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RESEARCH ARTICLE

THE $\mathcal{D}$-CATEGORY OF A MONOID

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One of the continuing themes for study in algebraic semigroups is the interplay between multiplication and the various structures relating to division. Even though the concept of division is initially dependent upon multiplication, the various types of division which have been defined for semigroups form important tools for the analysis of the multiplicative structure of a semigroup. In particular, the five equivalences of Green ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$) have occupied a central position in the development of semigroup theory. Corresponding to the equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ are four quasi-orders which provide four connected global division structures on a semigroup. (See 3.2 below.) In [5] the author introduced the division category of a semigroup S, denoted $\mathcal{D}(S)$, and in [6] showed it to be the proper global division structure corresponding to the equivalence $\mathcal{D}$. Then in [8], M. P. Loganathan presented an alternative division category for a semigroup. While not isomorphic with each other, both categories are equivalent in the case of a regular semigroup. From [5], [8] and [10] it is clear that division categories of one type or another are indispensable in (co-)extension theory. Yet it seems equally clear that division categories have an independent significance due to their role in the continuing development of global division theory for semigroups.

In this paper we continue the study of the category $\mathcal{D}(S)$, but operate under the mild restriction that the involved semigroups are monoids. The prime focus is on the connections between properties possessed by either the division category or the underlying monoid. The first section provides some general background material on $\mathcal{D}(S)$. The next section studies conditions on S which ensure that $\mathcal{D}(S)$ has an initial object. This leads to an examination in the third section of conditions under which $\mathcal{D}(S)$ is a quasi-order, in which case $\mathcal{D}(S)$ may be identified with the $\mathcal{J}$ quasi-order. This will occur precisely when the category $\mathcal{D}(S)$ has powers [or copowers]. Exact conditions are provided for $\mathcal{D}(S)$ to be identified with any of the $\mathcal{L}$, $\mathcal{R}$, or $\mathcal{H}$ quasi-orders. This is followed by a brief section on the existence of terminal objects in $\mathcal{D}(S)$.

In the final section we look at examples. The skeleton of $\mathcal{D}(S)$ is computed for some important classes of monoids. In the process, some connections with elementary combinatorics are uncovered. (See 5.4-6.) While most examples involve inverse semigroups, a general study of $\mathcal{D}(S)$ for inverse monoids occurs in other papers. We refer the reader to 1.6 below for extended comments and references on this subject.

The four natural quasi-orders seem to play a greater role in both compact semigroups and ordered semigroups. (However, terminology may vary. See, e.g., [2]A3.5-7.) This is hardly surprising. Indeed quasi-orders, like partial orders, may be considered primitive geometric structures. In this regard, the category $\mathcal{D}(S)$ may be regarded in part to be a directed graph which contributes to a global picture of the underlying monoid. For multiplicative semilattices, all four quasi-orders coincide with each other and with $\mathcal{D}(S)$, yielding the corresponding order semilattice. Indeed, much of what occurs in this paper and in [1] and [7] may be understood as partially generalizing the fundamental correspondence between the algebraic concept of a semilattice (as a semigroup) and the geometric concept of a semilattice (as a partially ordered set).

SECTION 1. SOME BACKGROUND

1.1 Let S be a monoid. As in [6], D(S) denotes the category whose set of objects is S, whose morphisms are ordered triples $\langle u,x,v\rangle : x \to uxv$, and whose composition is given by

$$\langle u',uxv,v'\rangle\langle u,x,v\rangle = \langle u'u,x,vv'\rangle.$$

Each morphism set, Hom(x,y), is thus the (possibly empty) set of distinct ways in which x divides y in S.

1.2 Between morphisms in D(S), let $\langle u,x,v\rangle \sim \langle u',x,v'\rangle$ stand for

$$uxv = uxv' = u'xv = u'xv',$$

and denote the congruence on D(S) generated from this relation by $\simeq$. By the *division category* of a monoid S, denoted $\mathcal{D}(S)$, is meant the quotient category $D(S)/\simeq$. The congruence class of $\langle u,x,v\rangle$ in D(S) is a morphism in $\mathcal{D}(S)$ to be denoted by [u,x,v]. In $\mathcal{D}(S)$, the morphism set Hom(x,y) represents the *essentially* distinct ways in which x divides y. The following theorem about isomorphisms in $\mathcal{D}(S)$ is a partial translation of standard $\mathcal{D}$-class theory into the setting of $\mathcal{D}(S)$.

1.3 *THEOREM. Let* S *be a monoid. In the division category,* $\mathcal{D}(S)$, *the following hold*:

i) $x \simeq y$ *if and only if* $x\mathcal{D}y$ *in* S.

ii) [u,x,v] *is an isomorphism if and only if* $ux\mathcal{L}x\mathcal{R}xv$ *in* S.

iii) *The class of all isomorphisms in* $\mathcal{D}(S)$ *is coprime: a composite of morphisms is an isomorphism precisely when each morphism is an isomorphism.*

iv) *Aut(x)* $\cong$ *H(x), the (left) Schützenberger group of* $\mathcal{H}_x$*, under the bijection* $[u,x,v] \to [u]$.

v) *S is stable (and thus* $\mathcal{J} = \mathcal{D}$*) precisely when Aut(x) = End(x) for each* $x \in S$.

Proof. See [6]2.1, 8, 9, 10, 17 and 20.

We now present some elementary facts about regular elements.

1.4 *THEOREM. Let S be a monoid and let* $x, y \in S$. *Then:*

i) *x is regular if there exists e in E(S) with* $x \cong e$ *in* $\mathcal{D}(S)$.

ii) *Let x and y be regular. If x divides y, then there exist e, f in E(S) with* $e\mathcal{D}x$, $f\mathcal{D}y$ *and* $e \geq f$. *Moreover,* $x \cong e$ *and* $y \cong f$ *in* $\mathcal{D}(S)$ *so that Hom(x,y) is bijective with Hom(e,f).*

iii) *In particular, if* $e \in E(S)$, *y is regular and e divides y, then there exists f in* $E(S) \cap \mathcal{D}_y$ *such that* $e \geq f$.

Proof. (i) follows from 1.3(i). Suppose that x and y are regular elements with $y \in SxS$. Pick $e \in \mathcal{D}_x$ and $g \in \mathcal{D}_y$ with $e, g \in E(S)$. From $y \in SxS$ it follows that $g \in SeS$. Let $g = uev$. Set $f = evgue$. It easily follows that $f \in E(S)$, $f \in \mathcal{D}_g$, and hence $f \in \mathcal{D}_y$. Clearly $e \geq f$. Again 1.3(i) implies that $x \cong e$ and $y \cong f$ in $\mathcal{D}(S)$ and (ii) follows.

1.5 *THE FULL EMBEDDING THEOREM. Let S be a monoid and let* $e \in E(S)$. *Then the inclusion* $eSe \subseteq S$ *induces a full embedding,* $\mathcal{D}(eSe) \subseteq \mathcal{D}(S)$. *The embedding is also dense, and hence an equivalence of categories, if and only if* $e \in \mathcal{J}_1$.

Proof. Let $x, y \in eSe$ and let $\langle u,x,v\rangle : x \to y$ in D(S). Then $\langle eue,x,eve\rangle : x \to y$ in D(eSe), and $\langle u,x,v\rangle \sim \langle eue,x,eve\rangle$. If $\langle u',x,v'\rangle : x \to y$ also in D(S), then $\langle u,x,v\rangle \sim \langle u',x,v'\rangle$ if and only if in D(eSe), $\langle eue,x,eve\rangle \sim \langle eu'e,x,ev'e\rangle$. The first part of the theorem is seen. If the embedding is dense, then $eSe \cap \mathcal{D}_1$ is nonempty so that $e \in \mathcal{J}_1$. If $e \in \mathcal{J}_1$, then by 1.4 there exists an indempotent f in $eSe \cap \mathcal{D}_1$. Let u,v in S be such that uv = 1 and vu = f. Then for all y in S, vyuεfSf and hence vyuεeSe. But vyu$\mathcal{D}$y for all y, so by 1.3(i) we are done.

1.6 For inverse monoids, the above theorem has a converse. In [1], Aznar and Sevilla show that _for a pair of inverse monoids_, S _and_ T, _the following are equivalent_: (i) $\mathcal{D}(S) \cong \mathcal{D}(T)$; (ii) $\mathcal{D}(S)$ _and_ $\mathcal{D}(T)$ _are equivalent categories_; (iii) _there exists a semigroup embedding_, $\phi: S \to T$ _such that_ $\mathcal{D}\phi: \mathcal{D}(S) \to \mathcal{D}(T)$ _is an equivalence functor_; _and_ (iv) _there exists_ $e \varepsilon E(T) \cap J_1^T$ _and an isomorphism_, $\phi: S \cong eTe$. Thus inverse monoids are largely determined by their division structure. In [1] Aznar and Sevilla provide a categorical description of the division categories of inverse monoids. This has been expanded upon by the author in [7] and used to extend the Clifford construction of bisimple inverse monoids to arbitrary inverse monoids. The work of Aznar and Sevilla arose in response to conjectures made along these lines by the author in the previous unrevised version of the present paper.

1.7 It is well known that if e,f in E(S) lie in the same $\mathcal{D}$-class, then $eSe \cong fSf$. But what if e and f only lie in the same J-class. One can use Theorem 1.5 to find a relationship between eSe and fSf, and thus demonstrate the extent to which regular J-classes are homogeneous.

1.8 _THEOREM_. _Let_ S _be a monoid and let_ e,f _in_ E(S) _lie in the same_ J-_class_. _Then_ $\mathcal{D}(eSe) \cong \mathcal{D}(fSf)$. _If_ $e \varepsilon J_1$, _then_ $\mathcal{D}(eSe) \cong \mathcal{D}(S)$.

Proof. Suppose that $e \varepsilon J_1$. Let f,u,v in $\mathcal{D}_1$ be as in the previous proof. Let $x \to vxu$ be the isomorphism from S onto fSf such that $x\mathcal{D}vxu$ for all x in S. The chain $1 \geq e \geq f$ induces a chain of full embeddings, $\mathcal{D}(fSf) \subsetneq \mathcal{D}(eSe) \subsetneq \mathcal{D}(S)$, which are all equivalence functors. Consider ascending chains of isomorphism classes:

$\mathcal{D}^{fSf}_{fxf} \subseteq \mathcal{D}^{eSe}_{fxf} \subseteq \mathcal{D}^{S}_{fxf}$. They all have a common cardinality, since by the above isomorphism and the fullness of the embedding, fxf $\mathcal{D}$ v(fxf)u, first in S and then in fSf. Thus $|\mathcal{D}^{S}_{fxf}| = |\mathcal{D}^{fSf}_{v(fxf)u}| = |\mathcal{D}^{fSf}_{fxf}|$. Hence there must exist an equivalence functor $E:\mathcal{D}(eSe) \to \mathcal{D}(S)$ which is bijective between isomorphism classes; that is, an isomorphism of small categories. For the general case, we use 1.4(iii) to find g in $E(S) \cap \mathcal{D}_f$ such that $e \geq g$ and $e\mathcal{J}g$. We have fSf $\cong$ gSg so that $\mathcal{D}(fSf) \cong \mathcal{D}(gSg)$. But gSg = g(eSe)g and $e\mathcal{J}g$ also in eSe. By the particular case, $\mathcal{D}(eSe) \cong \mathcal{D}(gSg)$ and the theorem follows.

SECTION 2. INITIAL OBJECTS

2.1 Recall that an initial object in a category $\underline{\underline{K}}$ is an object I such that for any object X, Hom(I,X) has exactly one morphism. Initial objects, if they exist, are unique to within isomorphism. In $\mathcal{D}(S)$, the obvious candidate for an initial object is 1.

2.2 THEOREM. *Let S be a monoid. If $\mathcal{D}(S)$ has an initial object, then it is unique and equals 1. Moreover S - {1} is an ideal. In general, 1 is the initial object in $\mathcal{D}(S)$ if and only if for all x in S, [x,1,1] = [1,1,x].*

Proof. Let I be an initial object of S. Then there exist $u,v \in S$ such that $[u,I,v]:I \to 1$. Since I is initial, [1,I,1] = [I,1,1][u,I,v] so that [I,1,1] is an isomorphism as isomorphisms are coprime. Hence $I\mathcal{L}1$ by 1.3(ii) and dually $I\mathcal{R}1$ so that $1\mathcal{H}I$. But $Aut(I) = 1_I$ implies $\mathcal{H}_I$ = I, i.e., 1 = I and 1 is the initial object of $\mathcal{D}(S)$. Let $x \in \mathcal{J}_I$. If uxv = 1, then [u,x,v][x,1,1] = [1,1,1] since End(1) is trivial. Again $[x,1,1]:1 \cong x$ so that x, being initial, is equal to 1. Hence $\mathcal{J}_I$ = 1 and S - {1} is an ideal. The given condition is necessary. If the condition holds and uv = x, then [u,1,v] = [u,v,1][1,1,v] = [u,v,1][v,1,1] = [uv,1,1] = [x,1,1]. Thus Hom(1,x) = {[x,1,1]}.

2.3 COROLLARY. *If* 1 *is initial in* $\mathcal{D}(S)$ *and* T *is an epimorphic image of* S, *then* 1_T *is initial in* $\mathcal{D}(T)$.

Proof. The equality of the theorem is inherited by T.

2.4 From this corollary it is easy to see that asserting that 1 is initial in $\mathcal{D}(S)$ is stronger than asserting that $J_1 = \{1\}$. There are many epimorphic pairs $S \to T$ with $J_1 = \{1\}$ in S, but not in T.

2.5 NOTATION. $V(S) = \{x \in S : [x,1,1] = [1,1,x]\}$. Clearly 1 is initial in $\mathcal{D}(S)$ if and only if $V(S) = S$.

2.6 THEOREM. $V(S)$ *is a submonoid of* S. *Moreover,* $\langle E\rangle \subseteq V(S)$, *where* $\langle E\rangle$ *is the submonoid generated by the set of idempotents,* E.

Proof. If $e \in E$, then $\langle e,1,1\rangle \sim \langle e,1,e\rangle \sim \langle 1,1,e\rangle$. Thus $E \subseteq V$. Let $x,y \in V$. Then:

$$\begin{aligned}[xy,1,1] &= [x,y,1][y,1,1] = [x,y,1][1,1,y] = [x,1,y] \\ &= [1,x,y][x,1,1] = [1,x,y][1,1,x] = [1,1,xy].\end{aligned}$$

Thus $xy \in V$. Since $1 \in E$ and $E \subseteq V$, V is a submonoid and the theorem follows.

2.7 From 2.2 and 2.6 it is clear that any idempotent-generated monoid has 1 initial in $\mathcal{D}(S)$. For many kinds of monoids this actually characterizes the case for 1 being an initial object.

2.8 THEOREM. *Let* X *be a regular* $\mathcal{D}$-*class. Then:*

i) $X \cap V(S) = X \cap \langle E\rangle$.

Moreover, the following are equivalent:

ii) *For all* $x \in X$, $Hom(1,x) = \{[x,1,1]\}$.

iii) $X \subseteq V(S)$.

iv) $X \subseteq \langle E\rangle$.

Proof. That $X \cap \langle E\rangle \subseteq X \cap V$ is clear. So let $x \in X \cap V$. Thus, $[x,1,1] = [1,1,x]$. Compose this equality with $[1,x,x']$ where x' is an inverse of x. We thus obtain $[x,1,x'] = [1,1,xx'] = [xx',1,xx']$ where $xx' \in X \cap E$. Thus by [6]3.6 there is a path of idempotents $e_1, e_2, \ldots, e_n$ where $e_1 = xx'$, $e_n = x'x$, for each $i \leq n-1$ either $e_i \mathcal{L} e_{i+1}$

or $e_i R e_{i+1}$ and $\langle e_1 e_2 \ldots e_n, 1, e_n \ldots e_2 e_1\rangle = \langle x, 1, x'\rangle$ in $D(S)$. Thus $x \in \langle E\rangle$, and (i) follows. Clearly (iii) and (iv) are now equivalent, with (ii) implying both. Assume (iii) and let $[u,1,v]: 1 \to x$ where $x \in X$. Pick $e, f \in E$ such that $e \in R_x$ and $f \in L_x$. Then $eu \in R_x$ and $vf \in L_x$ so that $eu, vf \in X$. By (iii) applied to vf we have:

$$\begin{aligned}[u,1,v] &= [eu,1,vf] = [eu,vf,1][1,1,vf] \\ &= [eu,vf,1][vf,1,1] = [euvf,1,1] \\ &= [x,1,1].\end{aligned}$$

Thus (ii) follows and we are done.

2.9 EXAMPLE. Let X be a finite set and let $T(X)$ be the full transformation monoid on X. It is well known that the set of nonunits of $T(X)$ is idempotent generated. Thus if x is a nonunit, since it is regular, we have $\mathrm{Hom}(1,x) = \{[x,1,1]\}$. 1 is not initial in $\mathcal{D}(T(X))$ unless X is a singleton, since $\mathrm{Aut}(1) \cong S(X)$, the full symmetric group on X.

2.10 COROLLARY. *Let S be a regular monoid. Then* $V(S) = \langle E\rangle$, *and* 1 *is initial in* $\mathcal{D}(S)$ *iff* $S = \langle E\rangle$.

2.11 COROLLARY. *Let* $e \geq f$ *in* $E(S)$. *Then* $\mathrm{Hom}(e,f) = \{[f,e,f]\}$ *precisely when* $\mathcal{D}_f \cap eSe \subseteq \langle E(eSe)\rangle$.

Proof. By 1.5, all calculations of Hom(e,f) can take place inside D(eSe). The corollary follows from the theorem, since $\mathcal{D}_f \cap eSe$ is a union of regular $\mathcal{D}$-classes in eSe. Indeed, let x belong to the intersection and let y be an inverse of x in S. Then eye is an inverse of x in eSe.

2.12 We now turn our attention to the situation where S may contain nonregular $\mathcal{D}$-classes. Recall that a monoid S is *stable* if inside each $\mathcal{D}$-class both the L quasi-order coincides with the L equivalence, and the R quasi-order coincides with the R equivalence. For stable monoids, $\mathcal{D} = \mathcal{J}$.

2.13 THEOREM. *Let S be stable and assume that every* $\mathcal{D}$*-class has only finitely many* $\mathcal{D}$*-classes between it and*

$\mathcal{D}_1$. *Then* 1 *is initial in* $\mathcal{D}(S)$ *if and only if* S *is idempotent generated.*

Proof. We need only prove one direction. Suppose 1 is initial in $\mathcal{D}(S)$. If $S \neq \langle E\rangle$, then let $x \notin \langle E\rangle$ be an element in a maximal $\mathcal{D}$-class containing such elements. By 2.8 the class $X = \mathcal{D}_x$ is not regular. Since $[x,1,1] = [1,1,x]$ let us look at a $\sim$-chain from $\langle x,1,1\rangle$ to $\langle 1,1,x\rangle$. The first morphism $\langle x,1,1\rangle$ is such that $x \in X$, but 1 is in a higher (in fact the highest) $\mathcal{D}$-class. Suppose $\langle u,1,v\rangle$ arises in this chain with $u \in X$, but v belonging to a higher $\mathcal{D}$-class. Let $\langle u,1,v\rangle \sim \langle u_1,1,v_1\rangle$, the next morphism in the chain. From $uv_1 = x$, since S is stable and X nonregular, we must have v_1 belonging to a higher $\mathcal{D}$-class. Thus $v_1 \in \langle E\rangle$. From $u_1v_1 = x \notin \langle E\rangle$ we obtain $u_1 \notin \langle E\rangle$. Thus $u_1 \in X$. No matter how far we go in our $\sim$-chain, the left entry of $\langle u,1,v\rangle$ will always be in X. Thus $\langle 1,1,x\rangle$ cannot arise, i.e., $[x,1,1] \neq [1,1,x]$. This is a contradiction. Thus $S = \langle E\rangle$.

2.14 COROLLARY. *If* S *is a finite monoid, then* 1 *is initial in* $\mathcal{D}(S)$ *if and only if* $S = \langle E\rangle$.

2.15 EXAMPLE. We construct an example of a monoid such that $E = \{1\}$, the monoid is stable with infinitely many elements, and 1 is an initial element. Let $S_1 = F(e_1,e_2)$ be a semigroup free on the idempotent generators $\{e_1,e_2\}$. Define S_2 by

$$S_2 = (S_1 - \{e_1,e_2\}) \cup F(e_{11},e_{12}) \cup F(e_{21},e_{22}).$$

Here we are replacing e_1 and e_2 by two copies of S_1. Multiplication on the three semigroup components is left alone, but each multiplication between the components is reduced to multiplication in the higher original component. Thus $e_{11}e_{21} = e_1e_2$ and $e_{11}e_2e_1 = e_1e_2e_1$. We keep repeating the process. Thus if S_n is generated by a set of 2^n idempotents, then replace each idempotent by a copy of $F(e_1,e_2)$ and extend the multiplication from all components as above. Let S be the countable union of the $S_n - E(S_n)$. S is an infinite semigroup without idempotents. But 1 is initial in $\mathcal{D}(S^1)$. We need to show $V(S^1) = S^1$. Clearly S^1 is generated by pairs of the form f_1f_2 where f_1f_2 is the ghost of some previous idempotent f. But

$$\begin{aligned}[f_1f_2,1,1] &= [f_{11}f_{12}f_{21}f_{22},1,1]\\ &= [f_{11}f_{12}f_{21}f_{22},1,f_{21}f_{22}]\\ &= [f_{11}f_{12},1,f_{21}f_{22}]\\ &= [f_{11}f_{12},1,f_{11}f_{12}f_{21}f_{22}]\\ &= [1,1,f_{11}f_{12}f_{21}f_{22}]\\ &= [1,1,f_1f_2].\end{aligned}$$

Thus $f_1f_2 \varepsilon V(S^1)$. Hence $S^1 = V(S^1)$ and we are done.

2.16 Let $\underline{\underline{Ab}}$ denote the category of abelian groups. Let $\underline{\underline{Ab}}^{\mathcal{D}(S)}$ denote the abelian category of all functors from $\mathcal{D}(S)$ to $\underline{\underline{Ab}}$ and natural transformations between them. Let $\underleftarrow{\text{Lim}}:\underline{\underline{Ab}}^{\mathcal{D}(S)} \to \underline{\underline{Ab}}$ be the inverse limit functor. We say that $\mathcal{D}(S)$ has <u>cohomological dimension zero</u> [or codimension zero] if $\underleftarrow{\text{Lim}}$ is an exact functor. We denote this by $cd\mathcal{D}(S) = 0$. If 1 is initial in $\mathcal{D}(S)$, then $\underleftarrow{\text{Lim}}F \cong F(1)$ is a natural isomorphism and $\underleftarrow{\text{Lim}}$ is exact. Thus if 1 is initial in $\mathcal{D}(S)$, then $cd\mathcal{D}(S) = 0$. We may ask whether or not the converse is true. Using a theorem of Laudal which characterizes small categories of codimension zero, it is not hard to see that $cd\mathcal{D}(S) = 0$ if and only if there is a morphism $\alpha\varepsilon\text{End}(1)$ such that for every pair of morphisms $\beta,\gamma:1 \to x$ with a common codomain x, $\beta\circ\alpha = \gamma\circ\alpha$. Clearly when $\mathcal{J}_1 = \{1\}$, $cd\mathcal{D}(S) = 0$ if and only if 1 is initial in $\mathcal{D}(S)$, for then [1,1,1] is the only choice for α. Laudal's Theorem is found in [4].

2.17 <u>THEOREM</u>. <u>Let</u> $\mathcal{J}_1 = \{1\}$. <u>Then</u> 1 <u>is initial</u> in $\mathcal{D}(S)$ <u>if and only if</u> $cd\mathcal{D}(S) = 0$.

SECTION 3. WHEN $\mathcal{D}(S)$ IS A QUASI-ORDER

3.1 Recall that a <u>quasi-ordered set</u> is a pair $(S, \geq)$ where S is a set and $\geq$ is a reflexive, transitive binary operation defined on S. Every quasi-ordered set $(S, \geq)$

may be turned into a small category with object set S by setting Hom(x,y) = $\{[{}^{x}_{y}]\}$ if $x \geq y$, but otherwise letting Hom(x,y) be empty. Reflexivity ensures that $1_x = [{}^{x}_{x}]$ exists for all $x \in S$, while transitivity yields the composition $[{}^{y}_{z}] \circ [{}^{x}_{y}] = [{}^{x}_{z}]$. Such a category has the property that each morphism set, Hom(x,y), has at most one morphism. This property provides a categorical definition of a quasi-ordered set. On the object set of any small category possessing this property, set $x \geq y$ precisely when Hom(x,y) is nonempty. In practice, we identify a quasi-ordered set with its corresponding category.

3.2 On a given monoid, S, four natural quasi-orders are defined: the $\mathcal{J}$, $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ quasi-orders. Here $x \geq y(\mathcal{J})$ iff $y \in SxS$; $x \geq y(\mathcal{L})$ iff $y \in Sx$; $x \geq y(\mathcal{R})$ iff $y \in xS$; and $x \geq y(\mathcal{H})$ iff $y \in Sx \cap xS$. We denote the corresponding small categories by $\mathcal{J}(S)$, $\mathcal{L}(S)$, $\mathcal{R}(S)$ and $\mathcal{H}(S)$. As categories, the quasi-orders $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are canonically embedded in $\mathcal{D}(S)$. $\mathcal{L}(S)$ ($\mathcal{R}(S)$) is identified with the subcategory of $\mathcal{D}(S)$ determined by choosing only morphisms of the form [u,x,1]([1,x,v]). Using either $\mathcal{L}(S)$ or $\mathcal{R}(S)$, there are two ways to embed $\mathcal{H}(S)$ into $\mathcal{D}(S)$. We identify $\mathcal{J}(S)$ with the maximal quasi-order image of $\mathcal{D}(S)$. Indeed, Hom(x,y) is nonempty in $\mathcal{D}(S)$ precisely when $y \in SxS$, that is, $x \geq y(\mathcal{J})$. We summarize our remarks as follows:

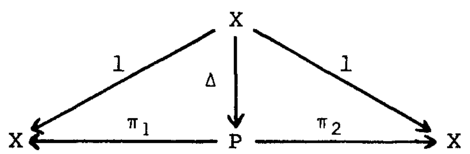
3.3 THEOREM. *Let S be a monoid. The maximal quasi-order image of $\mathcal{D}(S)$ is the quasi-order $\mathcal{J}(S)$. If $\mathcal{D}(S)$ is a quasi-order, then $\mathcal{D}(S) = \mathcal{J}(S)$. Both the quasi-orders, $\mathcal{L}(S)$ and $\mathcal{R}(S)$, lie (isomorphically) inside $\mathcal{D}(S)$. If $\mathcal{L}(S) = \mathcal{D}(S)$, then $\mathcal{L}(S) = \mathcal{J}(S)$ and $\mathcal{H}(S) = \mathcal{R}(S)$. Dually, if $\mathcal{R}(S) = \mathcal{D}(S)$, then $\mathcal{R}(S) = \mathcal{J}(S)$ and $\mathcal{H}(S) = \mathcal{L}(S)$. Finally, $\mathcal{H}(S)$ can be embedded in $\mathcal{D}(S)$ via either $\mathcal{L}(S)$ or $\mathcal{R}(S)$. If $\mathcal{H}(S) = \mathcal{D}(S)$, then $\mathcal{D}(S)$ and all four standard quasi-orders coincide.*

3.4 *COMMENT*. If S is a commutative monoid, then the four standard quasi-orders coincide with each other, but need not equal $\mathcal{D}(S)$. Conditions such as $\mathcal{D}(S) = \mathcal{J}(S)$ or $\mathcal{L}(S) = \mathcal{D}(S)$ are quite strong. By comparison, they are much stronger than $\mathcal{D} = \mathcal{J}$ or $\mathcal{L} = \mathcal{D}$.

3.5 We turn to examine conditions under which $\mathcal{D}(S)$ is a quasi-order. We begin by looking at some categorical equivalences. Recall that products of the form $X \times X$ in a category are called *powers*, while coproducts of the form $X + X$ are called *copowers*.

3.6 *LEMMA*. *Let* $\underline{K}$ *be a category whose class of isomorphisms is coprime, and let* X *be an object of* $\underline{K}$. *Then the power* $X \times X$, *exists if and only if for all objects* U *of* $\underline{K}$, Hom(U,X) *has at most one morphism, in which case* X *may be chosen as its own power. Dual remarks hold for the existence of the copower,* $X + X$.

Proof. If a product diagram $X \overset{\pi_1}{\leftarrow} P \overset{\pi_2}{\rightarrow} X$ exists, then there is a unique morphism $\Delta : X \to P$ such that



commutes. Hence $1_X = \pi_1\Delta = \pi_2\Delta$. Since isomorphisms are coprime in $\underline{K}$, Δ, π_1 and π_2 are isomorphisms with $\pi_1 = \Delta^{-1} = \pi_2$. Hence $X \overset{1}{\leftarrow} X \overset{1}{\rightarrow} X$ is also a product diagram. But this is equivalent to asserting that for all objects U, Hom(U,X) has at most one morphism.

3.7 *THEOREM*. *Let* S *be a monoid. Then the following are equivalent:*

i) $\mathcal{D}(S) = \mathcal{J}(S)$.

ii) *For all* $x \in S$, *the power* $x \times x$ *exists in* $\mathcal{D}(S)$.

iii) *For all* $x \in S$, *the copower* $x + x$ *exists in* $\mathcal{D}(S)$.

When these conditions hold each element x *is both its* own power and copower. Moreover $\mathcal{H}$ is trivial and S is stable.

Proof. All but the last assertion follow from the previous lemma. If the conditions are met, then for all x, End(x) = Aut(x) = {[1,x,1]}. By 1.3(iv) and (v), $\mathcal{H}$ is trivial and S is stable.

3.8 COROLLARY. If $\mathcal{D}$(S) has either finite products or finite coproducts then $\mathcal{D}$(S) = $\mathcal{J}$(S). If $\mathcal{D}$(S) has finite products, then $\mathcal{D}$(S) is equivalent to an upper semilattice. If $\mathcal{D}$(S) has finite coproducts, then $\mathcal{D}$(S) is equivalent to a lower semilattice. If $\mathcal{D}$(S) has both, then it is equivalent to a lattice.

For regular monoids we have:

3.9 THEOREM. If S is regular, then the following are equivalent:

i) $\mathcal{D}$(S) = $\mathcal{J}$(S).

ii) If e ≥ f in E(S), then Hom(e,f) = {[f,e,f]} in $\mathcal{D}$(S).

iii) For each e in E(S), e is initial in $\mathcal{D}$(eSe).

iv) For each e in E(S), eSe is generated by E(eSe).

Proof. Clearly (i) implies (ii). The converse is implied by 1.4(ii). The equivalence of (iii) with (iv) follows from 2.10. Applying 1.5 twice, we have (i) implies (iii) implies (ii).

If a monoid S is a band, then S = E(S). In this case all principal monoids eSe must be idempotent generated.

3.10 THEOREM. Let S be a band. Then:

i) $\mathcal{D}$(S) = $\mathcal{J}$(S).

ii) $\mathcal{D}$(S) is equivalent to the order semilattice, (S/$\mathcal{D}$, ≥).

iii) $\mathcal{D}$(S) has finite coproducts.

iv) $\mathcal{D}$(S) has finite products when (S/$\mathcal{D}$, ≥) is a lattice.

Proof. (i) follows from 3.9(iv). (ii) is a well known property of the $\mathcal{J}$ quasi-order of a band. (iii) and (iv) follow immediately from (ii).

We present two classes of monoids for which $\mathcal{D}(S) = \mathcal{J}(S)$, but the monoid is not a band.

3.11 EXAMPLE. Let $(X, \geq)$ be a finite, totally ordered set. Let $S = S(X, \geq)$ be the semigroup of all order preserving mappings on X. S is regular, all of its principal monoids are generated by their idempotents, and thus $\mathcal{D}(S) = \mathcal{J}(S)$. (See 5.3 below.) S is not a band whenever $|X| \geq 3$.

3.12 EXAMPLE. Let E be a nonempty set. Let T be the monoid freely generated from E as a set of idempotent generators. Thus if 1 is the empty word, then $E(T) = E \cup \{1\}$. Let I be the ideal in T consisting of all words having at least two nonconsecutive occurrences of some element in E. (Thus if e,f are in E, then efe is in I, but not eef = ef.) Let S be the Rees quotient, $S = T/I$. Then $\mathcal{D}(S) = \mathcal{J}(S)$. If $|E| \geq 2$, then S is not regular. Moreover, $\mathcal{D}(S)$ has neither products nor coproducts.

We now examine what happens when any of the quasi-order embeddings of $\mathcal{L}(S)$, $\mathcal{R}(S)$ or $\mathcal{H}(S)$ into $\mathcal{D}(S)$ become identifications.

3.13 THEOREM. Let 1 be initial in $\mathcal{D}(S)$ and let any of the following conditions hold: $\mathcal{R}(S) = \mathcal{J}(S)$, $\mathcal{L}(S) = \mathcal{J}(S)$, or $\mathcal{H}(S) = \mathcal{J}(S)$. Then S is a band and $\mathcal{D}(S) = \mathcal{J}(S)$.

Proof. Assume that $\mathcal{R}(S) = \mathcal{J}(S)$, or equivalently, that $\mathcal{L}(S) \subseteq \mathcal{R}(S)$. Thus for each pair u,v in S, there exists $\tilde{u} = \tilde{u}(u,v)$ such that $uv = v\tilde{u}$. Suppose that we are presented:

$$x = ux = uv = u'v.$$

Then $x = u^n x$ for all $n \geq 0$, and $x = u^n v$ for all $n \geq 1$. What is more, for $\tilde{u}$ given as above, $x = v\tilde{u}^n$ for all $n \geq 1$, and $x = x\tilde{u}^n$ for all $n \geq 0$. Thus

$$u'x = u'uv = u'v\tilde{u} = x\tilde{u} = x.$$

By assumption, 1 is initial in $\mathcal{D}(S)$. By 2.2, for any $x \in S$, $[1,1,x] = [x,1,1]$ and there exists a chain in $D(S)$,

$$\langle 1,1,x\rangle = \langle u_1,1,v_1\rangle \sim \langle u_2,1,v_2\rangle \sim \cdots \sim \langle u_n,1,v_n\rangle = \langle x,1,1\rangle .$$

But by what was just seen, we must have in succession:

$$u_1x = x,\ u_2x = x,\ \cdots,\ u_nx = x,\ xx = x.$$

Thus S is a band and by 3.10, $\mathcal{D}(S) = \mathcal{J}(S)$.

Recall that a band is said to be *right* (*left*) *regular*, if $\mathcal{D} = \mathcal{R}$ ($\mathcal{D} = \mathcal{L}$). The following corollary is an easy consequence of the previous two theorems (3.10, 13).

3.14 *COROLLARY*. *Let* S *be a monoid*. *Then*:

i) $\mathcal{R}(S) = \mathcal{D}(S)$ *iff* S *is a right regular band*.

ii) $\mathcal{L}(S) = \mathcal{D}(S)$ *iff* S *is a left regular band*.

iii) $\mathcal{H}(S) = \mathcal{D}(S)$ *iff* S *is a semilattice*.

If S is commutative, then $\mathcal{H}(S) = \mathcal{J}(S)$. The next theorem summarizes for this case many of the previous results.

3.15 *THEOREM*. *Let* S *be a commutative monoid*. *Then the following are equivalent*:

i) S *is a semilattice*.

ii) $\mathcal{D}(S) = \mathcal{J}(S)$.

iii) 1 *is initial in* $\mathcal{D}(S)$.

iv) $\mathcal{D}(S)$ *has finite coproducts*.

v) $\mathcal{D}(S)$ *has finite copowers*.

vi) $\mathcal{D}(S)$ *has finite powers*.

vii) $\mathrm{cd}\mathcal{D}(S) = 0$

Proof. The equivalence of (ii), (v) and (vi) comes from 3.7. Since $\mathcal{H}(S) = \mathcal{J}(S)$, the equivalence of (i), (ii) and (iii) comes from 3.13, 14. By 3.10, (i) implies (iv), which trivially implies (v). Thus (i) through (vi) are mutually equivalent, with (iii) implying (vii). Assume (vii). Since S is commutative, End(1) = Aut(1). But by the discussion in 2.16, there must exist $\alpha \in \mathrm{Aut}(1)$ such that $\beta\alpha = \gamma\alpha$ for all pairs $\beta, \gamma : 1 \to x$ in $\mathcal{D}(S)$ with a common codomain. Upon cancelling α to obtain $\beta = \gamma$, (iii) must follow.

3.16 In the process of studying relationships between various global division structures, we have obtained characterizations of left (right) regular bands and of semilattices in particular. But what of arbitrary bands? If S is a band, then by 3.10 $\mathcal{D}(S)$ is a quasi-order. Stated otherwise, if all x in S are idempotent as elements of a monoid, then they must also be idempotent as objects in the category $\mathcal{D}(S)$. (See 3.7.) By 3.11,12 the converse need not hold. Notice, however, that the property of being a band is <u>hereditary</u>: if S is a band, then all of its submonoids T are also bands. In this case, the property $\mathcal{D}(S) = \mathcal{J}(S)$ is inherited by all submonoids T of S. In general, the property $\mathcal{D}(S) = \mathcal{J}(S)$ is not hereditary, and to assert that it holds and is hereditary for a monoid S is stronger than the simple assertion that $\mathcal{D}(S) = \mathcal{J}(S)$.

In what follows, $T \leq_1 S$ denotes the fact that T is a submonoid of S.

3.17 <u>THEOREM</u>. <u>Let</u> S <u>be</u> <u>a</u> <u>monoid</u>. <u>Then</u> <u>the</u> <u>following</u> <u>are</u> <u>equivalent</u>:

i) S <u>is</u> <u>a</u> <u>band</u>.
ii) <u>If</u> $T \leq_1 S$, <u>then</u> $\mathcal{D}(T) = \mathcal{J}(T)$.
iii) <u>If</u> $T \leq_1 S$, <u>then</u> 1 <u>is</u> <u>initial</u> <u>in</u> $\mathcal{D}(T)$.
iv) <u>If</u> $T \leq_1 S$, <u>then</u> $cd\mathcal{D}(T) = 0$.

Proof. Assume (i). If $T \leq_1 S$, then T is also a band so that $\mathcal{D}(T) = \mathcal{J}(T)$ by 3.10. Thus (ii)-(iv) must follow from (i). To see the converses, pick x in S and set $T = \langle x \rangle$, the cyclic monoid on x which contains $1 = x^0$. T is clearly a commutative monoid. Moreover, T is a semilattice if and only if $x = x^2$, in which case $T = \{1,x\}$. By Theorem 3.15, any of the conditions (ii)-(iv) on S will first force T to be a semilattice and thus force x to be indempotent. Since this must hold for all x in S, the theorem follows.

A result similar to Theorem 3.15 holds for inverse monoids. But first a lemma.

3.18 LEMMA. *Let S be an inverse monoid and let* $e,f \in E(S)$. *Then* $\mathcal{R}_f \cap Se$ *is bijective with* Hom(e,f) *in* $\mathcal{D}(S)$ *under the map:* $u \to [u,e,u']$, *where* u' *is the inverse of* u *in* S. *In particular,* $\mathcal{R}_f$ *is bijective with* Hom(1,f), *so that* Hom(1,f) *reduces to* $\{[f,1,f]\}$ *precisely when* $\mathcal{R}_f = \{f\}$. *Finally, the above correspondence is multiplicative: if* $g \in E(S)$ *and* $v \in \mathcal{R}_g \cap Sf$, *then* $vu \in \mathcal{R}_g \cap Se$ *corresponds to* $[v,f,v'][u,e,u']$ *in* Hom(e,g).

Proof. The first assertion is just [6]4.10. The remaining assertions are clear.

3.19 THEOREM. *If S is an inverse monoid, then the following are equivalent:*

i) *S is a semilattice.*

ii) $\mathcal{D}(S) = \mathcal{J}(S)$.

iii) $\mathcal{D}(S)$ *has finite coproducts.*

iv) $\mathcal{D}(S)$ *has finite copowers.*

v) $\mathcal{D}(S)$ *has finite powers.*

vi) 1 *is initial in* $\mathcal{D}(S)$.

vii) $cd\mathcal{D}(S) = 0$.

Proof. The equivalence of (ii), (iv) and (v) again comes from 3.7. That (ii) implies (vi) is trivial, and by 3.18 above, (vi) implies (i). By 3.10, (i) implies (iii) which trivially implies (iv). Thus (i) - (vi) are equivalent, with (vi) implying (vii). Assume (vii). Again, as in the previous proof, there must exist $\alpha \in End(1)$ such that $\beta\alpha = \gamma\alpha$ for all pairs $\beta,\gamma:1 \to x$ with common codomain. Now End(1) is isomorphic with the submonoid, $\mathcal{R}_1$, by the above lemma. But the latter is right cancellative. (True of $\mathcal{R}_1$ in any monoid.) Thus from $\alpha\alpha = 1_1\alpha$ we obtain $\alpha = 1_1$, so that (vi) must follow.

Do surjective images of monoids for which the $\mathcal{D}$ category equals the $\mathcal{J}$ quasi-order also have this property? The answer is affirmative for regular monoids. To see this we need Howie's version of Lallement's Lemma.

(See [3]4.7.)

3.20 LEMMA. *Let $\phi: T \to S$ be a surjective homomorphism where T is a regular monoid. Then S is also a regular monoid. If $e \varepsilon E(S)$, then there exists $f \varepsilon E(T)$ such that $\phi f = e$.*

3.21 THEOREM. *Let T be a regular monoid for which $\mathcal{D}(T) = \mathcal{J}(T)$. Let $\phi: T \to S$ be a surjective homomorphism. Then S is also regular and $\mathcal{D}(S) = \mathcal{J}(S)$.*

Proof. By 3.20 above, S is regular. To see that $\mathcal{D}(S) = \mathcal{J}(S)$, by 3.9 we need only show that for each idempotent $e \varepsilon E(S)$ the principal monoid eSe is generated by its idempotents. So pick $e \varepsilon E(S)$ and $x \varepsilon eSe$. Next, using 3.20 again, pick $f \varepsilon E(T)$ and $y \varepsilon T$ such that $\phi f = e$ and $\phi y = x$. Now $fyf \ \varepsilon \ fTf$ and $\phi(fyf) = exe = x$. Since $\mathcal{D}(T) = \mathcal{J}(T)$, by 3.9(iv) $fyf \ \varepsilon < E(fTf) >$. But $\phi < E(fTf) > \subseteq < E(eSe) >$. Thus $x = \phi(fyf) \ \varepsilon \ < E(eSe) >$ and we are done.

3.22 At present, the author knows of neither proof nor counterexample to the irregular version of this theorem. The theorem clearly holds when S is a Rees quotient, $S = T/I$, where I is an ideal in T. It is the author's belief that further investigation into monoids and semigroups for which $\mathcal{D}(S) = \mathcal{J}(S)$ could yield other interesting classes of examples, and perhaps some new worthwhile results.

3.23 PROBLEM. A functor $F: \underline{\underline{K}} \to \underline{\underline{K}}'$ is *almost full* if whenever $Hom(x,y)$ is nonempty in $\underline{\underline{K}}$, the restriction $F: Hom(x,y) \to Hom(Fx,Fy)$ is surjective. Any functor between quasi-orders is almost full. Using Theorem 3.9, it is easy to see that for S regular the following are equivalent: (i) $\mathcal{D}(S) = \mathcal{J}(S)$; (ii) $\mathcal{L}(S) \subseteq \mathcal{D}(S)$ is almost full; (iii) $\mathcal{R}(S) \subseteq \mathcal{D}(S)$ is almost full; and (iv) $\mathcal{H}(S) \subseteq \mathcal{D}(S)$ is almost full. If we drop the regularity assumption, then (i) still implies (ii) - (iv). What about the converses?

SECTION 4. TERMINAL OBJECTS AND THE KERNEL

4.1 We turn our attention to the existence of terminal objects in $\mathcal{D}(S)$. Once again the idempotents in S will play a prominent role. Recall that a *terminal object* in a category $\underline{\underline{K}}$ is an object T such that for every object X in $\underline{\underline{K}}$, Hom(X,T) has exactly one morphism. Likewise recall that the *kernel* of a monoid S is the intersection of all its ideals and is denoted by K(S). If K(S) is nonempty, then it is the minimal ideal of S. Finally, by a *zeroid* of S we mean any element θ in S such that for all x in S, $\theta x \theta = \theta$.

4.2 THEOREM. *A monoid S has a zeroid if and only if its kernel K(S) is a nonempty rectangular band, in which case K(S) is the set of zeroids of S. In general, the set of zeroids of S is precisely the set of terminal objects of $\mathcal{D}(S)$.*

Proof. If K(S) is a nonempty rectangular band, then all elements of K(S) satisfy the equation $\alpha\beta\alpha = \alpha$. If $\theta \in K(S)$ and $x \in S$, then we have $\theta x \theta = \theta(\theta x \theta)\theta = \theta$ and θ is a zeroid. If θ is a zeroid of S then $\langle\theta,\theta,\theta\rangle \sim \langle 1,\theta,1\rangle$, and using $\sim$ we see that $uxv = \theta$ implies

$$\begin{aligned}[u,x,v] &= [\theta,\theta,\theta][u,x,v] = [\theta u,x,v\theta] \\ &= [\theta u,xv\theta,1][1,x,v\theta] \\ &= [\theta,xv\theta,1][1,x,v\theta] = [\theta,x,v\theta] \\ &= [1,\theta x,v\theta][\theta,x,1] \\ &= [1,\theta x,\theta][\theta,x,1] = [\theta,x,\theta].\end{aligned}$$

Hence $\mathrm{Hom}(x,\theta) = \{[\theta,x,\theta]\}$, and all zeroid elements of S are terminal objects of $\mathcal{D}(S)$. Finally, let T be the set of terminal objects of $\mathcal{D}(S)$. Clearly T must be a $\mathcal{D}$-class of S as terminal objects form an isomorphism class in a category. If $\theta \in T$, then since every element of S must divide θ, we have $\theta \in K(S)$ and hence $T \subseteq K(S)$. If $x \in K(S)$, then there exist a, a', b, b' $\in$ S such that $axb = \theta$ and $a'\theta b' = x$. Since $\theta \in T$, $[a,x,b][a',\theta,b'] = 1_\theta$. Since isomorphisms are coprime, $[a,x,b]: x \cong \theta$, x is a terminal object, and $K(S) = T$. Hence, for all $x \in T$,

$x^2 \varepsilon T$ also and both $[1,x,x]$ and $[x,x,1]$ are isomorphisms. But by 1.3(ii) $x^2 \mathcal{L} x \mathcal{R} x^2$ so that $x \mathcal{H} x^2$. However, $\mathrm{Aut}(x) \cong H_L(x)$ is trivial so that $x = x^2$ for all $x \varepsilon T$. Since $K(S) = T$ is a $\mathcal{D}$-class of idempotents, it is a rectangular band.

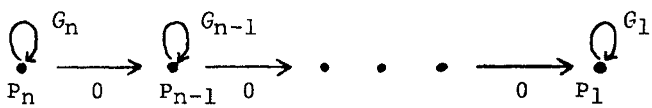
4.3 <u>COROLLARY</u>. <u>If</u> $\mathcal{D}(S)$ <u>has terminal objects and if</u> T <u>is an epimorphic image of</u> S, <u>then</u> $\mathcal{D}(T)$ <u>has a terminal object</u>.

4.4 <u>EXAMPLE</u>. If $S = \mathcal{T}(X)$, the full transformation monoid on a set X, then $\mathcal{D}(S)$ has terminal objects. This is because the set of all constant transformations is a nonempty set of zeroids of S.

SECTION 5. COMPUTATIONS OF $\mathcal{D}(S)$.

5.1 In this section we compute the structure of $\mathcal{D}(S)$ for several familiar classes of monoids. When we say that we will compute the structure of $\mathcal{D}(S)$, we mean that we will compute $\mathcal{D}(S)$ to within equivalence (not isomorphism) of categories. More precisely, from each isomorphism class of $\mathcal{D}(S)$ we will pick one object (i.e., from each $\mathcal{D}$-class of S we will choose one element). The full subcategory of $\mathcal{D}(S)$ that these choices determine is called a <u>skeleton</u> of $\mathcal{D}(S)$ and it is equivalent to $\mathcal{D}(S)$. Our choice of elements will be made to facilitate the computations. Since all are regular, it is to our advantage to choose an idempotent from each $\mathcal{D}$-class. However, even more is possible. The kinds of semigroups we are studying all have a finite number of $\mathcal{D}$-classes, and it is possible to actually choose idempotents from them so as to obtain a totally ordered set of idempotents (using the $\geq$ ordering). Such a choice will greatly simplify the computations.

5.2 Let $T = T_n$ be the full transformation semigroup on $\{1,2,\ldots,n\}$. There are n distinct $\mathcal{D}$-classes in T which are totally ordered by the common rank of their elements. (By the <u>rank</u> of a transformation or partial transformation we mean the size of its image.) If α and β are transformations on $\{1,2,\ldots,n\}$, then α divides β precisely when rank $\alpha \geq$ rank β. Let $e_n > e_{n-1} > \ldots > e_1$ be a $\mathcal{D}$ cross-section of idempotents with $e_n = 1$ and $\text{rank}(e_k) = k$. By 2.9, $\text{Hom}(e_n, e_k)$ is trivial for $k<n$, while $\text{Hom}(e_n, e_n) = \text{Aut}(e_n)$ is isomorphic with the full symmetric group on $\{1,2,\ldots,n\}$. Now $e_k T e_k$ is isomorphic with the full transformation semigroup on $\{1,2,\ldots,k\}$. Thus by Theorem 1.5, our remarks about $\text{Hom}(e_n, e_k)$ apply to the general case $\text{Hom}(e_k, e_j)$. $\text{Hom}(e_k, e_k)$ is isomorphic with the symmetric group on $\{1,2,\ldots,k\}$, while for $j<k$ $\text{Hom}(e_k, e_j)$ is trivial. Hence the skeleton of $\mathcal{D}(T)$ induced by the above chain of idempotents looks like the following n-point category where $\text{Aut}(P_k) = \mathcal{G}_k$, the full symmetric group on $\{1,2,\ldots k\}$, and between points we have only the zero morphisms.



5.3 Let Ω_n denote the semigroup of all order preserving transformations on the ordered set $\{1<2<\ldots<n\}$. Ω_n is a band when $n = 1,2$, but not when $n \geq 3$. In general, Ω_n is generated from its idempotents. This is clear for $n \leq 2$. Assume it for $n = 1,\ldots,m-1$ and let $\phi \in \Omega_m$. Since ϕ is order preserving there must exist a fixed point k such that $\phi(k) = k$. Let $\phi^{-1}(k) = \{i+1,\ldots, j-1\}$ where $i+1 \leq k \leq j-1$. Set $I = \max(i,\phi(i))$ and $J = \min(j,\phi(j))$. Then $I+1 \leq k \leq J-1$. Now ϕ factors in Ω_m as $\phi = \beta\gamma\delta$ where δ pointwise fixes $\{1,2,\ldots,i,j,\ldots, m\}$, but sends $\{i+1,\ldots,j-1\}$ to k; γ pointwise fixes $\{1,\ldots,J-1\}$, but behaves like ϕ on $\{J,\ldots,m\}$; while β pointwise fixes $\{I+1,\ldots,m\}$, but acts like ϕ on $\{1,\ldots,I\}$. Clearly $\delta = \delta^2$. By the induction hypothesis (on lower cases $n = I$, $m+1-J$), both β and γ are products of idempotents in Ω_m and hence so is ϕ. Hence for all n, $\Omega_n = \langle E(\Omega_n) \rangle$. It is not hard to show that for all α in $E(\Omega_n)$, $\alpha\Omega_n\alpha \cong \Omega_p$ where p is the rank of α as a

transformation. We also leave to the reader the easy verification that Ω_n is regular. Thus Ω_n is a regular monoid, all of whose principal monoids $\alpha\Omega_n\alpha$ are generated by their idempotents. By 3.9, $\mathcal{D}(\Omega_n) = \mathcal{J}(\Omega_n)$.

5.4 Let $I = I_n$ be the symmetric inverse semigroup on $\{1,\ldots,n\}$. Since the empty transformation is allowed, there are actually n+1 $\mathcal{D}$-classes, again totally ordered by division. For $0 \le k \le n$ let e_k be the identity transformation on $\{1,\ldots,k\}$. The e_k's are a totally ordered set of idempotents which form a $\mathcal{D}$-class cross section. $e_n = 1$, while e_0 is the empty bijection. Let $k \ge j$. By Lemma 3.18, the morphism set $\mathrm{Hom}(e_k,e_j)$ may be labelled by $\mathcal{R}_{e_j} \cap Se_k$, the set of all partial bijections from within $\{1,\ldots,k\}$ onto $\{1,\ldots,j\}$, and this labelling is multiplicative. But the latter set may be identified with Perm(k,j), the set of all (partial) permutations on $\{1,\ldots,k\}$ taken j at a time. For example, $\binom{32}{12}$ in $\mathcal{R}_{e_2} \cap Se_4$ is identified with 32 in Perm(4,2). To distinguish 32 in Perm(4,2) from 32 in Perm(7,2), we subscript the k to get 32_4 distinct from 32_7. Under this identification, the skeleton of $\mathcal{D}(I)$ obtained from the above chain of idempotents is isomorphic with the <u>Full Permutation Category</u> on $\{1,\ldots,n\}$, denoted <u>Perm</u>$_n$. The object class is $\{0,1,\ldots,n\}$ and for $k,j \le n$, Hom(k,j) = Perm(k,j) which is empty unless $k \ge j$. The composition is the natural composition of such permutations described by the following example. Let $2453_5 \in \mathrm{Perm}(5,4)$ and $32_4 \in \mathrm{Perm}(4,2)$. Then $(32_4)(2453_5) = 54_5 \in \mathrm{Perm}(5,2)$, because 54 is the third entry in 2453 followed by the second entry in 2453. Returning to I, this corresponds to the multiplication, $\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 4 & 5 & 3\\ 1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix}$.

5.5 Let $O = O_n$ be the inverse semigroup of all order preserving partial bijections on $\{1,\ldots,n\}$. O is a submonoid of I and forms an $\mathcal{H}$-class cross section of I. As in 5.4 we let the e_k, $0 \le k \le n$, form a $\mathcal{D}$-class cross section of O. The induced skeleton of $\mathcal{D}(O)$ must be isomorphic with a subcategory of $\underline{\underline{\text{Perm}}}_n$. This subcategory is in turn isomorphic with the <u>Full Combination Category</u> on $\{1,\ldots,n\}$, denoted $\underline{\underline{\text{Comb}}}_n$. The object class is again $\{0,1,\ldots,n\}$, while Hom(k,j) in $\underline{\underline{\text{Comb}}}_n$ equals Comb(k,j), the set of all combinations of $\{1,\ldots,k\}$ taken j at a time. That is, Comb(k,j) is the set of all subsets of $\{1,\ldots,k\}$ having exactly j elements. Again, to avoid ambiguity we will subscript the domain k. The composition of combinations is the natural one, illustrated as follows. Let $\{2,4,5,7\}_8 \in$ Comb(8,4) and $\{2,3\}_4 \in$ Comb(4,2). Then $\{2,3\}_4\{2,4,5,7\}_8 = \{4,5\}_8$ because 4 and 5 are the second and third smallest numbers that can be drawn out of $\{2,4,5,7\}$. The reason why $\underline{\underline{\text{Comb}}}_n$ must be a copy of the skeleton of $\mathcal{D}(\mathcal{O})$ is that combinations correspond bijectively with order preserving permutations. Thus, e.g., $\{2,3\}_4$ corresponds to 23_4. In this way $\underline{\underline{\text{Comb}}}_n$ becomes isomorphic with a subcategory of $\underline{\underline{\text{Perm}}}_n$, namely the category of all order preserving (partial) permutations on $\{1,\ldots,n\}$. But this is precisely the skeleton that we would expect to find upon restricting our attention to those partial bijections in $\mathcal{I}$ which preserve the natural ordering; that is, to partial bijections in $\mathcal{O}$.

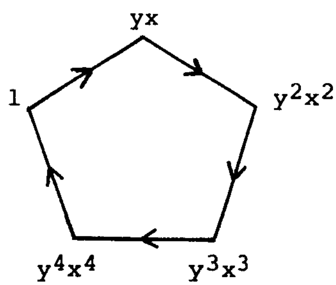
Let $k \geq j$. Recall that the <u>permutation symbol</u> P(k,j) denotes the number of permutations of k objects taken j at a time, and that this number is given by $P(k,j) = k!/(k-j)!$. Likewise the <u>binomial symbol</u> $\binom{k}{j}$ represents the number P(k,j)/j! which counts the number of combinations of k objects taken j at a time. Using 1.4(ii), we obtain from 5.4 and 5.5 the following result.

5.6 <u>THEOREM</u>. <u>Let $\mathcal{I}_n$ be the symmetric inverse semigroup on</u> $\{1,2,\ldots,n\}$, <u>and let $\mathcal{O}_n$ be the submonoid of all partial bijections which preserve the natural ordering on</u> $\{1,2,\ldots,n\}$. <u>Let Φ and Ψ be partial bijections in $\mathcal{I}_n$ with rank $\Phi = k$, rank $\Psi = j$ and $k \geq j$. Then</u> $\mathrm{Hom}(\Phi,\Psi)$ <u>in $\mathcal{D}(\mathcal{I}_n)$ has</u> P(k,j) <u>distinct morphisms. If both Φ and Ψ lie in $\mathcal{O}_n$, then</u> $\mathrm{Hom}(\Phi,\Psi)$ <u>in $\mathcal{D}(\mathcal{O}_n)$ has $\binom{k}{j}$ distinct morphisms</u>.

5.7 Let $\mathcal{B} = \{y^m x^n : m,n \geq 0\}$ be the <u>bicyclic</u> <u>semigroup</u>, where multiplication is juxtaposition followed by applying the identity, $xy = 1$, to reduce the middle. Thus, $y^3x^2yx^4 = y^3x^5$. $\mathcal{B}$ is a bisimple inverse monoid. Let d be a positive integer. Then $\mathcal{B}_d$ is the submonoid of $\mathcal{B}$ characterized by the condition: $y^m x^n \in \mathcal{B}_d$ iff $m \equiv n (\text{mod } d)$. $\mathcal{B}_d$ is a simple inverse monoid with d $\mathcal{D}$-classes. The idempotents $1, yx, \ldots, y^{d-1}x^{d-1}$ form a totally ordered $\mathcal{D}$-class cross section. For $0 \leq m,n < d$,

$$\text{Hom}(y^m x^m, y^n x^n) \cong R_{y^n x^n} \cap S y^m x^m = \{y^n x^{n+kd} : n + kd \geq m\}$$

by 3.18, with composition of morphisms corresponding to multiplication in $\mathcal{B}_d$. In particular, $\text{End}(y^m x^m)$ is free and cyclic on the endomorphism corresponding to $y^m x^{m+d}$. But even more is true. Let $\mathcal{D}_o$ be the skeleton determined by this chain of idempotents. $\mathcal{D}_o$ is free on the graph which is a directed d-gon, denoted Σ_d, whose vertices we label $1, yx, \ldots, y^{d-1}x^{d-1}$. The following diagram is Σ_5.

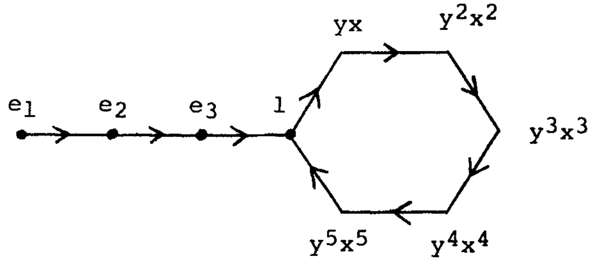


In this graph, the directed path from y^2x^2 to y^3x^3 in which the entire graph is covered twice before stopping at y^3x^3 corresponds to the morphism labelled by y^3x^{13}. In general, the morphism from $y^m x^m$ to $y^n x^n$ labelled by $y^n x^{n+kd}$ corresponds to the directed path of length $n + kd - m$ from $y^m x^m$ to $y^n x^n$.

5.8 Let S be a regular ω-semigroup with a nonempty minimal ideal. Suppose further that $\mathcal{H}$ is trivial. By [9], S is isomorphic with the disjoint union of a finite chain of idempotents $\{e_1 > \ldots > e_p\}$ and a generalized bicyclic semigroup $\mathcal{B}_d$, with multiplication between the two semigroups given by:

$$y^m x^n e_k = y^m x^n = e_k y^m x^n.$$

If $p = 0$, then $S \cong \mathcal{B}_d$. We identify S with this union. Choose $\{e_1, \ldots, e_p, 1, yx, \ldots, y^{d-1}x^{d-1}\}$ to be a $\mathcal{D}$-class cross section of totally ordered idempotents. We obtain as a skeleton for $\mathcal{D}(S)$ a category $\mathcal{D}_o$ which is free on the directed graph Γ obtained by adjoining a directed chain to Σ_d, the graph of 5.7, at 1. In the diagram below, Γ is given for $p = 3$, $d = 6$.



5.9 Let us drop the above assumption that $\mathcal{H}$ be trivial. So let $\overline{S}$ be a regular ω-semigroup with a nonempty minimal ideal. It follows from [9] that $\mathcal{H}$ is a congruence on $\overline{S}$ and that there exists a submonoid S of the type described in 5.8 above, such that S forms a complete $\mathcal{H}$-class cross section in $\overline{S}$. In the terminology of [5], $\overline{S}$ is a <u>split $\mathcal{H}$-coextension</u> of S. Conversely, every split $\mathcal{H}$-coextension of S is a regular ω-semigroup with nonempty minimal ideal. According to [5]4.2 each such coextension can be isomorphically described as a <u>generalized semidirect product</u>, SxF, where F is a group valued functor over $\mathcal{D}(S)$, $F:\mathcal{D}(S) \to \underline{\underline{Gr}}$. (Since S is regular, the covering equalities in [5]4.2 are automatically satisfied.) But F itself is determined (to within natural equivalence) by its restriction $F|\mathcal{D}_o$ where $\mathcal{D}_o$ is the skeleton of $\mathcal{D}(S)$ given in 5.8. Moreover, since $\mathcal{D}_o$ is free on the directed graph Γ also given in 5.8, the restriction $F|\mathcal{D}_o$ is completely determined by the further restriction $F|\Gamma$, which is simply a group valued diagram over the directed graph

Γ. Backing up, if we are presented a group valued diagram over Γ, $G:\Gamma \to \underline{\underline{Gr}}$, then G uniquely determines a group valued functor $F_o:\mathcal{D}_o \to \underline{\underline{Gr}}$ such that $F_o|\Gamma = G$, and F_o itself must extend to a functor $F:\mathcal{D}(S) \to \underline{\underline{Gr}}$ which is unique to within natural equivalence. Thus every split $\mathcal{H}$-coextension of S is uniquely determined to within isomorphism by a group valued diagram over Γ. Using the construction found in [7], it is clear that the structure of S itself is determined by the free category on Γ, namely $\mathcal{D}_o$. Thus the structure of S is determined by Γ, while the structure of any coextension SxF is determined from $F|\Gamma$. The group diagram $F|\Gamma$ may be viewed as a conjunction of two group diagrams at the common vertex 1. One diagram lies over the stem of Γ and corresponds to the finite chain of groups given in [9]2.7. It both describes the complement of the minimal ideal and tells how it is attached to the minimal ideal. The second diagram $F|\Sigma$ determines the structure of the minimal ideal and corresponds to the cycle of groups encountered in the third section of [9]. An efficient construction of the minimal ideal from $F|\Sigma$ is provided by the Bruck-Reilly construction as given in that section as well as in [3]V7.

5.10 PROBLEM. Let S be a free inverse monoid. What does a skeleton of $\mathcal{D}(S)$ look like?

5.11 We construct a bisimple monoid S for which $cd\mathcal{D}(S) = 0$, but 1 is not an initial object in $\mathcal{D}(S)$. Let S be defined on generators {a,b,c,d} and subject to relations ab = 1, cd = 1, cb = 1, and ad = b. Given the first relation, the last relation implies $a^2d = 1$. Both submonoids, <{a,c}> and <{b,d}>, are freely generated and thus noncommutative. Moreover, $\mathcal{R}_1 = \langle\{a,c\}\rangle$, $\mathcal{L}_1 = \langle\{b,d\}\rangle$ and $\mathcal{R}_1\mathcal{L}_1 \subseteq \mathcal{R}_1 \cup \mathcal{L}_1$, so that $S = \mathcal{L}_1\mathcal{R}_1$ and the monoid is bisimple. Hence End(1) is a skeleton of $\mathcal{D}(S)$. We claim that End(1) is isomorphic with the semilattice, 1^0, from which everything asserted about $\mathcal{D}(S)$ must follow. To begin,

$$[a,1,b] = [c,1,b] = [c,1,d] = [a^2,1,d] = [a^2,1,b^2]$$

so that $[a,1,b]$ is idempotent. Now take $[A,1,B]$ in End(1) with $A \in R_1$, $B \in L_1$ and length $A = n \geq 1$. Then

$$[A,1,B] = [A,1,b^n] = [a^n,1,b^n] = [a,1,b]^n = [a,1,b].$$

Thus End(1) = $\{[1,1,1],[a,1,b]\}$ and the claim is verified.

5.12 PROBLEM. In this section we have encountered various endomorphism monoids, End(x). What general facts must hold about endomorphism monoids? For example, $J_1 = H_1$ must hold by 1.3. What kinds of monoids can be endomorphism monoids in some division category $\mathcal{D}(S)$?

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