-finite.

Proof. Let x be Dedekind-finite and suppose x is not finite. Then by the preceeding lemma x has a countable infinite subset $y = \{z_1, z_2, z_3, \ldots\}$. Define a function f on y into y by: $f(z_n) = z_{n+1}$, then $f''y = \{z_2, z_3, z_4, \ldots\}$. Extend f to a function f^* defined on the whole of x by $f^*(u) = u$ iff $u \notin y$ and $f^*(u) = f(u)$ iff $u \in y$. Then f^* is a one-to-one mapping from x onto (the proper subset) $x - \{z_1\}$. Thus x would be Dedekind-infinite, a contradiction, q.e.d.

Now we shall construct a Cohen-extension $\mathfrak{N}$ of some ZF-model $\mathfrak{M}$ in which (AC^{ω}) fails by showing that in $\mathfrak{N}$ there are infinite sets which are Dedekind-finite. The model used is due to Halpern-Lévy [35] as indicated above.

Theorem. If ZF is consistent, then
"ZF + there exists an infinite set which is Dedekind-finite" is consistent too. Thus (AC^{ω}) is not provable in ZF.

Proof. Let $\mathfrak{M}$ be a countable standard model of ZF. Consider the following structure $\mathfrak{A} = \langle A,R \rangle$ in $\mathfrak{M}$, where A is ω and R is the unary predicate which holds for every $x \in A$; thus $\mathfrak{A} = \langle \omega,\omega \rangle$. Define a ramified language $\mathcal{L}$ in $\mathfrak{M}$ which has besides the usual ZF-symbols also constants $\underline{x}$ for every set x of $\mathfrak{M}$, constants $\dot{a}_i$ for every $i \in \omega = A$, a constant $\dot{b}$ and limited quantifiers $\bigvee^{\alpha}$ and limited comprehension operators E^{α} for all ordinals α of $\mathfrak{M}$, so that the sequences $\{\langle \alpha, \bigvee^{\alpha} \rangle ; \alpha \in On^{\mathfrak{M}}\}$ and $\{\langle \alpha, E^{\alpha} \rangle ; \alpha \in On^{\mathfrak{M}}\}$ are in $\mathfrak{M}$ (see section A for details). Define a condition p to be a partial finite function from $\omega \times \omega$ into $2 = \{0,1\}$ and define the forcing relation $\Vdash$ as in section A. Thus the key-clauses (3) and (7) read in the present context (t is a constant term):

$$p \Vdash t \in \dot{a}_j \Leftrightarrow (\exists n \in \omega)[p \Vdash t \simeq \underline{n} \ \& \ p(\langle n,j \rangle) = 1]$$

$$p \Vdash t \in \dot{b} \Leftrightarrow (\exists j \in \omega)[p \Vdash t \simeq \dot{a}_j].$$

Obtain a complete sequence $\mathcal{R}$ of conditions and thereby an interpretation $val_{\mathcal{R}}$ of the constant terms of the language $\mathcal{L}$, which defines the model $\mathfrak{N}$. Write $a_j = val(\dot{a}_j)$, $b = val(\dot{b})$; then $a_j \subseteq \omega$ for all $j \in \omega$ and $b = \{a_j ; j \in \omega\}$. By our "Hauptsatz", $\mathfrak{N}$ is a model of ZF. We want fo prove that in $\mathfrak{N}$, b is infinite while Dedekind-finite. This is done in several steps.

1 Step. $a_i \neq a_j$ if and only if $i \neq j$.

Proof. Suppose there are integers i and j such that $i \neq j$ and $a_i = a_j$ holds in $\mathfrak{N}$. Then (by lemma P) $a_i = a_j$ is forced by some p in the complete sequence $\mathcal{R}$, which defines $\mathfrak{N}$. Hence $p \Vdash a_i = a_j$. Since p is finite there exists a natural number n such that $\langle n,i \rangle \notin p$ and $\langle n,j \rangle \notin p$. Since $i \neq j$, we can extend p to a condition q by defining:

$$q = p \cup \{\langle\langle n,i \rangle ,1 \rangle ,\langle\langle n,j \rangle ,0 \rangle\}.$$

By the forcing definition $q \Vdash n \in \dot{a}_i$ and $q \Vdash \neg n \in \dot{a}_j$ where $p \leq q$.

Thus, by the 1[st] Extension-lemma, we have obtained a contradiction.

2.Step. It holds in $\mathfrak{N}$ that $b = \{a_j;\ j \in \omega\}$ is infinite.

Proof. Otherwise there would be a one-to-one function f in $\mathfrak{N}$ mapping b onto some member n of ω. This is impossible by the result proved in the first step.

3.Step. It holds in $\mathfrak{N}$ that b is Dedekind-finite.

Proof. Let f be any one-to-one function in $\mathfrak{N}$ which maps b onto some subset c of b. We claim that $(f''b =) c = b$. It is sufficient to show, that there exists a number $m \in \omega$ such that $f(a_j) = a_j$ for all $j \geqslant m$.

By definition of $\mathfrak{N}$, f is a limited comprehension term $E^{\alpha}x\Phi(x) = t_f$. By our assumption it holds in $\mathfrak{N}$ that f is one-to-one; thus by lemma P:

$$p \Vdash Fnc(t_f) \wedge Fnc(t_f^{-1}) \wedge Dom(t_f) = \dot{b} \wedge Range(t_f) \subseteq \dot{b}$$

for some p in the complete sequence $\mathcal{R}$. Let $occ(\Phi)$ be the $\mathfrak{M}$-set of numbers j such that $\dot{a}_j$ occurs in Φ, where $t_f = E^{\alpha}x\Phi(x)$. Let k be any (sufficient large) natural number such that $occ(\Phi) \subseteq k$ and $Dom(p) \subseteq \omega \times k$. This means: all $j \in occ(\Phi)$ are smaller than k and if $\langle\langle i,n\rangle ,e\rangle \in p$ for some $i \in \omega$, $e \in 2$, then $n < k$.
We claim that $p \Vdash^{*} \langle \dot{a}_j, \dot{a}_j\rangle \ \varepsilon\ t_f$ for $j \geqslant m = k + 1$.
Otherwise there would exist an extension q of p (by the definition of forcing) and natural numbers n_1, n_2 such that $n_1 \neq n_2$, $m \leqslant n_1$, $m \leqslant n_2$ and

(0) $\quad q \Vdash \langle \dot{a}_{n_1}, \dot{a}_{n_2}\rangle \ \varepsilon\ t_f$

Choose $h \in \omega$ such that $\langle\langle n,j\rangle ,e\rangle \in q$ implies $n < h$, $j < h$ and such that $n_1 \neq h$, $n_2 \neq h$. Define a permutation σ on ω by $\sigma(h) = n_2$, $\sigma(n_2) = h$, $\sigma(i) = i$ for $i \in \omega - \{h, n_2\}$. An application of the symmetry-lemma to (0) yields:

$$\sigma(q) \Vdash \langle \dot{a}_{n_1}, \dot{a}_h\rangle \ \varepsilon\ t_f$$

since $occ(\Phi) \subseteq k < m \leqslant n_1, n_2$, hence $\sigma(\Phi) = \Phi$, thus $\sigma(t_f) = \sigma(E^{\alpha}x\Phi(x)) = E^{\alpha}x\sigma(\Phi(x)) = E^{\alpha}x\Phi(x) = t_f$. By definition of σ, $q \cup \sigma(q)$ is a function and hence a condition extending both q and $\sigma(q)$. Therefore by the first extension lemma and lemma B:

$$q \cup \sigma(q) \Vdash \langle \dot{a}_{n_1}, \dot{a}_{n_2}\rangle \ \varepsilon\ t_f \wedge \langle \dot{a}_{n_1}, \dot{a}_h\rangle \ \varepsilon\ t_f \wedge Fnc(t_f) \wedge Fnc(t_f^{-1}).$$

Hence $q \cup \sigma(q) \Vdash \dot{a}_{n_2} = t_f(\dot{a}_{n_1}) = \dot{a}_h$, since t_f is a function. This contradicts the result proved in the first step. Thus in fact $p \Vdash^* t_f(\dot{a}_j) = \dot{a}_j$ for all $j \geq m$ and f must be surjectiv. This finishes the proof the theorem.

D) THE POWER OF THE CONTINUUM IN GENERIC EXTENSIONS

If we assume the axiom of choice (AC), then every set x is equipotent with precisely one aleph $\aleph_\alpha$. If $\mathbb{R}$ is the set of all reals then $\bar{\bar{\mathbb{R}}} = \aleph_\nu$ for a certain ordinal ν. Is it possible to determine this ordinal? It follows from Cantors theorem $\bigwedge_x(\bar{\bar{x}} < \overline{\overline{2^x}})$ that $\nu \geq 1$. G.Cantor has spent many years in order to solve this problem without arriving at the determination of the value for ν.

The natural approach to this problem is to determine the cardinalities of various subsets of $\mathbb{R}$. Cantor showed that every perfect set has cardinality $2^{\aleph_0}$ (a set is perfect iff it is a compact subset of $\mathbb{R}$, non-void and every element of it is an accumulation point of it). Moreover, the Cantor-Bendixion-theorem asserts that every closed subset of $\mathbb{R}$ is either countable or the union of a perfect set and a countable set. Thus no closed subset of $\mathbb{R}$ has a cardinal strictly between $\aleph_0$ and $2^{\aleph_0}$. Some further results of classical descriptive set theory read as follows:
(a) Every uncountable $\underset{\sim}{\Sigma}^1_1$-set of reals contains a perfect subset.
(b) Every $\underset{\sim}{\Pi}^1_1$-set of reals is the disjoint union of $\aleph_1$ many Borel sets.
It follows that every $\underset{\sim}{\Sigma}^1_1$-set has power $\leq \aleph_0$ or $= 2^{\aleph_0}$. Since Borel-sets are $\underset{\sim}{\Delta}^1_1$ (Souslin's theorem), hence $\underset{\sim}{\Sigma}^1_1$, it follows that every $\underset{\sim}{\Pi}^1_1$-set of reals has cardinality $\leq \aleph_1$ or $= 2^{\aleph_0}$ (For the notion $\underset{\sim}{\Sigma}^1_1$, etc,... see chapter II, page 44). For a treatment of these results see: [78] and:

[55] A.A.LJAPUNOW: Arbeiten zur deskriptiven Mengenlehre;
V.E.B.-Deutscher Verlag der Wissenschaften, Berlin 1955.

Since it was impossible to exhibit a subset of $\mathbb{R}$ of cardinality strictly between $\aleph_1$ and $2^{\aleph_0}$, Cantor conjectured in 1878 that

(CH) $\quad 2^{\aleph_0} = \aleph_1$,

called the "Continuum-Hypothesis". David Hilbert listed this problem as the first problem in his famous list of unsolved problems at the first international congress of Mathematicians in 1900 in Paris. Despite many attempts this problem remained for a long time unsolved. It was however used freely in proofs since it turned out to be a powerful assertion and also often symplified situations. W.Sierpiński deduced a large number of propositions (there called $C_1 - C_{82}$) from (CH),

[79] W.SIERPIŃSKI: Hypothèse du continu. Warszawa-Lwów 1934 (2nd edition, New York 1956).

In the lit erature there are many papers in which (CH) or the generalized continuum-hypothesis (GCH) is discussed and proved to be equivalent to other statements. W.Sierpiński contributed many papers concerning the (GCH).
H.Rubin has shown, e.g., that the (GCH) is equivalent in ZF to: "For all transfinite cardinals p and q, if p covers q, then for some r it holds that $p = 2^r$".
(see H.Rubin, Bull. AMS.65(1959)p.282-283). B.Sobociński has published a series of notes on the (GCH) in the Notre Dame Journal of formal Logic (parts I, II, III, vol. 3 and 4 (1962,63). K.Gödel has published in 1947 an article in which he gives a survey on results around the (GCH) and in which he discusses the more philosophic problem of the "truth" of the (GCH):

[26] K.GÖDEL: What is Cantor's Continuum Problem? Amer.Math. Monthly 54(1947)p.515-525, Corrections vol.55(1948)p.151.

Kurt Gödel showed in 1938 that the (GCH) is consistent with ZF, see chapter II of these lecture notes. Thus the (GCH) cannot be refuted in ZF. We have presented here a proof, that (AC) cannot be proved from the ZF-axioms. Since the (GCH) implies the (AC) - see page 24 - it follows, that also the (GCH) is not a theorem of ZF. Thus (GCH) is neither provable nor refutable in the system ZF. But now the following question arises: if we are willing to add the (AC) to the axioms of Zermelo-Fraenkel set theory ZF, is then the continuum-hypothesis (CH) or even the (GCH) derivable? P.J.Cohen [9]-[12] has shown that the (GCH) is not provable in ZF + (AC). Hence the truth or the falsity of the continuum hypothesis cannot be decided on the basis of the usual axioms of set theory, including the axiom of choice.

Theorem (P.J.Cohen). If ZF is consistent, then ZF + (AC) + $2^{\aleph_0} \geq \aleph_2$ is consistent too. Thus the continuum hypothesis (CH) is not a theorem of ZF + (AC).

Proof. Let $\mathfrak{M}$ be a countable standard model of ZF + (AC). We shall construct an extension $\mathfrak{N}$ of $\mathfrak{M}$ by adding generically so many new subsets of ω, such that $2^{\aleph_0} = \aleph_1$ is violated in $\mathfrak{N}$. We shall show below that it is sufficient to add $\aleph_2$-in the sense of $\mathfrak{M}$ -new subsets of ω to $\mathfrak{M}$. Since $\mathfrak{M}$ is countable, hence has only countably many (in the sense of the meta-language) subsets of ω (though in $\mathfrak{M}$ these sets have cardinality $\geq \aleph_1$), there is hope, that we will find $(\aleph_2)_{\mathfrak{M}}$ - many subsets of ω not yet in $\mathfrak{M}$, since $(\aleph_2)_{\mathfrak{M}}$ is countable outside of $\mathfrak{M}$.

Define in $\mathfrak{M}$ a ramified language $\mathcal{L}$ which has besides the usual ZF-symbols, the limited existential quantifiers $\bigvee^{\alpha}$, the limited comprehension operators E^{α} (for ordinals α in $\mathfrak{M}$), the constants $\underline{x}$ for $x \in \mathfrak{M}$, a further binary predicate symbol $\dot{a}$. Define $\mathcal{L}$ in such a way, so that the correspondences $x \mapsto \underline{x}$, $\alpha \mapsto E^{\alpha}$ and $\alpha \mapsto \bigvee^{\alpha}$ are all classes of $\mathfrak{M}$ (use e.g. the standard trick presented on p.79). Define a condition p to be a finite partial function from $\omega \times \aleph_2$ into $2 = \{0,1\}$. Define the forcing relation $\Vdash$ in the usual way (see page 81-82) containing the following key-clause:

$$p \Vdash \dot{a}(t_1,t_2) \Leftrightarrow (\exists\, n \in \omega)(\exists\, \nu \in \aleph_2)(p \Vdash t_1 = \underline{n}\ \&\ p \Vdash t_2 = \underline{\nu}\ \&\ p(\langle n,\nu\rangle) = 1).$$

This means in terms introduced in section **A**: We take as relational system $\mathfrak{A} = \langle A,R\rangle$ the very special case $\mathbf{A} = 1 = \{0\}$ and $R = \emptyset$, and choose a generic copy of $\mathfrak{A}$ in $\mathbf{S} = \omega \times \aleph_2$. Thus by choosing a complete sequence $\mathcal{R}$ of conditions and defining the valuation-function as in section B, our Hauptsatz tells us, that the model $\mathfrak{N}$ obtained in this way is a model of ZF which contains $\mathfrak{M}$ as a submodel and contains $\mathbf{a} = \mathrm{val}(E^{\alpha}\langle x,y\rangle\, \dot{a}(x,y)) \subseteq \omega \times \aleph_2$, for $\alpha = \omega_2^{\mathfrak{M}}$ (the superscript $\mathfrak{M}$ indicates that the concepts are understood in the sense of $\mathfrak{M}$).

By a theorem proved in section C, $\mathfrak{N}$ is also a model of (AC), since we have added to $\mathfrak{M}$, a model of ZF + (AC), only one new Cohen-generic set. Thus it remains to show, that in $\mathfrak{N}$ the continuum hypothesis is wrong.

Since $\aleph_2^{\mathfrak{M}}$ is the ordinal $\omega_2^{\mathfrak{M}}$ in $\mathfrak{M}$ and ordinals are preserved by the transition from $\mathfrak{M}$ to $\mathfrak{N}$, $\omega_2^{\mathfrak{M}}$ is an ordinal of $\mathfrak{N}$. Thus if we define for $\nu < \omega_2^{\mathfrak{M}} = \gamma$:

$$a_{\nu} = \{n;\ n \in \omega \wedge \langle n,\nu\rangle \in \mathbf{a}\}$$

then $a_{\nu} \subseteq \omega$ and $\nu_1 \neq \nu_2 \rightarrow a_{\nu_1} \neq a_{\nu_2}$ (as in the proof of the preceeding theorem) and we get

$$2^{\aleph_0} \geq \bar{\bar{\gamma}}^{\mathfrak{N}} \quad (\text{in } \mathfrak{N}).$$

γ is in $\mathfrak{M}$ the second infinite cardinal: $\gamma = \omega_2^{\mathfrak{M}}$; We shall show that cardinals are preserved in the extension, i.e. an ordinal which is a cardinal in $\mathfrak{M}$ is a cardinal in $\mathfrak{N}$ and vice versa. Then it will follow that $\gamma = \omega_2^{\mathfrak{M}}$ is also the second infinite cardinal in $\mathfrak{N}$: $\gamma = \omega_2^{\mathfrak{N}}$, and hence $2^{\aleph_0} \geq \aleph_2$ in $\mathfrak{N}$ as desired. To this end we need some lemmata.

<u>Lemma 1</u>. If B is in $\mathfrak{M}$ a set of conditions such that its elements are pairwise incompatible, then B is (in $\mathfrak{M}$) countable.
$(\forall B \subseteq \underline{\text{Cond}})[B \in \mathfrak{M} \;\&\; (\forall p_1, p_2 \in B)(p_1 \neq p_2 \rightarrow p_1 \cup p_2 \notin \underline{\text{Cond}}) \Rightarrow \bar{\bar{B}}^{\mathfrak{M}} \leq \omega]$.

<u>Proof</u>. <u>Cond</u> is the $\mathfrak{M}$-set of all conditions. Suppose the lemma is false, and let B be a set of $\mathfrak{M}$, such that $p_1, p_2 \in B \rightarrow (p_1 = p_2 \vee p_1 \cup p_2 \notin \underline{\text{Cond}})$ and $\bar{\bar{B}}^{\mathfrak{M}} > \omega$. Define $B_n = \{p \in B;\ \bar{\bar{p}} \leq n\}$. Since $\bigcup_{n=1}^{\omega} B_n = B$ and B is uncountable, there is a number $n \in \omega$ such that B_n is (in $\mathfrak{M}$) uncountable.

There are conditions $q \in \underline{\text{Cond}}$ such that $\{p \in B_n;\ q \subseteq p\}$ is in $\mathfrak{M}$ still uncountable, namely the empty condition $q = \emptyset$ has this property. On the other hand the cardinality of all such conditions q is bounded by n, since $q \subseteq p$. Thus we may define m to be the greatest natural number such that there exists a condition q such that $\bar{\bar{q}} = m$ and $\{p \in B_n;\ p \supseteq q\}$ is in $\mathfrak{M}$ uncountable. Let q_0 be such a condition of cardinality m having this property. Now choose in $\{p \in B_n;\ p \supseteq q_0\}$ any condition p_1. Since in B all conditions are pairwise incompatible, the elements of $\{p \in B;\ p \supseteq q_0\}$ are also pairwise incompatible.

$p_1 - q_0$ is not empty, since otherwise $p_1 = q_0$ and p_1 would be included in all conditions in $\{p \in B;\ p \supseteq q_0\}$, and hence compatible with them. Thus we can find $\langle\langle k,\nu\rangle, e\rangle \in p_1 - q_0$ such that $\langle\langle k,\nu\rangle, 1-e\rangle$ is contained in (in the sense of $\mathfrak{M}$) uncountably many conditions from $B^* = \{p \in B_n;\ p \supseteq q_0\}$. This follows, since p_1 is incompatible with every $p \in B^*$. It follows that $\{p \in B_n;\ p \supseteq q_0 \cup \{\langle\langle k,\nu\rangle, 1-e\rangle\}\}$ is uncountable in the sense of $\mathfrak{M}$ and $q_0 \cup \{\langle\langle k,\nu\rangle, 1-e\rangle\}$ has cardinality m+1, a contradiction to the choice of q_0 with maximal cardinality having this property. Thus lemma 1 is proved.

<u>Lemma 2</u>: If f is a function in $\mathfrak{N}$, such that $\text{Dom}(f) \in \mathfrak{M}$ and $\text{Range}(f) \subseteq x$ for some $x \in \mathfrak{M}$, then there exists a function

g in $\mathfrak{M}$ such that $Dom(f) = Dom(g)$, $Range(f) \subseteq \bigcup Range(g) \subseteq x$, and $g(s)$ is in $\mathfrak{M}$ countable for every $s \in Dom(f)$.

<u>Proof</u>. Since $f \in \mathfrak{N}$, there is by definition of $\mathfrak{N}$ a term t_f of the forcing language $\mathcal{L}$ such that $f \triangleq val(t_f)$. Thus the following holds in $\mathfrak{N}$ (for $x,z \in \mathfrak{M}$):

$$(*) \qquad \bigwedge_u \bigwedge_v \bigwedge_w [\langle u,v\rangle \in t_f \wedge \langle u,w\rangle \in t_f \rightarrow v = w] \wedge Dom(t_f) = \underline{z} \wedge Range(f) \subseteq \underline{x}.$$

Since $\mathfrak{N}$ is a generic extension, there is a condition p_0 in the complete sequence $\mathcal{R}$ (which defines $\mathfrak{N}$) such that p_0 forces (*) -see lemma P in section B. Using weak forcing and lemma A of section A, this entails:

$$(**) \qquad (\forall u,v,w \in \mathfrak{M})(\forall q \geqslant p_0)[q \Vdash^* \langle \underline{u},\underline{v}\rangle \in t_f \;\&\; q \Vdash^* \langle \underline{u},\underline{w}\rangle \in t_f \Rightarrow v = w].$$

Further, for every $u \in Dom(f)$ there is a condition p' in the complete sequence $\mathcal{R}$ such that $p' \Vdash^* \langle \underline{u},\underline{f(u)}\rangle \in t_f$ (this follows since $\langle \underline{u},\underline{f(u)}\rangle \in t_f$ holds in $\mathfrak{N}$). Since both p_0 and p' are in $\mathcal{R}$ and $\mathcal{R}$ is totally ordered by $\subseteq$ we obtain that $p_0 \cup p'$ is a condition. Hence,defining

$$g(s) = \{y;\ y \in x \;\&\; (\exists p' \geqslant p_0)(p' \Vdash^* \langle \underline{s},\underline{y}\rangle \in t_f)\}$$

for $s \in z = Dom(f)$, we obtain that $f(s) \in g(s)$. The function $g: z \mapsto x$ is in $\mathfrak{M}$ by lemma I of section A, and $Dom(g) = Dom(f) = z$ and $Range(f) \subseteq \bigcup Range(g) \subseteq x$.

We claim that $g(s)$ is countable in $\mathfrak{M}$ for $s \in z$. For $s \in z$ choose in $\mathfrak{M}$ for every $y \in g(s)$ a condition $p_y \supseteq p_0$ such that $p_y \Vdash^* \langle \underline{s},\underline{y}\rangle \in t_f$. We claim, that $\{p_y;\ y \in g(s)\}$ satisfies the hypothesis of lemma 1. In fact, if $y_1, y_2 \in g(s)$ and $p_{y_1} \cup p_{y_2}$ is a condition, then $p_{y_1} \cup p_{y_2} \Vdash^* \langle \underline{s},\underline{y_1}\rangle \in t_f \;\&\; p_{y_1} \cup p_{y_2} \Vdash^* \langle \underline{s},\underline{y_2}\rangle \in t_f$. But (**) entails $y_1 = y_2$. Hence $p_{y_1} = p_{y_2}$, since for every $y \in g(s)$ we have chosen <u>one</u> p_y. Now lemma 1 yields that $\{p_y; y \in g(s)\}$ is countable. This in turn implies, that $g(s)$ is in $\mathfrak{M}$ countable: $\overline{\overline{g(s)}}^{\mathfrak{M}} \leqslant \omega$, quod erat demonstrandum.

Notice, that we could interpolate between Range(f) and x only a "multivalued" function g, since the whole complete sequence $\mathcal{R}$ is <u>not</u> in $\mathfrak{M}$, and could thus not be used in order to find the interpolating function g (if $\mathcal{R}$ would be available in $\mathfrak{M}$, we could show $\mathfrak{M} = \mathfrak{N}$, hence $f \in \mathfrak{N}$, but this is contradictory).

But this defect is not too heavy since $g(s)$ is for $s \in z$ always countable, as we have shown.

Lemma 3. Cardinals are absolut in the extension from $\mathfrak{M}$ to $\mathfrak{N}$.

Proof. Let α be an ordinal of $\mathfrak{N}$ (and hence of $\mathfrak{M}$, by lemma T of section B), and let $\overline{\overline{\alpha}}^{\mathfrak{N}}$ be the cardinal of α in $\mathfrak{N}$ and let $\overline{\overline{\alpha}}^{\mathfrak{M}}$ be the cardinal of α in $\mathfrak{M}$ (i.e. the least ordinals equipotent with α). Since $\mathfrak{M} \subseteq \mathfrak{N}$, every function from ordinals $\beta \leqslant \alpha$ onto α which is in $\mathfrak{M}$ is also in $\mathfrak{N}$.
Hence $\overline{\overline{\alpha}}^{\mathfrak{N}} \leqslant \overline{\overline{\alpha}}^{\mathfrak{M}}$. We shall show that also $\geqslant$ holds.

Let f be a function in $\mathfrak{N}$ from $\delta_0 = \overline{\overline{\alpha}}^{\mathfrak{N}}$ onto $\delta_1 = \overline{\overline{\alpha}}^{\mathfrak{M}}$. If α is finite, then $f \in \mathfrak{M}$ and $\delta_0 = \delta_1$ follows trivially. Hence let us assume that α is infinite. By lemma 2 there exists in $\mathfrak{M}$ a function g such that $\mathrm{Dom}(g) = \mathrm{Dom}(f) = \delta_0$ and $\delta_1 = \mathrm{Range}(f) \subseteq \bigcup \mathrm{Range}(g) \subseteq \delta_1$ (hence $=$), and $s \in \delta_1 \to g(s)$ is countable in $\mathfrak{M}$. Hence:

$$\overline{\overline{\alpha}}^{\mathfrak{M}} = \delta_1 = \overline{\overline{\mathrm{Range}(f)}}^{\mathfrak{M}} = \overline{\overline{\bigcup \mathrm{Range}(g)}}^{\mathfrak{M}} \leqslant \overline{\overline{\mathrm{Dom}(g) \times \omega}}^{\mathfrak{M}} = \overline{\overline{\mathrm{Dom}(g)}}^{\mathfrak{M}}$$

(since $\mathrm{Dom}(g) = \mathrm{Dom}(f)$ is infinite and (AC) holds in $\mathfrak{M}$),

$$\overline{\overline{\mathrm{Dom}(g)}}^{\mathfrak{M}} = \overline{\overline{\mathrm{Dom}(f)}}^{\mathfrak{M}} = \overline{\overline{\delta_0}}^{\mathfrak{M}} = (\overline{\overline{\alpha}}^{\mathfrak{N}})^{\mathfrak{M}} = \overline{\overline{\alpha}}^{\mathfrak{N}}.$$

Hence $\overline{\overline{\alpha}}^{\mathfrak{N}} = \overline{\overline{\alpha}}^{\mathfrak{M}}$ and the lemma is proved.

The lemma implies that in particular — the notion of being $\aleph_2$ (the second infinite cardinal) is absolute in the extension from $\mathfrak{M}$ to $\mathfrak{N}$. Thus the continuum has in $\mathfrak{N}$ power $\geqslant \aleph_2$. This proves the theorem.

Theorem (P.J.Cohen). If ZF is consistent, then so is ZF + (AC) + (GCH) + $V \neq L$. Thus the axiom of constructibility is not provable in ZF + (GCH) + (AC).

Proof. Let $\mathfrak{M}$ be a countable standard model of ZF + $V = L$ and let $\mathfrak{N}$ be the model obtained by adding to $\mathfrak{M}$ one Cohen-generic real a, $\mathfrak{N} = \mathfrak{M}[a]$. Then $V \neq L$ holds in $\mathfrak{N}$ since the class of constructible sets of $\mathfrak{N}$ depends only on the class of ordinals which are in $\mathfrak{N}$. But $\mathfrak{M}$ and $\mathfrak{N}$ have the same ordinals, hence $\mathfrak{M}$ is in $\mathfrak{N}$ the class of constructible sets. Since $a \notin \mathfrak{M}$, we infer that a is not constructible in $\mathfrak{N}$. On the other hand (AC) holds in $\mathfrak{N}$ by the first theorem of ection C. Further the (GCH) holds in $\mathfrak{N}$ since $a \subseteq \omega$, and $V = L(a)$ holds in $\mathfrak{N}$. To see this, use the corresponding proof of $L \models$ (GCH) in chapter II, q.e.d.

The continuum hypothesis can be violated in generic extensions in various ways. R.M.Solovay has extended the result of Cohen and has shown that $2^{\aleph_0}$ can be anything it ought to be. The only values excluded for $2^{\aleph_0}$ are those excluded by König's theorem, which asserts that $2^{\aleph_0}$ is of cofinality greater than $\aleph_0$.

[81] R.M.SOLOVAY: $2^{\aleph_0}$ can be anything it ought to be; In: The Theory of Models, 1963 Symposium at Berkely; North Holland Publ. Comp. Amsterdam 1965, p.435.

<u>Theorem</u> (Solovay [61]): Let $\aleph_\alpha$ be an infinite cardinal in the countable standard model $\mathfrak{M}$ with $\aleph_0 < cf(\aleph_\alpha)$. Then there is an extension $\mathfrak{N}$ of $\mathfrak{M}$ such that the ordinals (cardinals) of $\mathfrak{N}$ are precisely the ordinals (cardinals, resp.) of $\mathfrak{M}$ and $2^{\aleph_0} = \aleph_\alpha$ in $\mathfrak{N}$.

The (GCH) can also be violated in various other forms. E.g. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for ordinals $\alpha < \gamma$ and $2^{\aleph_\gamma} \neq \aleph_{\gamma+1}$ (Solovay [81], Derrick-Drake, H.Schwarz et al.), thus answering a problem of Hajnal (Zeitschr.math. Logik u.Gr.Math.vol.2(1956)p.131-136). For a proof of this result see e.g. Jensen [40] p.68-74, the thesis of Schwarz (cited on p.76) and the article of Derrick-Drake in the same volume as Jensen [40].

Solovay has furthermore shown that there are generic extensions in which $2^{\aleph_i} = \aleph_{n_i}$ is consistent where $i < n_i$ and $n_0 \leqslant n_1 \leqslant \ldots \leqslant n_k$ ($k \in \omega$) is any sequence of natural numbers. W.B.Easton has extended this result. By means of forcing with a proper class of conditions he constructs a Cohen-generic extension $\mathfrak{N}$ of a countable standard model $\mathfrak{M}$ of NBG-set theory (with global choice) in which:

$$2^{\aleph_\alpha} = \aleph_{G(\alpha)} \quad \text{for every regular cardinal } \aleph_\alpha \text{ holds,}$$

where G is any function in $\mathfrak{M}$ from ordinals to ordinals satisfying the following two requirements: (1) $\alpha \leqslant \beta$ implies $G(\alpha) \leqslant G(\beta)$ and (2) $\aleph_{G(\alpha)}$ is not cofinal with any cardinal less than or equal to $\aleph_\alpha$. This result is contained in Easton's thesis (Princeton 1964), published partly in:

[14]:W.B.EASTON: Powers of regular cardinals. Annals of math. Logic, vol.1 (1970),

J.R.Shoenfield has developed a method for obtaining generic extensions of countable standard models of ZF without using ramified languages. Dana Scott told us, that Shoenfield's approach is equivalent with the Boolean-valued model approach (see the forthcoming

article of Scott-Solovay in the UCLA-set theory Symposium proceedings, vol.2).
Shoenfield presents the result of Easton in his article:

[77] J.R.SHOENFIELD: Unramified Forcing; Proceedings of the 1967-set theory symposium at UCLA, to appear in the AMS-publications.

E) THE INDEPENDENCE OT THE (BPI) FROM THE ORDERING - THEOREM

We consider the following statement:

(BPI) Boolean Prime Ideal Theorem: Every Boolean algebra has a prime ideal.

A Boolean algebra B is a distributive, complemented lattice $\mathbb{B} = \langle B, \sqcup, \sqcap, - \rangle$, where $x \sqcup y$ is the join of x and y, $x \sqcap y$ the meet, and -x the complement of x. B can be partially ordered by defining $x \leq y \leftrightarrow x \sqcup y = y$. Then $x \sqcup z$ is the least upper bound for x and z in B, and $x \sqcap y$ is the greatest lower bound for x and y. The maximal element in B is denoted by 1_B and the minimal element by 0_B. An ideal I in $\mathbb{B}$ is a subset of B satisfying the following three conditions:

(i) $0_B \in I$,
(ii) $x \in I \wedge y \leq x \rightarrow y \in I$,
(iii) $x \in I \wedge y \in I \rightarrow x \sqcup y \in I$.

A prime ideal is an ideal with the additional property:

(iv) $x \in I \leftrightarrow (-x) \notin I$.

In a Boolean algebra the prime ideals are just the maximal proper ideals.

The Boolean Prime ideal theorem (BPI) has a considerable number of equivalent forms in several branches of mathematics and in logic, although by far not as many as the axiom of choice (AC). The (BPI) is thus an interesting and natural principle of set theory.

Lemma. The following statements are all equivalent (in ZF) with the Boolean-Prime-Ideal theorem (BPI):

(a) The Stone representation theorem: Every Boolean algebra $\mathbb{B}$ is isomorphic to a field of sets.
(b) The Tychonoff-theorem for T_2-spaces: The product of compact Hausdorff-spaces is compact in the product topology.
(c) In every commutative ring with unit, every proper ideal is included in some prime ideal.

(d) The Stone-Čech compactification theorem.

(e) Alaoglu's theorem: The unit sphere of the adjoint of a Banach-space is a compact Hausdorff space.

(f) In every Boolean algebra, there exists a 2-valued measure.

(g) The principle of consistent choices.

(h) The completeness theorem for 1^{st}-order languages: Let Σ be a set of 1^{st}-order sentences with arbitrary many non-logical constants. If Σ is consistent, then it has a model.

(i) The compactness theorem for 1^{st}-order languages: Let Σ be as in (h). If every finite subset of Σ has a model, then Σ has a model.

For a proof see the following papers: Łos-Ryll Nardzewski: Fund. Math. 38 (1951) and Fund. Math. vol. 41 (1954); D.Scott: Bull.AMS 60 (1954) p. 390, L.Henkin: Bull.AMS. 60 (1954) p. 390; H.Rubin - D.Scott: Bull. AMS. 60 (1954) p.389; R.Sikorski: Boolean Algebras (Springer-Verlag Berlin 1964), Appendix.

We are interested here in one of the consequences of the (BPI), namely the ordering principle:

(OP) Every set x can be totally ordered.

We shall use the notions "totally ordered" and "linearly ordered" synonymously (i.e. equivalently). A somehow stronger principle is the following:

(OE) Order-Extension-Principle: If x is a set and r a partial ordering on x, then there exists a linear ordering t on x such that $r \subseteq t$.

The (OE) has been discovered by Banach, Kuratowski and Tarski (see: W.Sierpiński: Zarys terji mnogosci, Warszawa 1928, p. 158). The first proof which appeared in print is due to E.Marczewski (Szpilrajn) (Fund. Math. 16 (1930) p. 386-389). Marczewski used the lemma of Zorn-Kuratowski in order to deduce (OE). Łos, Ryll-Nardzewski and L.Henkin observed, that (OE) is already a consequence of the (BPI) (proof either via the compactness theorem, or directly using the ultrafilter theorem). Thus we have:

$$ZF \vdash (AC) \to (BPI) \to (OE) \to (OP).$$

We are interested in the problem, whether the converses of these implications also hold. It is not known, whether (OE) → (BPI) is provable in ZF or not. In this section we shall present a result of Adrian R.Mathias, which says, that (OP) → (OE) is not a theorem of ZF. In the next section we shall present the proof of J.D.Halpern-A.Lévy, that (BPI) does not imply the axiom of choice (AC).

Preparatory remarks. Mostowski has constructed in his paper [64] a model $\mathfrak{M}$ containing urelements (atoms) in which the ordering principle (OP) holds while (AC) fails in it. Mostowski takes a countable set of atoms linearly ordered of type η_0 (i.e. the order type of the rationals). A set is called symmetric iff it is mapped onto itself by some finite-support subgroup of G, the group of all order-preserving mappings from η_0 onto η_0. A set x is in the model $\mathfrak{M}$ iff x is hereditarily symmetric. The proof that (OP) holds in $\mathfrak{M}$ is based on the fact, that every set x of $\mathfrak{M}$ has a unique minimal support, supp(x), where supp(x) is a finite subset of η_0 (the set of atoms). The correspondence $x \mapsto \mathrm{supp}(x)$ is in $\mathfrak{M}$ and hence the lexicographic ordering of supp(y) for $y \in x$ together with wellorderings of the sets $K_e = \{y \in x;\ \mathrm{supp}(y) = e\}$ can be used to obtain a (symmetric) totalordering of x.

This idea can be carried over to Cohen-generic extensions. The rôle which was played by the urelements in Mostowski's model $\mathfrak{M}$, will be played by generic reals in the Cohen-extension. However, instead of adding a generic copy of η_0 (the rationals) to some countable standard model $\mathfrak{M}$ of ZF + V = L (in this case we would not know, how to destroy (OE) in the extension $\mathfrak{N}$) we shall add a generic copy of a certain partially ordered set $\langle I,\leqslant\rangle$ to $\mathfrak{M}$. If $\langle I,\leqslant\rangle$ has sufficiently enough automorphisms, then the generic copy of $\langle I,\leqslant\rangle$ will not have in the extension $\mathfrak{N}$ a total ordering, which extends $\leqslant$ (the symmetry-lemma will be used here).

What are the properties, $\langle I,\leqslant\rangle$ has to fulfill, so that in the extension $\mathfrak{N}$ the ordering principle (OP) remains true. The typical property of η_0, which was used by Mostowski in [64] in order to prove the existence of a unique, minimal, finite support of every set x of his model $\mathfrak{M}$ (the supports are sets of urelements!) was the homogeneous ordering of η_0. We shall show, that, if we require that $\langle I,\leqslant\rangle$ is a countable, homogeneous, $\aleph_0$-universal partially ordered set in $\mathfrak{M}$, and $\mathfrak{M} \models$ ZF + V = L, then in $\mathfrak{N}$ (the extension of $\mathfrak{M}$) every set $x = \mathrm{val}(E^{\alpha}x\Phi(x))$ has a unique, minimal, finite support supp(x), such that the correspondence $x \mapsto \mathrm{supp}(x)$ is $\mathfrak{N}$-definable. The verification of (OP) in $\mathfrak{N}$ is then standard. We need here, that $\mathfrak{M}$ is a model of V = L for two reasons, first in order to establish (using a theorem of B.Jónsson) that there are in $\mathfrak{M}$ homogeneous, universal partially ordered sets, and second in order to ensure that $K_e = \{x;\ \mathrm{supp}(x) \subseteq e\}$, e a finite subset of I, has a definable wellordering in $\mathfrak{N}$. Finally let us notice that, in contrast to Mostowski's permutation model $\mathfrak{M}$, the fact, that we have chosen a partially ordered set $\langle I,\leq\rangle$ and not a linearly ordered set (like Mostowski's η_0) will not cause any troubles when we want to linear order lexicographically the supports, since the generic sets a_i (for $i \in I$) are subsets of ω and $\{a_i; i \in I\}$ has thus in $\mathfrak{N}$ a definable ordering (namely the ordering of the real-line).

Having clarified the basic ideas behind the construction of a model $\mathfrak{N}$ of ZF + (OP) + $\neg$ (OE) + $\neg$ (BPI), we start to present the details of the proof. First we define the notion of a homogeneous, universal relational system. The notion is a generalization of Hausdorff's notion of an η_α-set. For more information we refer our reader to the following publications:

[41] B.JONSSON: Homogeneous universal relational systems; Math. Scand. vol.8 (1960) p. 137-142.

[3] J.L.BELL - A.B.SLOMSON: Models and Ultraproducts; North-Holland publ. Comp. Amsterdam 1969. (chapter 10).

<u>Definition</u>. The <u>type</u> τ of a relational system $\mathfrak{A} = \langle A,R_i\rangle_{i\in\zeta}$ where ζ is a finite ordinal) is a sequence $\langle n_0, n_1, \ldots n_{\zeta-1}\rangle$ of natural numbers such that for $0 \leq i \leq \zeta-1$ the relation R_i is n_i-ary.

<u>Definition</u>. The relational system $\mathfrak{B} = \langle B,S_i\rangle_{i\in\zeta}$ is a <u>subsystem</u> of $\mathfrak{A} = \langle A,R_i\rangle_{i\in\zeta}$ iff $B \subseteq A$ and $S_i = R_i \cap B^{n_i}$ (restriction of R_i to B).

<u>Definition</u>. Let $\mathcal{K}$ be a class of relational systems all of the same type τ. A system $\mathfrak{A}$ is <u>$\mathcal{K}$-homogeneous</u>, iff the following holds:

(1) $\mathfrak{A} \in \mathcal{K}$

(2) If $\mathfrak{B} = \langle B,S_i\rangle_{i\in\zeta}$, $\mathfrak{B} \in \mathcal{K}$, and if $\mathfrak{B}$ is a subsystem of $\mathfrak{A} = \langle A,R_i\rangle_{i\in\zeta}$ such that $\bar{\bar{B}} < \bar{\bar{A}}$, and if ϕ is an isomorphism of $\mathfrak{B}$ into $\mathfrak{A}$, then ϕ can be extended to an automorphism of $\mathfrak{A}$.

<u>Definition</u>. Let $\mathcal{K}$ be a class of relational systems all of the same type and let α be an ordinal. A system $\mathfrak{A} \in \mathcal{K}$ is called $(\aleph_\alpha,\mathcal{K})$-universal iff every system $\mathfrak{B} = \langle B,S_i\rangle_{i\in\zeta}$, such that $\bar{\bar{B}} \leq \aleph_\alpha$, is isomorphic to a subsystem of $\mathfrak{A}$, and A

has itself cardinality $\aleph_\alpha$.

Bjarni Jónsson has proved in [41] under the assupmtion of the (GCH), that under certain conditions on $\mathcal{K}$ and $\aleph_\alpha$ there are $(\aleph_\alpha, \mathcal{K})$-universal, homogeneous relational systems. It follows in particular that there are countable $\aleph_0$-universal, homogeneous partially ordered sets (here $\mathcal{K}$ is the class of all partially ordered sets).

Theorem (A.R.D.Mathias): Let $\mathfrak{M}$ be a countable, standard model of ZF + V = L; then $\mathfrak{M}$ can be extended to a countable standard model $\mathfrak{N}$ of ZF + (OP) + $\neg$ (OE). Thus the orderextension principle (OE) and a fortiori the Boolean prime ideal theorem (BPI) is independent from the ordering principle (OP) in the system ZF.

This result is contained in

[58] A.R.D.MATHIAS: The Order Extension Principle; Proceedings of the 1967-set theory symposium at UCLA. To appear.

Proof. By the theorem of B.Jónsson [41] there exists in $\mathfrak{M}$ a countable, $\aleph_0$-universal, homogeneous parially ordered set $\langle I, \leqslant \rangle$. We shall extend $\mathfrak{M}$ by adding to $\mathfrak{M}$ a generic copy of $\langle I, \leqslant \rangle$. We emphasize that if we write $i < j$ then $i \leqslant j$ and $i \neq j$. Thus $<$ is irreflexive, while $\leqslant$ is reflexive: $i \leqslant j \leftrightarrow (i < j \vee i = j)$.

We construct in $\mathfrak{M}$ a ramified language $\mathcal{L}$ with the usual limited quantifiers $\bigvee^\alpha$ and limited comprehension operators E^α (for ordinals α in $\mathfrak{M}$), the ZF-symbols, constants $\underline{x}$ for each set x of $\mathfrak{M}$, individual constants $\dot{a}_i$ for each $i \in I$ and two further constants $\dot{A}$ and $\dot{<}$. The wellformed formulae and limited comprehension terms are defined as usual, with the restriction that if $\dot{A}$ or $\dot{<}$ occurs in the $\mathcal{L}$-formula Φ then $E^\alpha x\Phi(x)$ is a limited comprehension term only if $\alpha \geqslant \omega + 1$.

A condition p is a finite partial function from $\omega \times I$ into $2 = \{0,1\}$. Define the (strong) forcing relation $\Vdash$ as in section A of this chapter. In our present case clause (3) reads:

$$p \Vdash t \varepsilon \dot{a}_i \Leftrightarrow (\exists n \in \omega)(p \Vdash \underline{n} \simeq t \ \& \ p(\langle n,i\rangle) = 1)$$

where t is any constant term of $\mathcal{L}$. Clause (7) reads:

$$p \Vdash t \varepsilon \dot{A} \Leftrightarrow (\exists i \in I)(p \Vdash t \simeq \dot{a}_i)$$

$$p \Vdash t_1 \dot{<} t_2 \Leftrightarrow (\exists i_1 \in I)(\exists i_2 \in I)(i_1 < i_2 \ \& \ p \Vdash t_1 \simeq \dot{a}_{i_1} \ \& \ p \Vdash t_2 \simeq \dot{a}_{i_2})$$

Let again $\Vdash^*$ denote the weak forcing relation. It follows

(1) $i_1 < i_2 \Rightarrow \emptyset \Vdash^* \dot{a}_{i_1} \dot{<} \dot{a}_{i_2}$,

(2) $(\forall i \in I)(\emptyset \Vdash^* \dot{a}_i \varepsilon \dot{A})$.

Obtain a complete sequence $\mathcal{R}$ of conditions and thereby a valuation $val_{\mathcal{R}}(t)$ of the constant terms t of $\mathcal{L}$, which defines the model $\mathfrak{N}$. Write $a_i = val(\dot{a}_i)$, $A = val(\dot{A})$, $\nabla = val(\dot{<})$, then $a_i \subseteq \omega$,

$A = \{a_i; i \in I\}$ and ∇ is an irreflexive parial ordering (in $\mathfrak{N}$) of A. By our Hauptsatz, $\mathfrak{N}$ is a model of ZF. All what remains to show is, that the ordering principle (OP) holds in $\mathfrak{N}$ while (OE) fails in $\mathfrak{N}$. To this end we need the following restriction lemma and the symmetry - lemma which we have proved in its full generality in section C (see page 99), so that it is available in the present situation.

<u>Restriction lemma:</u> Suppose that $p \Vdash \Phi$ and let $occ(\Phi)$ be the finite set of elements of I such that $i \in occ(\Phi)$ iff $\dot{a}_i$ occurs in Φ. Further let $p/occ(\Phi) = \{\langle\langle n,i\rangle,e\rangle \in p; i \in occ(\Phi)\}$, then $p/occ(\Phi) \Vdash^* \Phi$.

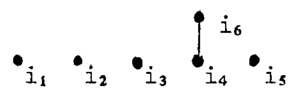
<u>Proof</u>. Define $c_0 = occ(\Phi)$, $d = \{i \in I ; \bigvee_{n\in\omega} \bigvee_{e\in 2} (\langle\langle n,i\rangle,e\rangle \in p \wedge i \notin c_0)\}$, and $q = p/occ(\Phi)$. Suppose $\sim q \Vdash^* \Phi$. Then by lemma A (i) (see section A) there exists an extension q' of q such that $q' \Vdash \neg \Phi$. q' mentions names of reals in c_0, and also others, say those in the finite set c_1 :

$$c_1 = \{i \in I; \bigvee_{n\in\omega} \bigvee_{e\in 2} (\langle\langle n,i\rangle,e\rangle \in q' \wedge i \notin c_0)\}.$$

By defintion $c_0 \cap d = c_0 \cap c_1 = \emptyset$. By the univerality of $\langle I,\leq\rangle$ we can find a subset c_2 of I, such that $\langle c_0 \cup d,\leq\rangle$ and $\langle c_0 \cup c_2,\leq\rangle$ are isomorphic, $c_2 \cap (c_0 \cup d) = \emptyset$. and there exists an isomorphism τ which is identical on c_0. By the homogeneity of $\langle I,\leq\rangle$, τ can be extended to an automorphism σ of $\langle I,\leq\rangle$. Hence: $\sigma(i) = i$ for $i \in c_0$ and $\sigma(j) \in c_2$ for $j \in c_1$. By the symmetry lemma, $q' \Vdash \neg\Phi$ implies $\sigma(q') \Vdash \sigma(\neg\Phi)$. But σ is the identity on $c_0 = occ(\Phi)$, hence $\sigma(\Phi) \triangleq \Phi$, and we obtain $\sigma(q') \Vdash \neg \Phi$. By construction of σ, the domain of the functions p and $\sigma(q')$ coincides only on a (finite subset of $\omega \times c_0$, where both have the same values, since $q \subseteq p$ and $q \subseteq q'$, $q \subseteq \sigma(q')$. Hence $p \cup \sigma(q')$ is a function and therefore a condition. By the first extension lemma (see section A): $p \cup \sigma(q') \Vdash \neg \Phi$. On the other hand $p \Vdash \Phi$ entails also $p \cup \sigma(q') \Vdash \Phi$, a contradiction to the consistency - lemma (see section A). This proves $q \Vdash^{*} \Phi$, q.e.d.

Lemma: The Orderextension Principle (OE) does not hold in $\mathfrak{N}$. In particular $\nabla = \mathrm{val}(\dot{<})$ cannot be extended in $\mathfrak{N}$ to a total ordering of $A = \mathrm{val}(\dot{A})$.

Proof. Suppose ∇ can be extended in $\mathfrak{N}$ to a linear ordering of A. Let R be such an orderextension of ∇. By the definition of $\mathfrak{N}$ there exists a limited conprehension term $t \triangleq E^{\alpha}x\Phi(x)$ of $\mathcal{L}$ such that $R = \mathrm{val}(t)$. Let $\Psi(t)$ be the $\mathcal{L}$-sentence "t is a totalordering of $\dot{A}$ extending $\dot{<}$" Let $c = \mathrm{occ}(t)$ be the set of indices $i \in I$ such that $\dot{a}_i$ occurs in t. Hence $c = \mathrm{occ}(\Psi(t))$. Let $S(c) = \{i \in I; \bigvee_{j \in c} (i \leqslant j \vee j \leqslant i)\}$ be the "shadow of c in $\langle I, \leqslant \rangle$".
By the universality of $\langle I, \leqslant \rangle$ we may embed the following partially ordered set into $I - S(c)$:



[all elements are pairwise incomparable, only i_4 is smaller than i_6].

Since $R = \mathrm{val}(t)$ is a linear ordering of $\dot{A}$ in, it holds in $\mathfrak{N}$ that $\{a_{i_1}, a_{i_2}, a_{i_3}\}$ are ordered by R. Assume e.g. that $\langle a_{i_1}, a_{i_2} \rangle \in R$, $\langle a_{i_2}, a_{i_3} \rangle \in R$. Hence we have obtained six generic terms $\dot{a}_{i_1}, \ldots, \dot{a}_{i_6}$ such that: (i) $\emptyset \Vdash^{*} \dot{a}_{i_4} \dot{<} \dot{a}_{i_6}$

(ii) $\emptyset \Vdash^{*} \neg(\dot{a}_{i_m} \dot{<} \dot{a}_{i_k} \vee \dot{a}_{i_k} \dot{<} \dot{a}_{i_m})$
for $m,k = 1,2,\ldots 6$ with $\langle m,k \rangle \neq \langle 4,6 \rangle$,

(iii) $\emptyset \Vdash^{*} \neg(\dot{a}_{i_m} \dot{<} \dot{a}_j \vee \dot{a}_j \dot{<} \dot{a}_{i_m})$
for $m = 1,2,\ldots,6$ and $j \in c = \mathrm{occ}(t)$

(iv) $\mathfrak{N} \models \langle a_{i_1}, a_{i_2} \rangle \in R \wedge \langle a_{i_2}, a_{i_3} \rangle \in R$.

Since everything which holds in $\mathfrak{N}$ must be forced (strongly or weakly) by some condition in the corresponding complete sequence $\mathfrak{R}$, we obtain, that there is a condition p in $\mathfrak{R}$ such that

$$p \Vdash^{*} \Psi(t) \wedge \langle \dot{a}_{i_1}, \dot{a}_{i_2} \rangle \,\varepsilon\, t \wedge \langle \dot{a}_{i_2}, \dot{a}_{i_3} \rangle \,\varepsilon\, t$$

By the restriction lemma, we may assume that p contains finitely many ordered pairs $\langle\langle n,i \rangle, e \rangle$ (with $n \in \omega$, $e \in 2$) only with $i \in \mathrm{occ}(t) \cup \{i_1, i_2, i_3\}$. Define

$$p_1(occ(t),i_1,i_2) = p/_{occ(t)} \cup \{i_1,i_2\}$$
$$p_2(occ(t),i_2,i_3) = p/_{occ(t)} \cup \{i_2,i_3\}.$$

Then by the restriction lemma:

(+) $p_1(occ(t),i_1,i_2) \Vdash^* \Psi(t) \wedge \langle \dot{a}_{i_1},\dot{a}_{i_2}\rangle \in t$

(++) $p_2(occ(t),i_2,i_3) \Vdash^* \Psi(t) \wedge \langle \dot{a}_{i_2},\dot{a}_{i_3}\rangle \in t$.

Now define mappings τ_1 and τ_2 on $occ(t) \cup \{i_1,i_2,i_6,i_5\}$ and on $occ(t) \cup \{i_2,i_3,i_4,i_5\}$, respectively, by:

τ_1 : $i_1 \mapsto i_6$, $i_2 \mapsto i_5$, τ_1 identical on $occ(t)$,

τ_2 : $i_2 \mapsto i_5$, $i_3 \mapsto i_4$, τ_2 identical on $occ(t)$;

Hence τ_1 maps $\langle occ(t) \cup \{i_1,i_2\},\leqslant\rangle$ isomorphically on $\langle occ(t) \cup\{i_5, i_6\},\leqslant\rangle$ and similar τ_2 maps $\langle occ(t) \cup \{i_2,i_3\},\leqslant\rangle$ isomorphically on $\langle occ(t) \cup \{i_4,i_5\},\leqslant\rangle$. By the homogeneity of $\langle I,\leqslant\rangle$, τ_1 and τ_2 can by extended to automorphisms σ_1 and σ_2, respectively, of $\langle I,\leqslant\rangle$. Using σ_1 we obtain from (+) by the symmetry-lemma:

(0) $q_1 = p_1(occ(t),i_6,i_5) \Vdash^* \Psi(t) \wedge \langle \dot{a}_{i_6},\dot{a}_{i_5}\rangle \in t$

and using σ_2 from (++):

(00) $q_2 = p_2(occ(t),i_5,i_4) \Vdash^* \Psi(t) \wedge \langle \dot{a}_{i_5},\dot{a}_{i_4}\rangle \in t$

(notice, that the symmetry-lemma also holds with respect to weak forcing $\Vdash^*$). Since $q_1 \cup q_2$ is a condition and $\emptyset \Vdash^* \dot{a}_{i_4} \dot{<} \dot{a}_{i_6}$ we obtain:

$$q_1 \cup q_2 \Vdash^* \Psi(t) \wedge \dot{a}_{i_4} \dot{<} \dot{a}_{i_6} \wedge \langle \dot{a}_{i_6},a_{i_5}\rangle \in t \wedge \langle \dot{a}_{i_5},\dot{a}_{i_4}\rangle \in t$$

But t extends $\dot{<}$, hence $\langle \dot{a}_{i_4},\dot{a}_{i_6}\rangle \in t$. But $\Psi(t)$ says, that t is a linear ordering on $\dot{A}$, a contradiction! since what we have shown is the following: p is in $\mathcal{R}$, the complete sequence, which defines $\mathfrak{N}$. Define $\mathcal{R}^*$ to be the sequence starting with $p/occ(t)$, having p as its second element, and containing then all conditions q of $\mathcal{R}$ which extend p. Then $\mathcal{R}^*$ defines obviously the same model $\mathfrak{N}$. Since $p/_{occ(t)} \Vdash^* \Psi(t)$ we infer that every complete sequence $\mathcal{R}$ starting with $p/_{occ(t)}$ must force $\Psi(t)$, and $\Psi(t)$ has to hold, hence also in the model $\mathfrak{N}_0$ defined by some complete sequence $\mathcal{R}_0$ which starts with $p/_{occ(t)}$ and has $q_1 \cup q_2$ as second element. But we have just shown, that in this model $val_{\mathcal{R}_0}(t)$ cannot define a total ordering on $val_{\mathcal{R}_0}(\dot{A})$ which extends $val_{\mathcal{R}_0}(\dot{<})$. In this way we have thus obtained a contradiction. This proves the lemma.

Next we want to prove, that in $\mathfrak{N}$ every set can be totally ordered. The idea behind the proof is the following. It can happen that for different $\mathcal{L}$-formula $\Phi_1(x)$ and $\Phi_2(x)$ we have that

$val(E^{\alpha}x\Phi_1(x)) \hat{=} val(E^{\alpha}x\Phi_2(x))$ where $occ(\Phi_1) \neq occ(\Phi_2)$. We want to show that for every x in $\mathfrak{N}$ there is a $\mathcal{L}$-formula $\Psi(x)$ such that $u = val(E^{\alpha}x\Psi(x))$ and

$$occ(\Psi) \subseteq \bigcap \{occ(\Phi);\ u = val(E^{\alpha}x\Phi(x))\}.$$

Then it follows, that every set u of $\mathfrak{N}$ is the valuation of a limited comprehension term $E^{\alpha}x\Psi(x)$ with minimal set $occ(\Psi)$, called the support of u. We then have to show, that the correspondence $u \mapsto$ support of u is definable in $\mathfrak{N}$. Then the rest of the proof that (OP) holds in $\mathfrak{N}$ is standard.

Notation: If $t = E^{\alpha}x\Phi(x)$ is a limited comprehension term and $occ(\Phi) = c$, then we shall write $t \hat{=} t(c)$ in order to indicate that t mentions reals $\dot{a}_i$ if and only if $i \in c$.

Lemma. Let $t(c,d_1)$ and $t'(c,d_2)$ be limited comprehension terms mentioning only reals $\dot{a}_i$ for $i \in c \cup d_1$, $i \in c \cup d_2$ respectively where c,d_1,d_2 are finite disjoint subsets of I. Suppose that

$$\mathfrak{N} \models t(c,d_1) = t'(c,d_2),$$

then there is a limited comprehension term $t'' \hat{=} t''(c)$ mentioning reals $\dot{a}_i$ for $i \in c$ such that

$$\mathfrak{N} \models t(c,d_1) = t''(c).$$

Proof. All what holds in $\mathfrak{N}$ is forced by some p in the complete sequence $\mathcal{R}$. Hence there exists a condition $p \in \mathcal{R}$ such that

$$p \Vdash^* t(c,d_1) = t'(c,d_2).$$

By the restriction lemma we may assume that $\langle\langle n,i\rangle,e\rangle \in p$ implies $i \in c \cup d_1 \cup d_2$.

[more precisely, the restricted condition $p/_{c \cup d_1 \cup d_2} = p_0$ need not to be in $\mathcal{R}$, a priori, but the sequence $\mathcal{R}_0$ which starts with p_0, has p as its second element and contains then all conditions q of $\mathcal{R}$ which extend p defines obviously the same model $\mathfrak{N}$. Thus we may assume, that we have already chosen $\mathcal{R}_0$ as the sequence which defines $\mathfrak{N}$].

Define $p_0 = p_0(c) = p/_c$, $p_1 = p_1(d_1) = p/_{d_1}$ and $p_2 = p_2(d_2) = p/_{d_2}$. Hence $p_0 \cup p_1 \cup p_2 \Vdash^* t(c,d_1) = t'(c,d_2)$. A limited comprehension term $t''(c)$ will be found for which $p_1 \cup p_2 \cup p_3 \Vdash^* t(c,d_1) = t''(c)$. It will be enough to consider the case when d_2 contains only one element, say i_0. The general case follows by induction.

We shall write $i \parallel y$ for $\neg(i \leq j \vee j \leq i)$ and similarly $x \dot{\parallel} y$ for the $\mathcal{L}$-formula $\neg(x \dot{<} y \vee y \dot{<} x \vee x = y)$.

Let i_1, i_2, i_3 be elements of I, but not in $c \cup d_1 \cup \{i_0\}$, such that $i_1 < i_0 < i_3 \wedge i_2 \not\parallel i_0$ and such that for $\nu = 1,2,3$ there are automorphisms τ_1, τ_2, τ_3 of $\langle I, \leqslant \rangle$ with $\tau_\nu(i_0) = i_\nu$ and $i \in c \cup d_1 \rightarrow \tau_\nu(i) = i$. Then by the symmetry-lemma (for $\nu = 1,2,3$):

$$p_0(c) \cup p_1(d) \cup p_2(i_\nu) \Vdash^* t(c,d_1) = t'(c,i_\nu)$$

where $\tau_\nu(p_2(i_0)) = p_2(i_\nu)$ for notational simplicity. This together with $p_0(c) \cup p_1(d) \cup p_2(i_0) \Vdash^* t(c,d_1) = t'(c,i_0)$ implies (using the restriction lemma):

$$(*) \qquad p_0(c) \cup p_2(i_0) \cup p_2(i_\nu) \Vdash^* t'(c,i_0) = t'(c,i_\nu)$$

for $\nu = 1,2,3$. We introduce the following notations. For a condition q define C_q ("the content of q") to be the following $\mathcal{L}$-sentence:

$$\bigwedge\!\!\!\bigwedge\{\underline{n} \in \dot{a}_j;\ \langle\langle n,j\rangle,1\rangle \in q\} \wedge \bigwedge\!\!\!\bigwedge\{\neg\,\underline{m} \in \dot{a}_j;\ \langle\langle m,j\rangle,0\rangle \in q\},$$

where $\bigwedge\!\!\!\bigwedge$ denotes conjunction. Let $C_q(^j/_x)$ be the result of replacing the generic constant $\dot{a}_j$ in C_q by the variable x.

For a finite subset s of I let D_s be the diagram of $\langle\{\dot{a}_j; j \in s\}, \dot{<}\rangle$ i.e.:

$$D_s \triangleq \bigwedge\!\!\!\bigwedge\{\dot{a}_{j_1} \dot{<} \dot{a}_{j_2};\ j_1 < j_2 \wedge j_1, j_2 \in s\} \wedge \bigwedge\!\!\!\bigwedge\{\neg\,\dot{a}_{j_1} \dot{<} \dot{a}_{j_2};\ j_1, j_2 \in s \wedge \neg\, j_1 < j_2\}.$$

Let $D_s(^j/x)$ be the result of replacing the constant $\dot{a}_j$ in D_s by the variable x. We now claim that the following continuity-property holds:

$$(**)\ p_0(c) \cup p_2(i_0) \Vdash^* \bigwedge_x [x \in \dot{A} \wedge C_{p_2}(^{i_0}/x) \wedge D_{c\cup\{i_0\}}(^{i_0}/x).\rightarrow.\ t'(c,i_0) = t'(c,x)],$$

where $t'(c,x)$ results from $t'(c,i_0)$ by replacing $\dot{a}_{i_0}$ in t' by x. Let k be a limited comprehension term. We have to show that, if q is any extension of $p_0(c) \cup p_2(i_0)$ and $q \Vdash^* k \in \dot{A} \wedge C_{p_2}(^{i_0}/k) \wedge D_{c\cup\{i_0\}}(^{i_0}/k)$, then there is a $q' \supseteq q$ with $q' \Vdash^* t'(c,i_0) = t'(c,k)$.

Assume that k and q are given such that the just mentioned hypothesis is fullfilled. Since in particular $q \Vdash^* k \in \dot{A}$ we know by the definition of forcing, that there exists a $q' \supseteq q$ and a $j \in I$ such that $q' \Vdash^* k = \dot{a}_j$. Pick $\nu = 1,2$ or 3, so that $\langle\{i_0, j\}, \leqslant\rangle$ and $\langle i_0, i_\nu\}, \leqslant\rangle$ are isomorphic. We claim that there is an automorphism τ of $\langle I, \leqslant\rangle$ so that for this choice of ν, τ maps $\langle c \cup \{i_0, i_\nu\}, \leqslant\rangle$ isomorphically on $\langle c \cup \{i_0, j\}, \leqslant\rangle$ in such a way that τ restricted to $c \cup \{i_0\}$ is the identical mapping.

In fact, since q' extends q the first extension lemma (see page 82) yields:

$$q' \Vdash^* k = \dot{a}_j \wedge C_{p_2}(^{i_0}/_j) \wedge D_{c\cup\{i_0\}}(^{i_0}/_j),$$

which means in particular, that q' (weakly) forces that $c \cup \{i\}$ and $c \cup \{j\}$ are isomorphic. Since already $\emptyset \Vdash^* D_{c\cup\{i_0\}}(^{i_0}/_j)$ (obviously) we obtain that in fact $\langle c \cup \{i_0\},\leqslant\rangle$ and $\langle c \cup \{j\},\leqslant\rangle$ are isomorphic such that there is an isomorphism σ_1 leaving c pointwise fixed. But by the choice of ν there is also an isomorphism σ_2 from $\langle c \cup \{i_\nu\},\leqslant\rangle$ onto $\langle c \cup \{i_0\},\leqslant\rangle$ leaving c pointwise fixed. Hence $\tau_0 = \sigma_1\sigma_2$ is an isomorphism from $\langle c \cup \{i_\nu\},\leqslant\rangle$ onto $\langle c \cup \{j\},\leqslant\rangle$ leaving c pointwise fixed. It follows that $\langle c \cup \{i_0,j\},\leqslant\rangle$ can be mapped isomorphically on $\langle c \cup \{i_0,i_\nu\},\leqslant\rangle$ so that $c \cup \{i_0\}$ is left pointwise fixed. Let τ be such an isomorphism.
Since $q' \Vdash^* C_{p_2}(^{i_0}/_j)$ it must hold that $p_2(j) = \sigma_1(p_2(i_0)) \subseteq q'$.
Hence

$$p_0(c) \cup p_2(i_0) \cup p_2(j) \subseteq q'$$

and applying τ to (*) the symmetry lemma and the 1st extension lemma entail:

$$q' \Vdash^* t'(c,i_0) = t'(c,j).$$

Since $q' \Vdash^* k = \dot{a}_j$, this gives us $q' \Vdash^* t'(c,i_0) = t'(c,k)$ as desired and (**) is proved.

The limited comprehension term t"(c) can now be constructed. Suppose that $t'(c,i_0)$ is $E^\alpha y\Phi(y)$ where Φ is a formula of $\mathcal{L}$ containing the constants $\dot{a}_{i_0}$ and $\dot{a}_i$ for $i \in c$. Let $\Phi^*(x,y)$ be the $\mathcal{L}$-formula obtained from $\Phi(y)$ by replacing $\dot{a}_{i_0}$ by the variable x (at all places of occurrence) - it is assumed, that x does not occur in $\Phi(y)$.
Then define

$$t''(c) = E^\alpha y(\bigvee_x[x \in \dot{A} \wedge C_{p_2}(^{i_0}/_x) \wedge D_{c\cup\{i_0\}}(^{i_0}/_x) \wedge \Phi^*(x,y)]).$$

By (**), $p_0(c) \cup p_2(i_0) \Vdash^* t'(c,i_0) = t''(c)$. Since $p_0(c) \cup p_2(i_0) \subseteq p \in \mathcal{R}$ and everything forced by p is true in $\mathfrak{N}$, the lemma is proved.

Lemma: The ordering principle (OP) holds in $\mathfrak{N}$.

Proof. Set up in $\mathfrak{N}$ a ramified language $\mathcal{L}^*$ with a name a^* for each $a \in A$, names A^* (for A), ∇^* (for $\nabla = val(\dot{<})$), names x^* for each $x \in \mathfrak{M}$, limited quantifiers $\bigvee^\alpha$, limited comprehension operators E^α (for all ordinals α of $\mathfrak{N}$) and the usual ZF-symbols. Do this in a way so that $\{\langle a,a^*\rangle ; a \in A\}$ is a set of $\mathfrak{N}$ and $\{\langle \alpha, \bigvee^\alpha\rangle ; \alpha \in On^{\mathfrak{N}}\}$ and $\{\langle \alpha,E^\alpha\rangle ; \alpha \in On^{\mathfrak{N}}\}$ are classes of $\mathfrak{N}$. Notice, that we assume here that $\mathfrak{M}$ is $\mathfrak{N}$-definable -this assumption can be made by the remarks of page 96-97. Notice furthermore that we could not use symbols like a_i^* for $i \in I$ as names for the elements of A since the correspondence $i \mapsto a_i$ $(i \in I)$ is <u>not</u> in $\mathfrak{N}$ (for each $i \in I$ we added a generic real $a_i = \mathrm{val}(\dot{a}_i)$ to $\mathfrak{M}$, but we did <u>not</u> add the correspondence $\{\langle i,\dot{a}_i\rangle ; i \in I\}$ generically to $\mathfrak{M}$). Define an interpretation Ω^* for the constant terms of $\mathcal{L}^*$ inductively by setting:

$$\Omega^*(a^*) = a \qquad \text{for each } a \in A,$$
$$\Omega^*(A^*) = A$$
$$\Omega^*(\nabla^*) = \nabla \text{ , and}$$
$$\Omega^*(x^*) = x \qquad \text{for each } x \in \mathfrak{M},$$

and then extending to all limited comprehension terms of $\mathcal{L}^*$, so that Ω^* is $\mathfrak{N}$-definable. Now define $\mathrm{supp}^*(u)$, for $u \in \mathfrak{N}$, as the finite subset, call it d, of A of minimal cardinality such that there is a term $t^*(d)$ of $\mathcal{L}^*$ mentioning only the names a^* for $a \in d$ with $\Omega^*(t^*(d)) = u$. Read $\mathrm{supp}^*(u)$ as: the support of u. By the previous lemma, supp^* is always well defined. Notice, that there is a clear one-to-one-correspondence * between constant terms t of $\mathcal{L}$ and terms t^* of $\mathcal{L}^*$ so that $\Omega^*(t^*) = \mathrm{val}(\bar{t})$. Further, supp^* is $\mathfrak{N}$-definable. For each finite subset d of A let

$$V_d = \{x \in \mathfrak{N} ; \mathrm{supp}^*(x) = d\}$$

Each class V_d has an $\mathfrak{N}$-definable wellordering: the symbols from the alphabet of $\mathcal{L}^*$used to construct constant terms t^* with $\mathrm{supp}^*(\Omega^*(t^*)) = d$ have an $\mathfrak{N}$-definable wellordering and thus the constant terms t^* of $\mathcal{L}^*$ with $\mathrm{supp}^*(\Omega^*(t^*)) = d$ have an $\mathfrak{N}$-definable wellordering (e.g. the lexicographic ordering as modified by Gödel **[25]** p.36). Using the interpretation Ω^* one obtains an induced well-ordering of V_d. Each d is a finite subset of A, which is linearly ordered, being a subset of the real line. The set of finite subsets d of A can be linearly ordered e.g. by the usual lexicographic method. [For more details see the proof of the theorem on page 97-98].

Hence let <u>Lex</u> be the lexicographic ordering of finite subsets of A and for each finite subset d of A let W_d be the wellordering of V_d. Now if u is any set of $\mathfrak{N}$, then

$$\{\langle x_1,x_2\rangle ; x_1,x_2 \in u \wedge [\langle \mathrm{supp}^*(x_1),\mathrm{supp}^*(x_2)\rangle \in \underline{\mathrm{Lex}} \vee$$
$$\vee (\mathrm{supp}^*(x_1) = d = \mathrm{supp}^*(x_2) \wedge \langle x_1,x_2\rangle \in W_d)]\}$$

is a linear ordering of u. This proves the lemma and hence the theorem of A.R.D. Mathias is proved.

Let (AC_{wo}) (in the notation of **A**.Lévy [39]p.223) be the following consequence of the usual axiom of choice:

$$(AC_{wo}): \bigwedge_x [\bigwedge_y (y \in x \to y \neq \emptyset \wedge y \text{ can be wellordered}) \to \bigvee_f (Fnc(f) \wedge Dom(f) = x \wedge \bigwedge_{y \in x} f(y) \in y)].$$

Theorem (A.R.D. Mathias): If ZF is consistent, then ZF + (OP) + (AC_{wo}) + $\neg$ (AC) is consistent too. Thus the Axiom of choice (AC) is independent from (AC_{wo}) + Ordering principle (OP) in ZF.

Proof ([58]): Take the model $\mathfrak{N}$ constructed above. A set z is in $\mathfrak{N}$ wellorderable iff there is a finite subset d of A such that for all y, if $y \in z$, then $supp^*(y) \subseteq d$. Hence, if z is wellorderable in $\mathfrak{N}$, then the $\mathfrak{N}$-definable linear-ordering turns out to be a well-ordering. Thus, if x is a set o well-orderable sets, then there is a function f in $\mathfrak{N}$ which assigns to each $z \in x$ a wellordering. Hence (AC_{wo}) holds in $\mathfrak{N}$. As it was shown in preceeding lemmata, (OP) and $\neg$ (AC) are true in $\mathfrak{N}$. This proves the theorem.

F) THE KINNA-WAGNER CHOICE PRINCIPLE

An interesting weakened version of the axiom of choice has been considered by W.Kinna and K.Wagner in their paper:

[44] W.KINNA-K.WAGNER: Ueber eine Abschwächung des Auswahlpostulates; Fund. Math. 42(1955)p.75-82.

In contrast to the usual (AC) where the choice function selects one element, the functions considered by Kinna-Wagner select non-empty, but proper subsets:

(KW-AC): The Kinna-Wagner choice principle: If x is a set all of whose elements have at least two elements, then there exists a function f, defined on x, such that for all $y \in x$, $\emptyset \neq f(y) \subsetneq y$.

$$\bigwedge_x [\bigwedge_y (y \in x \to 2 \leqslant \bar{\bar{y}}) \to \bigvee_f (Fnc(f) \wedge Dom(f) = x \wedge \bigwedge_{y \in x} (\emptyset \neq f(y) \subseteq y \wedge y \neq f(y)))].$$

Obviously (KW-AC) is a consequence of (AC). W.Kinna and K.Wagner have shown, that (KW-AC) implies the ordering principle (OP) "Every set can be totally ordered":

Lemma (Kinna-Wagner[44]): ZF $\vdash$ (KW-AC) $\rightarrow$ (OP).

Sketch of Proof. Let M be a given set and let f be a function, defined on P(M), the power set of M, such that $\emptyset \neq f(y) \neq y$, $f(y) \subset y$ whenever $y \in P(M)$ and y contains at least two elements. Following the $\bigcap$-method of Zermelo's second proof for (AC) $\rightarrow$"Well-ordering-theorem" (see: Math. Amm. 65 (1908)p.107-128) one starts the proof with the following definition:

Let F be the least family of subsets of M satisfying:

(1) $M \in F$,

(2) $(x \in F \wedge 2 \leqslant \bar{\bar{x}}) \rightarrow (f(x) \in F \wedge x - f(x) \in F)$,

(3) $t \subseteq F \rightarrow \bigcap t \in F$.

Define an element e of F to be normal iff the following holds:

$$\bigwedge_{x \in F}[e \subseteq x \wedge e \neq x \rightarrow e \subseteq f(x) \vee e \subseteq x - f(x)].$$

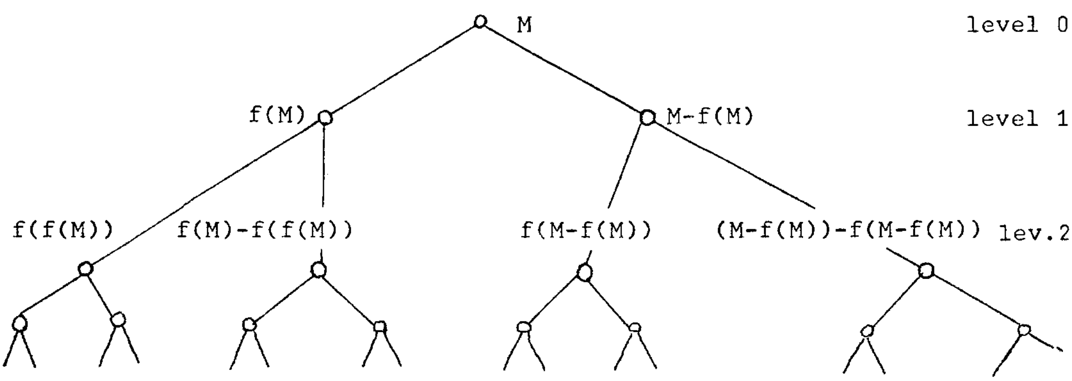
There are normal elements in F, e.g. M is normal. For e normal, define:

$$g_e = \{x \in F;\ x \cap e = \emptyset \vee [x \cap e \neq \emptyset \rightarrow (e \subseteq x \vee x \subseteq f(e) \vee x \subseteq e-f(e)]\}.$$

One proves that g_e satisfies the conditions (1), (2), (3) and is therefore (by the minimality of F) equal to F. This shows that if e is normal, then

$$(*) \quad \bigwedge_{x \in F}[x \cap e = \emptyset \vee (x \cap e \neq \emptyset \wedge [e \subseteq x \vee x \subseteq f(e) \vee x \subseteq e - f(e)])]$$

holds. Now, let N be the set of normal elements of F, then N satisfies conditions (1), (2), (3) and thus N = F by the minimality of F. This shows that every element of F is normal. Intuitively this means the following: by means of an iterated application of f to M, then to f(M) and M-f(M), then to f(f(M)), f(M)-f(f(M)), f(M-f(M)), (M-f(M)) - f(M-f(M)), etc.... one obtains by transfinite induction the following binary tree:



where at each point x the tree splits into two branches, if $\bar{\bar{x}} \geqslant 2$, where $f(x)$ is the successor of x at the left branch and $x - f(x)$ the successor of x on the right branch. The information, that every element is normal means that if $e \in F$ and $x \in F$, then either e and x are on different branches, **or**, if there is a branch going through x and e then either the level of x is smaller than the level of e (case: $e \subseteq x$) or x appears on the branch after one of the two successors of e. With this in mind, it is not difficult to prove, that the binary relation ρ, defined on M by:

$$p \,\rho\, q \leftrightarrow p \in f(\bigcap\{x \in F;\ p,q \in x\} \vee q \notin f(\bigcap\{x \in F;\ p,q \in x\}).$$

is a linear ordering of M. This proves the lemma.

The proof shows, that a binary, wellfounded tree can be embedded into P(M), the powerset of M, such that the image forms a chain C in P(M). The tree itself is equipotent with some ordinal α, hence $\overline{\overline{P(M)}} \geqslant \bar{\bar{\alpha}}$. On the other hand, if D(C) is the Dedekind completion of C, then D(C) is a maximal chain in P(M), linking $\emptyset$ and M. Hence $\bar{\bar{M}} \leqslant 2^{\alpha}$. This proves the following:

<u>Theorem</u> (Kinna-Wagner **[44]**) In ZF the statement (KW-AC) is equivalent to the statement:
(KW-O): For every set M there is an ordinal α such that there exists a one-to-one function mapping M into $P(\alpha)$, the powerset of α.

(KW-O) is a strong form of the Ordering-principle, since it asserts that every set M has a linear ordering ρ which is **a** subset of the canonical ordering $\leqslant$ on $P(\alpha)$, for some ordinal α, defined by $x \leqslant y \leftrightarrow \text{Min}\{(x \cup y) - (x \cap y)\} \in y$. Since ZF $\vdash$ (AC) $\rightarrow$ (KW-AC) $\leftrightarrow$ (KW-O) $\rightarrow$ (OP) we ask, whether the first or the last arrow can be reversed. It is known, that **both** cannot be reversed. J.D.Halpern and A.Lévy have first shown, that (AC) does not follow from (KW-AC) in ZF (see **[35]**, and:

[34] J.D.HALPERN-A.LÉVY: the Ordering Theorem does not imply the axiom of choice; Notices of the Amer. Math. Soc. 11 (January 1964)p.56.

The problem, whether (OP) $\rightarrow$ (KW-O) can be proved in ZF remained for some years open, though Mostowki has shown already in 1958, that in ZF^{0} (id est ZF but without the axiom of foundation) (OP) does not imply (KW-O), see:

[67] A.MOSTOWSKI: On a problem of W.Kinna and K.Wagner; Coll. Math. vol.6(1958)p.207-208.

The independence of (KW-0) from (OP) in ZF^0 follows more directly already from the fact, that (PW) holds in every Fraenkel-Mostowski-Specker model (see p.62), hence in particular in Mostowski's model of [64] in which (OP) holds, while (AC) fails. If (KW-0) would hold there, then (AC) would also hold. In 1969 U.Felgner has shown, that also in full ZF, (KW-0) is independent from (OP).

[18] U.FELGNER: Das Ordnungstheorem impliziert nicht das Kinna-Wagnersche Auswahlprinzip. To appear.

Here we shall sketch the proofs for both results. But,in order to prove that (AC) is independent from (KW-AC) we shall not use the model $\mathfrak{M}[a_0, a_1, \ldots, A]$ (a_n generic reals for $n \in \omega$, $A = \{a_n; n \in \omega\}$) which was used originally by Halpern and Lévy (for a more detailed description of this model, see sections C and G). We shall use the model of Mathias,described in section E, since this will give us the additional information, that (KW-AC) does not imply the order-extension principle (OE).

<u>Theorem</u>. The choice principle (KW-AC) of Kinna and Wagner holds in the model of Mathias. Hence, if ZF is consistent, then (KW-AC) does not imply the orderextension principle (OE), though (KW-AC) implies the ordering principle (OP).

<u>Proof</u>. We have shown in section E, that Mathias' model $\mathfrak{N}$ has the following features: for every set u of $\mathfrak{N}$, the following relation is a linear ordering of u:

$$\{\langle x_1, x_2\rangle ; x_1, x_2 \in u \wedge [\langle supp^*(x_1), supp^*(x_2)\rangle \in \underline{Lex} \vee (supp^*(x_1) = supp^*(x_2) = d \wedge \langle x_1, x_2\rangle \in W_d)]\}$$

Here <u>Lex</u> was the lexicographic ordering of finite subsets of A, the set of generic reals a_i, and W_d was the wellordering of $V_d = \{x; supp^*(x) = d\}$. Since $A \subseteq 2^{\aleph_0}$, every set u of $\mathfrak{N}$ splits into $\leq 2^{\aleph_0}$ many wellordered subsets. Thus we can compute an ordinal λ such that u can be embedded (i.e. mapped into, by a one-to-one function) into $\aleph_\lambda \times 2^{\aleph_0}$. Thus u can be embedded into $2^{\aleph_\alpha}$, the powerset of $\aleph_\alpha$, for a sufficient large ordinal α. Thus (KW-0) holds in $\mathfrak{N}$. The rest follows from results, proved in section E, q.e.d. (take e.g. $\alpha = \lambda$)

<u>Theorem</u>.(U.Felgner): If ZF is consistent, then (KW-0) is independent from the ordering principle (OP).

<u>Sketch of proof</u>. Let $\mathfrak{M}$ be a countable standard model of ZF + V = L. Let Q be the set in $\mathfrak{M}$ of rational numbers and let $\leqslant$ be the usual ordering (of type η_0) of Q. Define in $\mathfrak{M}$ a ramified language $\mathcal{L}$ which contains besides the usual ZF-symbols and the useful limited quantifiers $\bigvee^\alpha$ and limited comprehension terms E^α ($\alpha \in On^{\mathfrak{M}}$), constants $\dot{a}_{i,j}$, $\dot{b}_i$, for $i \in Q$ and $j \in \omega$, a constant $\dot{c}$ and a binary predicate $\dot{\leqslant}$. Let $\mathcal{L}$ further contain constants $\underline{x}$ for each $x \in \mathfrak{M}$ and a unary predicate $\dot{g}$, which will be used to make $\mathfrak{M}$ an $\mathfrak{N}$-definable class.

A condition p is a finite partial function from $\omega \times Q \times \omega$ into 2. A forcing relation $\Vdash$ is defined as usual containing the following key-clauses (here let t be any term of $\mathcal{L}$):

$$p \Vdash t \in \dot{a}_{i,j} \Leftrightarrow (\exists n \in \omega)(p \Vdash \underline{n} \simeq t \ \&\, p(\langle n,i,j\rangle) = 1);$$

$$p \Vdash t \in \dot{b}_i \Leftrightarrow (\exists j \in \omega)(p \Vdash t \simeq \dot{a}_{i,j});$$

$$p \Vdash t \in \dot{c} \Leftrightarrow (\exists i \in Q)(p \Vdash t \simeq \dot{b}_i);$$

$$p \Vdash t_1 \dot{\leqslant} t_2 \Leftrightarrow (\exists i_1, i_2 \in Q)(i_1 \leqslant i_2 \ \&\ p \Vdash t_1 \simeq \dot{b}_{i_1} \ \&\ p \Vdash t_2 \simeq \dot{b}_{i_2}).$$

Obtain a complete sequence $\mathcal{R}$ of conditions and thereby a valuation val of the constant terms of $\mathcal{L}$, which gives the model $\mathfrak{N}$. It holds that $a_{i,j} = val(\dot{a}_{i,j}) \subseteq \omega$, $b_i = val(\dot{b}_i) = \{a_{i,j};\ j \in \omega\}$, $c = val(\dot{c}) = \{b_i;\ i \in Q\}$ and $val(E^{\omega+3}x(\bigvee_{y_1}\bigvee_{y_2}(x = \langle y_1,y_2\rangle \wedge y_1 \dot{\leqslant} y_2))) = \trianglelefteq$ is a linear ordering of c. Notice, that inside of $\mathfrak{N}$ there is no isomorphism between $\langle Q,\leqslant\rangle$ and $\langle c,\trianglelefteq\rangle$, but outside of $\mathfrak{N}$ both are isomorphic.

<u>Symmetries</u>. Let $\mathfrak{G}$ be the group in $\mathfrak{M}$ of all orderpreserving one-to-one mappings from Q onto Q and let $\mathcal{H}$ be in $\mathfrak{M}$ the group of all permutations of ω, which move only finitely many elements of ω.

(1) $\mathfrak{G}$ operates on $\mathcal{L}$ by setting: if $\pi \in \mathfrak{G}$, then $\pi(\dot{a}_{i,j}) = \dot{a}_{\pi(i),j}$, $\pi(\dot{b}_i) = \dot{b}_{\pi(i)}$, π acts as the identity on all other symbols of $\mathcal{L}$.

(2) $\mathcal{H}$ operates on $\mathcal{L}$ by setting: if $\langle\tau,i\rangle \in \mathcal{H} \times Q$, then define $\tau_i = \langle\tau,i\rangle$ and then: $\tau_i(\dot{a}_{i,j}) = \dot{a}_{i,\tau(j)}$, $\tau_i(\dot{a}_{k,j}) = \dot{a}_{k,j}$ if $k \neq i$, τ_i acts identical on all other symbols of $\mathcal{L}$.

(3) If p is a condition, $\pi \in \mathfrak{G}$, $\tau_i = \langle\tau,i\rangle \in \mathcal{H} \times Q$, then define:
$\pi(p) = \{\langle\langle n,\pi(i),j\rangle,e\rangle;\ \langle\langle n,i,j\rangle,e\rangle \in p\}$
$\tau_i(p) = \{\langle\langle n,i,\tau(j)\rangle,e\rangle;\langle\langle n,i,j\rangle,e\rangle \in p\} \cup \{\langle\langle n,k,j\rangle,e\rangle \in p;\ k \neq i\}$.

We have two symmetry- lemmata:

<u>First symmetry-lemma</u>: If $\pi \in \mathfrak{G}$, then $p \Vdash \Phi \leftrightarrow \pi(p) \Vdash \pi(\Phi)$.

Second symmetry-lemma: If $\tau_i \in \mathcal{H} \times Q$, then $p \Vdash \Phi \Leftrightarrow \tau_i(p) \Vdash \tau_i(\Phi)$.

Restriction lemma: If $p \Vdash \Phi$ and $p_0 = \{\langle\langle m,i,j\rangle ,e \rangle \in p;\ \dot{a}_{i,j}$ occurs in $\Phi\}$, then $p_0 \Vdash^* \Phi$.

For a $\mathcal{L}$-formula Φ define $occ_1(\Phi)$ to be the set of ordered pairs $\langle i,j \rangle \in Q \times \omega$ such that $\dot{a}_{i,j}$ occurs in Φ, and let $occ_2(\Phi)$ be the set of rationals i such that $\dot{b}_i$ occurs in Φ. For a limited comprehension term $t = E^{\alpha} \times \Phi$ write $t = t(\Delta_1, \Delta_2)$ in order to indicate that $\Delta_1 = occ_1(\Phi)$ and $\Delta_2 = occ_2(\Phi)$. Further, let $pr_1(\Delta_1) = \{i \in Q; \bigvee_{j\in\omega} \langle u,j\rangle \in \Delta_1\}$ be the projection of Δ_1 to the first coordinate . The following support-lemma is a generalization of the corresponding support-lemmata of Mostowski [64] and Mathias [58].

Support lemma. Let $t(\Delta_1, \Delta_2)$ and $t^*(\Delta_1^*, \Delta_2^*)$ be limited comprehension terms of $\mathcal{L}$ such that

$$\mathfrak{N} \models t(\Delta_1, \Delta_2) = t^*(\Delta_1^*, \Delta_2^*);$$

then there exists a limited comprehension term $t^0 = t^0(\Delta_1^0, \Delta_2^0)$ such that $\Delta_1^0 = \Delta_1 \cap \Delta_1^*$ and $\Delta_2^0 = (\Delta_2 \cap \Delta_2^*) \cup pr_1(\Delta_1^0)$, and $\mathfrak{N} \models t(\Delta_1, \Delta_2) = t^0(\Delta_1^0, \Delta_2^0)$.

The proof is along the lines of the support lemma proved in section E, but in the present case slightly more complicated. Now, it can be verified, that the ordering principle (OP) holds in $\mathfrak{N}$: define $V_{\langle \Delta_1, \Delta_2 \rangle} = \{x;\ supp_1^*(x) = \Delta_1 \wedge supp_2(x) = \Delta_2\}$, where $supp_1^*(u)$, and $supp_2^*(u)$ are the finite sets such that $supp_1^*(u)$ is of minimal cardinality, $supp_2^*(u)$ is of minimal cardinality modulo $supp_1^*(u)$, such that there is a term $t = E^{\alpha}x\Phi(x)$ with $occ_1(\Phi) = supp_1^*(u)$, $occ_2(\Phi) = supp_2^*(u)$ and t is interpreted by u. Each class $V_{\langle \Delta_1, \Delta_2 \rangle}$ has a definable wellordering. This, together with a linear ordering of the ordered pairs $\langle \Delta_1, \Delta_2 \rangle$ can be used to obtain total orderings for any set of $\mathfrak{N}$ (see the corresponding proof in section E).

In a next lemma one shows that in $\mathfrak{N}$ every subset of $c = val(\dot{c})$ is a finite union of open, closed or at one side open, at the other side closed intervals from $\langle c, \trianglelefteq \rangle$ (use the 2nd-symmetry-lemma and the restriction lemma to see this). It follows, that in $\mathfrak{N}$ every subset of c is definable by a formula $\Phi(x)$ of $\mathcal{L}$ in which none of the symbols $\dot{a}_{i,j}$ occurs, since only $\dot{\leqslant}$ and names $\dot{b}_i$ for the endpoints of the intervals are used. Now the argument of Mostowski [67] can be used to show, that (KW-AC) fails in $\mathfrak{N}$ by showing, that there is no function f in $\mathfrak{N}$ selecting from each proper, closed interval of

$\langle c, \trianglelefteq \rangle$ a non-empty, proper subset. The arguments given above show, that in a transscription of Mostowski's continuity argument, by the restriction lemma, the forcing conditions do not pose additional problems. This proves the theorem.

Notice, that in contrast to all our examples of generic extensions, here we have used a somehow different approach. We did not add a Cohen generic copy of η_0 as a subset of $2^{\aleph_0}$ to the groundmodel (this would yield a model, in which (KW-AC) holds), but a copy of η_0 as a subset of the powerset of $2^{\aleph_0}$. Thus we have shown that neither (KW-AC) $\rightarrow$ (AC) nor (OP) $\rightarrow$ (KW-AC) is provable in **ZF**.

G) THE INDEPENDENCE OF THE AXIOM OF CHOICE (AC) FROM THE BOOLEAN PRIME IDEAL THEOREM (BPI)

In section E we considered the Boolean Prime Ideal theorem (BPI) and noticed that in ZF the (BPI) is a consequence of (AC). Here we shall prove, that the converse is not true, namely that (BPI) $\rightarrow$ (AC) is not a theorem of ZF. As we mentioned proviously, this result is due to J.D.Halpern - A.Lévy **[35]**. A short outline of this prove is contained in:

[31] J.D.HALPERN: The Boolean Prime Ideal Theorem; Lecture Notes prepared in connection with the Summer Institute on Axiomatic Set Theory at UCLA, July 10 - August 4, 1967 (informally distributed manuscripts), 7 pages.

Historically Halpern first showed in 1962 in his doctoral dissertation that in the model $\mathfrak{M}$ of Mostowski **[64]** the (BPI) holds, thus proving that in ZF^0 (i.e. ZF without foundation) the (BPI) does not imply (AC). Mostowski had shown in 1939 in **[64]** that in $\mathfrak{M}$ the ordering principle (OP) holds while (AC) is violated in $\mathfrak{M}$. Halpern's result appeared in print:

[32] J.D.HALPERN: The independence of the axiom of choice from the Boolean prime ideal theorem; Fund. Math. 55(1964) p.57-66.

After Cohen's invention of the generic ZF-model's in 1963, Halpern proved in collaboration with A-Lévy and H.Läuchli (via generic models), that also in full ZF the (AC) is independent from the (BPI) - see **[35]** and **[31]**.

According to the tradition of our lecture notes we shall start to explain the ideas behind the construction of a model of set theory in which (BPI) + $\neg$ (AC) holds by discussing Halpern's original proof [32], that in Mostowski's model $\mathfrak{M}$ (BPI) holds. But first of all we need two lemmata.

Lemma 1([32] p.62): Let $\mathbb{B} = \langle B, \sqcup, - \rangle$ be a Boolean algebra, Aut($\mathbb{B}$) be the group of all automorphisms of $\mathbb{B}$ and let H be any subgroup of Aut($\mathbb{B}$). If I is an ideal of $\mathbb{B}$, closed under H, and if $b \in B$ and J is the smallest ideal closed under H which includes I and $\{b\}$ and if $1_B \in J$, then there is a finite subset S of H such that

$$\bigsqcap_{\phi \in S}\left(\phi(-b)\right) \in I.$$

Proof. $\bigsqcap\{\phi(-b);\ \phi \in S\}$ is the greatest lower bound of the set of elements $\phi(-b)$ of B for $\phi \in S$. Let I(b) be the ideal generated by I and $\{b\}$; then $I(b) = \{x \in B; (\exists y \in I)(x \leqslant y \sqcup b)\}$. Close I(b) under automorphisms ϕ of H: $J_0 = \{z \in B;\ (\exists\, y \in I(b))(\exists\, \phi \in H)[z \leqslant \phi(y)]\}$ and let J be the ideal generated by J_0, id est: $J = \{\bigsqcup e;$ e is a finite subset of $J_0\}$. By assumption $1_B \in J$, hence there is a finite subset $e = \{x_1, \ldots, x_n\}$ of J_0 such that $1 = \bigsqcup e$. Since $x_i \in J_0$, there are $y_i \in I$ and $\phi_i \in H$ $(i = 1,2,\ldots,n)$ such that $x_i \leqslant \phi_i(y_i \sqcup b)$. Hence

$$1_B = \bigsqcup_{i=1}^{n} x_i = \bigsqcup_{i=1}^{n}\phi_i(y_i \sqcup b) = \bigsqcup_{i=1}^{n}\phi_i(y_i) \sqcup \bigsqcup_{i=1}^{n}\phi_i(b)$$

and hence by taking the complements:

$$-(\bigsqcup_{i=1}^{n}\phi_i(y_i) \sqcup \bigsqcup_{i=1}^{n}\phi_i(b)) = \bigsqcap_{i=1}^{n}\phi_i(-y_i) \sqcap \bigsqcap_{i=1}^{n}\phi_i(-b) = 0_B \in I$$

Since I is closed under H, it follows that $\bigsqcup\{\phi_i(y_i);\ i = 1,\ldots,n\} \in I$. Since $u \sqcap v = o$ implies $v \leqslant -u$ in a boolean algebra, put $u = \bigsqcap\{\phi_i(-y_i);\ 1 \leqslant i \leqslant n\}$ and $v = \bigsqcap\{\phi_i(-b);\ 1 \leqslant i \leqslant n\}$, then it follows $v \leqslant -u \in I$, hence $v \in I$ and if we define $S = \{\phi_i;\ 1 \leqslant i \leqslant n\}$, then the lemma is proved.

Lemma 2 ([32] p.62). If $\mathbb{B}$ is a Boolean algebra, X a finite subset of B and P is the set of all functions f on X such that $f(x) \in \{x,-x\}$, then

$$\bigsqcup_{f \in P}\left(\bigsqcap_{x \in X} f(x)\right) = 1_B$$

Proof by induction on the cardinality of X.

The (BPI) holds in Mostowski's model $\mathfrak{M}$. Take countably many atoms (urelemants of reflexive sets $x = \{x\}$) ordered of type η_0. Let G be the group of orderpreserving mappings and F be the filter of subgroups generated by the finite support subgroups. Mostowski showed in [64], that in $\mathfrak{M} = \mathfrak{M}[G,F]$, (OP) holds while (AC) fails (see section E of this chapter and chapter III for details and notation). J.D.Halpern extended in [32] this result by showing that in $\mathfrak{M}$ even the (BPI) holds. He proceeds as follows. Let $\mathbb{B}$ be a boolean algebra in $\mathfrak{M}$. Then $H[\mathbb{B}] = \{\phi \in G;\ \phi(\mathbb{B}) = \mathbb{B}\} \in F$ and by the definition of $\mathbf{F}$ there is a finite support subgroup $K[e] = \{\phi \in G;\ \phi$ is identical on $e\}$ (e a finite set of atoms) such that $K[e] \leqslant H[\mathbb{B}]$. Thus $\mathbb{B}$ is e-symmetric and every $\phi \in K[e]$ is an automorphism of $\mathbb{B}$. Halpern shows, that among the e-symmetric ideals of B there is a prime ideal I. In fact consider (outside of $\mathfrak{M}$) the set

$$Z = \{J;\ J \text{ is an ideal of } \mathbb{B} \wedge J \in \mathfrak{M} \wedge 1_B \notin J \wedge K[e] \leqslant H[J]\}.$$

Z is inductively ordered by $\subseteq$ (see the proof of a similar situation on p.65), and has hence by Zorn-Kuratowski's lemma a maximal element, say I_0. We claim, that I_0 is a prime ideal of B (I_0 is e-symmetric). Suppose not, then there exists $b \in B$ such that $b \notin I_0$ and $(-b) \notin I_0$. Let I_1 be the smallest ideal of B which includes I_0 and b and is closed under $K[e]$, and let I_2 be the smallest ideal of B which includes I_0 and $-b$ and is closed under $K[e]$. Since both, I_1 and I_2, are e-symmetric ideals, hence in $\mathfrak{M}$, they cannot satisfy the hypothesis not to include 1_B, since otherwise $I_1 \in Z$ and $I_2 \in Z$, contradicting the maximality of I_0. Hence $1_B \in I_1$ and $1_B \in I_2$ and by lemma 1 there are finite subsets S_1, S_2 of $K[e]$, such that

(1) $$\prod \{\phi(-b);\ \phi \in S_1\} \in I_0$$

and

(2) $$\prod \{\phi(b);\ \phi \in S_2\} \in I_0.$$

Let $r = \bar{\bar{e}}$, then r determines $r + 1$ open intervals K_i $(0 \leqslant i \leqslant r)$ of A (in the ordering of type of the rationals). We want to get (via lemma 2) the contradiction that $1_B \in I_0$. To do so, we need a certain finite subset X of B such that $\prod \{f(x);\ x \in X\} \in I_0$ for all fuctions f on X such that $f(x) \in \{x,-x\}$; then lemma 2 entails that $1_B \in I_0$. Since one wants to derive $\prod \{f(x);\ x \in X\} \in I_0$ by some permutation arguments from (1) and (2), Halpern finds (using a combinatorial theorem of R.Rado) a certain finite subset W of the set A of atoms and takes then as $X = \{x \in B;\ \exists \phi \in K[e]:\ \phi^*(b) = x \wedge \phi(g) - e \subseteq W\}$, where ϕ^* is the unique extension of ϕ to an automorphism of the whole universe (see p,53), and g is the support of $b \in B$. Rado's theorem gives only in dependence of $\bar{\bar{S}}_1$, $\bar{\bar{S}}_2$ and $\overline{\overline{K_i \cap g}}$ $(i \in r + 1)$ a certain finite cardinal number q. The property, that the atoms are totally ordered of type η_0 is essentially used to conclude, that between all points of $\mathbf{W}$ we can embed q points. W has a certain partition property which is used to find an automorphism ψ of B such that either all elements of $\{\phi(b);\ \phi \in S_2\}$ or $\{\phi(-b);\ \phi \in S_1\}$ can be mapped into $\{f(x);\ x \in X\}$. Then $\prod\{f(x);\ x \in X\} \leqslant \psi(\prod\{\phi(-b);\ \phi \in S_1\})$ or $\prod\{f(x);\ x \in X\} \leqslant \psi(\prod\{\phi(b);\ \phi \in S_2\})$ for $\psi \in K[e]$. Since I_0 is closed under $K[e]$, it follows from (1) and (2) that $\psi(\prod\{\phi(-b);\ \phi \in S_1\}) \in I_0$ and $\psi(\prod\{\phi(b);\ \phi \in S_2\}) \in I_0$. Hence, since I_0 is an ideal, $\prod\{f(x);\ x \in X\} \in I_0$ for all f under consideration. As indicated above, this yields $1_B \in I_0$, a contradiction! Hence I_0 is prime. For all details of the proof, the reader must be referred to Halpern's paper [32].

It is possible to carry over these ideas to the construction of a Cohen-generic model $\mathfrak{N}$ of ZF + (BPI) + $\neg$ (AC). This has been done in collaboration by Halpern, Lévy and Läuchli. The construction of the model has been carried out by Halpern-Lévy [35]. In their construction a combinatorial argument was used (different from Rado's theorem), which has been established by Halpern and Läuchli in:

[33] J.D.HALPERN - H.LÄUCHLI: A partition theorem; Transactions Amer. Math. Soc. vol.124(1966)p.360-367.

The model constructed by Halpern-Lévy is a boolean valued model. We shall, however, construct a Cohen-generic extension by means of forcing. Our remark, that in the case of permutation models the atoms have to be linearly ordered of type η_0, suggests that in the case of Cohen generic models $\mathfrak{N}$ the generic reals $\dot{a}_i$ (for $i \in \omega$) have to form a dense subset of the real line of $\mathfrak{N}$. Following this idea we construct $\mathfrak{N}$ by adding to some countable standard model $\mathfrak{M}$ infinitely many Cohen-generic reals $\dot{a}_i$ $(i \in \omega)$ and generically a set A which just collects these reals $\dot{a}_i$. We shall use the notation $\mathfrak{N} \triangleq \mathfrak{M}[a_0, a_1, \ldots, A]$. This model $\mathfrak{N}$ has been described in section C of this chapter on pages 101 - 103.

<u>Theorem</u> (Halpern-Lévy): $\mathfrak{M}[a_0, a_1, \ldots, A]$ is a model of ZF + (BPI) + $\neg$ (AC). Hence, if ZF is consistent, then (BPI) $\rightarrow$ (AC) is not provable in ZF.

<u>Proof</u>. A detailed presentation of $\mathfrak{M}[a_0, a_1, \ldots, A]$ has been given in section C, where it was shown, that in this model A is an infinite, but Dedekind-finite set. Hence (AC) does not hold in hold in this

model. The proof, that (BPI) holds will require several lemmata.

First we remind our reader to the following (see p.99 and 102):

Symmetry-lemma: Let G be in $\mathfrak{M}$ the group of all permutations of ω. for any $\sigma \in G$, any condition $p: \omega \times \omega \nrightarrow 2$ and any $\mathcal{L}$-sentence Φ, $p \Vdash \Phi \Leftrightarrow \sigma(p) \Vdash \sigma(\Phi)$.

Restriction Lemma. Let p be a condition, Φ a sentence of $\mathcal{L}$, occ(Φ) the finite set of natural numbers i such that $\dot{a}_i$ occurs in Φ, and let $p_0 = p/_{occ(\Phi)}$ be $\{\langle\langle n,i\rangle, e\rangle \in p;\ i \in occ(\Phi)\}$. If $p \Vdash \Phi$, then $p_0 \Vdash^* \Phi$.

Proof. Suppose $p \Vdash \Phi$ and $\sim p_0 \Vdash^* \Phi$. Then there is a condition $q \supseteq p_0$, such that $q \Vdash \neg \Phi$. Define $c_0 = occ(\Phi)$,

$$d_1 = \{i \in \omega; \bigvee_{n\in\omega} \bigvee_{e\in 2} (\langle\langle n,i\rangle, e\rangle \in p \wedge i \notin c_0)\}$$

$$d_2 = \{i \in \omega; \bigvee_{n\in\omega} \bigvee_{e\in 2} (\langle\langle n,i\rangle, e\rangle \in q \wedge i \notin c_0)\}$$

Now let σ be any permutation of ω which leaves c_0 pointwise fixed, and maps d_1 in $\omega-(c_0 \cup d_2)$. Then $Dom(q) \cap Dom(\sigma(p)) \subseteq \omega \times c_0$. Since both, q and $\sigma(p)$, extend p_0, they coincide on the common part of their domain. Hence $\sigma(p) \cup q$ is a function, hence also a condition. Since σ leaves c_0 pointwise fixed: $\sigma(\Phi) = \Phi$. Thus the symmetry lemma tells us, that $\sigma(p) \Vdash \Phi$. Together with $q \Vdash \neg \Phi$, the extension lemma yields $\sigma(p) \cup q \Vdash \Phi \wedge \neg \Phi$, a contradiction. Hence $p_0 \Vdash^* \Phi$ must be true.

The next lemma will say, that $A \triangleq \{a_i; i \in \omega\}$ is a dense subset of 2^ω in the product topology of 2^ω. The following relation < is the usual linear ordering of 2^ω:

$$s_1 < s_2 \Leftrightarrow Min((s_1 - s_2) \cup (s_2 - s_1)) \in s_2$$

Since 2^ω is considered as the product of $\aleph_0$ copies of the two-point discrete space $2 = \{0,1\}$, we may endow 2^ω with the product topology, i.e. the basic open sets are of the form

$$b_r = \{f \in 2^\omega; f \supseteq r\}$$

where r is a finite partial function from ω into 2. Hence b_r is the set of functions (=real numbers) from ω into 2 which extend r. The space 2^ω endowed with this topology is called the Cantor-space, since 2^ω is homeomorphic to the Cantor-discontinuum (considered as a subspace of the real line with the usual interval-topology) - see e.g. Ph.Dwinger: Introduction to Boolean Algebras, Physica-Verlag Würzburg 1961, p.49-50, or R.Sikorski: Boolean algebras, Springer Verlag Berlin 1964, p.43, and textbooks on Topology, e.g.Kelley. The space 2^ω with the topology as given above is a totally disconnected compact Hausdorff-space and the basic open sets b_r are also closed, and hence regular open sets.

Lemma 3. Every basic open set b_r of 2^ω contains a generic real a_i, or better stated: $\mathfrak{N} \models$ "$b_r \cap A \neq \emptyset$ for every finite partial function $r : \omega \to 2$".
Hence it holds in $\mathfrak{N}$ that A is a dense subset of 2^ω.

Proof. suppose $b_r \cap A \neq \emptyset$ does not hold in $\mathfrak{N}$ for every basic open set b_r. Hence, there is a finite partial function $r : \omega \to 2$ in such that $b_r \cap A = \emptyset$ holds in $\mathfrak{N}$. Since everything which holds in $\mathfrak{N}$ is forced by some condition in the complete sequence $\mathcal{R}$ (which defined $\mathfrak{N}$), there is $p \in \mathcal{R}$ such that

$$p \Vdash \underline{b_r} \cap A = \emptyset$$

Since p is finite, there is a natural number i_0 such that for every $n \in \omega$, $\langle n,i_0\rangle \notin Dom(p)$. Define the following extension q of p:

$$q = p \cup \{\langle\langle m,i_0\rangle,1\rangle ; m \in \omega \wedge r(m) = 1\} \cup \{\langle\langle m,i_0\rangle,0\rangle ; m \in \omega \wedge r(m)=0\}.$$

Identify subsets a of ω with their corresponding characteristic function

$$\chi_a(m) = \begin{cases} 0 \text{ if } m \notin a \\ 1 \text{ if } m \in a \end{cases}$$

then $q \Vdash^* \underline{r} \subseteq \dot{a}_{i_0}$ and hence $q \Vdash^* \dot{a}_{i_0} \in \underline{b_r}$. It follows, that in the model $\mathfrak{N}_0$ defined by any complete sequence $\mathcal{R}_0$ which has p as first and q as second element, it holds that $b_r \cap A \neq \emptyset$. This is in contradiction to the assumption, that p (and hence $\mathcal{R}_0$) forces $b_r \cap A = \emptyset$. Thus, every basic open set (i.e. every absolute interval) of 2^ω contains an element of A and A is in $\mathfrak{N}$ a dense subset of 2^ω, q.e.d.

Continuity-Lemma. Let $\Phi(x_1,\ldots,x_n)$ be an $\mathcal{L}$-formula with no free variables other than $x_1,\ldots,x_n$ and suppose that Φ contains none of the symbols $\dot{a}_i$ (for $i \in \omega$), but Φ may contain constants $\underline{x}$ for $x \in \mathfrak{M}$ or the constant $\dot{A}$. Let $g = \langle g_1,\ldots,g_n\rangle$ be a sequence of different members of A. If $\Phi(g_1,\ldots,g_n)$ holds in $\mathfrak{N}$, then there exists a sequence $\langle b_{r_1},\ldots,b_{r_n}\rangle$ of pairwise disjoint basic open sets of 2^ω, such that $g_\nu \in b_{r_\nu}$ (for $1 \leq \nu \leq n$) and the following holds: if $h = \langle h_1,\ldots,h_n\rangle$ is any sequence of different members of A such that $h_\nu \in b_{r_\nu}$ for $1 \leq \nu \leq n$, then

$$\mathfrak{N} \models \Phi(h_1,\ldots,h_n).$$

Proof. Suppose that for sets $g_1,\dots,g_n \in A$ the sentence $\Phi(g_1,\dots,g_n)$ holds in $\mathfrak{N}$. Let $t_1,\dots,t_n$ be constant terms of $\mathcal{L}$, such that $g_\nu = \mathrm{val}(t_\nu)$ for $1 \leqslant \nu \leqslant n$. Consider the following $\mathcal{L}$-sentence:

$$(0)\quad \begin{cases} (\Phi(t_1,\dots,t_n) \wedge t_1 \varepsilon \dot{A} \wedge \dots \wedge t_n \varepsilon \dot{A}) \rightarrow \\ \bigvee_{r_1} \cdots \bigvee_{r_n} [\bigwedge_{\nu=1}^{n} (r_\nu \varepsilon \bigcup_{k=0}^{\omega} 2^k) \wedge \text{the basic open intervals} \\ b_{r_\nu} \text{ (for } 1 \leqslant \nu \leqslant n) \text{ are pairwise disjoint} \wedge \bigwedge_{\nu=1}^{n} t_\nu \varepsilon b_r \wedge \\ \bigwedge_{x_1} \cdots \bigwedge_{x_n} (\bigwedge_{\nu=1}^{n} (x_\nu \varepsilon b_{\underline{r_\nu}} \wedge x_\nu \varepsilon \dot{A}) \rightarrow \Phi(x_1,\dots,x_n))]. \end{cases}$$

The continuity lemma is proved as soon as we have verified that (0) holds in $\mathfrak{N}$. So suppose that the sentence (0) does not hold in $\mathfrak{N}$. Then the negation of (0) holds in $\mathfrak{N}$, and since everything which holds in $\mathfrak{N}$ is forced by some condition of the complete sequence $\mathcal{R}$ (which defines $\mathfrak{N}$), there exists $p \in \mathcal{R}$, such that $p \Vdash^* \neg (0)$. The statement (0) has the form $\Psi_1 \rightarrow \Psi_2$. Hence $p \Vdash^* \neg (0)$ is equivalent to $p \Vdash^* \Psi_1 \wedge \neg\Psi_2$. Thus $p \Vdash^* \Psi_1$ and $p \Vdash^* \neg \Psi_2$. We shall obtain a contradiction by showing that there exists an extension p_0 of p such that $p_0 \Vdash^* \Psi_2$. First, $p \Vdash^* \Psi_1$ is:

$$(*)\qquad p \Vdash^* \Phi(t_1,\dots,t_n) \wedge \bigwedge_{\nu=1}^{n} t_\nu \varepsilon \dot{A}$$

It follows from the forcing definition, that there are $i_1,\dots,i_n \in \omega$ such that $p' \Vdash t_1 = \dot{a}_{i_1} \wedge \dots \wedge t_n = \dot{a}_{i_n}$ for some extension p' of p. Hence:

$$(**)\quad p' \Vdash^* \Phi(\dot{a}_{i_1},\dots,\dot{a}_{i_n}).$$

Extend p' to a further condition p'' such that

(1) If $\langle m,i\rangle \in \mathrm{Dom}(p'')$ and $m' < m$, then $\langle m',i\rangle \in \mathrm{Dom}(p'')$,

(2) If $1 \leqslant \nu \leqslant n$, then there is $m \in \omega$ such that $\langle m,i_\nu\rangle \in \mathrm{Dom}(p'')$,

(3) If $j_1 \neq j_2$ then there is $m \in \omega$ such that $\langle m,j_1\rangle \in \mathrm{Dom}(p'')$, $\langle m,j_2\rangle \in \mathrm{Dom}(p'')$ and $p''(\langle m,j_1\rangle) \neq p''(\langle m,j_2\rangle)$,

(4) p'' extends p'.

It is possible to find conditions p'' satisfying (1), (2), (3) and (4). We shall not explicitely describe such a condition, but assume that we have obtained such a p''. Define interval designators (i.e. functions from finite proper initial segments of ω into 2) for $1 \leqslant \nu \leqslant n$, called r_ν, by

$$r_\nu(m) = p''(\langle m,i_\nu\rangle)$$

for $m \in \{m';\ \langle m',i_\nu\rangle \in \mathrm{Dom}(p'')\}$. By condition (1), r_ν is a function defined on some initial segment of ω. By condition (2), r_ν is for no ν with $1 \leqslant \nu \leqslant n$ the empty function and by (3) the basic open sets $b_{r_\nu} = \{f \in 2^\omega;\ f \supseteq r_\nu\}$ are pairwise disjoint (in the sense of the meta-language). Hence this is weakly forced by every condition, thus:

(α) $\emptyset \Vdash^* \bigwedge\!\!\!\bigwedge_{\nu=1}^{n} (\mathrm{Fnc}(r_\nu) \wedge r_\nu \neq \emptyset) \wedge \bigwedge_\nu \bigwedge_\mu (1 \leq \nu, \mu \leq n \wedge \nu \neq \mu \rightarrow b_{r_\nu} \cap b_{r_\mu} = \emptyset)$.

Since $\mathrm{occ}(\Phi(\dot{a}_{i_1},\ldots,\dot{a}_{i_n})) = \{i_1,\ldots,i_n\}$ by our assumption on Φ, the restriction lemma, applied to (**) with p' replaced by p", yields:

(β) $p_1 \Vdash^* \Phi(\dot{a}_{i_1},\ldots,\dot{a}_{i_n})$

where p_1 is the restriction of p" to $\{i_1,\ldots,i_n\}$, id est $p_1 = \{\langle\langle m,i\rangle, e\rangle \in p";\ i \in \{i_1,\ldots,i_n\}\}$. Now we claim that the following holds:

(γ) $p" \Vdash^* \bigwedge_{x_1} \ldots \bigwedge_{x_n} [\bigwedge\!\!\!\bigwedge_{\nu=1}^{n} (x_\nu \varepsilon \dot{A} \wedge x_\nu \varepsilon b_{\underline{r_\nu}}) \rightarrow \Phi(x_1,\ldots,x_n)]$.

We have to prove that if $t'_1,\ldots,t'_n$ are constant term of $\mathcal{L}$ and q is any extension of p" such that $q \Vdash^* \bigwedge\!\!\!\bigwedge_{\nu=1}^{n} (t'_\nu \varepsilon \dot{A} \wedge t'_\nu \varepsilon b_{\underline{r_\nu}})$, then there exists a further extension q' of q such that $q' \Vdash^* \Phi(t'_1,\ldots,t'_n)$.

Hence suppose terms $t'_1,\ldots,t'_n$ and a condition q are given, where q extends p" and q (weakly) forces the conjunction of the statements $t'_\nu \varepsilon \dot{A} \wedge t'_\nu \varepsilon b_{\underline{r_\nu}}$. Then there exists an extension q' of q and numbers $j_1,\ldots,j_n \in \omega$, such that

$$q \subseteq q' \Vdash^* t'_1 = \dot{a}_{j_1} \wedge \ldots \wedge t'_n = \dot{a}_{j_n}.$$

Hence $q' \Vdash \dot{a}_{j_1} \varepsilon b_{r_1} \wedge \ldots \wedge \dot{a}_{j_n} \varepsilon b_{r_n}$. Thus, if we identify subsets of ω with their corresponding characteristic functions : $q' \Vdash^* \underline{r_1} \subseteq \dot{a}_{j_1} \wedge \ldots \wedge \underline{r_n} \subseteq \dot{a}_{j_n}$. This implies $q_1 \subseteq q'$, where

$$q_1 = \{\langle\langle m,j_\nu\rangle, e\rangle;\ 1 \leq \nu \leq n \wedge r_\nu(m) = e\}.$$

Define the following permutation σ of ω: $\sigma(i_\nu) = j_\nu$ for $1 \leq \nu \leq n$, $\sigma(j_\nu) = i_\nu$, $\sigma(i) = i$ for all other natural numbers i. It follows that

$$q_1 = \sigma(p_1)$$

Hence $p_1 \subseteq p" \subseteq q \subseteq q'$ and $\sigma(p_1) = q_1 \subseteq q'$, and it follows that $p_1 \cup \sigma(p_1)$ is contained in the condition q'. Thus, $p_1 \cup \sigma(p_1)$ is a condition. Applying σ to (β) we get by the symmetry-lemma

$$\sigma(p_1) = q_1 \Vdash^* \Phi(\dot{a}_{j_1},\ldots,\dot{a}_{j_n})$$

Hence, by the extension lemma $q' \Vdash^* \Phi(\dot{a}_{j_1},\ldots,\dot{a}_{j_n})$. Now, since q' forces $t'_1 = \dot{a}_{j_1} \wedge \ldots \wedge t'_n = \dot{a}_{j_n}$ we infer that q' also (weakly) forces $\Phi(t'_1,\ldots,t'_n)$. This proves (γ).

Both, (α) and (γ), show that p" (weakly) forces that there are non-empty, pairwise disjoint absolute intervals b_{r_ν} such that

$\dot{a}_{i_\nu} \in b_{r_\nu}$, and if $\Phi(\dot{a}_{i_1},\ldots,a_{i_n})$, then $\Phi(t'_1\ldots t'_n)$ for every sequence $\langle t'_1,\ldots,t'_n\rangle$ with $t'_\nu \in \dot{A} \wedge t_\nu \in b_{r_\nu}$ $(1 \leq \nu \leq n)$. Hence $p'' \Vdash^* \Psi_2$. Since $p \subseteq p' \subseteq p''$, we have obtained the desired contradiction. This proves the continuity-lemma.

Lemma 4. Let r be a function from some finite ordinal into 2. Then $b_r \cap A$ is infinite. Further, A is Dedekind-finite (though infinite in the usual sense).

Proof. By lemma 3 every absolut interval $b_r = \{f \in 2^\omega;\ f \supseteq r\} = \{f \in 2^\omega;\ r_0 \leq f \leq r_1\}$ contains a generic real, where $r_0 = \{\langle m,e\rangle;(m \in Dom(r) \wedge e = r(m)) \vee (m \notin Dom(r) \wedge e = 0)\}$ and $r_1 = \{\langle m,e\rangle;\ (m \in Dom(r) \wedge e = r(m)) \vee (m \notin Dom(r) \wedge e = 1)\}$.
Since every absolut interval includes countably many pairwise disjoint absolute intervals, it follows that $b_r \cap A$ is infinite. Hence A is infinite. That A is Dedekind-finite was proved in section C, page 102. Notice, that the Dedekind-finiteness also follows from the continuity-lemma (see e.g. [35], th.10).

Remark. Since we want to be able to prove that every class V_d of sets u of $\mathcal{N}$ such that there exists a constant term $E^\alpha x\ \Phi(x)$ with $u = val(E^\alpha x\ \Phi(x))$ and $occ(\Phi) \subseteq d$, where d is a finite subset of ω, has an $\mathcal{N}$-definable wellordering, we proceed as in the proof of Mathias' theorem (see p.120-121) and define in $\mathcal{N}$ a ramified language $\mathcal{L}^*$. The alphabet of $\mathcal{L}^*$ contains besides the usual ZF-symbols and the useful limited existential quantifiers $\bigvee^\alpha$ and limited comprehension operators E^α (for ordinals α of $\mathcal{N}$), a name a^* for each $a \in A$, a name A^* for A and names x^* for each $x \in \mathcal{M}$. Define an interpretation Ω^* for the constant terms of $\mathcal{L}^*$ by induction: $\Omega^*(a^*) = a$, $\Omega^*(A^*) = A$, $\Omega^*(x^*) = x$ and then extending to all limited comprehension terms of $\mathcal{L}^*$. Hence Ω^* is $\mathcal{N}$-definable (for more details see the analoguous situation on page 121). In order to obtain $\mathcal{N}$ -definable wellorderings of the $\mathcal{N}$-classes $V_d = \{x \in \mathcal{N};\ supp^*(x) \subseteq d\}$ we need a support-lemma. The proof of the support-lemma will depend on the following generalization of the continuity-lemma.

Lemma 5. Let $\Phi(x_1,\ldots,x_n)$ be any $\mathcal{L}$-formula with no free variables other than $x_1,\ldots,x_n$ and let $c^* = \{a_i;\ \dot{a}_i$ occurs in $\Phi\}$. If $g_1,\ldots,g_n$ is a sequence of different members of $A-c^*$ and if $\Phi(g_1,\ldots,g_n)$ holds in $\mathcal{N}$, then there exists a sequence $b_{r_1},\ldots,b_{r_n}$ of absolute intervals of A, pairwise disjoint and disjoint from c^*, such that $g_\nu \in b_{r_\nu}$ $(1 \leq \nu \leq n)$ and $\Phi(g_1',\ldots,g_n')$ holds for every sequence $g_1',\ldots,g_n'$ of different members of A such that $g_\nu' \in b_{r_\nu}$ for $1 \leq \nu \leq n$.

Proof. Let $\Psi(x_1,\ldots,x_n, y_1,\ldots,y_m)$ be the formula obtained from Φ by replacing each occurrence of $\dot{a}_{i_\nu}$ by the variable y_ν, where different variables are used for different constants (it is assumed, that the variables $y_1,\ldots,y_m$ do not occur in Φ). If we suppose that $\Phi(g_1,\ldots,g_n)$ holds in $\mathfrak{N}$ for different members of $A-c^*$, then $\Psi(g_1,\ldots,g_n, a_{i_1},\ldots,a_{i_m})$ holds in $\mathfrak{N}$ for different members of A. By the continuity lemma there are pairwise disjoint absolute intervals $b_{r_1},\ldots,b_{r_n}$, $b_{r_{n+1}},\ldots,b_{r_{n+m}}$ of A, such that $\Psi(g_1',\ldots,g_{n+m}')$ holds, whenever $g_\nu' \in b_{r_\nu}$ for $1 \leq \nu \leq n+m$. Hence $\Phi(g_1',\ldots,g_n')$ holds whenever $g_\nu' \in b_{r_\nu}$ for $1 \leq \nu \leq n$, since $\Psi(g_1',\ldots,g_n',a_{i_1},\ldots,a_{i_m})$ holds in $\mathfrak{N}$. Since $a_{i_\nu} \in b_{r_{n+\nu}}$ for $1 \leq \nu \leq m$ and all b_{r_ν} $(1 \leq \nu \leq n+m)$ are pairwise disjoint, it follows that the b_{r_ν} for $1 \leq \nu \leq n$ are pairwise disjoint and disjoint from c^*, q.e.d.

Support-lemma. Let $t_1 = E^\alpha x\ \Phi_1(x)$ and $t_2 = E^\alpha x\ \Phi_2(x)$ be limited comprehension terms, and suppose that $\mathfrak{N} \models t_1 = t_2$. There exists an $\mathcal{L}$-formula $\Phi_3(x)$ such that $occ(\Phi_3) = occ(\Phi_1) \cap occ(\Phi_2)$ and and for $t_3 = E^\alpha x\ \Phi_3(x)$ the following holds: $\mathfrak{N} \models t_1 = t_3$.

Proof. Since $p \Vdash^* t_1 = t_2$ for $p \in \mathcal{R}$, we may assume by the restriction lemma, that $\langle m,i\rangle \in Dom(p)$ only if $i \in occ(\Phi_1) \cup occ(\Phi_2)$. Define

$$c = occ(\Phi_1) \cap occ(\Phi_2),\ d_1 = occ(\Phi_1) - c,$$

and assume (without loss of generality), that $d_2 = occ(\Phi_2) - c = \{i_0\}$ has only one element (the general case follows by induction). Split p into subconditions $p_0 = p_0(c) = p/c$, $p_1 = p_1(d_1) = p/d_1$ and $p_2 = p_2(i_0) = p/d_2$. Let $j_0 \in \omega$, $j_0 \notin c \cup d_1 \cup \{i_0\}$ and let τ be a permutation of which leaves $c \cup d_1$ pointwise fixed and maps i_0 onto j_0. Then the symmetry-lemma applied to $p_0(c) \cup p_1(d_1) \cup p_2(i_0) = p$: $p \Vdash^* t_1(c,d_1) = t_2(c,i_0)$ yields

$$p_0(c) \cup p_1(d_1) \cup p_2(j_0) \Vdash^* t_1(c,d_1) = t_2(c,j_0)$$

where $p_2(j_0) = \tau(p_2(i_0))$ and $t_1 = t_1(c,d_1)$, $t_2(c,j_0) = E^\alpha x\ \tau(\Phi(x))$. Hence both relations together, using the restriction lemma entail:

$$p' = p_0(c) \cup p_2(i_0) \cup p_2(j_0) \Vdash^* t_2(c,i_0) = t_2(c,j_0).$$

Obtain an extension p" of p' as in the proof of the continuity lemma, such that p" satisfies conditions (1), (2), (3) and (4) (listed on page 134), where condition (2) reads in the present context as follows:

(2) If $i \in c \cup \{i_0, j_0\}$, then there is $m \in \omega$ such that $\langle m,i\rangle \in Dom(p'')$. Define r_i by: $r_i(m)) : p''(\langle m,i\rangle)$ and $b_{r_i} = \{f \in 2^\omega;\ r_i \subseteq f\}$. Then the b_{r_i} are pairwise disjoint and $a_i \in b_{r_i}$ for $i \in c \cup \{i_0, j_0\}$. In particular $b_{r_{i_0}} \cap b_{r_{j_0}} = \emptyset$ and $b_{r_{i_0}}$ as well as $b_{r_{j_0}}$ are disjoint from $\{a_i;\ i \in c\}$, see lemma 5.

It was shown in the proof of the continuity-lemma, that

(#) $p'' \Vdash^* \bigwedge_y [y \varepsilon \dot{A} \wedge y \varepsilon b_{r_{j_0}} \rightarrow t_2(c,i_0) = t_2(c,y)]$

where $t_2(c,i_0) = t_2(c,x)$ is to be taken as $\Phi(x)$. Now define $\Phi_2^*(y,x)$ to be the $\mathcal{L}$-formula obtained from $\Phi_2(x)$ by replacing $\dot{a}_{i_0}$ by the variable y at all places of occurrence (it is assumed, that y does not occur in $\Phi_2(x)$). Define

$$t_3 = E^\alpha x\ (\bigvee_y [y \varepsilon \dot{A} \wedge y \varepsilon b_{r_{i_0}} \wedge \Phi_2^*(x,y)]).$$

By (#): $p'' \Vdash^* t_2(c,i_0) = t_3$, where $occ(t_3) = c$. Since p" can be chosen to be included in $\bigcup \mathcal{R}$, hence in some $q \in \mathcal{R}$, it follows that $q \Vdash t_1 = t_3$, and the lemma is proved.

Lemma 6. The ordering principle (OP) holds in $\mathfrak{N}$.

Proof. For $u \in \mathfrak{N}$ define $supp^*(u)$ as the finite subset of A of minimal cardinality such that there is a term t^* of $\mathcal{L}^*$ mentioning only names a^* for $a \in d$ with $\Omega^*(t^*) = u$. By the support-lemma, $supp^*(u)$ is always defined. Put

$$V_d^0 = \{u \in \mathfrak{N};\ supp^*(u) = d\}$$

then the $\mathfrak{N}$-class V_d has an $\mathfrak{N}$-definable wellordering. Together with the lexicographic ordering of the set of finite subsets of A one concludes, that in $\mathfrak{N}$ every set has a total ordering (for all details see the corresponding proof in section E, page 120-121).

Notice, that every $\mathfrak{N}$-class $V_d = \{u \in \mathfrak{N};\ supp^*(u) \subseteq d\}$ has also an $\mathfrak{N}$-definable wellordering, namely the one inducec (via Ω^*) by the $\mathfrak{N}$-definable well-ordering of the constant terms t of $\mathcal{L}^*$ with $occ(t) \subseteq d$. Notice further, that the proof of lemma 6 shows that more that (OP) holds in $\mathfrak{N}$, namely the Kinna-Wagner ordering principle (KW-O) holds in $\mathfrak{N}$. Next we want to show that also the Boolean Prime Ideal theorem (BPI) holds in $\mathfrak{N}$. First we need the following:

Lemma 7. Let $\Phi(x_0, x_1, \ldots, x_n)$ be a formula of $\mathcal{L}$ with no free variables other than $x_0, \ldots, x_n$ and suppose that $occ(\Phi) \subseteq d$. If $t_1, \ldots, t_n$ are constant terms of $\mathcal{L}$ such that $\bigvee^1_{x_0} \Phi(x_0, t_1, \ldots, t_n)$ holds in $\mathfrak{N}$ and if $val(t_\nu) \in V_d$ for $1 \leq \nu \leq n$ and some finite subset d of A, then there is a constant term t_0 of $\mathcal{L}$ such that $\Phi(t_0, t_1, \ldots, t_n)$ holds in $\mathfrak{N}$ and $val(t_0) \in V_d$.

Proof. Since $\bigvee^1_{x_0} \Phi(x_0, t_1, \ldots, t_n)$ holds in $\mathfrak{N}$, there is a constant term t' of $\mathcal{L}$ such that $\Phi(t', t_1, \ldots, t_n)$ holds in $\mathfrak{N}$ (since sets of $\mathfrak{N}$ are valuations of terms of $\mathcal{L}$). Suppose $val(t') \notin V_d$. Let $c = \{i_1, \ldots, i_m\}$ be the set of numbers such that $\dot{a}_{i_\nu}$ occurs in t' and $1 \leq \nu \leq m$ and $a_{i_\nu} \notin d$. Hence $c \neq \emptyset$. Let $t' = E^\alpha\ x\ \Psi(x)$ and let $\Psi^*(x, y_1, \ldots, y_m)$ be the formula obtained from $\Psi(x)$ by replacing the constants $\dot{a}_{i_\nu}$ by y_ν for $1 \leq \nu \leq m$, different variables for different constants. Hence $t' = E^\alpha\ x\ \Psi^*(x, \dot{a}_{i_1}, \ldots, \dot{a}_{i_m})$. By lemma 5 we find absolut intervals $b_{r_1}, \ldots, b_{r_m}$ such that $a_{i_\nu} \in b_{r_\nu}$ $(1 \leq \nu \leq m)$ and by lemma 3 generic sets a_{j_ν} such that $a_{j_\nu} \in b_{r_\nu}$ and $a_{j_\nu} \neq a_{i_\nu}$ $(1 \leq \nu \leq m)$ and further the b_{r_ν} are pairwise disjoint and all are disjoint from d. Hence $\mathfrak{N} \models E^\alpha\ x\ \Psi^*(x, a_{i_1}, \ldots, a_{i_m}) = E^\alpha\ x\ \Psi^*(x, a_{j_1}, \ldots, a_{j_m})$ and by the support lemma there is a term t_0 such that $\mathfrak{N} \models t' = t_0$ and t_0 mentions only generic reals in d, hence $val(t_0) \in V_d$, and lemma 7 is proved.

Lemma 8. If $\mathbb{B}$ is a Boolean algebra in $\mathfrak{N}$, then there is a prime ideal J of $\mathbb{B}$ in $\mathfrak{N}$ such that

$$supp^*(J) \subseteq supp^*(B).$$

Hence the (BPI) holds in $\mathfrak{N}$.

Proof. Let $\mathbb{B} = \langle B, \sqcap, - \rangle$ be a boolean algebra in $\mathfrak{N}$, where $\sqcap$ is the meetoperation (i.e. product, or greatest lower bound) and - is the complementation operation ($\sqcup$ is definable by means of $\sqcap$ and -). Let $d = supp^*(\mathbb{B})$, hence $\mathbb{B} \in V_d$. Since the operations: projection to the 1st (2nd, 3rd resp.) coordinate, are single valued it follows from lemma 7, that B, $\sqcap$ and - are sets of V_d. Further 1_B and 0_B (the largest and the smallest, resp.) are in V_d, since they are unique.

Consider the set Z of all proper ideals of $\mathbb{B}$, which are in V_d. Since $\{0_B\} \in Z$, $Z \neq \emptyset$. Since V_d has an $\mathfrak{N}$-definable wellordering and $Z \subseteq V_d$, it follows that Z has maximal elements. Let I be a maximal proper ideal of $\mathbb{B}$ with $I \in V_d$ (id est: $I \in Z$). We want to prove that I is prime

Suppose I is not prime. Then for some $b \in B$, $b \notin I$ and $(-b) \notin I$. Since $b \in \mathfrak{N}$, there is a constant term t_b of $\mathcal{L}$, such that $b = val(t_b)$. We shall derive a contradiction by showing that $1_B \in I$.

It holds that $supp^*(b) - d \neq \emptyset$, since otherwise the ideal generated by I and {b} would be in Z, contradicting the maximality of I.

Hence let t_b a constant term of $\mathcal{L}$ such that $b = val(t_b)$ and, if $t_b = E^{\alpha}\, x\, \Psi(x)$ then $occ(\Psi) = \{i \in \omega;\ \dot{a}_i \text{ occurs in } \Psi\}$ is of minimal cardinality. Write

$$occ(\Psi) - d = \{i \in occ(\Psi);\ i \notin d\} = \{i_1, \ldots, i_k\}$$

The case k = 1 is especially simple, and, as an illuminating example for the proof-procedure, is discussed in detail in Halpern-Lévy [35] and Halpern [31]. We, however, shall start directly with handling the general case, but recommend our reader to look at the discussion of the illuminating example k = 1 in [35] and [31] .

We shall need a combinatorial theorem of Halpern-Läuchli ([33], theorem 2). Before we formulate a particular case of that theorem, we have to introduce some notation.

A <u>tree</u> <u>T</u> = $\langle T, \leq \rangle$ is a partially ordered set such that for each $x \in T$, $\{y \in T;\ y < x\}$ is totally ordered by $\leq$. The cardinality of this set is called the <u>order</u> of x, or the <u>level</u> at which x occurs. A fan (Fächer) is a non-empty tree such that all elements of it have finite order and each level is a finite set. Hence, if ord(x) is the order of x in the tree $\langle T, \leq \rangle$, then

$\langle T, \leq \rangle$ is a fan $\Leftrightarrow \bigwedge_{x \in T}(ord(x) \in \omega \wedge \bigwedge_{n \in \omega}\{x \in T;\ ord(x) = n\}$ is finite).

Define T/n (restriction of T to n):

$$T/n = \{x \in T;\ ord(x) \leq n\}.$$

<u>Definition</u>. Let D_1 and D_2 be subsets of the fan $\langle T, \leq \rangle$.

D_1 dominates $D_2 \Leftrightarrow \bigwedge_{x \in D_2} \bigvee_{y \in D_1} (x \leq y)$.

D_1 supports $D_2 \Leftrightarrow \bigwedge_{x \in D_2} \bigvee_{y \in D_1} (y \leq x)$.

A subset D of the fan $\langle T, \leq \rangle$ is called (m,1)-dense if there is an element x of T, such that $ord(x) = m$ and $\{y \in T;\ ord(y) = m + 1 \wedge x < y\}$ is dominated by D. Let $\langle T_i, \leq_i \rangle$ be fans for $i < k \in \omega$ and let D_i be a (m,1)-dense subset of T_i (with respect to $\leq_i$) for $i < k$. Then the cartesian product $\prod_{i<k} D_i$ is called an (m,1)-matrix.

<u>Theorem</u> (Halpern-Läuchli). Let $\langle T_i, \leq_i \rangle$, for $i < k$, be finitely many fans without maximal elements. There is a positive integer n such that for every 2-partition $Q = \langle Q_1, Q_2 \rangle$ of $\prod_{i<k}(T_i/n)$, id est $\prod_{i<k}(T_i/n) = Q_1 \cup Q_2$, $Q_1 \cap Q_2 = \emptyset$, either Q_1 or Q_2 includes an (m,1)-matrix for some $m < n$.

For a proof see [33]. (Correct in [33], p.364, the following two misprints: in lemma 1 the second quantifier on the left side of $\models_d$ is an existential quantifier, and two (lines) below in 1.1. the third quantifier on the left side of $\models_d$ is a universal quantifier).

Notice that in [33] and [35] the terminology "finitistic tree" is used. We do not like this philosophically sounding word "finitistic" and use the word "fan" which is also used in Intuitionism.

Now we return to the proof of lemma 8. Let t_B, $t_\sqcap$ and t_- be terms of $\mathcal{L}$ such that $\mathrm{val}(t_B) = B$, $\mathrm{val}(t_\sqcap) = \sqcap$ (the meet operation of the boolean algebra B) and $\mathrm{val}(t_-) = -$ (the complement operation of B), such that $\mathrm{occ}(t_B) \subseteq d$, $\mathrm{occ}(t_\sqcap) \subseteq d$ and $\mathrm{occ}(t_-) \subseteq d$. Further, let $t_b = E^\alpha\, x\, \Psi(x)$ be the term obtained above, and let t_I be the $\mathcal{L}$-term with $\mathrm{occ}(t_I) \subseteq d$ such that $\mathrm{val}(t_I) = I$. Define:

$$\&(x_1, \ldots, x_k)$$

to be the $\mathcal{L}$-formula obtained from $(t_b \,\varepsilon\, t_B \wedge \neg\, t_b \,\varepsilon\, t_I) \wedge (t_-(t_b) \,\varepsilon\, t_B \wedge \neg\, t_-(t_b) \,\varepsilon\, t_I)$ by replacing each occurrence of $\dot{a}_{i_\nu}$ in t_b (i.e. $E^\alpha\, x\, \Psi(x)$) by the variable x_ν for $1 \leqslant \nu \leqslant k$. In the formula used above $t_-(t_b)$ denotes the "complement" of t_b, id est $\mathrm{val}(t_-(t_b)) = (-b)$.

Since $\&(\dot{a}_{i_1}, \ldots, \dot{a}_{i_k})$ holds in $\mathfrak{N}$, there exists k absolute intervals $b_{r_1}, \ldots, b_{r_k}$ which are pairwise disjoint and disjoint from d, such that

(1) $$\bigwedge_{\nu=1}^{k} (x_\nu \,\varepsilon\, \dot{A} \wedge x_\nu \,\varepsilon\, b_{r_\nu} \rightarrow \&(x_1, \ldots, x_k))$$

holds in $\mathfrak{N}$ (this follows by lemma 5). We put $S^0_\nu = \{b_{r_\nu}\}$, for $1 \leqslant \nu \leqslant k$, and $S^0 = \langle S^0_1, \ldots, S^0_k\rangle$. We continue and define k-termed sequences of absolute intervals, S^n, for every $n \in \omega$. Simultanously we prove that the sequences S^n have the following properties (S^n_ν denotes the ν-the coordinate of the sequence S^n):

(P1) S^m_ν is a finite set of absolute subintervals of the members of S^{m-1}_ν for $m \geqslant 1$ and $1 \leqslant \nu \leqslant k$. Therefore, by the definition of S^0 and by induction on m, the members of S^m_ν are subintervals of b_{r_ν} for $1 \leqslant \nu \leqslant k$.

(P2) Every member of S^{m-1}_ν has at least two subintervals in S^m_ν for $m \geqslant 1$ and $1 \leqslant \nu \leqslant k$.

(P3) The members of S^m_ν are pairwise disjoint, for $m \geqslant 0$, $1 \leqslant \nu \leqslant k$.

(P4) If $\Sigma \in \Pi\, S^{m-1}$ and G is a finite set of elements of A which contains exactly one member out of each member of $\bigcup \mathrm{Range}(S^m)$, then $\sqcap\{\mathrm{val}(t_b(u_1, \ldots, u_k)); \langle u_1, \ldots, u_k\rangle \,\varepsilon\, \prod_{\nu=1}^{k} (\Sigma_\nu \cap G)\} \in I$, and

$\sqcap\{-\mathrm{val}(t_b(u_1, \ldots, u_k)); \langle u_1, \ldots, u_k\rangle \,\varepsilon\, \prod_{\nu=1}^{k} (\Sigma_\nu \cap G)\} \in I$, where

$t_b(u_1, \ldots, u_k)$ results from t_b by replacing $\dot{a}_{i_\nu}$ by u_ν for $1 \leqslant \nu \leqslant k$.

The only one of (P1),...,(P4) which applies in the case $m = 0$, is (P3); but this holds since the b_{r_ν}'s are pairwise disjoint.

Induction. Let us assume now that for $m \leq n$, S^m is defined and (p1),...,(P4) hold. We shall define S^{n+1} and prove that (P1),... ...,(P4) hold for $m = n+1$. By (P1) the members of S^n_ν are subintervals of b_{r_ν} (for $1 \leq \nu \leq k$). Therefore, if $\Sigma \in \Pi S^n$, then $\Pi\Sigma \subseteq \Pi S^*$, where $S^* = \langle b_{r_1},\dots,b_{r_k}\rangle$. (The symbol Π is used to denote the cartesian product and sequences are understood to be 1-1-functions with domain some element of ω). Hence, if $\Sigma \in \Pi S^n$ then by (1):

$$(2)\quad \Sigma \in \Pi S^n \wedge \langle u_1,\dots,u_k\rangle \in \Pi(\Sigma_\nu \cap A) \rightarrow \quad \&(u_1,\dots,u_k)$$

Consider the ideal J generated by $I \cup \{val(t_b(u_1,\dots,u_k));$ $\langle u_1,\dots,u_k\rangle \in \Pi(\Sigma_\nu \cap A)\}$. Since $J \in V_d$ and I is a maximal proper ideal in V_d and $I \subseteq J$ we must have $1_B \in J$. Hence, by lemma 1, there is a finite subset $G_1(\Sigma)$ of $\Pi(\Sigma_\nu \cap A)$ such that $\prod\{-val(t_b(u_1,\dots,u_k));$ $\langle u_1,\dots,u_k\rangle \in G_1(\Sigma)\} \in I$. By (2) $G_1(\Sigma)$ has at least two members. By considering, in the same way, the ideal generated by $I \cup \{-val(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \Pi(\Sigma_\nu \cap A)\}$ one obtains a finite subset $G_2(\Sigma)$ of $\Pi(\Sigma_\nu \cap A)$ which has at leat two members such that $\prod\{val(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in G_2(\Sigma)\} \in I$.

Define $G'_\mu(\Sigma)$ to be the set of those elements which occur on some place in an k-tuple of $G_\mu(\Sigma)$, $\mu = 1,2$. $G'_\mu(\Sigma)$ is a finite subset of A and it has at least two members in common with each Σ_ν for $1 \leq \nu \leq k$. Put

$$G^* = \bigcup\{G'_1(\Sigma) \cup G'_2(\Sigma);\ \Sigma \in \Pi S^n\}.$$

Then G^* is a finite subset of A which has at least two members in common with each member of $\bigcup \mathrm{Range}(S^n)$. Since $G_\mu(\Sigma) \subseteq \prod_{\nu=1}^{k}(\Sigma_\nu \cap G^*)$ for $1 \leq \mu \leq 2$, we get

$$(3)\quad \Sigma \in \Pi S^n \rightarrow \begin{cases} \prod\{-val(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \prod_{\nu=1}^{k}(\Sigma_\nu \cap G^*)\} \in I \\ \text{and } \prod\{val(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \prod_{\nu=1}^{k}(\Sigma_\nu \cap G^*)\} \in I, \end{cases}$$

since $Y_1 \subseteq Y_2 \subseteq B \rightarrow \prod Y_2 \leq \prod Y_1$, and I is an ideal. By lemma 5 we obtain from (3) that there are absolut intervals $b_1,\dots,b_\lambda$, pairwise disjoint (and disjoint from d) such that, if $G^* = \{a_{j_1},\dots,a_{j_\lambda}\}$, then $a_{j_\nu} \in b_\nu$ for $1 \leq \nu \leq \lambda$ and

$$(4)\quad \Sigma \in \Pi S^n \wedge \bigwedge_{\nu=1}^{\lambda}[(x_\nu \in \dot{A} \wedge x_\nu \in b_\nu) \rightarrow$$
$$\prod\{-val(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \prod_{\nu=1}^{k}(\Sigma_\nu \cap \{x_1,\dots,x_\lambda\})\} \in I$$
$$\wedge \prod\{val(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \prod_{\nu=1}^{k}(\Sigma_\nu \cap \{x_1,\dots,x_\lambda\})\} \in I.$$

We can assume that each b_ν (for $1 \leq \nu \leq \lambda$) is a subinterval of some member of $\bigcup \mathrm{Range}(S^n)$ since this can be attained always by taking appropriate intersections. Moreover it follows from our construction of G^*, that if $s \in \bigcup \mathrm{Range}(S^n)$, then s includes at least two subintervals out of the sequence $\langle b_1,\dots,b_\lambda\rangle$.

Let us define now the k-sequence S^{n+1} by (for $1 \leq \nu \leq k$):

$$S_\nu^{n+1} = \{b_\gamma;\ 1 \leq \gamma \leq \lambda \wedge b_\gamma \text{ is a subset of a member of } S_\nu^n\}.$$

What we just said concerning the sequence $\langle b_1,\dots,b_\lambda\rangle$ shows that S^{n+1} satisfied the requirements (P1), P2), (P3) and by (4) also (P4).

Notice, that we defined the infinite sequence $S^0, S^1, \dots, S^n, \dots$ by induction in $\mathfrak{M}$ where at each step we made arbitrary choices, namely by selecting $G_\mu(\Sigma)$ for $\mu = 1,2$. But at each step we made only finitely many of them and each set S^n is in V_d which has a definable wellordering (in terms of members of d) as we have shown previously. Hence the construction of $S^0, S^1, \dots, S^n \dots$ as given above can be performed inside of $\mathfrak{M}$.

To apply the Halpern-Läuchli theorem we define the following trees.

$$T_\nu = \bigcup_{n\in\omega} S_\nu^n$$

$\leq_\nu$ is the converse of the inclusion relation $\subseteq$.

(for $1 \leq \nu \leq k$). It follows from (P1),...(P4) that the n-th level of $\langle T_\nu, \leq_\nu\rangle$ is exactly S_ν^n and $\langle T_\nu, \leq_\nu\rangle$ is a fan and by (P2) has no tree-tops. Hence all the requirements of the Halpern-Läuchli theorem hold in the present case. Let n be a natural number as in the consequence of that theorem.

Let H be a choice function on the finite set $W = \bigcup\{S_\nu^m;\ m \leq n \wedge 1 \leq \nu \leq k\}$ such that $\mathrm{Range}(H) \subseteq A$. Let y be the k-sequence given by $y_\nu = \{H(s);\ s \in \bigcup\{S_\nu^m;\ m \leq n\}\}$ for $1 \leq \nu \leq k$. We shall show that for every $z \subseteq \Pi y$ either

(5) $\prod\{\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in z\} \in I$,
or $\prod\{-\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in z\} \in I$.

Once (5) is proved then we shall obtain the desired contradiction $1_B \in I$. Let us prove(5)

We define the following 2-partition of $\prod_{\nu=1}^k (T_\nu/n)$:

$$Q_1 = \{g \in \prod_{\nu=1}^k (T_\nu/n);\langle H(g_1),\dots,H(g_k)\rangle \in z\}$$

$$Q_2 = \prod_{\nu=1}^k (T_\nu/n) - Q_1,$$

where g is the sequence $\langle g_1,\dots,g_k\rangle$. By our choice of n the Halpern-Läuchli theorem asserts that there exists a positive integer $m_0 < n$ such that either Q_1 or Q_2 includes an $(m_0,1)$-matrix M. Suppose $M \subseteq Q_1$. By definition there are $(m_0,1)$-dense subsets A_ν of T_ν such that

$$M = \prod_{\nu=1}^k A_\nu.$$

Since all the sets A_ν are $(m_0,1)$-dense, we may choose a k-sequence of intervals $\langle \tau_1,\dots,\tau_k\rangle$ such that $\tau_\nu \in S_\nu^{m_0}$ and for all $s \in S_\nu^{m_0+1}$, if $s \subseteq \tau_\nu$, then $r \subseteq s$ for some $r \in A_\nu$. With this choice of $\langle \tau_1,\dots,\tau_k\rangle$ define a function f on $\bigcup \mathrm{Range}(S^{m_0+1})$ (using lemma 3) by:

$$f(s) = \begin{cases} \text{if for all } \nu \text{ with } 1 \leq \nu \leq k,\ s \nsubseteq \tau_\nu, \text{ then let } f(s) \text{ be an} \\ \text{arbitrary member of } s \cap A. \\ \text{if } (\exists \nu)(1 \leq \nu \leq k \wedge s \subseteq \tau_\nu), \text{ then take an arbitrary member} \\ r \text{ of } A_\nu \text{ for which } r \subseteq s \text{ holds and let } f(s) \text{ be } H(r). \end{cases}$$

Thus for every $s \in \bigcup \mathrm{Range}(S^{m_0+1})$, $f(s) \in s$. Define

$$G = \{f(s);\ s \in \bigcup \mathrm{Range}(S^{m_0+1})\}.$$

By requirement (P4) for m_0+1 we get

(6) $\prod\{\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \prod_{\nu=1}^{k}(\tau_\nu \cap G)\} \in I.$

We shall now prove that $\prod_{\nu=1}^{k}(\tau_\nu \cap G) \subseteq z$. Once this is proved, then (5) obviously follows, since $x_1 \leq x_2 \in I_1 \rightarrow x_1 \in I$ is a property of ideals. Hence let us prove that the cartesian product of $\tau_\nu \cap G$ is included in z, and let $h = \langle h_1,\dots,h_k\rangle \in \prod_{\nu=1}^{k}(\tau_\nu \cap G)$. Since $h_\nu \in \tau_\nu \cap G$, there are $s_\nu \in \bigcup \mathrm{Range}(S^{m_0+1})$ for $1 \leq \nu \leq k$ such that $f(s_\nu) = h_\nu$. Since f is a choice function, $h_\nu \in s_\nu$.
Since also $h_\nu \in \tau_\nu \in S_\nu^{m_0}$, it follows from (P1) and (P3) that $s_\nu \subseteq \tau_\nu$. But in this case $h_\nu = f(s_\nu) = H(r_\nu)$ where $r_\nu \in A_\nu$. Thus $h = \langle H(r\ ),\dots,H(r_k)\rangle$ where $\langle r_1,\dots,r_k\rangle \in \prod_{\nu=1}^{k} A_\nu \subseteq Q_1$. But by definition of Q_1 this implies $h = \langle H(r_1),\dots,H(r_k)\rangle \in z$ and hence $\Pi(\tau_\nu \cap G) \subseteq z$ holds.

To deal with the other case, namely $M \subseteq Q_2$, let us write $z^* = (\Pi y) - z$ and proceed exactly as in the case $M \subseteq Q_1$, replacing z by z^* and Q_1 by Q_2. Where (P4) was used to obtain (6), we use (P4) now to obtain $\prod\{-\mathrm{val}(t_b(u_1,\dots,u_k));\dots\} \in I$ and get $\prod\{-\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in z^*\} \in I$, so that (5) holds.

Thus we have shown that for every $z \subseteq \Pi y$, (5) holds. This will be used to obtain by means of lemma 2 the desired contradiction that 1_B would be in I.

Let P be the set of all functions ϕ defined on

$$X = \{\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in \Pi y\}$$

such that for $x \in X$, $\phi(x) \in \{x,-x\}$. Consider

$$z = \{\langle u_1,\dots,u_k\rangle \in \Pi y;\ \phi(\mathrm{val}(t_b(u_1,\dots,u_k)) = \mathrm{val}(t_b(u_1,\dots,u_k))\}.$$

Then $z \subseteq \Pi y$ and by (5): $\prod\{\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in z\} \in I$ or $\prod\{-\mathrm{val}(t_b(u_1,\dots,u_k));\langle u_1,\dots,u_k\rangle \in z\} \in I$. Since one of these elements is in I, their intersection is in any case in I. Thus $\prod\{\phi(x);\ x \in X\} \in I$ for every ϕ. Since P is finite, the union of these intersections is again in I. It was shown in lemma 2, that this union equals 1_B. Hence $1_B \in I$. The assumption, that I is not prime leads, hence, to a contradiction, id est: I is prime. This proves lemma 8.

This proves the theorem, that $\mathfrak{M}[a_0, a_1, \ldots, A] \models ZF + (BPI) + \neg(AC)$. The axiom of choice (AC) is therefore not provable from (BPI) in ZF.

We may use the model constructed above in order to obtain a further independence result. We consider the following two definitions of continuity:

<u>Definition</u> (L.Cauchy): The function f from reals to reals is continuous at x_0 iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

<u>Definition</u> (Heine - Borel): f is continuous at x_0 iff $\lim_{n\to\infty} f(x_n) = f(x_0)$ for every sequence $\{x_n\}_{n\in\omega}$ convergent to x_0.

In elementary analysis one proves, that both definition are equivalent, but the proof uses the axiom of choice. That the equivalence is no longer true if we drop (AC) has been discovered by Halpern-Lévy in [35] and independently by

[37] M.JAEGERMANN: The axiom of choice and two definitions of Continuity. Bull.Acad.Polon.Sci.vol.13(1965)p.699-704.

<u>Theorem</u> (Halpern-Lévy; Jaegermann): It is not provable in ZF, that every function from reals to reals which is continuous (at x_0) in the sense of Heine-Borel is also continuous (at x_0) in the sense of Cauchy.

Notice, that it is obviously provable in ZF that every Cauchy-continuous function is also Heine-Borel continuous.

<u>Proof</u>. By lemma 4 (see page 136) A is dedekind-finite, while infinite and by lemma 3 (see p.133) A is a dense subset of 2^{ω} (in the sense of the product topology). The function

$$\phi : x \mapsto \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}$$

for $x \in A$, is a one-one mapping of A into the interval $]0,1]$ (right closed, left open). ϕ is one-to-one since no $x \in A$ is finite (finite subsets of ω are all in the groundmodel $\mathfrak{M}$). Since absolut intervals of A are non-empty, the image $\phi(A)$ is a dense subset of $[0.1]$, and

$\phi(A)$ is a dedekind-finite, while infinite, subset of the closed interval $[0,1]$. Define the following function f:

$$f(x) = \begin{cases} 0 \text{ if } x \in \phi(A) \\ 1 \text{ otherwise} \end{cases}$$

for $x \in [0,1]$. Then $f(0) = 1$, since $0 \notin \phi(A)$. Since $\phi(A)$ is dedekind-finite, every sequence $\{x_n\}_{n\in\omega}$ convergent to 0, for $0 < x_n < 1$, can contain at most finitely elements from $\phi(A)$. Hence $\lim(f(x_n)) = 1 = f(0)$ and f is continuous in the sense of Heine-Borel. Since $\phi(A)$ is dense in $[0,1]$, f is obviously not continuous in the sense of Cauchy. q.e.d.

H) THE AXIOM OF DEPENDENT CHOICES

In his paper "Axiomatic and algebraic aspects of two theorems on sums of cardinals" (Fund.Math.35(1948)p.79-104, in particular p.96) A.Tarski considered the following **axiom, which was first formulated by P.Bernays (J.S.L. 7 (1942) p.86):**

(DC) AXIOM OF DEPENDENT CHOICES: Let R be a binary relation on the set x such that $(\forall y \in x)(\exists z \in x)(\langle y,z\rangle \in R)$, then there exists a countable sequence $y_0, y_1, \ldots, y_n, \ldots$ $(n \in \omega)$ of elements of x such that $\langle y_n, y_{n+1}\rangle \in R$ for all $n \in \omega$.

The name "dependent choices" is used, since (DC) asserts that there exists a choice sequence where y_n is chosen in dependence of the choice of y_{n-1}. **Bernays** mentions that (DC) follows from (AC) and that (DC) implies the countable axiom of choice (AC^ω) obviously (see p.100 of these notes for the formulation of (AC^ω)).

Both axioms, (AC^ω) and (DC), are powerful weakened forms of (AC); e.g. in analysis (AC^ω) is sufficient to prove most of the "positive" results such as the first fundamental theorem of Lebesgue-measure. In addition, (DC) is sufficient to prove such results as the Baire category theorem. Further, we mention, that A.Lévy has shown in his paper "A Hierarchy of formulas in set theory" [48], that (DC) is equivalent (in ZF) with some forms of the Löwenheim-Skolem theorem (see [48] p.72-74).

In the formulation of (DC) the choice of y_n is made in dependence of the choice of y_{n-1}. In this formulation, (DC) can not be generalized to yield the existence of sequences of length larger than ω (if certain hypothesis are fullfilled), since e.g. y_ω can not depend on a "predecessor". But we get the idea to let depend y_ω on the set

$\{y_n;\ n \in \omega\}$. More generally we formulate for cardinal numbers α (i.e. finite or an aleph):

(DC^α) Dependent Choices: Let R be a binary relation between subsets and elements of a set x, such that for every $y \subsetneq x$ with $\mathrm{card}(y) < \alpha$ there is an $z \in x$ with $\langle y,z\rangle \in R$, then there is a function $f : \alpha \to x$ such that $\langle f''\beta, f'\beta\rangle \in R$ for every ordinal $\beta < \alpha$.

Here $f'\beta = f(\beta)$ and $f''\beta = \{f(\gamma);\ \gamma < \beta\}$. The formulation of ($DC^\alpha$) is due to A.Lévy:

[52] A.LÉVY: The Interdependence of certain consequences of the axiom of choice; Fund.Math. 54(1964)p.135-157.

Lemma 1: Lévy's axiom (DC^ω) is in ZF equivalent with Bernays' axiom of dependent choices (DC).

Proof. (1) Suppose Bernays' (DC) and let R be a binary relation defined between subsets and elements of a set x such that $(\forall y \subset x)$ $(\mathrm{card}(y) < \omega \to (\exists z \in x)(\langle y,z\rangle \in R)$. Consider $S = \{u;\ u$ is a finite subset of $x\}$ and define the following relation R^* on S:

$$\langle u_1,u_2\rangle \in R^* \Leftrightarrow (\exists z \in x)(u_1 \cup \{z\} = u_2 \wedge \langle u_1,z\rangle \in R).$$

By Bernays' (DC) there exists a sequence $u_0, u_1, \dots, u_n, \dots$ $(n \in \omega)$ of elements of S such that $\langle u_n,u_{n+1}\rangle \in R^*$ for all $n \in \omega$. Define a function $f : \omega \to x$ by: $f(n)$ is the only element of $u_{n+1} - u_n$. then f satisfies Lévy's (DC^ω).

(2) Now suppose (DC^ω) and let R be a relation on x such that for every $y \in x$ there is $z \in x$ such that $\langle y,z\rangle \in R$. Define

$$S_n = \{\langle z_0,\dots,z_n\rangle;\langle z_0,z_1\rangle \in R \wedge \langle z_1,z_2\rangle \in R \wedge \dots \wedge \langle z_{n-1},z_n\rangle \in R\}$$

and $S = \bigcup\{S_n;\ n \in \omega\}$; Define the following relation R^* between subsets T and elements t of S by:

$$\langle T,t\rangle \in R^* \Leftrightarrow (\exists z \in x)(\exists t_0 \in T)(t = t_0 * \langle z\rangle)$$

where $t_0 * \langle z\rangle$ denotes concatenation, i.e. $t_0 * \langle z\rangle = \langle v_0,\dots,v_n,z\rangle$ if $t_0 = \langle v_0,\dots,v_n\rangle$. By ($DC^\omega$) there is a function $f : \omega \to S$ such that for all $n \in \omega$: $\langle f''n, f'n\rangle \in R^*$. Since $f(0) = f'0 = \langle v_0,\dots,v_k\rangle \in S$ define $g : \omega \to x$ by $g(0) = v_0, \dots, g(k) = v_k, g(n) =$ the last coordinate of $f(n-k)$, for $n \geq k+1$. Then g satisfies Bernays' (DC), q.e.d.

Lemma 2 (A.Lévy [52]): $ZF^0 \vdash \bigwedge_\alpha (DC^\alpha) \to (AC)$.

Proof. Let x be any set. Define $W = \{y \subseteq x;\ y$ is wellorderable$\}$ and a binary relation R between subsets of x and elements of x by:

$\langle y,z\rangle \in R \Leftrightarrow (y \in W \wedge z \in x \wedge z \notin y)$.
Let $\alpha = \aleph(x) = \sup\{\lambda;\ \lambda$ is an ordinal and embeddable into $x\}$.
α is a cardinal number. Suppose x is not well-orderable; then R satisfies the hypothesis of (DC^{α}), and hence, by (DC^{α}), there exists a function f from α into x such that $\langle f''\beta, f(\beta)\rangle \in R$ for all ordinals $\beta < \alpha$. By definition of R this implies, that α is embeddable into x. Hence $\alpha \in \aleph(x) = \alpha$, a contradiction, since ordinals are allways well-founded, q.e.d.

Corollary 3. $ZF^{0} \vdash \bigwedge_{\alpha}(DC^{\alpha}) \leftrightarrow (AC)$.

Next we shall discuss another family of weakened forms of the axiom of choice and investigate the interdependences, resp. indepences between them and the family of statements $\{(DC^{\alpha});\ \alpha$ a cardinal$\}$.
Let α be an ordinal: (AC^{α}): if x is a set of non-empty sets, such that $card(x) = \alpha$, then there exists a function f defined on x such then $f(y) \in y$ for all $y \in x$.
A.Lévy has obtained in [52] (among other things) the following results:

Lemma 4: Let α be an aleph, then (in ZF): $(DC^{\alpha}) \rightarrow (AC^{\alpha})$. Let α_1 and α_2 be alephs such that $\alpha_1 < \alpha_2$, then (in ZF) $(DC^{\alpha_2}) \rightarrow (DC^{\alpha_1})$ and $(AC^{\alpha_2}) \rightarrow (AC^{\alpha_1})$.

For a proof see Lévy [52] p.138, p.140 and p.142.

The following nice result was obtained by R.B.Jensen in 1965 (unpublished). We are grateful to A.R.D.Mathias for telling us Jensen's proof.

Theorem (R.B.Jensen): ZF $\vdash \bigwedge_{\alpha}(AC^{\alpha}) \rightarrow (DC^{\omega})$.

Proof. Suppose, this is not true. Then there exists a set X and a binary relation R on X such that $(\forall y \in X)(\exists z \in X)(\langle y,z\rangle \in R)$, but there is no sequence $x_0, x_1, \dots, x_n, \dots$ $(n \in \omega)$, such that $\langle x_n, x_{n+1}\rangle \in R$ for all $n \in \omega$. (Notice, that we consider Bernays' (DC) rather than (DC^{ω})). Consider

$$W = \{Y;\ Y \subseteq X \wedge Y \text{ is wellorderable}\}.$$

Then for all $Y \in W$ we get that $\langle Y, R^{-1} \upharpoonright Y\rangle$ is well-founded and we can introduce a rank-notion ρ_Y on Y by induction on the well-founded relation R^{-1} (restricted to Y). Since $\bigcup W = X$ (since all singletons are in W) we can define for every $z \in X$:

$$\lambda(z) = \sup\{\rho_Y(z);\ Y \in W\}.$$

Step 1: If $z \in X$, then $\omega \leq \lambda(z)$.

Proof. Suppose not; then $\lambda(z) = n < \omega$ for some $z \in X$, and hence $\lambda(z) = \rho_Y(z)$ for some $Y \in W$. Thus there are elements $x_1, \ldots, x_n$ in Y, such that $\langle x_1, x_2\rangle \in R^{-1}$, $\langle x_2, x_3\rangle \in R^{-1}, \ldots, \langle x_{n-1}, x_n\rangle \in R^{-1}$ where $x_n = z$. But for given $x_1 \in X$, there exists $x_0 \in X$ such that $\langle x_1, x_0\rangle \in R$, and hence $\langle x_0, x_1\rangle \in R^{-1}$. Hence for $Y^* = Y \cup \{x_0\}$ we get $\rho_{Y^*}(z) = n+1$, a contradiction.

Step 2: For every $z \in X$ there exists $Y \in W$ such that $\lambda(z) = \rho_Y(z)$.

Proof. For ordinals $\gamma \leq \lambda(z)$ consider the following sets:

$$T_\gamma = \{\langle Y, M\rangle;\ Y \in W \wedge M \text{ wellorders } Y \wedge \rho_Y(z) = \gamma\}$$

and define $K(z) = \{T_\gamma;\ \gamma \leq \lambda(z)\}$. The set $K(z)$ is wellordered of type $\leq \lambda(z) + 1$. Hence by $(AC^{\lambda(z)+1})$ there exists a function f which selects from each T_γ one element. Write $f(T_\gamma) = \langle Y_\gamma, M_\gamma\rangle$, and define $Y^* = \bigcup\{Y_\gamma;\ \gamma \leq \lambda(z)\}$. We claim that Y^* is wellorderable. In fact, put $Y_0 = A_0$, $A_1 = Y_1 - Y_0, \ldots, A_\gamma = Y_\gamma - (\bigcup\{Y_\beta;\ \beta < \gamma\}), \ldots$ for $\gamma \leq \lambda(z)$ and $N_0 = M_0, \ldots, N_\gamma = M_\gamma \cap (A_\gamma \times A_\gamma)$; then N_γ wellorders A_γ, and $Y^* = \bigcup\{A_\gamma;\ \gamma \leq \lambda(z)\}$ and the A_γ are pairwise disjoint. Define

$$M^* = \{\langle a, b\rangle;\ \bigvee_\gamma \bigvee_\delta (a \in A_\gamma \wedge b \in A_\delta \wedge [\gamma < \delta \vee (\gamma = \delta \wedge \langle a, b\rangle \in N_\gamma)])\}.$$

then M^* wellorders Y^*. Hence $Y^* \in W$, and $\rho_{Y^*}(z)$ is defined. It follows that $\rho_{Y^*}(z) = \lambda(z)$, q.e.d.

Step 3. The final step: Take an element $z_0 \in X$ such that $\lambda(z_0)$ is minimal in $\{\lambda(z);\ z \in X\}$. By the Hypothesis on R, there exists $z_1 \in X$ such that $\langle z_0, z_1\rangle \in R$. For any $Y \in W$ such that $z_1 \in Y$ define $Y^+ = Y \cup \{z_0\}$. Then always $\rho_{Y^+}(z_0) = \rho_{Y^+}(z_1) + 1$. By the result of step 2, there exists $Y \in W$, such that $\lambda(z_1) = \rho_Y(z_1)$. Hence for this Y:

$$\lambda(z_1) = \rho_{Y^+}(z_1) < \rho_{Y^+}(z_1) + 1 = \rho_{Y^+}(z_0) \leq \lambda(z_0).$$

A contradiction! $\lambda(z_0)$ would not be minimal! This proves Jensen's theorem.

Independence Results

Mostowski has shown in 1948 by means of models containing urelements that (DC) does not imply (AC) in ZF^0, or even: $ZF^0 \nvdash (DC^\omega) \rightarrow (AC^{\omega_1})$:

[68] A.MOSTOWSKI: On the principle of dependent choices; Fund. Math.35 (1948)p.127-130.

We shall describe briefly Mostowski's model. Consider a set theory in which there are $\aleph_1$ many reflexive sets. Let A be the set, of reflexive sets $x = \{x\}$, of cardinality $\aleph_1$. Define $R_0 = A$, $R_\gamma = \bigcup\{P(R_\beta);\ \beta < \gamma\}$ and $V = \bigcup_\gamma R_\gamma$. Take an enumeration of the set of reflexive sets (called "atoms" in the sequel): $R_0 = A = \{a_\gamma;\gamma < \omega_1\}$. Every ordinal γ can be written (in a unique way) as $\beta + n$, where β is a limit ordinal and $n \in \omega$. Write $\gamma \equiv 0$ iff $\gamma = \beta + n$ with n even, and write $\gamma \equiv 1$ iff $\gamma = \beta + n$ with n odd. A permutation f of R_0 (i.e. one-one-mapping of R_0 onto R_0) is called admissible iff f preserves pairs, id est:

Def. f admissible $\Leftrightarrow \bigwedge_{\gamma<\omega_1}[f(a_\gamma) \neq a_\gamma \rightarrow [(\gamma \equiv 0 \wedge f(a_\gamma) = a_{\gamma+1}) \vee (\gamma \equiv 1 \wedge f(a_\gamma) = a_{\gamma-1})]]$.

Hence admissible permutations on R_0 leave $B = \{\{a_\gamma,a_{\gamma+1}\};\gamma \in \omega_1 \wedge \gamma \equiv 0\}$ pointwise fixed. Let G be the group of all admissible permutations on R_0. Call a subgroup H of G a countable support subgroup if there is a countable subset e of R_0 such that $H = \{f \in G;$ f leaves e pointwise fixed$\}$. Let F be the filter of subgroups of G which has the set of countable support subgroups as filter basis. Define $\mathfrak{M} = \mathfrak{M}[G,F]$ as in chapter III, section B, page 54. Then $\mathfrak{M}$ obviously violates (AC), since e.g. $B \in \mathfrak{M}$, but B has no choice set in $\mathfrak{M}$. Moreover (AC^{ω_1}) is false in $\mathfrak{M}$, since B is wellorderable in $\mathfrak{M}$ of type ω_1. Mostowski shows that (DC), id est (DC^ω), holds in $\mathfrak{M}$. In fact, if R is a binary relation in $\mathfrak{M}$, such that R satisfies the hypothesis of (DC). Choose (outside of $\mathfrak{M}$) a countable sequence $x_0, x_1, \ldots$ such that $\langle x_n,x_{n+1}\rangle \in R$ for all $n \in \omega$. Each x_n has a countable support S_n. Since $\bigcup_{n\in\omega} S_n$ is again countable, the subgroup $H = \{f \in G;$ f leaves $\bigcup S_n$ pointwise fixed$\}$ is in F and fixes the sequence $x_0, x_1, \ldots, x_n, \ldots$ which is hence in $\mathfrak{M}$, q.e.d.

Thus we have proved the following:

<u>Theorem</u> (A.Mostowski [68]): There exists a permutation model of ZF^0 containing atoms in which (DC^ω) holds, but (AC^{ω_1}) is violated. Thus (DC) $\rightarrow$ (AC), or better (DC) $\rightarrow$ (AC^{ω_1}), is not a theorem of ZF^0.

A.Lévy's paper [52] contains further independence results. Since Lévy's paper is "pre-Cohen", as Mostowski's [68], the results apply only to ZF^0 and the method is by construction of Fraenkel-Mostowski-

Specker models.

Lévy asks in [52] p.137 and in [49] p.224, whether $(AC^{\omega}) \rightarrow (DC^{\omega})$ is provable. R.B.Jensen has solved this problem in 1965 (unpublished). We are grateful to F.R.Drake for sending us his abstract of Jensen's proof. Jensen presented his result during the Logic-Colloquium 1965 in Leicester. He first gave a permutation model containing urelements in order to illuminate the basic idea in his independence proof, and then translated the method to the construction of a Cohen generic model. We follow Jensen and present first his permutation model. In the sequel we make, of course, the tacit assumption, that ZF^0 is consistent.

Theorem (R.B.Jensen): There exists a permutation model $\mathfrak{M}$ of ZF^0 containing atoms in which (AC^{ω}) holds, while (DC^{ω}) is not true in $\mathfrak{M}$. Hence $(AC^{\omega}) \rightarrow (DC^{\omega})$ is not provable in ZF^0.

Proof. Take a set theory with choice in which there is a set R_0 of reflexive sets (called atoms), such that R_0 has cardinality $\aleph_1$. We want to define a certain tree-ordering on R_0.

Consider first the well-ordered set ω_1 and consider

$$\omega_1^m = \{\langle x_1, \dots, x_m\rangle ;\ x_1 < x_2 < \dots < x_m \in \omega_1\}$$

and $\underset{\sim}{H} = \bigcup\{\omega_1^m;\ 1 \leqslant m < \omega\}$. There is a natural partial ordering $\ll$ on $\underset{\sim}{H}$ defined by: $s_1 \ll s_2 \Leftrightarrow$ the sequence s_1 is an initial segment of s_2. More precisely one defines first $s_1 \lessdot s_2$ to express that s_1 immediately preceeds s_2:

$$s_1 \lessdot s_2 \Leftrightarrow (s_1 = \langle x_1, \dots, x_m\rangle \wedge s_2 = \langle x_1, \dots, x_m y\rangle)$$

Hence $s_1 \lessdot s_2 \Leftrightarrow (\exists y \in \omega_1)(s_2 = s_1 * \langle y\rangle)$ if $*$ denotes concatenation of sequence. Now define

$s_1 \ll s_2 \Leftrightarrow$ there are finitely many elements of $\underset{\sim}{H}$, say $h_1, \dots, h_n$, such that $s_1 = h_1 \lessdot h_2 \lessdot \dots \lessdot h_n = s_2$.

Since $\underset{\sim}{H}$ has cardinality $\aleph_1$, there exists a one-to-one mapping f from R_0 onto $\underset{\sim}{H}$. Via f one carries the tree-structure $\ll$ over to R_0 by:

$$a < b \Leftrightarrow f(a) \ll f(b).$$

Hence $<$ is a strict partial ordering on R_0 and $\langle R_0, \leqslant\rangle$ is a tree. Notice that $\langle R_0, \leqslant\rangle$ has no tree-tops, that the order of each element of R_0 is finite and that for every $a \in R_0$ the set of immediate successors $\{b \in R_0 ;\ a < b \wedge ord(b) = ord(a) + 1\}$ has cardinality $\aleph_1$.

Let G be the group of all orderpreserving one-to-one mappings from R_0 onto R_0. In order to define an interesting filter F of subgroups we define the notion of a "small subtree". First:

B is a <u>complete branch</u> in $\langle R_0, \leqslant \rangle$ iff $\langle B, \leqslant \rangle$ is totally ordered and if $a < b \in B$, then $a \in B$ and there does not exists an element $d \in R_0$ such that $a < d$ for all $a \in B$.

<u>Definition</u>. If T is a subset of R_0, then $\langle T, \leqslant \rangle$ is a small subtree of $\langle R_0, \leqslant \rangle$ iff T is countable, $\langle T, \leqslant \rangle$ is a subtree (id est: $a < b \in T$ implies $a \in T$) and no branch in $\langle T, \leqslant \rangle$ is a complete branch in $\langle R_0, \leqslant \rangle$.

<u>Definition</u>. A subgroup H of G is called a <u>nice</u> subgroup iff there exists a small subtree $\langle T, \leqslant \rangle$ such that

$$H = \{\pi \in G;\ \pi \text{ leaves } T \text{ pointwise fixed}\} = K[T]$$

Define F to be the set of those subgroups of G which include a nice subgroup. F is a filter of subgroups. Define $H[x] = \{\pi \in G;\ \pi(x) = x\}$ and let $TC(x) = \{x\} \cup x \cup \bigcup x \cup \ldots$ be the transitive closure of x. Define as in chapter III, p.54: $\mathfrak{M} = \mathfrak{M}[G,F] = \{x;\ \bigwedge_y (y \in TC(x) \rightarrow H[y] \in F)\}$. Specker's theorem (see p.54) shows that $\mathfrak{M}$ is a model of ZF^0. We shall show that $(AC^{\omega}) \wedge \neg (DC^{\omega})$ holds in $\mathfrak{M}$.

<u>Lemma</u>: The axiom of dependent choices (DC^{ω}) does not hold in $\mathfrak{M}$.

<u>Proof</u>. Since for $a \in R_0$, a has finite order, and hence $\{b \in R;\ b \leqslant a\}$ is a small subtree it follows that $K[a] = H[a] \in F$ and hence R_0, every element of R_0 and the tree-ordering $\leqslant$ is in $\mathfrak{M}$. Obviously $\leqslant$ is a binary relation satisfying $(\forall x \in R_0)(\exists y \in R_0)(x < y)$. If (DC^{ω}), and hence (DC), would hold in $\mathfrak{M}$, then there would be a countable sequence $S = \{a_0, a_1, \ldots, a_n, \ldots\}$ of elements of R_0 in $\mathfrak{M}$, such that $a_0 < a_1 < \ldots < a_n < \ldots$. If S is in $\mathfrak{M}$, then there is a small subtree T of R_0 such that if $\pi \in K[T] = \{\sigma \in G;\ \sigma$ leaves T pointwise fixed$\}$ (by def.see p.57) entails $\pi(S) = S$, id est $K[T] \leqslant H[S]$. But $S^* = \{b \in R_0;\ (\exists a \in S)(b \leqslant a)\}$ is a complete branch in $\langle R_0, \leqslant \rangle$. Since T is small, $T \cap S^*$ is finite (or even possibly empty). Hence there exists $b_0 \in S$ with $b_0 \notin T$. Define the following permutation τ of R_0: pick any element $c_0 \in R_0 - T$ such that, if b_1 is an immediate predecessor of b_0, then $b_1 < c_0$ and $ord(c_0) = ord(b_1) + 1$ (hence $ord(c_0) = ord(b_0)$ and c_0 and b_0 are in the same set of immediate successors of b_1). Let τ be the identity on $\{x \in R_0;\ \neg (x \geqslant b_0 \vee x \geqslant c_0)\}$ and $\tau(b_0) = c_0$, $\tau(c_0) = b_0$ and τ maps the subset $\{x \in R_0;\ x \geqslant b_0\}$ onto $\{x \in R_0;\ x \geqslant c_0\}$ and reversely. Hence $\tau \in K[T]$ and since $K[T] \leqslant H[S]$, it follows that $\tau(S) = S$.

But $b_0 \in S$ implies $c_0 = \tau(b_0) \in \tau(S) = S$, a contradiction. This proves the lemma.

Lemma. The countable axiom of choice (AC^{ω}) holds in $\mathfrak{M}$.

Proof. Let $z = \{x_i ; i \in \omega\}$ be a countable set in $\mathfrak{M}$ such that $i \neq j \rightarrow x_i \cap x_j = \emptyset$, $x_i \neq \emptyset$ for all $i \in \omega$ and the sequence $\{\langle x_i, i\rangle ; i \in \omega\} = \Sigma$ is in $\mathfrak{M}$. Hence there is a small subtree T such that $K[T] \leqslant K[\Sigma] \lneqq H[\Sigma] \leqslant H[z]$ (since Σ is a wellordering of z, see p.57-58). Write $T = T_z$. Proceed outside of $\mathfrak{M}$. Since (AC) holds in the surrounding set theory, there exists a choice set $C = \{y_i ; i \in \omega\}$ such that $y_i \in x_i$ for all $i \in \omega$. The set C need **not** be in $\mathfrak{M}$, but $C \subseteq \mathfrak{M}$ by the transitivity of $\mathfrak{M}$. We are looking for mappings (not in G) which transform C into some choice set C^* which is in $\mathfrak{M}$.

Since $y_i \in \mathfrak{M}$, there are small subtrees T_i of $\langle R_0, \leqslant\rangle$ such that $K[T_i] \leqslant H[y_i]$ for $i \in \omega$. Proceed by cases.

Case 1. $T^* = \bigcup\{T_i ; i \in \omega\}$ is a small subtree. Then obviously $K[T^*] \leqslant K[C] \lneqq K[C]$ and C is in $\mathfrak{M}$ and we are done since we have obtained a choice-set C for z in $\mathfrak{M}$.

Case 2. T^* is not a small subtree. We shall construct a sequence of permutations $\pi_i \in K[T_z]$ such that $T^0 = \bigcup\{\pi_i(T_i) ; i \in \omega\}$ is a small subtree. We construct these permutations π_i by induction on i.

I) Let π_0 be the identity on R_0.
II) Suppose that for $0 \leqslant i < n$ permutations $\pi_i \in K[T_z]$ are defined.
III) Construction of π_n.

Of course, the construction of π_n takes place outside of $\mathfrak{M}$. Remember that the tree structure $\leqslant$ on R_0 has the property, that the set of immediate successors of any element of R_0 has (outside of $\mathfrak{M}$!) cardinality $\aleph_1$ while small subtrees always have cardinality $\leqslant \aleph_0$. Hence we may shift (displace) the tree T_n into a tree $\pi_n(T_n)$ so that $T_z \cap T_n = T_z \cap \pi_n(T_n)$ but $\pi_n(T_n) - T_z$ is disjoint from $\bigcup\{\pi_i(T_i) ; 0 \leqslant i < n\}$. In details:

For $a \in R_0$ let $[a]$ be $\{x \in R_0 ; a \leqslant x\}$. Further call the cardinality of $\{y \in R_0 ; y \leqslant a \wedge y \neq a\}$ the order of a, in symbols ord(a). Hence ord(a) is for $a \in R_0$ always finite. Define $L(k)$ to be the k-th level:

$$L(k) = \{x \in R_0 ; \mathrm{ord}(x) = k\}$$

For $y \in L(k)$ let $I(y)$ be the set of immediate successors of y:

$$I(y) = \{x \in R_0 ; y < x \wedge \mathrm{ord}(x) = \mathrm{ord}(y) + 1\}$$

Thus $y_1 \neq y_2$, $y_1, y_2 \in L(k)$ implies $I(y_1) \cap I(y_2) = \emptyset$ (and $I(y)$) always has cardinality $\aleph_1$ and further $L(k+1) = \bigcup\{I(y);\ y \in L(k)\}$.

When we consider $\langle [a], \leqslant \rangle$ we mean of course the ordered pair consisting of $[a]$ and $\leqslant$ restricted to $[a]$. If $a_1, a_2 \in L(k)$, then $\langle [a_1], \leqslant \rangle$ and $\langle [a_2], \leqslant \rangle$ are isomorphic (and isomorphic with $\langle R_0, \leqslant \rangle$). Let $\sigma^{a_1}_{a_2}$ be such an isomorphism which maps $[a_1]$ one-to-one onto $[a_2]$ in an orderpreserving way. If $\mathrm{ord}(a_1) \neq \mathrm{ord}(a_2)$ then $\sigma^{a_1}_{a_2}$ is undefined.

For $m < \mathrm{ord}(x)$ define $\lambda(m,x)$ to be the element $z \in R_0$ with $\mathrm{ord}(z) = m$ and $z < x$. Since $\leqslant$ is a tree-ordering of height ω, $\lambda(m,x)$ is unique.

With this amount of notation we are able to define by induction (on $k \in \omega$) a sequence of permutations $g^k_n \in G$ which approximate π_n. g^0_n is the identical mapping on R_0. Suppose g^m_n is defined for $0 \leqslant m \leqslant k$. Define g^k_n by means of g^{k-1}_n, σ and the following function ϕ which we are going to characterize:

Since T_z, T_n and $\bigcup\{\pi_i(T_i);\ 0 \leqslant i < n\}$ are all countable and each set $I(y)$ for $y \in L(k-1)$ is uncountable we may find a function ϕ from $L(k)$ onto $L(k)$ which preserves the partition $\{I(y);\ y \in L(k-1)\}$ id est $\phi''I(y) = \{\phi(x);\ x \in I(y)\} = I(y)$ for all $y \in L(k-1)$, and leaves $L(k) \cap T_z$ pointwise fixed and

$$\{g(x);\ x \in L(k) \cap T_n \wedge x \notin T_z\} \cap \{x \in L(k);\ x \in T_z \cup \bigcup_{i=0}^{n-1} \pi_i(T_i)\} = \emptyset.$$

Now define g^k_n : let x be in R_0:

$$g^k_n(x) = \begin{cases} g^{k-1}_n(x) & \text{if } \mathrm{ord}(x) < k, \\ \phi(g^{k-1}_n(x)) & \text{if } \mathrm{ord}(x) = k, \\ \sigma^{\lambda(k,x)}_{\phi(\lambda(k,x))}(x) & \text{if } \mathrm{ord}(x) > k. \end{cases}$$

Thus the sequence g^k_n for $k \in \omega$ is defined by induction for all $k \in \omega$ and we define π_n : let $x \in R_0$:

$$\pi_n(x) = g^{\mathrm{ord}(x)}_n(x)$$

It follows that $\pi_n \in K[T_z]$ and $\pi_n(T_n)$ is "disjoint from $\bigcup_{i=0}^{n-1} \pi_i(T_i)$ modulo T_z". Since we have defined the sequence $\pi_0, \pi_1, \ldots, \pi_n, \ldots$ by induction it follows that every branch B which is included in

$$T^0 = \bigcup\{\pi_i(T_i);\ i \in \omega\}$$

is either included in T_z or in one of the small subtrees $\pi_i(T_i)$. Hence T^0 itself is small. Hence $K[T^0] \in F$.

Define $C^0 = \{\pi_n(y_n);\ n \in \omega\}$; then $K[T^0] \leqslant K[\pi_n(T_n)] \leqslant H[\pi_n(y_n)]$ since $K[T_n] \leqslant H[y_n]$. Thus $K[T^0] \leqslant K[C^0] \leqslant H[C^0]$ and it follows that C^0 is a set of $\mathfrak{M}$.

Since $\pi_n \in K[T_z] \leq K[z]$ it follows from $y_n \in x_n \in z$ that $\pi_n(y_n) \in \pi_n(x_n) = x_n \in z$. Hence C^0 is a choice set. This proves the lemma and Jensen's theorem is established.

Discussion. The function ord can be given in $\mathfrak{M}$ and $\{L(k); k \in \omega\}$ is a countable set in $\mathfrak{M}$. By (AC^ω) in $\mathfrak{M}$, $\mathfrak{M}$ contains a choice set C for this set. It follows that $\{x \in R_0;\ (\exists y \in C)(x \leq y)\}$ is a small subtree!

Another point is, that the method of proof given above can be used to yield the following generalization:

Theorem: If α is an aleph, then $(AC^\alpha) \to (DC^\omega)$ is not provable in ZF^0 (provided ZF^0 is consistent).

The proof is analoguous to the one given above. Instead of defining $\underset{\sim}{H}$ to be $\bigcup\{\omega_1^m;\ m \in \omega\}$ one takes $\bigcup\{(\alpha^+)^m;\ m \in \omega\}$ as $\underset{\sim}{H}$, where α^+ is the successor aleph of α. Small subtrees of $\langle R_0, \leq \rangle$ are subtrees without complete branches of cardinality $\leq \alpha$. Notice, that here R_0 has cardinality α^+. The proof can be carried over to the present case, since α^+ is a regular cardinal.

This generalization shows, that the other result of Jensen, namely $ZF^0 \vdash \bigwedge_\alpha (AC^\alpha) \to (DC^\omega)$ is the "best possible" result.

Translation to a Cohen-generic model

Theorem (R.B.Jensen 1965): Every countable standard model $\mathfrak{M}$ of $ZF + V = L$ can be extended to a countable standard model $\mathfrak{N}$ of ZF such that:

(a) The ordinals of $\mathfrak{N}$ are exactly the ordinals of $\mathfrak{M}$,

(b) the alephs of $\mathfrak{N}$ are exactly the alephs of $\mathfrak{M}$,

(c) (AC^ω) holds in $\mathfrak{N}$,

(d) (DC^ω) does not hold in $\mathfrak{N}$.

Proof. Let $\mathfrak{M}$ be a countable standard model of $ZF + V = L$. We extend $\mathfrak{M}$ by adding to $\mathfrak{M}$ a generic copy of $\underset{\sim}{H}$ (the tree defined on p.151) and generically all small subtrees T of $\underset{\sim}{H}$ together with wellorderings on each T.

Hence take the first uncountable ordinal $\omega_1^{\mathfrak{M}}$ of $\mathfrak{M}$ and define $\underset{\sim}{H}$ in $\mathfrak{M}$ and the tree-ordering $\leq$ on $\underset{\sim}{H}$ as before. Next define a ramified language $\mathcal{L}$ in $\mathfrak{M}$ which contains besides the usual ZF-symbols, the limited quantifiers $\bigvee^\alpha$, the limited comprehension terms E^α (for ordinals α of $\mathfrak{M}$), constants $\underline{x}$ for each set x of $\mathfrak{M}$, constants $\dot{a}_h$ for each $h \in \underset{\sim}{H}$, a binary predicate $\dot{\leqslant}$ and binary predicates $\dot{S}_T$ for each small subtree T of $\underset{\sim}{H}$.

A condition p is a finite partial function from $\omega \times \underset{\sim}{H}$ into 2. Define a strong forcing relation $\Vdash$ containing the following key-clauses:

$p \Vdash t \,\varepsilon\, \dot{a}_h \Leftrightarrow (\exists\, n \in \omega)(p \Vdash t \approx \underline{n} \;\&\; p(\langle n,h\rangle) = 1)$,

$p \Vdash t_1 \dot{\leqslant} t_2 \Leftrightarrow (\exists\, h_1, h_2 \in \underset{\sim}{H})(h_1 \leqslant h_2 \;\&\; p \Vdash t_1 \approx \dot{a}_{h_1} \;\&\; p \Vdash t_2 \approx \dot{a}_{h_2})$,

$p \Vdash \dot{S}_T(t_1, t_2) \Leftrightarrow (\exists\, h \in \underset{\sim}{H})(h \in T \;\&\; p \Vdash t_1 \approx \dot{a}_h \;\&\; p \Vdash t_2 \approx \underline{h})$.

where t, t_1, t_2 are constant terms of $\mathcal{L}$. Obtain a complete sequence $\mathcal{R}$ of conditions and thereby a valuation val(t) of the constant terms t. By the Hauptsatz, $\mathfrak{N}$ = {val(t); t constant term of $\mathcal{L}$} is a model of ZF which contains $\mathfrak{M}$ as a complete submodel. Further $\mathfrak{M}$ and $\mathfrak{N}$ have precisely the same ordinals (by lemma T, see page 90); hence (a) holds.

Notice that

1) $a_h = \mathrm{val}(\dot{a}_h) \subseteq \omega$,

2) $\leqslant\, = \mathrm{val}\,(\dot{\leqslant})$ is a tree-ordering the field of which is just the set $\{a_h;\, h \in \underset{\sim}{H}\}$,

3) $S_T = \mathrm{val}\,(\dot{S}_T) = \{\langle a_h, h\rangle;\, h \in T \subseteq \underset{\sim}{H}\}$

more precisely $\leqslant$ is the valuation of $E^{\omega+1}x(\bigvee_y \bigvee_z (x = \langle y,z\rangle \wedge y \dot{\leqslant} z))$ and similar for S_T. Since we took symbols $\dot{S}_T$ into $\mathcal{L}$ only for small subtrees T which are in $\mathfrak{M}$ and $\mathfrak{M}$ satisfies V = L, hence (AC), there is in $\mathfrak{M}$ a wellordering W_T for T. Since S_T is a one-one-function from $\{a_h;\, h \in T\}$ onto T, and val $(\underline{W_T}) = W_T$ is in $\mathfrak{N}$ it follows that in $\mathfrak{N}$ the sets $\{a_h;\, h \in T\}$ (T a small subtree of $\underset{\sim}{H}$ in $\mathfrak{M}$) are wellorderable.

Let G be the group (in $\mathfrak{M}$) of all orderpreserving one-to-one-mappings π from $\underset{\sim}{H}$ onto $\underset{\sim}{H}$. For a condition p define

$\pi(p) = \{\langle\langle n,\pi(h)\rangle, e\rangle;\, \langle\langle n,h\rangle, e\rangle \in p\}$

and define the action of G on $\mathcal{L}$ in the following way: if $\pi \in G$ and $h \in \underset{\sim}{H}$, then $\pi(\dot{a}_h) = \dot{a}_{\pi(h)}$, $\pi(\dot{S}_T) = \dot{S}_{\pi''T}$, where

$\pi''T = \{\pi(h);\, h \in T\}$

and for an $\mathcal{L}$-formula Φ let $\pi(\Phi)$ be result obtained from Φ by replacing any occurence of $\dot{a}_h$ by $\pi(\dot{a}_h)$ and of $\dot{S}_T$ by $\pi(\dot{S}_T)$ (the other symbols of Φ remain unchanged). Then the following holds:

Symmetry-Lemma: If $\pi \in G$, p is a condition and Φ an $\mathcal{L}$-sentence, then
$p \Vdash \Phi \Leftrightarrow \pi(p) \Vdash \pi(\Phi)$.

Proof by induction (see page 99).
This lemma enables us to prove:

Lemma: The countable axiom of dependent choices (DC^{ω}) does not hold in $\mathfrak{N}$.

Proof. One shows that no condition p can force that $\{a_h; h \in \underset{\sim}{H}\}$, partially ordered by $\leqslant$, has a complete branch. See the proof of $\neg(DC^{\omega})$ in the preceeding FraenkelMostowski-Specker model.

In order to show that (AC^{ω}) holds in $\mathfrak{N}$ we need a lemma which says, that any set of $\mathfrak{M}$ which is $\mathfrak{N}$-countable (i.e. countable in $\mathfrak{N}$) is also $\mathfrak{M}$-countable. More generally we shall prove that cardinals are preserved in the transition from $\mathfrak{M}$ to $\mathfrak{N}$, thus proving (b).

Combinatorial Lemma. Let B be in $\mathfrak{M}$ a set of conditions. There exists in $\mathfrak{M}$ a subset B' of B such that B' is in $\mathfrak{M}$ countable and for every $p \in B$ there is a $p' \in B'$ such that p and p' are compatible.

For a proof see e.g. Jensen's lecture notes [39] (Springer-Berlin) page 147, or these notes page 106-107. The lemma implies obviously that if B is in $\mathfrak{M}$ a set of conditions whose elements are pairwise incompatible, then B is countable. We have shown on p.106-108 (these notes), that this implies that cardinals (i.e. alephs or finite ordinals) are absolut in the extension from $\mathfrak{M}$ to $\mathfrak{N}$; thus we have shown:

Lemma. For every ordinal γ of $\mathfrak{N}$ (and hence of $\mathfrak{M}$) the cardinality of γ in the sense of $\mathfrak{N}$ is equal to the cardinality of γ in the sense of $\mathfrak{M}$:

$$\overline{\overline{\gamma}}^{\mathfrak{M}} = \overline{\overline{\gamma}}^{\mathfrak{N}} ,$$

hence $\mathfrak{M}$ and $\mathfrak{N}$ have precisely the same alephs.

In order to show that (AC^{ω}) holds in $\mathfrak{N}$ we introduce the following notation: for a small subtree T of $\langle \underset{\sim}{H}, \leqslant \rangle$ let V(T) be the $\mathfrak{N}$ -class of sets which are explicit definable from T. Since $\mathcal{L}$ is $\mathfrak{M}$ -definable and $\mathfrak{M}$ is a complete submodel of $\mathfrak{N}$ (see the necessary remark on p.96-97), $\mathcal{L}$ is $\mathfrak{N}$-definable. Since the correspondence $h \to a_h$ is not in $\mathfrak{N}$ (for all $h \in \underset{\sim}{H}$), we cannot interpret $\mathcal{L}$ inside of $\mathfrak{N}$, but what we can do is to interpret certain sublanguages $\mathcal{L}(T)$ of $\mathcal{L}$

in $\mathfrak{N}$. Namely let for a small subtree T be $\mathcal{L}(T)$ the language which contains besides the ZF-symbols, the symbols V^{α} and E^{α}, further $\underline{x}$ for $x \in \mathfrak{M}$ and $\dot{\leqslant}$, only constants $\dot{a}_h$ for $h \in T$ and symbols $\dot{S}_D$ for small subtrees D for $D \subseteq T$.

Notation: we say that the set s of $\mathfrak{N}$ is explicit definable from T if there is a constant term t_s of $\mathcal{L}(T)$ such that $s = val(t_s)$.

Now let V(T) be the collection of all sets of $\mathfrak{N}$ which are explicit definable from T. We claim that V(T) is $\mathfrak{N}$ -definable and has an $\mathfrak{N}$ -definable wellordering.

In fact, since $\mathcal{L}(T)$ is $\mathfrak{M}$-definable, hence $\mathfrak{N}$-definable, and the correspondences $a_h \to h$ for $h \in T$ and $S_D \to D$ (since $D = \mathrm{Range}(S_D)$) for $D \subseteq T$ are in $\mathfrak{N}$ we can define an interpretation Ω for constant terms t of $\mathcal{L}(T)$ in $\mathfrak{N}$ by setting

$$\Omega(\dot{a}_h) = a_h,\ \Omega(\dot{S}_D) = S_D \quad \text{for } h \in T,\ D \subseteq T,\ D \text{ small subtree},$$

$$\Omega(\dot{\leqslant}) = \leqslant \text{ and } \Omega(\underline{x}) = x,$$

and then extending by induction to all t's of $\mathcal{L}(T)$. Thus

$$V(T) = \{\Omega(t); t \text{ is a constant term of } \mathcal{L}(T)\}$$

and V(T) is an $\mathfrak{N}$-definable class.

Lemma. If T is a small subtree of $\langle \underset{\sim}{H}, \leqslant \rangle$, then V(T) has an $\mathfrak{N}$-definable wellordering.

Proof. Since T and $\{D \subseteq T;\ D$ a small subtree$\}$ are wellorderable in $\mathfrak{M}$ and further $\{\underline{x};\ x$ a set of $\mathfrak{M}\}$, $\{V^{\alpha};\ \alpha \in On^{\mathfrak{M}}\}$ and $\{E^{\alpha};\ \alpha \in On^{\mathfrak{M}}\}$ have $\mathfrak{M}$-definable wellorderings, it is clear, that the alphabet of $\mathcal{L}(T)$ is wellorderable in $\mathfrak{M}$. Hence the class of all constant terms t of $\mathcal{L}(T)$ (considered as finite sequences of symbols from the alphabet of $\mathcal{L}(T)$) has a (lexicographic) wellordering which is carried over to V(T) via Ω.

Notice, that Ω can not be defined for all terms of $\mathcal{L}$, but only for terms of $\mathcal{L}(T)$; but this is sufficient in the present case, q.e.d.

Lemma. For every term t of $\mathcal{L}$ there is a small subtree T such that for every condition p it holds that $p \Vdash^* t \neq \emptyset \to t \cap C(T) \neq \emptyset$.

Sketch of Proof. C(T) is the unlimited term of $\mathcal{L}$ so that $V(T) = val(C(T))$. For given t consider $B = \{q \in Cond;\ q \Vdash^* \bigvee_x x \,\varepsilon\, t\}$, where Cond is the $\mathfrak{M}$-set of conditions. Now apply the combinatorial lemma and obtain a $\mathfrak{M}$-countable subset B' of B so that for every $q \in B$ there is $q' \in B'$ with $q \cup q' \in Cond$, and construct the tree T. Then for $p' \in B'$ there is t' of $\mathcal{L}(T)$ so that $p' \Vdash^* t \neq \emptyset \to t' \,\varepsilon\, t$.

Lemma. (AC^{ω}) holds in $\mathfrak{N}$.

Sketch of proof. Let z be a countable set in $\mathfrak{N}$, $z = \{z_i ; i \in \omega\}$ so that each $z_i \in z$ is not empty. Let t_z, t_i ($i \in \omega$) be constant terms of $\mathcal{L}$ so that $z = val(t_z)$ and $z_i = val(t_i)$. By the preceeding lemma there is a sequence of small subtrees T_i such that $\emptyset \Vdash^* t_i \neq \emptyset \rightarrow t_i \cap C(T_i)$. Obtain a sequence of permutations π_i leaving t_z invariant but such that $T^* = \bigcup\{\pi_i(T_i); i \in \omega\}$ is a small subtree (use the construction presented in the Fraenkel-Mostowski-Specker version of the model). Hence in $\mathfrak{N}$: $z_i \cap V(T^*) \neq \emptyset$ for all $i \in \omega$. Since $V(T^*)$ has an $\mathfrak{N}$-definable wellordering we obtain a choice sequence in $\mathfrak{N}$. This proves Jensen's theorem. Again, as in the case of the permutation model, the theorem can be strengthened to

Theorem (Jensen): Let α be an infinite cardinal in a countable standard model $\mathfrak{M}$ of ZF + V = L; then $\mathfrak{M}$ can be extended to a countable standard model $\mathfrak{N}$ of ZF in which (AC^{α}) is true but (DC^{ω}) is false and furthermore $On^{\mathfrak{M}} = On^{\mathfrak{N}}$ and $\mathfrak{M}$ and $\mathfrak{N}$ have the same alephs.

The Independence of the Axiom of Choice from the Principle of Dependent choices

Mostowski showed in 1948 that (AC) is independent from (DC) in ZF^0 . W.Marek in Warsaw translated Mostowski's construction of a model $\mathfrak{M}$ of ZF^0 + (DC) + $\neg$ (AC^{ω_1}) to yield a Cohen-generic model $\mathfrak{N}$ of ZF plus (DC) + $\neg$ (AC). Thus (DC^{ω}) $\rightarrow$ (AC) is not provable in full ZF.

[57] W.MAREK: A remark on independence proofs; Bull. Acad. Polon. Sci. vol.14(1966)p.543-545.

Marek just immitates Mostowski's model by adding to a given countable standard model $\mathfrak{M}$ of ZF + V = L generically a set X of $\aleph_1$ many unordered pairs B_μ ($\mu \in \omega_1$) where each B_μ contains two Cohen-generic reals U_μ^0 and U_μ^1. The extension $\mathfrak{N}$ is obtained as the constructible closure (using ordinals of $\mathfrak{M}$), but besides Gödel's eight fundamental operations Marek takes a nineth one which will serve to add countable sequences. Unfortunately the proof in [57] is only briefly sketched.

We shall consider in the sequel Feferman's model $\mathfrak{M}[a_0, a_1, .. a_n ..]$. Feferman showed that in this model the (BPI) fails and Dana Scott pointed out that even the axiom of choice for unordered pairs (AC_2) is violated - see Feferman [16] (we cited this paper on p.98). Several people observed that (DC) holds in Feferman's model. Probably R.M.Solovay was the first who made this observation. A proof is published in:

[73] G.E.SACKS: Measure - Theoretic Uniformity in Recursion Theory and Set Theory; Transactions Amer.Math.Soc.vol.142(1969)p.381-420

The proof is presented in [73] in the language of measure-theoretic uniformity rather than in the forcing approach. The results of Sacks [73] are announced in: G.E.Sacks: Measure Theoretic Uniformity; Bull. Amer. Math. Soc. vol.73(1967)p.169-174, and (under the same title) in the Gödel-Festschrift (Springer-Verlag, Berlin 1969)p.51-57.

FEFERMAN'S MODEL $\mathfrak{M}[a_0, a_1, \ldots]$.

Let $\mathfrak{M}$ be a countable standard model of ZF + V = L. Define in $\mathfrak{M}$ a ramified language $\mathcal{L}$, which contains besides the usual ZF-symbols ($\neg, \vee, \bigvee$, $=$, ε and variables), limited existential quantifiers $\bigvee^\alpha$ and limited comprehension operators E^α (for ordinals α of $\mathfrak{M}$), constants $\underline{x}$ for each set x of $\mathfrak{M}$ and unary (generic) predicates $\dot{a}_i$ for $i \in \omega$. We suppose that this is done in such a way that the correspondences $x \to \underline{x}$, $i \to \dot{a}_i$, $\alpha \to \bigvee^\alpha$ and $\alpha \to E^\alpha$ are all $\mathfrak{M}$-definable. This can be attained e.g. by the standard-method - see page 79. Define conditions p to be finite partial functions from $\omega \times \omega$ into $2 = \{0,1\}$. The (strong) forcing relation $\Vdash$ between conditions p and $\mathcal{L}$-sentences is defined as usual. The definition contains the following key-clause:

$$p \Vdash \dot{a}_i(t) \Leftrightarrow (\exists\, n \in \omega)(p \Vdash t = \underline{n} \;\&\; p(\langle n,i\rangle) = 1)$$

where t is a constant term of $\mathcal{L}$. Obtain a complete sequence $\mathcal{R}$ and thereby a valuation val(t) of the constant terms t of $\mathcal{L}$. Let $\mathfrak{N} = \{\mathrm{val}(t);\ t$ a constant term of $\mathcal{L}\}$, then $\mathfrak{N}$ is a countable standard model of ZF. It holds that $a_i = \mathrm{val}(\dot{a}_i) \subseteq \omega$. We use the following notation

$$\mathfrak{N} \doteq \mathfrak{M}[a_0, a_1, \ldots]$$

i.e. $\mathfrak{N}$ results from $\mathfrak{M}$ by adding countably many Cohen-generic reals a_i $(i \in \omega)$ to $\mathfrak{M}$. Notice, that the correspondence $a_i \to i$, id est $\{\langle a_i, i\rangle;\ i \in \omega\}$, is not added and that we did not add a set A which just collects these generic reals a_i. Concerning the latter This is the main difference to the model of Halpern-Lévy: $\mathfrak{M}[a_0, a_1, \ldots, A]$ (see pages 101-103 and p.131). Halpern-Lévy's model satisfies the (BPI) and since

$(BPI) \to (OP) \to (AC_2)$ (in ZF),

where (BPI) is the Boolean prime ideal theorem, (OP) the ordering principle and (AC_2) the statement which says that every set of unordered pairs has a choice-function, it follows, that (AC_2) holds in $\mathfrak{M}[a_0, a_1, \ldots, A]$. We shall show in the sequel that (AC_2) does not hold in $\mathfrak{M}[a_0, a_1, \ldots]$. Moreover it will be shown that there is no set in $\mathfrak{M}[a_0, a_1, \ldots]$ which just collects the reals $a_0, a_1, \ldots, a_n, \ldots$. Though the models of Feferman and of Halpern-Lévy seem to be very similar, they are considerably different and have extremely different features.

Symmetry-properties of Feferman's model

Let G be the group (in $\mathfrak{M}$) of all one-to-one mappings π of ω onto ω. Let $\pi(\Phi)$ be the result of substituting $\dot{a}_{\pi(i)}$ for $\dot{a}_i$ in the $\mathcal{L}$-formula Φ, and define $\pi(p) = \{\langle\langle n,\pi(i)\rangle,e\rangle; \langle\langle n,i\rangle,e\rangle \in p\}$ for conditions p. Then the "classical" Symmetry-lemma says:

"If $\pi \in G$, Φ is an $\mathcal{L}$-sentence and p a condition, then

$p \Vdash \Phi$ iff $\pi(p) \Vdash \pi(\Phi)$"

Feferman considers in [16] p.330-331, a different kind of transformation (we use Lévy's notation in [51] p.147-149).

If r is a set of $\mathfrak{M}$ and $r \subseteq \omega \times \omega$, then r defines a transformation:

Definition: Let Q be a function on a subset of $\omega \times \omega$ into 2 (in particular Q may be a condition). We define

$$[r,Q] = \{\langle\langle n,i\rangle,e\rangle ; (\langle\langle n,i\rangle,e\rangle \in Q \wedge \langle n,i\rangle \in r) \vee \vee(\langle\langle n,i\rangle,1-e\rangle \in Q \wedge \langle n,i\rangle \notin r)\}.$$

Definition. Let Φ be an $\mathcal{L}$-formula. We write $[r,\Phi]$ for the result of replacing each occurrence of $\dot{a}_i(\zeta)$ (where ζ is a variable or a constant term of $\mathcal{L}$) in Φ by

$$[r,\zeta] \ \varepsilon\ \omega \wedge (\dot{a}_i([r,\zeta]) \leftrightarrow \langle[r,\zeta],\underline{i}\rangle \ \varepsilon\ r),$$

where $[r,\zeta] \triangleq \zeta$ if ζ is a variable and $\langle[r,\zeta],\underline{i}\rangle \ \varepsilon\ r$ stands for:

$$\bigvee\nolimits_x^\omega[x \ \varepsilon\ r \wedge \bigwedge\nolimits_y^\omega(y \ \varepsilon\ x \leftrightarrow (\bigwedge\nolimits_z^\omega(z \ \varepsilon\ y \leftrightarrow z \simeq [r,\zeta]) \vee \bigwedge\nolimits_z^\omega(z \ \varepsilon\ y \leftrightarrow (z \simeq [r,\zeta] \vee z \simeq \underline{i}))))$$

(the Kuratowski-definition of an ordered pair as a limited sentence of $\mathcal{L}$) and $z \simeq [r,\zeta]$ for $\bigwedge\nolimits_x^\omega(v \ \varepsilon\ z \leftrightarrow v \ \varepsilon\ [r,\zeta])$.

It follows, that if p is a condition (in $\mathfrak{M}$) and $r \subseteq \omega \times \omega$, r in $\mathfrak{M}$, then $[r,p]$ is a condition (in $\mathfrak{M}$). Further if Φ is a limited $\mathcal{L}$-formula, then $[r,\Phi]$ is a limited $\mathcal{L}$-formula; if Φ is unlimited, then so is $[r,\Phi]$. If t is a limited comprehension term, say $E^{\alpha}x\Phi(x)$, then $[r,t] \triangleq E^{\alpha}x[r,\Phi(x)]$ is a limited comprehension term (this is used above for $[r,\zeta]$ if ζ is a constant term).

<u>Lemma I.</u> Let Q_1 and Q_2 be functions with $\mathrm{Dom}(Q_1) \subseteq \omega \times \omega$, $\mathrm{Dom}(Q_2) \subseteq \omega \times \omega$ and $\mathrm{Range}(Q_1) \subseteq 2$, $\mathrm{Range}(Q_2) \subseteq 2$ and let $r \subseteq \omega \times \omega$ and suppose that Q_1, Q_2 and r are in $\mathfrak{M}$. If $Q_1 \subseteq Q_2$, then $[r,Q_1] \subseteq [r,Q_2]$. Further it holds that $[r,[r,Q_i]] = Q_i$ (for i = 1,2). If $\Phi(x)$ is an $\mathcal{L}$-formula and $\Phi'(x) = [r,\Phi(x)]$, then $[r,\Phi(u)] = \Phi'([r,u])$ if u is a variable of $\mathcal{L}$ or a constant term of $\mathcal{L}$. Further it holds that $\delta[r,t] = \delta(t)$ for any constant term t of $\mathcal{L}$.

(see p.79 for definition of δ).

<u>Lemma II.</u> If $\mathcal{R} \triangleq (p_0, p_1, \ldots, p_n, \ldots)$ is a complete sequence of conditions and if $r \subseteq \omega \times \omega$ is a set in $\mathfrak{M}$, then

$$[r,\mathcal{R}] \triangleq_{\mathrm{Def}} ([r,p_0],[r,p_1],\ldots,[r,p_n],\ldots)$$

is a complete sequence.

<u>Lemma III.</u> For every constant term t: $\mathrm{val}_{\mathcal{R}}(t) = \mathrm{val}_{[r,\mathcal{R}]}([r,t])$ for r and $\mathcal{R}$ as in the preceeding lemma. For every $\mathcal{L}$-sentence ,

$$\mathfrak{N}_{\mathcal{R}} \models \Phi \quad \text{iff} \quad \mathfrak{N}_{[r,\mathcal{R}]} \models [r,\Phi].$$

For a detailed proof see Lévy's paper [51] p.148. Lemma III can be strengthened to:

<u>Lemma IV.</u> (Feferman [16]): Let Φ be an $\mathcal{L}$-sentence, p a condition (in $\mathfrak{M}$) and $r \subseteq \omega \times \omega$, r a set of $\mathfrak{M}$. Then

$$p \Vdash \Phi \quad \text{iff } [r,p] \Vdash [r,\Phi].$$

Next we shall present Feferman's lemma, which says, that (BPI) does not hold in $\mathfrak{N}$. We need some lemmata.

<u>Lemma V.</u> Let $P(\omega)$ be the powerset of ω and I a prime ideal in $P(\omega)$. If I is not principal, then I contains all finite subsets of ω.

Proof. Suppose, there is a finite subset $S = \{b_1, \ldots, b_n\} \subseteq \omega$ such that $S \notin I$. Then $\omega - S \in I$ since I is prime. Write $B_1 = \omega - \{b_1\}, B_2 = \omega - \{b_2\}, \ldots, B_n = \omega - \{b_n\}$. Then $\bigcap\{B_i;\ 1 \leq i \leq n\} \in I$ and since I is prime there exists i with $1 \leq i \leq n$ such that $B_i \in I$. Since I is a proper ideal, $I = \{x \subseteq \omega;\ x \subseteq B_i\}$. Hence I is a principal ideal,

q.e.d.

Lemma (Feferman [16] p.343): Let $P^{\mathfrak{N}}(\omega)$ be the powerset of ω in the sense of $\mathfrak{N}$. Every prime-ideal $I \in \mathfrak{N}$ in the Boolean algebra $\langle P^{\mathfrak{N}}(\omega), \subseteq \rangle$ is a principal ideal.

Proof. Suppose there exists in $\mathfrak{N}$ a non-principal prime ideal I of $\langle P^{\mathfrak{N}}(\omega), \subseteq \rangle$. Thus I contains all finite subsets of ω. Let t_I be a constant term of $\mathcal{L}$ such that $I \triangleq val(t_I)$ and $t_I \triangleq E^{\alpha}x\Phi(x)$.
Let $occ(\Phi) = \{i \in \omega;\ \dot{a}_i$ occurs in $\Phi\}$ and $n \in \omega$ such that $i \in occ(\Phi)$ implies $i < n$. We shall show that neither a_n nor $\omega - a_n$ is in I (a_n generic). Proceed by cases:

Case 1. Suppose $a_n \in I$ holds in $\mathfrak{N}$. Since everything that holds in $\mathfrak{N}$ is forced by some conditions in $\mathfrak{R}$, the complete sequence which defines $\mathfrak{N}$, there exists $p \in \mathfrak{R}$ such that

$$p \Vdash \Phi(\dot{a}_n).$$

Let k_0 be chosen so that for all $k \geq k_0$, $\langle k,n \rangle \notin Dom(p)$.
Define in $\mathfrak{M}$:

$$r = \{\langle k,m \rangle;\ m \neq n \vee (m = n \wedge k < k_0)\} \subseteq \omega \times \omega.$$

It follows that $[r,p] = p$. Let $\Phi'(x) = [r,\Phi(x)]$. Lemma IV implies $p \Vdash [r,\Phi(\dot{a}_n)]$, hence $\mathfrak{N} \models [r,\Phi(\dot{a}_n)]$ since $[r,p] = p \in \mathfrak{R}$. But $\mathfrak{N} \models [r,\Phi(\dot{a}_n)] \leftrightarrow \Phi'([r,\dot{a}_n])$. By construction of r: $\mathfrak{N} \models \Phi'([r,\dot{a}_n]) \leftrightarrow \Phi([r,\dot{a}_n])$. Hence

$$\mathfrak{N} \models \Phi([r,\dot{a}_n])$$

and therefore $val([r,\dot{a}_n]) \in I$. But

$$val([r,\dot{a}_n]) = [(a_n \cap k_0) \cup (\omega - a_n)] \cap (\omega - k_0).$$

Since prime ideals J satisfy: $a \sqcap b \in J$ then $a \in J$ or $b \in J$, it follows that either $(a_n \cap k_0) \cup (\omega - a_n)$ or $(\omega - k_0)$ is in I. But I contains all finite subsets of ω, hence in particular k_0. Hence $\omega - k_0$ cannot be in I since otherwise $k_0 \cup (\omega - k_0) = \omega \in I$. Thus we get that $(a_n \cap k_0) \cup (\omega - a_n) \in I$. Hence $\omega - a_n \in I$. But by our assumption $a_n \in I$, a contradiction, ω would be in I.

Case 2. $\omega - a_n \in I$ holds in $\mathfrak{N}$. Proceed in a similar way and obtain a contradiction. This proves the lemma.

Corollary: The Boolean Prime Ideal Theorem does not hold in $\mathfrak{N}$.

Proof. It is well-known that (in ZF) the (BPI) is equivalent to the statement: "Every infinite Boolean algebra has a non-principal prime ideal" (see e.g. Tarski's abstract in the Bull. AMS 60(1954)p.390-391). It follows, hence, from the previous lemma, that (BPI) does not hold in $\mathfrak{N}$.

Remark: Dana Scott showed that in Feferman's model $\mathfrak{N}$ even there does not exist a choice set selecting reals from the cosets of the rationals in the reals - see Feferman's paper [16] p.343-344.

Let $C(\alpha,\mathring{a}_{i_1},\mathring{a}_{i_2},\ldots,\mathring{a}_{i_k})$ be the $\mathfrak{M}$-set of all constant terms t of $\mathcal{L}$ such that $\delta(t) \leqslant \alpha$ and symbols $\mathring{a}_j$ occur in t only for $j \in \{i_1,\ldots,i_k\}$. Let $\mathfrak{M}[\alpha,a_{i_1},\ldots,a_{i_k}]$ be the $\mathfrak{N}$-set of those sets (of $\mathfrak{N}$) which are denoted by members of $C(\alpha,\mathring{a}_{i_1},\ldots,\mathring{a}_{i_k})$. It is clear that $\mathcal{L}$ has a constant term $\underset{\sim}{t}(\alpha,\mathring{a}_{i_1},\ldots,\mathring{a}_{i_n})$ which denotes $\mathfrak{M}[\alpha,a_{i_1},\ldots,a_{i_k}]$. Define

$$C(\mathring{a}_{i_1},\ldots,\mathring{a}_{i_k}) = \bigcup\{C(\alpha,\mathring{a}_{i_1},\ldots,\mathring{a}_{i_k});\ \alpha \in On^{\mathfrak{M}}\}$$

and let $\underset{\sim}{t}(\mathring{a}_{i_1},\ldots,\mathring{a}_{i_k})$ be (an unlimited) constant term of $\mathcal{L}$ denoting

$$\bigcup\{\mathfrak{M}[\alpha,a_{i_1},\ldots,a_{i_k}];\ \alpha \in On^{\mathfrak{M}}\} = \mathfrak{M}[a_{i_1},\ldots,a_{i_k}]$$

Lemma. For each finite subset $\{i_1,\ldots,i_k\}$ of ω, $\mathfrak{M}[a_{i_1},\ldots,a_{i_k}]$ has an $\mathfrak{N}$-definable well-ordering.

For the proof use the techniques presented on pages 97-98, 138 and 158 .

Lemma. Let $\Phi(x)$ be an $\mathcal{L}$-formula whose only free variable is x such that $occ(\Phi(x)) = \{i \in \omega;\ \mathring{a}_i \text{ occurs in } \Phi(x)\} \subseteq k \in \omega$. Then for every condition p:

$$p \Vdash \bigvee\nolimits_x^\alpha \Phi(x) \leftrightarrow (\bigvee\nolimits_x^\alpha \Phi(x) \wedge x \,\varepsilon\, \underset{\sim}{t}(\alpha,\mathring{a}_0,\mathring{a}_1,\ldots,\mathring{a}_k)).$$

The idea for the proof is the following: if u is a constant term of $\mathcal{L}$, say $E^\beta x\Psi(x)$, with $\beta < \alpha$, such that $\Phi(u)$ and u mentions (names for) generic reals $\mathring{a}_j$ at most for $j \in \{0,1,\ldots,m\}$, then transform u into a term $u^* = E^\beta x\Psi^*(x)$ such that $\Phi(u^*)$ and u^* mention (names for) generic reals $\mathring{a}_j$ at most for $j \in \{0,1,\ldots,k\} = k + 1$. This can be achieved by replacing $a_k,a_{k+1},\ldots,a_m$ in u by pairwise disjoint subsets of a_k. More precisely one defines (in $\mathfrak{M}$) the following function r from constant terms to constant terms (assume $k \leqslant m$):

$$r(E^\omega x\mathring{a}_i(x)) = E^\omega x\mathring{a}_i(x) \quad \text{if } 0 \leqslant i < k,$$

$$r(E^\omega x\mathring{a}_{i+k}(x)) = E^\omega x\mathring{a}_k((m-k+1)\cdot x+\underline{i}) \quad \text{if } 0 \leqslant i \leqslant m-k,$$

$r(E^{\omega}x\mathring{a}_{i+m+1}(x)) = E^{\omega}x\mathring{a}_{i+k+1}(x)$ if $0 \leqslant i$.

Extend r to act on all constant terms of $\mathcal{L}$ in the following way. If $E^{\gamma}x\Gamma(x)$ is any constant term of $\mathcal{L}$, then replace first in $\Gamma(x)$ every occurence of $\mathring{a}_j(x)$ by $x \varepsilon E^{\omega}z\mathring{a}_j(z)$ and call the resulting formula $\Gamma'(x)$. Then replace every occurrence of $E^{\omega}z\mathring{a}_j(z)$ in $\Gamma'(x)$ by $r(E^{\omega}z\mathring{a}_j(z))$ and call the resulting formula $r(\Gamma(x))$. Finally define $r(E^{\gamma}x\Gamma(x))$ to be $E^{\gamma}xr(\Gamma(x))$. With these definitions let u^* be $E^{\beta}xr(\Psi(x))$, id est $r(u)$. It follows from the construction, that u^* mentions (names of) generic reals $\mathring{a}_j$ at most for $j \in k+1$. Hence $val(u^*) \in \mathfrak{M}[a_0, a_1, \ldots, a_k]$. A symmetry argument shows that $\Phi(u^*)$ holds.

<u>Lemma</u> (R.Solovay): The axiom of dependent choices (DC^{ω}) holds in Feferman's model $\mathfrak{N} \triangleq \mathfrak{M}[a_0, a_1, \ldots, a_n, \ldots]$.

<u>Outline of proof</u>. Let $\Phi(x,y)$ be an $\mathcal{L}$-formula whose only free variables are x and y, such that if $\mathring{a}_j$ occurs in Φ, then $j < m$. Let us assume for simplicity that $m = 1$. Suppose $E^{\alpha}\langle x,y\rangle\Phi(x,y)$ defines in $\mathfrak{N}$ a binary relation R on a set s such that for all $x \in s$ there exists $y \in x$ with $\langle x,y\rangle \in R$ in $\mathfrak{N}$. We intend to find in $\mathfrak{N}$ a function f from ω into s such that for all $n \in \omega$, $\langle f(n), f(n+1)\rangle \in R$ and $f \in \mathfrak{M}(a_0, a_1)$.

By the previous lemma it holds in $\mathfrak{N}$ that the following two formulae are equivalent:

(1) $\bigwedge_x \bigvee_y [x \varepsilon \underset{\sim}{t}(\mathring{a}_0, \mathring{a}_1, \ldots, \mathring{a}_n) \rightarrow \Phi(x,y)]$,

(2) $\bigwedge_x \bigvee_y [x \varepsilon \underset{\sim}{t}(\mathring{a}_0, \mathring{a}_1, \ldots, \mathring{a}_n) \rightarrow (\Phi(x,y) \wedge y \varepsilon \underset{\sim}{t}(\mathring{a}_0, \mathring{a}_1, \ldots, \mathring{a}_{n+1}))]$.

Consider the following transformation r_{n+1}:

$$r_{n+1}(E^{\omega}x\mathring{a}_0(x)) = E^{\omega}x\mathring{a}_0(x),$$
$$r_{n+1}(E^{\omega}x\mathring{a}_{i+1}(x)) = E^{\omega}x\mathring{a}_1(2^{i+1}\cdot x + 2^{i} - 1) \quad \text{for } 0 \leqslant i \leqslant n,$$
$$r_{n+1}(E^{\omega}x\mathring{a}_{n+2}(x)) = E^{\omega}x\mathring{a}_1(2^{n+1}\cdot x + 2^{n+1} - 1),$$
$$r_{n+1}(E^{\omega}x\mathring{a}_{n+j}(x)) = E^{\omega}x\mathring{a}_{1+j}(x) \quad \text{for } 3 \leqslant j \in \omega.$$

This transformation r can be extended to act on the whole $\mathfrak{M}$-class of constant terms in the same way as it was done in the preceeding proof. A symmetry argument yields that

(3) $\bigwedge_x \bigvee_y [x \varepsilon r_{n+1}(\underset{\sim}{t}(\mathring{a}_0, \ldots, \mathring{a}_n)) \rightarrow \Phi(x,y)]$, and

(4) $\bigwedge_x \bigvee_y [x \varepsilon r_{n+1}(\underset{\sim}{t}(\mathring{a}_0, \ldots, \mathring{a}_n)) \rightarrow \Phi(x,y) \wedge y \varepsilon r_{n+1}(\underset{\sim}{t}(\mathring{a}_0, \ldots, \mathring{a}_{n+1}))]$

are equivalent (namely, apply r_{n+1} to (1) $\leftrightarrow$ (2)). Notice that $r_{n+1}(\underset{\sim}{t}(\mathring{a}_0,\ldots,\mathring{a}_n))$ and $r_{n+1}(\underset{\sim}{t}(\mathring{a}_0,\ldots,\mathring{a}_{n+1}))$ mention (names of) generic reals $\mathring{a}_j$ only for j = 0 or j = 1. Hence $val(r_{n+1}(\underset{\sim}{t}(\mathring{a}_0,\ldots,a_k)) \subseteq \mathfrak{M}[a_0,a_1]$ for $k \in \{n,n+1\}$. But $\mathfrak{M}[a_0,a_1]$ has an $\mathfrak{N}$-definable wellordering. Hence there is in $\mathfrak{N}$ a function $f : \omega \to s$ such that $f(0) \in \mathfrak{M}[a_0,a_1]$,

$$f(n + 1) \in val(r_{n+1}(\underset{\sim}{t}(\mathring{a}_0,\ldots,\mathring{a}_{n+1}))) \subseteq \mathfrak{M}[a_0,a_1],$$

and $\bigwedge_{n\in\omega}\langle f(n),f(n + 1)\rangle \in R$ holds in $\mathfrak{N}$. We can insist that $f \in \mathfrak{M}[a_0,a_1]$, since the constant-terms that denote the well-orderings needed in the definition of the term t_f (for $f = val(t_f)$) are members of $C(\mathring{a}_0,\mathring{a}_1)$. This proves the lemma.

<u>Theorem</u> (R.Solovay): If ZF is consistent, then so is ZF + $\bigwedge_\alpha(AC^\alpha)$ + + (DC^ω) + $\neg(DC^{\omega_1})$.

The model used by Solovay is the Cohen-generic extension $\mathfrak{N} \triangleq \mathfrak{M}[a_0,\ldots,a_\gamma,\ldots]_{\gamma<\omega_1^{\mathfrak{M}}}$ of a countable standard model $\mathfrak{M}$ of **ZF + V = L which results from $\mathfrak{M}$ by adding ω_1 many (in the sense of** $\mathfrak{M}$) generic reals a_γ ($\gamma < \omega_1^{\mathfrak{M}}$) to $\mathfrak{M}$ but no set collecting these reals.

Another result in this area is due to Tomás Jech from Prague:

[38] T.JECH: Interdependence of weakened forms of the axiom of choice; Comment.Math.Univ.Carolinae, Prague, vol.7(1966)p.359-371, Corrections, vol.8(1967) page 567.

<u>Theorem</u> (J.Jech **[38]**): Let $\mathfrak{M}$ be a countable standard model of ZF + (AC) and α a regular infinite cardinal in $\mathfrak{M}$. Then there is an extension $\mathfrak{N}$ of $\mathfrak{M}$ with the same ordinals such that $\mathfrak{N}$ is a ZF-model satisfying (AC^β) and (DC^β) for every $\beta < \alpha$ but neither (AC^α) nor (DC^α) hold in $\mathfrak{N}$.

H.C.Doets asked, whether there is any interdependence between (DC^ω) and the (BPI). The answer follows from results presented in this chapter. In fact, Halpern and Lévy showed that $\mathfrak{M}[a_0,a_1,\ldots,A]$ satisfies (BPI) + $\neg(AC^\omega)$ and hence $\neg(DC^\omega)$ – see pages 100-103 and section G, p.131. On the other hand (DC^ω) + $\neg$(BPI) hold in Feferman's model $\mathfrak{M}[a_0,a_1,\ldots]$.