§ 16. Segal algebras on abelian groups

In § 9, Theorem 1, we established a bijective correspondence between the closed ideals of symmetric Segal algebras $S^1(G)$ and those of $L^1(G)$; if G is abelian, we can prove that this correspondence preserves the existence of approximate units.

THEOREM 1. Let G be any l.c. abelian group, $S^1(G)$ a Segal algebra on G. A closed ideal I_S of $S^1(G)$ belongs to $\mathcal{J}_a(S^1(G))$ if and only if the closure of I_S in $L^1(G)$ belongs to $\mathcal{J}_a(L^1(G))$ (for the notation, see § 14, (i)).

The proof is based on the method of linear functionals (§ 11) and requires some preparation; moreover, the assumption that G is abelian is needed only at one point of the proof. We shall, therefore, carry out our work for general l.c. groups, as long as possible, and introduce the commutativity of G only at the very end.

Let $S^1(G)$ be any Segal algebra, G being a general l.c. group. $S^1(G)$ acts in a natural way on the dual space $S^1(G)'$: for $f \in S^1(G)$ we define the operator $f^* \ast$ on $S^1(G)'$ by the relation (cf. § 11, (6) for the notation)

(1) $$\langle g , f^* \ast \phi_S \rangle_S = \langle f \ast g , \phi_S \rangle_S \quad \text{for all } g \in S^1(G).$$

This means that the operator $f^* \ast$ is the adjoint of the left multiplication operator on $S^1(G)$; this is one reason for the notation. We note that

(2) $$(f_1 \ast f_2)^* \ast \phi_S = f_2^* \ast f_1^* \ast \phi_S.$$

Another reason is provided by the following lemma which will be essential for the proof of Theorem 1.

LEMMA 1. Let $S^1(G)$ be a symmetric Segal algebra. Then the functional $f^* \ast \phi_S$ in $S^1(G)'$, obtained from $\phi_S \in S^1(G)'$ by means of the operator $f^*\ast$ $(f \in S^1(G))$, corresponds to a bounded continuous function $x \to f^* \ast \phi_S(x)$ on G (abuse of notation!) and we have

$$(3) \qquad f^* \ast \phi_S(x) = \overline{\langle R_x f , \phi_S \rangle_S} \qquad x \in G.$$

In other words, if $f^* \ast \phi_S(x)$ is defined by (3), then we have

$$(4) \qquad \langle g , f^* \ast \phi_S \rangle_S = \int g(x)\overline{f^* \ast \phi_S(x)}\, dx \qquad g \in S^1(G).$$

In particular, if the functional ϕ_S corresponds to $\phi \in L^\infty(G)$, i.e. if ϕ_S is given by

$$f \to \int f(x)\overline{\phi(x)}\, dx \qquad f \in S^1(G),$$

then (3), with ϕ in the place of ϕ_S, agrees with the customary definition of the function $f^* \ast \phi$ as the convolution of f^* with ϕ.

Proof. The relation (4) has already been proved: it is another formulation of § 11, Lemma 1, relation (9), with f and g interchanged. The customary definition of the function $f^* \ast \phi$, for $\phi \in L^\infty(G)$, is

$$\int f^*(y)\phi(y^{-1}x)\, dy = \int \overline{f(y)}\phi(yx)\, dy,$$

which agrees with (3) when $\phi_S \in S^1(G)'$ corresponds to $\phi \in L^\infty(G)$ (cf. also § 11, (7)). Thus Lemma 1 is proved.

We note that (3) gives for $x = e$

$$(5) \qquad f^* \ast \phi_S(e) = \overline{\langle f , \phi_S \rangle_S}.$$

Another lemma useful in the proof of Theorem 1 is given below. We shall write $\phi_S \perp I$ ($\phi_S \in S^1(G)'$, $I \subset S^1(G)$) to indicate that $\langle f , \phi_S \rangle_S = 0$ for all $f \in I$.

LEMMA 2.

(i) Let $S^1(G)$ be any Segal algebra; let I be a left ideal (closed or not) of $S^1(G)$. Then $\phi_S \perp I$ holds if and only if $f^* * \phi_S \perp I$ holds for all $f \in S^1(G)$.

(ii) Let $S^1(G)$ be <u>symmetric or pseudosymmetric</u>; let I be a <u>right</u> ideal (closed or not) of $S^1(G)$. Then $\phi_S \perp I$ holds if and only if $f^* * \phi_S = 0$ for all $f \in I$.

<u>Proof</u>.

(i) If $\langle f_o , \phi_S \rangle_S = 0$ for each $f_o \in I$, then

$$\langle f * f_o , \phi_S \rangle_S = 0$$

for all $f \in S^1(G)$, since I is a left ideal, whence

$$\langle f_o , f^* * \phi_S \rangle_S = 0$$

for all $f \in S^1(G)$, and each $f_o \in I$. Conversely, if this last relation holds for each $f_o \in I$ and all $f \in S^1(G)$, then

$$\langle f * f_o , \phi_S \rangle_S = 0;$$

for fixed f_o we then take $f = u_n$, with $(u_n)_{n \geq 1}$ in $S^1(G)$ such that

$$u_n * f_o \to f_o \text{ in } S^1(G) \quad (n \to \infty),$$

which yields $\langle f_o , \phi_S \rangle_S = 0$ for all $f_o \in I$.

(ii) Suppose $\langle f , \phi_S \rangle_S = 0$ for all $f \in I$. Then, since I is a right ideal, we have

$$\langle f * g , \phi_S \rangle_S = 0 \quad \text{for all } g \in S^1(G),$$

whence

$$\langle g , f^* * \phi_S \rangle_S = 0 \quad \text{for all } g \in S^1(G),$$

that is

$$f^* * \phi_S = 0 \quad \text{for all } f \in I.$$

Conversely, if this last condition holds, then - working backwards -

we obtain

$$\langle f * g \, , \, \phi_S \rangle_S \;=\; 0 \quad \text{for every } g \in S^1(G).$$

Now $S^1(G)$, being symmetric or pseudosymmetric, has right approximate units (§ 8, Proposition 1, (ii), (iii)); thus we can, for fixed $f \in I$, put $g = u_n$, with $(u_n)_{n \geq 1}$ in $S^1(G)$ such that

$$f * u_n \to f \text{ in } S^1(G) \quad (n \to \infty).$$

This yields $\langle f \, , \, \phi_S \rangle_S = 0$ for all $f \in I$.

We can now prove some propositions from which Theorem 1 will follow.

PROPOSITION 1. Let $S^1(G)$ be a symmetric Segal algebra; let I_S be a closed right ideal of $S^1(G)$. Then I_S has approximate right units if and only if the following condition (C_S) is satisfied:

(C_S) Whenever $f \in I_S$ and $\phi_S \in S^1(G)'$ are such that $f^* * \phi_S \perp I_S$ holds, then $f^* * \phi_S = 0$.

Proof. Suppose I_S has approximate right units and let $f \in I_S$ and $\phi_S \in S^1(G)'$ be such that

$$f^* * \phi_S \perp I$$

holds. Choose $(u_n)_{n \geq 1}$ in I_S so that

$$f * u_n \to f \text{ in } S^1(G) \quad (n \to \infty);$$

then

$$u_n^* * (f^* * \phi_S) \;=\; 0 \quad \text{for each } n$$

(cf. Lemma 2 (ii)), or

$$(f * u_n)^* * \phi_S \;=\; 0$$

(cf. (2)), whence for $n \to \infty$ results:

$$f^* * \phi_S = 0.$$

Conversely, suppose (C_S) holds. Given any $f \in I_S$, consider the right ideal

$$I_f = \{f * g \mid g \in I_S\}$$

of $S^1(G)$; we want to show: f lies in the $S^1(G)$-closure of I_f. Consider any $\phi_S \perp I_f$: thus, by Lemma 2, (ii) again,

$$(f * g)^* * \phi_S = 0 \quad \text{for all } g \in I_S ,$$

or $g^* * (f^* * \phi_S) = 0$ for all $g \in I_S$, whence

$$f^* * \phi_S \perp I_S ,$$

by the same lemma. Thus, by the assumed condition (C_S), we have

$$f^* * \phi_S = 0 ,$$

whence $\langle f , \phi_S \rangle_S = 0$ (cf. Lemma 1 and relation (5)). Since $\phi_S \perp I_f$ was arbitrary, the desired result follows. Thus the proof is complete.

REMARK 1. In the case $S^1(G) = L^1(G)$, we simply write <u>condition</u> (C) instead of (C_S); explicitly this reads, for a closed <u>right</u> ideal I of $L^1(G)$:

$$\text{(C)} \quad \begin{cases} \text{Whenever } f \in I \text{ and } \phi \in L^\infty(G) \text{ are such} \\ \text{that } f^* * \phi \perp I \text{ holds, then } f^* * \phi = 0. \end{cases}$$

In this context we mention again that here $f^* * \phi$ may be interpreted in the customary way (cf. Lemma 1).

PROPOSITION 2. Let $S^1(G)$ be a <u>symmetric</u> Segal algebra and let I be a closed <u>two-sided</u> ideal of $L^1(G)$; put $I_S = I \cap S^1(G)$, so that I_S

is a closed, two-sided ideal of $S^1(G)$. If I_S satisfies (C_S), then I satisfies (C).

<u>Proof</u>. Let $f \in I$ and $\phi \in L^\infty$ be such that $f^* * \phi \perp I$. There is a sequence $(u_n)_{n \geq 1}$ in $S^1(G)$ such that

$$f * u_n \to f \text{ in } L^1(G) \quad (n \to \infty);$$

cf. part (i) of the proof of Theorem 1 in § 9. Now $f * u_n$ lies in $S^1(G)$ (since $S^1(G)$ is a left ideal in $L^1(G)$) and also in I (since I is, in particular, a right ideal); thus $f * u_n$ lies in I_S. Now we also have

$$u_n^* * (f^* * \phi) \perp I,$$

by Lemma 2, (i) (since I is also a left ideal), or

$$(f * u_n)^* * \phi \perp I,$$

and in particular

$$(f * u_n)^* * \phi \perp I_S.$$

Since $f * u_n \in I_S$, we can apply (C_S) which yields

$$(f * u_n)^* * \phi = 0,$$

and for $n \to \infty$ we obtain

$$f^* * \phi = 0,$$

i.e. (C) holds for I.

PROPOSITION 3. Let G be a l.c. <u>abelian</u> group and $S^1(G)$ a Segal algebra on G. Let I_S be a closed ideal of $S^1(G)$ and let I be the closure of I_S in $L^1(G)$. If I satisfies (C), then I_S satisfies (C_S).

<u>Proof</u>. Let $f \in I_S$ and $\phi_S \in S^1(G)'$ be such that

$$f^* * \phi_S \perp I_S.$$

There is a sequence $(u_n)_{n\geq 1}$ in $S^1(G)$ such that

$$u_n * f \to f \text{ in } S^1(G) \quad (n \to \infty).$$

Now we also have, for each n,

$$u_n^* * (f^* * \phi_S) \perp I_S,$$

by Lemma 2 (i); hence, <u>since</u> G <u>is abelian</u>,

$$f^* * (u_n^* * \phi_S) \perp I_S. \tag{6}$$

Note that here $u_n^* * \phi_S$ may be considered as a function ϕ_n in $L^\infty(G)$ (cf. Lemma 1), and $f^* * \phi_n$ is an ordinary convolution, by the same lemma, with f^* defined in the usual way. Passing now to I, the closure of I_S in $L^1(G)$, we obtain from (6)

$$f^* * (u_n^* * \phi_S) \perp I.$$

Since $f \in I_S \subset I$ and $u_n^* * \phi_S \in L^\infty(G)$, we can apply the assumed condition (C): it follows that, for each n,

$$f^* * (u_n^* * \phi_S) = 0,$$

or

$$(u_n * f)^* * \phi_S = 0.$$

Letting here $n \to \infty$, we obtain

$$f^* * \phi_S = 0,$$

i.e. (C_S) holds for I_S.

<u>Proof of Theorem</u> 1. This now results from § 9, Theorem 1 (or Ch. 6, § 2.4), and from Propositions 1, 2, 3 above.

Theorem 1 reduces the investigation of $\mathcal{I}_a(S^1(G))$, for <u>abelian</u> G, to that of $\mathcal{I}_a(L^1(G))$; incidentally, already for abelian G <u>not all</u>

closed ideals of $L^1(G)$ belong to $\mathcal{J}_a(L^1(G))$ if G is not compact (cf. § 17), in contrast to § 15, Theorem 1.

It is an open question whether for Segal algebras on <u>abelian</u> l.c. groups G every ideal in $\mathcal{J}_a(S^1(G))$ possesses approximate units having a positive Fourier transform with compact support: this would be some analogy to § 15, Theorem 1. In the other direction, it is also an open question how far Theorem 1 above can be extended to <u>general</u> l.c. groups.

In connection with Theorem 1 let us restate a familiar definition.

DEFINITION. Let G be a l.c. abelian group, $S^1(G)$ a Segal algebra. A closed set $\hat{E}$ in the dual group $\hat{G}$ is said to be a <u>Wiener-Ditkin set</u> for $S^1(G)$ if there is only one closed ideal I_S of $S^1(G)$ such that cosp $I_S = \hat{E}$ (i.e. if $\hat{E}$ is a <u>Wiener set</u> for $S^1(G)$) and if this ideal I_S belongs to $\mathcal{J}_a(S^1(G))$.

This definition agrees with that given in Ch. 2, § 5.2 (cf. especially the Remark loc. cit.).

We now have immediately the following result:

COROLLARY of Theorem 1. For any Segal algebra $S^1(G)$ the Wiener-Ditkin sets are the same as those for $L^1(G)$.

<u>Proof</u>. This is a simple consequence of § 9, Theorem 1 (or Ch. 6, § 2.4), and Theorem 1 above.

In particular, single points of the dual group $\hat{G}$ are Wiener-Ditkin sets for $S^1(G)$. It is useful to verify directly that <u>a Segal algebra satisfies the condition of Wiener-Ditkin</u>: given any $\hat{a} \in \hat{G}$ and $f \in S^1(G)$ such that $\hat{f}(\hat{a}) = 0$, there is for every $\varepsilon > 0$ a $\tau \in S^1(G)$ such that

$$\hat{\tau}(\hat{x}) = 1 \ \underline{\text{near}}\ \hat{a} \text{ and } \|f * \tau\|_S < \varepsilon.$$

This can be verified very simply by reduction to $L^1(G)$, as follows. There is an $h \in S^1(G)$ such that $\hat{h}$ has compact support and $\hat{h}(\hat{x}) = 1$ near $\hat{a}$ (cf. Ch. 6, § 2.2 (iii)). $L^1(G)$ satisfies the condition of Wiener-Ditkin (cf. Ch. 6, § 1.4), thus there is a $\tau_1 \in L^1(G)$ such that $\hat{\tau}_1(\hat{x}) = 1$ near $\hat{a}$ and

$$\|f * \tau_1\|_1 < \varepsilon/\|h\|_S.$$

Let us put $\tau = \tau_1 * h$; then $\tau \in S^1(G)$ (since $h \in S^1(G)$), $\hat{\tau}(\hat{x}) = 1$ near $\hat{a}$, and

$$\|f * \tau\|_S \leqslant \|f * \tau_1\|_1 . \|h\|_S < \varepsilon.$$

REMARK 2. By the Corollary above, we may now simply speak of ' <u>Wiener-Ditkin sets in</u> $\hat{G}$ ', without any ambiguity. Likewise we may say ' Wiener set in $\hat{G}$ ': this is already familiar from Ch. 6, § 2.4. In practice, we are thus free to consider only $L^1(G)$.

Let us finally discuss anew the <u>injection theorem for Wiener-Ditkin sets</u> which was given in Ch. 7, § 4.5 and reads as follows. Let Γ be a closed subgroup of the dual group $\hat{G}$ and let $\hat{E}$ be a closed subset of Γ. Then $\hat{E}$ is a Wiener-Ditkin set in Γ if and only if $\hat{E}$ is a Wiener-Ditkin set in $\hat{G}$.

With the tools at our disposal, we can give the <u>proof</u> as follows. First, $\hat{E}$ is a Wiener set in Γ if and only if $\hat{E}$ is a Wiener set in $\hat{G}$: this is the injection theorem for Wiener sets (cf. Ch. 7, § 3.8). Now let $H \subset G$ be the orthogonal subgroup of $\Gamma \subset \hat{G}$, i.e. $(G/H)\hat{} = \Gamma$. Let I be the (unique) closed ideal of $L^1(G/H)$ with cospectrum $\hat{E}$; then the (unique) closed ideal of $L^1(G)$ with cospectrum $\hat{E}$ is clearly $T_H^{-1}(I)$ (cf. Ch. 4, § 4.3). If I lies in $\mathcal{J}_a(L^1(G/H))$, then $T_H^{-1}(I)$ lies in $\mathcal{J}_a(L^1(G))$: see § 14, Corollary 1 (with $S^1(G) = L^1(G)$ and $n = 1$); the converse is obvious, which completes the proof.

REMARK 3. The proof of the injection theorem for Wiener-Ditkin sets given here should be compared with that in Ch. 7, §§ 4.4 and 4.7. Corollary 1 of § 14 (cf. also Lemma 2 of § 8) represents, as it were, the 'non-commutative part' of that proof. A comparison of the two proofs will show that by separating the non-commutative and the strictly commutative parts one obtains a simpler proof and a clearer insight into the structure. Similarly, the conditions (C_S) and (C) used in the proof of Theorem 1 above embody the non-commutative part of the criterion of Herz-Glicksberg (Ch. 7, § 4.9).

In connection with Remark 3 the following may be mentioned. It often occurs that the proof of a result in classical harmonic analysis may be divided into two parts: one that admits of an extension to non-abelian groups and another, strictly abelian one. A clear recognition of these two components is of considerable interest: it leads not only to more general results, but also to greater simplicity in the proofs.