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On the construction of the three dimensional polymer measure

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Summary. The three dimensional polymer measure was first constructed by Westwater in 1980 with a very complicated proof. We give an alternative construction for small coupling parameter which is based on the approach by Brydges-Fröhlich-Sokal in quantum field theory and Bovier-Felder-Fröhlich, using skeleton inequalities. The main new features are the proof of convergence which had been open in the Brydges-Fröhlich-Sokal construction, and the construction of the measure on the space of paths with fixed time length.

1 Introduction

If $T > 0$, $d \in \mathbb{N}$, let C_T be the set of continuous functions f: $[0, T] \to \mathbb{R}^d$ satisfying $f(0) = 0$, and P_T the Wiener measure on C_T. Later on, we will deal only with the case $d = 3$. If $\lambda > 0$, the polymer measure $\hat{P}_T^\lambda$ on C_T with coupling constant λ is formally defined by

$$(d\hat{P}_{T,\lambda}/dP_T)(\omega) = \exp(-\lambda J_{0,T}(\omega))/z_{T,\lambda}$$

where $J_{0,T}(\omega) = \int_0^T ds \int_s^T dt\ \delta(\omega_s - \omega_t)$. δ is the Dirac function and $z_{T,\lambda}$ is the appropriate norming factor in order that $\hat{P}_{T,\lambda}$ becomes a probability measure. The difficulty is, of course, that $J_{0,T}$ is not well defined. For $d = 1$, it can be expressed as $J_{0,T} = \frac{1}{2}\int l_T(x)^2 dx$ where l_T is the local time up to time T, i.e. the density of the sojourn measure $A \to \int_0^T 1_A(\omega_s) ds$ for which a continuous version is known to exist for almost all ω. However, for $d \geqq 2$, this is no longer possible.

The only essential problem with the definition of $J_{0,T}$ is the integration near the diagonal. If one leaves out a gap near it, then everything behaves nicely. If $0 < \varepsilon \leqq T$, let $J_{0,T}^\varepsilon = \int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^T dt\, \delta(\omega_t - \omega_s)$ which still is only a formal definition but which can be given a good sense for $d \leqq 3$, see [5] and the Proposition 2.1 below.

One can then define $\hat{P}_{T,\lambda}^\varepsilon$ by replacing $J_{0,T}$ by $J_{0,T}^\varepsilon$. The problem is then to show that $\lim_{\varepsilon \to 0} \hat{P}_{T,\lambda}^\varepsilon$ exists. For $d = 2$, Varadhan (with a slightly different

regularization) has shown in [6] that

$$\lim_{\varepsilon \to 0} (J^{\varepsilon}_{0,T} - E(J^{\varepsilon}_{0,T})) = Y \tag{1.1}$$

exists in L_2 and that $E(e^{-\lambda Y}) < \infty$ for all $\lambda > 0$. One can then prove that

$$\lim_{\varepsilon \to 0} \hat{P}^{\varepsilon}_{T,\lambda} = \hat{P}^{0}_{T,\lambda} \tag{1.2}$$

where $(d\hat{P}^{0}_{T,\lambda}/dP_T)(\omega) = e^{-\lambda Y}/E(e^{-\lambda Y})$. This so called Varadhan renormalisation has been the starting point for many research papers on self-intersection properties. See e.g. [3, 9].

For $d = 3$ (1.1) is no longer true and the problem becomes much more difficult. Westwater proved in [7, 8] that $\lim_{\varepsilon \to 0} \hat{P}^{\varepsilon}_{T,\lambda}$ exists. His approach is based on cluster expansion techniques and is very complicated. There is, however, a much simpler approach by Bovier et al. [1] which is based on the approach in [2] on the ϕ^4_d quantum field model. Actually nothing like (1.2) is proved in [1]. It is shown there that the Laplace transform in time of certain transition densities remain bounded as $\varepsilon \to 0$ if λ is small. (Also a different regularization is used). It is notoriously difficult to invert such statements about Laplace transforms and to get the desired results for each fixed T. In order to avoid discussions of this point, we altogether avoid the use of Laplace transforms, by using a direct recursion Ansatz. In this way, we get quite precise estimates e.g. of the two point functions.

However, the main difficulty is to prove convergence as the regularization goes to 0. This had also been an open point in [1] and [2], which was also connected with the failure to prove rotational invariance of the quantum fields in [2]. We prove convergence here by bounding the derivative with respect to the regularisation parameter ε. The necessary cancellation of the divergencies here is much more subtle than in the proof of boundedness. I have actually been unable to prove the convergence of the two point functions at fixed time T. However, given the boundedness and tightness properties, only convergence of certain integrals over smooth test functions is needed. (The convergence of the Laplace transforms in T would also follow by an easy modification of our approach.)

Despite of the above mentioned shortcomings, the approach in [1] is both ingenious and essentially simple. It is the aim of this paper to give a full proof of the convergence in (1.2) based on this method for small coupling parameter λ. We also give rather sharp estimates on decay properties of the one point functions at fixed time and on short time intervals, which are essential for proving tightness.

(1.3) Theorem. *Let $d = 3$. If $T_0 > 0$, then there exist $\lambda_0 > 0$ such that for $\lambda \in [0, \lambda_0]$, $\hat{P}_{T_0,\lambda} = \lim_{\varepsilon \to 0} \hat{P}^{\varepsilon}_{T_0,\lambda}$ exists in the weak sense.*

For convenience, we will assume $T_0 = 1$.

Convergence is achieved by using a different "Schwinger Dyson equation" than that used in [1]. For this, the gap regularization is particularly well adapted. It might be difficult to handle with the lattice regularization used in [1].

The main shortcoming of the approach up to now is the restriction to small λ. There is some hope that a suitable extension of the method could prove the result for any $\lambda > 0$.

The method can probably be developed further to give more precise information on the polymer measures. An example of such results are in papers by Kusuoka [4] and more recently by Zhou [10] which are based on the Westwater approach.

The paper is organised as follows: In Sect. 2, we derive the basic inequalities, which are essentially the same as those used in [1]. We use these inequalities in Sect. 3 to prove some rather precise estimates for the so-called one point functions at fixed time. This implies tightness immediately. In Sect. 4, we derive our alternative inequalities by differentiating with respect to ε. This is used to prove convergence. Probably, one could use this different approach also to prove the estimates on the one-point functions of Sect. 3 directly. However, as remarked above, the required cancellation properties of the divergencies is much more delicate, and given the results of Sect. 3, only a relatively small additional information is needed.

The situation in dimension 3 is somewhat simplified by the fact that it is the border case where a socalled mass renormalisation is necessary, i.e. where (1.1) does not work. In fact the really virulent divergency is only logarithmic. This allows at several places to use rather crude estimates.

No knowledge of [1] or [2] is assumed.

2 The basic inequalities

For fixed ε, $\hat{P}^{\varepsilon}_{T,\lambda}$ has a bounded density with respect to Wiener measure. Therefore $\hat{P}^{\varepsilon}_{T,\lambda}(\omega_T \in A)$ has a density with respect to Lebesgue measure, which we denote by $p^{\varepsilon}_{T,\lambda}(x)$. p_T denotes the transition density of the Wiener measure, i.e.

$$p_T(x) = \frac{1}{(2\pi T)^{3/2}} \exp(-|x|^2/2T) .$$

We introduce some notations:

If $f(s_1, \ldots, s_n)$ is a bounded measurable function, defined on $t \leqq s_1 \leqq \ldots \leqq s_n \leqq T$, we put by an abuse of notation:

$$I_{t,T}(f(s_1, \ldots, s_n)) = \int_t^T ds_1 \int_{s_1}^T ds_2 \ldots \int_{s_{n-1}}^T ds_n f(s_1, \ldots, s_n) .$$

We will use $s_1, \ldots, s_n$ exclusively in such expressions. Further pushing the abuse: If e.g. s_1 does not appear on the left hand side, then nevertheless the s_1-integration is performed on the right hand side (giving of course just a factor $(s_2 - t)$). We also write I_T for $I_{0,T}$. Let $j^{\varepsilon}(s) = 1_{[\varepsilon,\infty)}(s)$, $\varepsilon > 0$.

If $0 \leqq s < t \leqq T$, $0 \leqq u < v \leqq T$; $\varepsilon, a > 0$, $\omega \in C_T$, let

$$J^{\varepsilon,a}_{s,t;u,v}(\omega) = \int_s^t d\sigma \int_{(u \vee (\sigma+\varepsilon)) \wedge v}^{v} d\tau p_a(\omega_\sigma - \omega_\tau) .$$

We put $J^{\varepsilon,a}_{s,t} = J^{\varepsilon,a}_{s,t;s,t}$. We need also a joint continuous version of $J^{\varepsilon}_{0,T}$ for the Brownian bridge, tied down at $x \in \mathbb{R}^3$. To formulate this, we have to define the bridges for various T and x on a single probability space. Let $\psi_{T,x}$: $C_1 \to C_T$ be defined by

$$\psi_{T,x}(\omega)(t) = \sqrt{T}\omega\left(\frac{t}{T}\right) + \frac{t}{T}(x - \sqrt{T}\omega(1)) .$$

$P_1\psi_{T,x}^{-1}$ is the law of the tied down Brownian motion. If $a > 0$, $0 < \varepsilon \leqq T$, $x \in \mathbb{R}^3$, $\omega \in C_1$, let

$$Y_{T,x}^{\varepsilon,a}(\omega) = \int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^{T} dt\, p_a(\psi_{T,x}(\omega)(s) - \psi_{T,x}(\omega)(t)) .$$

Obviously, $Y_{T,x}^{\varepsilon,a}$ depends continuously on (ε, a, T, x) on ε, $a > 0$. If one considers the intersection local times of independent Brownian motions or bridges then no ε-gap is needed. We will use this for a bridge and a Brownian motion. So we define $W_{T,S;x,y}^{a}: C_1 \times C_S \to [0, \infty)$ by

$$W_{T,S;x,y}^{a}(\omega, \omega') = \int_0^T dt \int_0^S ds\, p_a(\psi_{T,x}(\omega)(t) - \omega'(s) - y), \quad (\omega, \omega') \in C_1 \times C_S ,$$

$C_1 \times C_S$ is equipped with the probability measure $P_1 \times P_S$.

(2.1) Proposition. *There exist versions of $J_{s,t;u,v}^{\varepsilon,a}$, $Y_{T,x}^{\varepsilon,a}$, $W_{T,S;x,y}^{a}$ which are jointly continuous in all the parameters on $\varepsilon > 0$, $a \geqq 0$.*

Especially, we claim that there exists a continuous limit as $a \to 0$ (but not for $\varepsilon \to 0$). If $a = 0$, we just drop the index a. The proposition can be proved by the methods in [5]. We will give the proof for Y in an Appendix. The other cases are similar.

It is better, not to normalise $\exp(-J_{0,T}^{\varepsilon})dP_T$ immediately to a probability measure, but only to a measure which remains bounded. This is achieved by so called counterterms.

$$\kappa_1(\varepsilon) = \int_\varepsilon^1 p_t(0)dt = 2(2\pi)^{-3/2}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right),$$

$$\kappa_2(\varepsilon) = I_1(j^\varepsilon(s_2)j^\varepsilon(s_3 - s_1)\,\|p_{s_1}p_{s_2-s_1}p_{s_3-s_2}\|_1)$$

where $\| \quad \|_1$ denotes the L_1-norm with respect to Lebesgue measure. Remark that $\|p_u p_v p_t\|_1 = (2\pi)^{-3}(uv + ut + vt)^{-3/2}$. The asymptotic behaviour of $\kappa_2(\varepsilon)$ for $\varepsilon \sim 0$ is then calculated as

$$\kappa_2(\varepsilon) \sim (2\pi)^{-2}\,|\log \varepsilon| . \tag{2.2}$$

If $s < t$, we define

$$\bar{J}_{s,t}^{\varepsilon,\lambda} = \lambda J_{s,t}^{\varepsilon} - \lambda(t-s)\kappa_1(\varepsilon) + \lambda^2(t-s)\kappa_2(\varepsilon) . \tag{2.3}$$

In order to not overburden the notation, we usually drop the dependence of $\bar{J}$ on λ and ε if no confusion can arise, but this dependence should always be kept in mind. We will use ε exclusively for the regularization parameter and λ for the coupling constant.

We define the measure $G_T^{\varepsilon,\lambda}$ on $\mathbb{R}^3$ by $G_T^{\varepsilon,\lambda}(A) = E(\exp(-\bar{J}_{0,T}^{\varepsilon,\lambda})1_{\omega_T \in A})$. As $\exp(-\bar{J}_{0,T})$ is, for fixed ε, $\lambda > 0$, a positive and bounded random variable, $G_T^{\varepsilon,\lambda}$ has a density $g_T^{\varepsilon,\lambda}$ with respect to Lebesgue measure.

We will derive two basic inequalities for $p_T(x) - g_T(x)$ for fixed ε. We will derive them by somewhat cavalier looking calculations using the Dirac δ-function. This, however, is only a notational convenience. We have to replace δ by p_a, letting $a \to 0$, and using Proposition 2.1.

More precisely, we make the following convention which is also in force in Sect. 4: If we write $A \geqq B$, where A, B are real valued expressions containing δ-functions,

J, $\bar{J}$ etc., this means that

$$\liminf_{a\to 0}(A(a)-B(a))\geqq 0\,,$$

where $A(a)$ and $B(a)$ are obtained by replacing δ by p_a and also J, $\bar{J}$ are replaced by the corresponding quantities with δ replaced by p_a. $A=B$ of course means that $A\geqq B$ and $B\geqq A$. If A, B do not contain the δ-function explicitly but possibly J, then the Proposition 2.1 guarantees that $A\geqq B$ has the usual meaning, at least if they are bounded.

With this notation, we have (dropping ε, λ for convenience) $g_T(x)=E(\exp(-\bar{J}_{0,T})\delta(x-\omega_T))$. Therefore

$$\begin{aligned} p_T(x)-g_T(x) &= \int_0^{T-\varepsilon}\frac{d}{dv}E(e^{-\bar{J}_{v,T}}\delta(x-\omega_T))dv \\ &= -\int_0^{T-\varepsilon}E(e^{-\bar{J}_{v,T}}\frac{d}{dv}\bar{J}_{v,T}\delta(x-\omega_T))dv \end{aligned}$$

$$\frac{d}{dv}\bar{J}_{v,T}=-\lambda\int_{(v+\varepsilon)\wedge T}^{T}dt\,\delta(\omega_v-\omega_t)+\lambda\kappa_1(\varepsilon)-\lambda^2\kappa_2(\varepsilon)\,.$$

Therefore, with our I_T-notation, we have:

$$\begin{aligned} p_T(x)-g_T(x) &= \lambda I_T(E(e^{-\bar{J}_{s_1,T}}j^\varepsilon(s_2-s_1)\delta(\omega_{s_1}-\omega_{s_2})\delta(x-\omega_T))) \\ &\quad +(-\lambda\kappa_1(\varepsilon)+\lambda^2\kappa_2(\varepsilon))I_T((p_{s_1}*g_{T-s_1}))\,. \end{aligned} \tag{2.4}$$

We now split the interaction $\bar{J}_{s_1,T}$ into the interaction on the time intervals $[s_1,s_2]$ and $[s_2,T]$ and the interaction between these intervals: If $0\leqq u\leqq v\leqq T$, then

$$\bar{J}_{u,T}=\bar{J}_{u,v}+\bar{J}_{v,T}+\lambda J_{u,v;v,T}\,. \tag{2.5}$$

Without the third summand which takes care of the mutual interaction, we could just use the Markov property in the first summand on the right hand side of (2.4) and get a convenient splitting. We will expand the factor with the mutual interaction. Roughly, if we write

$$e^{-\lambda J_{s_1,s_2;s_2,T}}=1-\lambda J_{s_1,s_2;s_2,T}+\cdots$$

and implement this into (2.4), then the 1-contribution gives a divergency (as $\varepsilon\to 0$) which cancels with the κ_1-divergency and the $\lambda J_{s_1,s_2;s_2,T}$-contribution cancels with the κ_2-divergency, and the rest stays bounded as $\varepsilon\to 0$. We will crucially use the fact that $J_{u,v;v,T}$ is positive. This gives us simple inequalities for the right hand side of (2.4) in terms of simple finite expansions.

We first use $e^{-x}\geqq 1-x$ for $x\geqq 0$, and get

$$\begin{aligned} &E(e^{-\bar{J}_{u,T}}j^\varepsilon(v-u)\delta(\omega_u-\omega_v)\delta(x-\omega_T)) \\ &\geqq E(e^{-\bar{J}_{u,v}-\bar{J}_{v,T}}j^\varepsilon(v-u)\delta(\omega_u-\omega_v)\delta(x-\omega_T)) \\ &-\lambda\int_u^v dt_1\int_v^T dt_2 j^\varepsilon(t_2-t_1)j^\varepsilon(v-u)E(e^{-\bar{J}_{u,v}-\bar{J}_{v,T}}\delta(\omega_u-\omega_v)\delta(\omega_{t_2}-\omega_{t_1})\delta(x-\omega_T))\,. \end{aligned} \tag{2.6}$$

Now the first summand on the right hand side looks fine, but in the second, we are faced with the same problem. Therefore, we do the same trick again: For $0 \leqq u \leqq t_1 \leqq v \leqq t_2 \leqq T$, we have

$$\bar{J}_{u,v} + \bar{J}_{v,T} = \bar{J}_{u,t_1} + \bar{J}_{t_1,v} + \bar{J}_{v,t_2} + \bar{J}_{t_2,T} + \lambda J_{u,t_1;t_1,v} + \lambda J_{v,t_2;t_2,T} \tag{2.7}$$

and therefore, by using just $e^{-x} \leqq 1$ for $x \geqq 0$, we get

$$\begin{aligned} &E(e^{-\bar{J}_{u,v}-\bar{J}_{v,T}}\delta(\omega_u - \omega_v)\delta(\omega_{t_2} - \omega_{t_1})\delta(x - \omega_T)) \\ &\leqq E(e^{-\bar{J}_{u,t_1}-\bar{J}_{t_1,v}-\bar{J}_{v,t_2}-\bar{J}_{t_2,T}}\delta(\omega_u - \omega_v)\delta(\omega_{t_2} - \omega_{t_1})\delta(x - \omega_T)) \\ &= (p_u * (g_{t_1-u} g_{v-t_1} g_{t_2-v}) * g_{T-t_2})(x) . \end{aligned} \tag{2.8}$$

We introduce some abbreviations:

$$\begin{aligned} q_t^{(1)}(x) &= j^\varepsilon(t) g_t(x) \\ q_t^{(2)}(x) &= I_t(j^\varepsilon(s_2) j^\varepsilon(t - s_1) g_{s_1}(x) g_{s_2-s_1}(x) g_{t-s_2}(x)) \\ A_T^{(1)}(x) &= I_T((p_{s_1} * g_{T-s_2})(x) q_{s_2-s_1}^{(1)}(0)) - \kappa_1(\varepsilon) I_T((p_{s_1} * g_{T-s_1})(x)) \\ A_T^{(2)}(x) &= I_T((p_{s_1} * q_{s_2-s_1}^{(2)} * g_{T-s_2})(x)) - \kappa_2(\varepsilon) I_T((p_{s_1} * g_{T-s_1})(x)) . \end{aligned}$$

We take the divergent κ_1-part into $A_T^{(1)}$ in order to cancel the divergency in the first part, and similarly with the κ_2-part in $A_T^{(2)}$, but it will take some calculations to prove that this works. (2.4), (2.6) and (2.8) give us the first of our basic inequalities:

$$p_T(x) - g_T(x) \geqq \lambda A_T^{(1)}(x) - \lambda^2 A_T^{(2)}(x) . \tag{2.9}$$

The second, which gives an estimate from above is obtained in exactly the same way by using $e^{-x} \leqq 1 - x + x^2/2$ for the last summand in (2.5) and $e^{-x} \geqq 1 - x$ for the last two summands in (2.7). This easily leads to

$$p_T(x) - g_T(x) \leqq \lambda A_T^{(1)}(x) - \lambda^2 A_T^{(2)}(x) + \lambda^3 A_T^{(3)}(x) \tag{2.10}$$

where

$$\begin{aligned} A_T^{(3)}(x) &= 3 I_T((p_{s_1} * q_{s_2-s_1}^{(3)} * g_{T-s_2})(x)) \\ q_T^{(3)} &= I_T([(g_{s_1} g_{s_4-s_3}) * (g_{T-s_4} g_{s_2-s_1})] g_{s_3-s_2}) . \end{aligned}$$

Remark that $A_T^{(3)}$ is nonnegative. There is no counterterm necessary to balance a further divergency, just because $A_T^{(3)}$ will turn out to be bounded. A consequence of (2.9) and (2.10) is

$$|p_T(x) - g_T(x)| \leqq \lambda |A_T^{(1)}(x)| + \lambda^2 |A_T^{(2)}(x)| + \lambda^3 A_T^{(3)}(x) \tag{2.11}$$

and

$$\left| \int_0^1 p_T(0) dT - \int_0^1 g_T(0) dT \right| \leqq \lambda \left| \int_0^1 A_T^{(1)}(0) dT \right| + \lambda^2 \int_0^1 |A_T^{(2)}(0)| dT + \lambda^3 \int_0^1 A_T^{(3)}(0) dT . \tag{2.12}$$

Looking at the definition of $q_t^{(1)}(x)$ and $q_t^{(2)}(x)$, one sees that if g is replaced by p, their appearance in $A_T^{(1)}$ and $A_T^{(2)}$ resemble the counterterms κ_1 and κ_2. Of course, there are some differences which have to be taken care of. We want to apply a recursion argument in order to prove, with the help of (2.10)–(2.12), that $p - g$ remains

bounded. This is the main idea which is borrowed from [1] and [2]. In the next section, we will get a rather precise estimate.

Already at this stage, one sees the difficulty with proving convergence as $\varepsilon \to 0$: Although the $A_T^{(i)}(x)$ essentially stay bounded, as will be proved in the next section, they certainly will not convergence to 0. Also the inequalities will not become equalities in the $\varepsilon \to 0$ limit. Therefore, it is not possible to prove convergence by just using (2.10)–(2.11). One could try to get a complete expansion, but the expansion will not convergence. The way out of these problems is to get alternative inequalities, essentially for $g_T^{\varepsilon_1} - g_T^{\varepsilon_2}$ which become sharp as $\varepsilon_1, \varepsilon_2 \to 0$. This is done in Sect. 4.

3 Recursive estimates of g and proof of tightness

We use c as a generic constant >0, not necessarily the same at different occurrences. Also $\phi(x)$ is a generic polynom in x with nonnegative coefficients, which may also vary from formula to formula. We usually drop the dependence on ε, λ in our notation.

(3.1) Proposition. *There exist $K > 0$ and $\lambda_0 > 0$ such that for $0 \leqq \lambda \leqq \lambda_0$ and $T \leqq 1$ one has*

$$|p_T(x) - g_T(x)| \leqq K\lambda\sqrt{T}p_{2T}(x)\,.$$

We need some joint continuity and decay properties of g depending on T, x, ε. g can be expressed as

$$g_T^{\varepsilon,\lambda}(x) = e^{+\lambda T\kappa_1(\varepsilon) - \lambda^2 T\kappa_2(\varepsilon)} E_1(e^{-\lambda Y_{T,x}^{\varepsilon}})p_T(x)$$

where E_1 refers to the expectation with respect to the Wiener measure on C_1 and $Y_{T,x}^{\varepsilon}$ was defined at the beginning of Sect. 2. By (2.2) it follows that $E_1(\exp(-\lambda Y_{T,x}^{\varepsilon}))$ depends continuously on (ε, T, x), $0 < \varepsilon \leqq T$ and is, of course, bounded by 1. Then

$$K_0(\varepsilon, \lambda) = \sup_{0 \leqq T \leqq 1} \sup_{x \in \mathbb{R}^3} \frac{|g_T^{\varepsilon,\lambda}(x) - p_T(x)|}{\sqrt{T}p_{2T}(x)}$$

is finite and depends continuously on $\varepsilon > 0$ because p_T decays faster at ∞ than p_{2T}. Let

$$K(\varepsilon, \lambda) = K_0(\varepsilon, \lambda) \vee \left| \int_0^1 (p_s(0) - g_s^{\varepsilon,\lambda}(0))ds \right|$$

which also is continuous in ε. ($\vee$ denotes max). Remark that for $T \leqq \varepsilon$, $g_T^{\varepsilon} = \left(1 + O\left(\frac{\lambda T}{\sqrt{\varepsilon}}\right)\right)p_T$. Therefore, for $T < \varepsilon$, the desired bound is trivially true.

The main task is to prove that $K(\varepsilon, \lambda)$ remains bounded as $\varepsilon \to 0$. We always keep $\varepsilon \leqq T \leqq 1$. We start now with the estimation of $A_T^{(1)}$. Abbreviating $\int_0^{T-\varepsilon} ds \int_{s+\varepsilon}^T dt$ just as $\int$, we have

$$\begin{aligned} A_T^{(1)} &= \int (p_t * g_{T-t})(g_{t-s}(0) - p_{t-s}(0)) + \int g_{t-s}(0)((p_s - p_t) * g_{T-t}) \\ &\quad + \left[\int p_{t-s}(0)(p_t * g_{T-t}) - \kappa_1(\varepsilon) \int_0^T dt p_t * g_{T-t} \right] \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned}$$

$$I_1 = \int_\varepsilon^T dt(p_t * g_{T-t}) \int_\varepsilon^t du(g_u(0) - p_u(0))$$

$$\left|\int_\varepsilon^t du(g_u(0) - p_u(0))\right| \leqq \left|\int_\varepsilon^1 du(g_u(0) - p_u(0))\right| + \int_t^1 du|g_u(0) - p_u(0)|$$

$$\leqq K(1 + |\log t|)$$

and so, as $g_s \leqq p_s + K\sqrt{s}p_{2s} \leqq c(1+K)p_{2s}$:

$$|I_1| \leqq K \int_\varepsilon^T dt(1 + |\log t|)(p_t * g_{T-t}) \leqq \phi(K) \int_0^T dt(1 + |\log t|)p_{2T} \tag{3.2}$$

$$\leqq \phi(K)T^{3/4}p_{2T}\,.$$

Estimating $g_{t-s}(0) \leqq |g_{t-s}(0) - p_{t-s}(0)| + p_{t-s}(0) \leqq \phi(K)p_{2t-2s}(0)$, we get

$$|I_2| \leqq c\phi(K) \int p_{2t-2s}(0)(|p_t - p_s| * g_{T-t})$$

$$\leqq \phi(K) \int_{\{s<t/2\}} t^{-3/2}(p_t + p_s) * p_{2T-2t} + \phi(K) \int_{\{s>t/2\}} p_{2t-2s}(0)|p_t - p_s| * p_{2T-2t}$$

$$= L_1 + L_2, \text{ say}\,.$$

$$L_1 \leqq \phi(K)\left\{\int_{s<t/2} t^{-3/2}p_{2T} + \int_{\substack{s<t/2\\ t<T/2}} t^{-3/2}p_{2T-2t+s} + \int_{\substack{s<t/2\\ t>T/2}} t^{-3/2}p_{2T-2t+s}\right\}$$

$$\leqq \phi(K)\left[\sqrt{T}p_{2T} + T^{-3/2}\int_0^{2T} du\, up_u\right] \leqq \phi(K)\sqrt{T}p_{2T}\,.$$

As to L_2, we remark that for $s > t/2$, we have

$$|p_s - p_t| \leqq c|t - s|t^{-1}p_{2t} \tag{3.3}$$

and therefore

$$L_2 \leqq \phi(K) \int_{\{s>t/2\}} \frac{1}{\sqrt{t-s\,t}}\, p_{2t} * p_{2T-2t} \leqq \phi(K)\sqrt{T}p_{2T}\,.$$

So, we have

$$|I_2| \leqq \phi(K)\sqrt{T}p_{2T}\,. \tag{3.4}$$

$$|I_3| \leqq \int_\varepsilon^T dt(p_t * g_{T-t}) \int_t^1 p_s(0)ds + \kappa_1(\varepsilon) \int_0^\varepsilon dt p_t * g_{T-t} \leqq \phi(K)\sqrt{T}p_{2T} + c\sqrt{\varepsilon}p_{2T}$$

$$\leqq \phi(K)\sqrt{T}p_{2T}$$

as $\varepsilon \leqq T$. Together with (3.2) and (3.4) this yields

$$|A_T^{(1)}| \leqq \phi(K)\sqrt{T}p_{2T}\,. \tag{3.5}$$

In order to estimate $A_T^{(2)}$ and $A_T^{(3)}$, it is convenient to formulate the following simple result:

(3.6) Lemma. *Let $a(u_1, \ldots, u_{n-1})$ and $\sigma(u_1, \ldots, u_{n-1})$ be continuous functions depending on $u_1, \ldots, u_{n-1} \geqq 0$ which satisfy $0 < a(u_1, \ldots, u_{n-1})$, $0 \leqq \sigma(u_1, \ldots, u_{n-1}) \leqq u_1 + \cdots + u_{n-1}$. Then*

$$I_T(a(s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}) p_{s_1} * p_{\sigma(s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1})} * p_{T - s_n})$$
$$\leqq p_T T^{3/2} I_T\left(\frac{1}{\sqrt{s_1}} a(s_2 - s_1, s_3 - s_2, \ldots, T - s_{n-1})\right).$$

Proof. Let $u = T - s_n + s_1$. Performing the integration on the left hand side with respect to s_1, leaving $s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}$ and therefore u fixed, just gives a factor $up_u = (2\pi)^{-3/2} u^{-1/2} \exp(-|x|^2/2u) \leqq (2\pi)^{-3/2} u^{-1/2} T^{3/2} p_{T-\sigma}$. Implementing this and convoluting with p_σ gives the desired bound.

To estimate $A_T^{(2)}$, let $\tilde{q}_t(x) = I_t(j^\varepsilon(s_2) j^\varepsilon(t - s_1) p_{s_1}(x) p_{s_2 - s_1}(x) p_{t - s_2}(x))$, i.e. the $q^{(2)}$ with the q's replaced by the p's. It turns out that the error by making this replacement is bounded as $\varepsilon \to 0$. It is important here that we have a $\sqrt{T}$-factor in our recursion Ansatz. To be precise, we make the following splitting:

$$A_T^{(2)} = I_T(p_{s_1} * (q^{(2)}_{s_2 - s_1} - \tilde{q}_{s_2 - s_1}) * g_{T - s_2}) + \{I_T(p_{s_1} * \tilde{q}_{s_2 - s_1} * g_{T - s_2}) - \kappa_2(\varepsilon) I_T(p_{s_1} * g_{T - s_1})\}$$
$$= B_1 + B_2, \text{ say}.$$

B_1 can be estimated in absolute value by expressions of the form $I_T(p_{s_1} * (\tilde{p}_{s_2 - s_1} \tilde{p}_{s_3 - s_2} \tilde{p}_{s_4 - s_3}) * \tilde{p}_{2T - 2s_4})$ where $\tilde{p}_u$ is either p_u or $K\sqrt{u} p_{2u}$ and where at least one of is the second possibility. Using the fact that $p_u \leqq c p_{2u}$, we can estimate this by

$$(3.7) \quad \phi(K) I_{2T}(p_{s_1} * ((s_2 - s_1)^a p_{s_2 - s_1} (s_3 - s_2)^b p_{s_3 - s_2} (s_4 - s_3)^c p_{s_4 - s_3}) * p_{2T - s_4})$$

where $a, b, c = 0$ or $\frac{1}{2}$ and at least one is $\frac{1}{2}$. $p_{u_1} p_{u_2} p_{u_3} = (2\pi)^{-3} (u_1 u_2 + u_1 u_3 + u_2 u_3)^{-3/2} \times p_{\sigma(u_1, u_2, u_3)}$ where $\sigma(u_1, u_2, u_3) = (u_1 u_2 u_3)(u_1 u_2 + u_1 u_3 + u_2 u_3)^{-1} \leqq u_1 + u_2 + u_3$.

Using (3.6), we can estimate (3.7) by

$$\phi(K) p_{2T} T^{3/2}$$
$$\times I_{2T}\left(\frac{1}{\sqrt{s_1}} \frac{\sqrt{s_2 - s_1}}{((s_2 - s_1)(s_3 - s_2) + (s_2 - s_1)(2T - s_3) + (s_3 - s_2)(2T - s_3))^{3/2}}\right)$$
$$\leqq \phi(K) p_{2T} T^{3/2},$$

which is much better than required. Remark, however, that if the $\sqrt{s_2 - s_1}$ factor is missing, then the integral is divergent. We thus obtain

$$(3.8) \qquad |B_1| \leqq \phi(K) T^{3/2} p_{2T}.$$

Both parts in B_2 are divergent for $\varepsilon \to 0$, and here we have to take into account the cancellation. It suffices to use relatively crude arguments as the divergency is only a logarithmical one. The somewhat tricky point is that $\tilde{q}$ enters by a convolution, but κ_2 just stands as a factor. So we have to operate the divergency $\tilde{q}$ out and "glue" the free ends in the convolution together. (This kind of surgery is much more tricky

in Sect. 4.) Coming to precise calculations, this procedure leads to the following splitting:

$$B_2(x) = \iint dy\,dz\, I_T((p_{s_1}(y) - p_{s_2}(z))\tilde{q}_{s_2-s_1}(z-y)g_{T-s_2}(x-z)) - I_T\left((p_{s_1} * g_{T-s_1})(x) \int_{s_1}^{1} du \|\tilde{q}_u\|_1\right) = C_1(x) - C_2(x), \text{ say}. \tag{3.9}$$

The effective cancellation has now been done within C_2. This part is easily seen to be bounded:

$$C_2 \leqq cI_T((p_{s_1} * g_{T-s_1})|\log s_1|) \leqq \phi(K)T^{3/4}p_{2T}. \tag{3.10}$$

However, we still have to check, that the surgery did do no harm, that is, we have to estimate C_1.

Let

$$\tau = ((s_2 - s_1)(s_3 - s_2) + (s_2 - s_1)(s_4 - s_3) + (s_3 - s_2)(s_4 - s_3))^{-3/2},$$

$$\sigma = (s_2 - s_1)(s_3 - s_2)(s_4 - s_3)\tau^{2/3}.$$

Then

$$C_1 \leqq cI_T(1_{\{s_1+\sigma<s_4/2\}}\tau|p_{s_1+\sigma} - p_{s_4}| * g_{T-s_4}) + cI_T(1_{\{s_1+\sigma>s_4/2\}}\tau|p_{s_1+\sigma} - p_{s_4}| * g_{T-s_4}) = C_{1,1} + C_{1,2}, \text{ say}. \tag{3.11}$$

$$\begin{aligned} |C_{1,1}| &\leqq cI_T(1_{\{s_1<s_4/2\}}\tau(p_{s_1+\sigma} + p_{s_4}) * g_{T-s_4}) \\ &\leqq \phi(K)I_T(1_{\{s_1<s_4/2\}}\tau(p_{2T-s_4} + p_{2T-2s_4+s_1+\sigma})) \\ &\leqq \phi(K)I_T(1_{\{s_1<s_4/2\}}\tau)p_{2T} + \phi(K)I_T(1_{\{s_1<s_4/2\}}1_{\{s_4>T/2\}}\tau p_{2T-2s_4+s_1+\sigma}). \end{aligned} \tag{3.12}$$

Performing the integration over τ in the first summand for fixed s_1, s_4, with respect to s_2, s_3 gives a factor $\leqq c(s_4 - s_1)^{-1}$ and $I_T(1_{\{s_1<s_2/2\}}(s_2 - s_1)^{-1}) \leqq cT$. The second summand on the right hand side of (3.12) at x is

$$\begin{aligned} &\leqq \phi(K)I_T\left(1_{\{s_1<s_4/2\}}1_{\{s_4>T/2\}}\tau\frac{1}{(T-(s_4-s_1))^{3/2}}\right)e^{-|x|^2/4T} \\ &\leqq \phi(K)e^{-|x|^2/4T}\int_0^{3T/4} du \frac{1}{\sqrt{u}(T-u)} \leqq \phi(K)Tp_{2T} \end{aligned}$$

and so

$$|C_{1,1}| \leqq \phi(K)Tp_{2T}. \tag{3.13}$$

As to $C_{1,2}$ we use (3.3) and get

$$\begin{aligned} |C_{1,2}| &\leqq I_T(\tau(s_4 - s_1 - \sigma)\frac{1}{s_4}p_{s_4} * g_{T-s_4}) \leqq \phi(K)I_T\left(\tau(s_4 - s_1)\frac{1}{s_4}\right)p_{2T} \\ &\leqq \phi(K)I_T\left(\frac{1}{s_2}\right)p_{2T} \leqq \phi(K)Tp_{2T}. \end{aligned} \tag{3.14}$$

(3.9)–(3.14) give $B_2 \leqq \phi(K)T^{3/4}p_{2T}$ and together with (3.8), this proves

$$|A_T^{(2)}| \leqq \phi(K)T^{3/4}p_{2T}. \tag{3.15}$$

As to $A_T^{(3)}$, we simply estimate by $g_t \leqq \phi(K)p_{2t}$ and get

$$A_T^{(3)} \leqq \phi(K) I_{2T}(p_{s_1} * [(p_{s_2-s_1} p_{s_3-s_2}) * (p_{s_5-s_4} p_{s_6-s_5})] * p_{s_4-s_3}) = \phi(K) I_{2T}(\tau p_\sigma)$$

where, with $u_i = s_i - s_{i-1}$, $1 \leqq i \leqq 7$ ($s_0 = 0$, $s_7 = 2T$)

$$\tau = (2\pi)^{-9/2}\{(u_2+u_3)(u_5+u_6)u_4 + u_2u_3(u_5+u_6) + u_5u_6(u_2+u_3)\}^{-3/2}$$

$$\sigma = \{u_1 + u_7 + u_4[u_2u_3(u_5+u_6) + u_5u_6(u_2+u_3)]\}\tau^{2/3}\,.$$

We get

$$\begin{aligned} I_{2T}(\tau p_\sigma) &\leqq c I_{2T}\left(\frac{1}{(u_1+u_7)^{3/2}}\tau\right) T^{3/2} p_{2T} \\ &\leqq c T^{3/2} p_{2T} \int_0^{2T} dw \frac{1}{\sqrt{2T-w}} \iiiint du_1\, du_2\, du_3\, du_4\, 1_{\{u_2+u_3+u_5+u_6<w\}} \\ &\qquad 1_{\{u_2<u_3\}} 1_{\{u_5<u_6\}} \frac{1}{(u_3u_6)^{3/2}[w-u_3-u_6]^{3/2}} \\ &\leqq c T^{3/2} p_{2T} \int_0^{2T} dw \frac{1}{\sqrt{w(2T-w)}} \leqq c T^{3/2} p_{2T}\,, \end{aligned}$$

and so we get

$$A_T^{(3)} \leqq \phi(K) T^{3/2} p_{2T}\,. \tag{3.16}$$

Combining (3.5), (3.15) and (3.16) yields for $\lambda \leqq 1$:

$$K_0(\varepsilon, \lambda) \leqq \lambda\phi(K(\varepsilon, \lambda))\,. \tag{3.17}$$

In order to get a similar estimate for K itself, one uses the following estimate:

$$\begin{aligned} \left|\int_0^1 (p_T(0) - g_T(0))dT\right| &\leqq \lambda \left|\int_0^1 A_T^{(1)}(0)dT\right| + \lambda^2 \int_0^1 |A_T^{(2)}(0)|\,dT + \lambda^3 \int_0^1 A_T^{(3)}(0)dT \\ &\leqq \lambda \left|\int_0^1 A_T^{(1)}(0)dT\right| + \lambda\phi(K), \text{ by (3.15) and (3.16)}\,. \end{aligned} \tag{3.18}$$

$$\begin{aligned} \left|\int_0^1 A_T^{(1)}(0)dT\right| &= \left| \iiint_{\{s+v+u\leqq 1\}} ds\, du\, dv (p_s * g_v)(0) j^\varepsilon(u) p_u(0) \right. \\ &\qquad \left. - \iint_{\{s+v\leqq 1\}} ds\, dv (p_s * g_v)(0) \int_\varepsilon^1 p_u(0)du \right| \\ &\leqq \iint_{\{s+v\leqq 1\}} ds\, dv (p_s * g_v)(0) \int_{1-s-v}^1 du\, p_u(0) \\ &\leqq \phi(K) \iint_{\{s+v\leqq 1\}} ds\, dv (s+v)^{-3/2}(1-s-v)^{-1/2} = \phi(K)\,. \end{aligned}$$

Therefore, together with (3.17) and (3.18), this yields $K(\varepsilon, \lambda) \leqq \lambda\phi(K(\varepsilon, \lambda))$. As $K(1, \lambda) = 0$ and $K(\varepsilon, \lambda)$ depends continuously on $\varepsilon > 0$, as remarked at the beginning of this section, we immediately conclude that for small $\lambda > 0$ $K(\varepsilon, \lambda) \leqq \rho(\lambda)$, where $\rho(\lambda) = \mathbf{inf}\{x \geqq 0\colon x = \lambda\,\phi(x)\}$, and for which obviously $\rho(\lambda) \leqq c\lambda$ holds for small enough λ, i.e. we have proved the Proposition 3.1.

Proof of tightness. We already know from (3.1) that $|E(\exp(-\bar{J}^{\varepsilon}_{0,1}))-1| \leqq c\lambda$. For $0 \leqq t < t+h \leqq 1$, we have

$$E(e^{-\bar{J}_{0,1}}|\omega_t-\omega_{t+h}|^4) \leqq E(e^{-\bar{J}_{0,t}-\bar{J}_{t,t+h}-\bar{J}_{t+h,1}}|\omega_t-\omega_{t+h}|^4)$$

$$= \|g_t\|_1 \, \|g_{1-t-h}\|_1 \int |x|^4 g_h(x)dx \leqq ch^2 .$$

This implies $\hat{E}^{\varepsilon}_{1,\lambda}(|\omega_t-\omega_{t+h}|^4) \leqq ch^2$ for small $\lambda > 0$, which proves tightness.

4 Proof of convergence in Theorem 1.3

As remarked at the end of Sect. 2, the procedure in Sect. 3 cannot be used to prove convergence directly. However, given the boundedness results of this section, it essentially suffices to have one-sided bounds of the derivative with respect to ε of certain quantities. For this differentiation, the gap-regularization we use is much more convenient than the lattice regularization used in [1]. (However, I believe, that the procedure used here could be used in this context, too, and maybe also for the construction of the ϕ^4_3 quantum field, see [2].)

The proof is quite involved, owing mainly to the fact that differentiation with respect to ε leads to expressions, in which, although they look of the same type as those encountered in Sect. 2 and 3, the cancellation of the divergencies is much more delicate.

We introduce an additional shorthand: $i(\varepsilon)$ is a generic nonnegative function of $\varepsilon > 0$ which is integrable in ε at 0.

Let ψ: $[0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be bounded and infinitely often differentiable with all derivatives bounded, and for $0 \leqq s < t \leqq 1$, let $\Psi_{s,t}$: $\Omega \to (0, \infty)$ be defined by

$$\Psi_{s,t} = \exp\left(\int_s^t \psi(u, \omega(u))du\right).$$

For short, we write $\Psi = \Psi_{0,1}$.

Let $\rho(\varepsilon) = E(\exp(-\bar{J}^{\varepsilon}_{0,1})\Psi)$. To prove the theorem, it suffices now to prove that $\lim_{\varepsilon\to 0}\rho(\varepsilon)$ exists for all choices of ψ and small enough λ. We already know from Proposition 3.1 that $\rho(\varepsilon)$ remains bounded as $\varepsilon \to 0$. Convergence is therefore implied by a one sided bound of the form

$$\rho(\varepsilon_2)-\rho(\varepsilon_1) \geqq -\int_{\varepsilon_1}^{\varepsilon_2} i(\varepsilon)d\varepsilon \tag{4.1}$$

for $0 < \varepsilon_1 < \varepsilon_2$. We write the left hand side as the integral over the derivative. Actually, we don't want to show that this derivative exists. All we need is a lower bound for $\liminf_{a\to 0}\frac{d}{d\varepsilon}\rho_a(\varepsilon)$ where ρ_a is obtained by replacing J by J^a (with δ replaced by p_a, differentiability is obvious). In fact, one easily gets

$$\rho(\varepsilon_2)-\rho(\varepsilon_1) \geqq \int_{\varepsilon_1}^{\varepsilon_2}\left(\liminf_{a\to 0}\frac{d}{d\varepsilon}\rho_a(\varepsilon)\right)d\varepsilon .$$

For notational convenience, we write $\frac{d}{d\varepsilon}\rho(\varepsilon)$ instead of $\liminf_{a\to 0}\frac{d}{d\varepsilon}\rho_a(\varepsilon)$ and use the $\delta - p_a$-convention of Sect. 2.

$$(4.2)\quad \frac{d}{d\varepsilon}\rho(\varepsilon) = \lambda \int_0^{1-\varepsilon} ds\, E(e^{-\bar{J}_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) + \left(\lambda \frac{d}{d\varepsilon}\kappa_1(\varepsilon) - \lambda^2 \frac{d}{d\varepsilon}\kappa_2(\varepsilon)\right)\rho(\varepsilon)$$

$$\bar{J}_{0,1} = \tilde{J}^s_{0,1} + \lambda J_{0,s;s,s+\varepsilon} + \lambda J_{s,s+\varepsilon;s+\varepsilon,1} - \lambda\varepsilon\kappa_1(\varepsilon) + O(\varepsilon^{3/4})$$

where $\tilde{J}^s_{0,1} = \bar{J}_{0,s} + \bar{J}_{s+\varepsilon,1} + \lambda J_{0,s;s+\varepsilon,1}$.

$$(4.3)\quad E(e^{-\bar{J}_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) \geqq E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi)e^{\lambda\varepsilon\kappa_1(\varepsilon) - O(\varepsilon^{3/4})}$$

$$- \lambda E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})(J_{0,s;s,s+\varepsilon} + J_{s,s+\varepsilon;s+\varepsilon,1})\Psi)(1 + O(\sqrt{\varepsilon}))\,.$$

In a similar way as in Sect. 3 there is a cancellation between the first summand on the right hand side of (4.3) and the $\frac{d}{d\varepsilon}\kappa_1$ summand in (4.2) and between the second summand in (4.3) and the $\frac{d}{d\varepsilon}\kappa_2$ summand in (4.2). The argument is however subtler than there. Then main difficulty is that the factors which are relevant for the divergency in (4.3), namely $\delta(\omega_s - \omega_{s+\varepsilon})$ in the first summand and $\delta(\omega_s - \omega_{s+\varepsilon})$ $(J_{0,s;s,s+\varepsilon} + J_{s,s+\varepsilon;s+\varepsilon,1})$, cannot readily be extracted from the expectation, the reason being that $\tilde{J}^s_{0,1}$ still contains the interaction between the time interval $[0, s]$ and $[s+\varepsilon, 1]$. This interaction gives a finite contribution which however is not going to 0 for $\varepsilon \to 0$. Therefore, it would do no good to expand it as one would loose all control over the cancellation with the $\frac{d}{d\varepsilon}\kappa_i$ summands in (4.2). It is therefore necessary to extract these divergencies, without doing much harm to the interactions on the whole path. This is a subtle procedure and some care is needed, otherwise all kind of uncancelled divergencies pop up.

We start with proving the partial cancellation of the first summand on the right hand side of (4.3) with the $\frac{d}{d\varepsilon}\kappa_1$ summand in (4.2).

$$\begin{aligned} E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi) &= E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\Psi_{0,s}\Psi_{s+\varepsilon,1}(1 + O(\varepsilon))) \\ &= p_\varepsilon(0)E(e^{-\bar{J}_{0,1-\varepsilon} - \lambda Y_s}\Psi_{0,s}(\Psi_{s+\varepsilon,1}\circ\theta_\varepsilon)(1 + O(\varepsilon))) \\ (4.4)\qquad &= p_\varepsilon(0)E(e^{-\bar{J}_{0,1-\varepsilon} - \lambda Y_s}\Psi(1 + O(\varepsilon))) \\ &\geqq p_\varepsilon(0)(1 + O(\varepsilon))\{E(e^{-\bar{J}_{0,1-\varepsilon}}\Psi) - \lambda E(Y_s e^{-\bar{J}_{0,1-\varepsilon}}\Psi)\} \end{aligned}$$

where $Y_s = Y^\varepsilon_s = \iint du\, dv\, 1_{\{0\leqq u\leqq s\leqq v\leqq 1-\varepsilon,\, v-u\leqq\varepsilon\}}\delta(\omega_u - \omega_v)$, and $\theta_\varepsilon(\omega)(t) = \omega_{t-\varepsilon}$. The justification of the second equality is by Proposition 3.1. Indeed for a fixed s, one calculates $E\{\exp[-\lambda J^{\varepsilon,a}_{0,s} - \lambda J^{\varepsilon,a}_{s+\varepsilon,1} - \lambda J^{\varepsilon,a}_{0,s;s+\varepsilon,1}]\ p_a(\omega_s - \omega_{s+\varepsilon})\Psi_{0,s}\Psi_{s+\varepsilon,1}\}$ by conditioning the Wiener measure on C_1 on the positions at s and at $s+\varepsilon$ which leads to a disintegration with a bridge on $[0, s]$ and a Wiener path on $[s+\varepsilon, 1]$. Proposition 3.1 (with the W) then easily proves the desired equality.

If $v-u \leqq \varepsilon$, then

$$E(\delta(\omega_u-\omega_v)e^{-\bar{J}_{0,1-\varepsilon}}\Psi)$$

$$\leqq E(\delta(\omega_u-\omega_v)e^{-\bar{J}_{0,u}-\bar{J}_{v,1-\varepsilon}-\lambda J_{0,u;v,1-\varepsilon}}(1+O(\sqrt{\varepsilon}))\Psi)$$

$$\leqq p_{v-u}(0)E(e^{-\bar{J}_{0,1-\varepsilon-(v-u)}}\Psi)(1+O(\sqrt{\varepsilon}))\,. \tag{4.5}$$

We claim that for $r \leqq 2\varepsilon$, we have

$$|E(e^{-\bar{J}_{0,1-r}}\Psi)-E(e^{-\bar{J}_{0,1}}\Psi)| \leqq c\varepsilon^{\frac{1}{2}+\delta} \tag{4.6}$$

for some $\delta>0$. Probably, the left hand side of (4.6) can be bounded by $\leqq c\varepsilon$, but this is not important for us. We postpone the proof of (4.6) for the moment. Using (4.4)–(4.6), we get for $\varepsilon \leqq s \leqq 1-\varepsilon$

$$E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})\Psi) \geqq p_\varepsilon(0)\rho(\varepsilon)(1-O(\varepsilon^{\frac{1}{2}+\delta}))$$

$$-\lambda p_\varepsilon(0)\iint du\,dv\,1_{\{u\leqq 0\leqq v,\,v-u\leqq\varepsilon\}}p_{v-u}(0)\rho(\varepsilon)(1+O(\sqrt{\varepsilon}))\,.$$

Remark that $\iint du\,dv\,1_{\{u\leqq 0\leqq v,\,v-u\leqq\varepsilon\}}p_{v-u}(0)=\varepsilon\kappa_1(\varepsilon)$. Therefore, we get for the first summand on the right hand side of (4.3) an estimate $\geqq p_\varepsilon(0)\rho(\varepsilon)-i(\varepsilon)$, as $\rho(\varepsilon)$ remains bounded. $p_\varepsilon(0)=\dfrac{d}{d\varepsilon}\kappa_1(\varepsilon)$, and so we see that we have obtained the required cancellation of the first part, and we get from (4.2)

$$\frac{d}{d\varepsilon}\rho(\varepsilon) \geqq -\lambda^2\int_0^{1-\varepsilon} ds\,E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})[J_{0,s;s,s+\varepsilon}+J_{s,s+\varepsilon;s+\varepsilon,1}]\Psi)(1+O(\sqrt{\varepsilon}))$$

$$-\lambda^2\frac{d}{d\varepsilon}\kappa_2(\varepsilon)\rho(\varepsilon)-i(\varepsilon)\,. \tag{4.7}$$

Let us now look at the first summand on the right hand side of (4.7) which should cancel with the second up to an integrable rest (integrable in ε). In a sense, this is much more delicate than the cancellation of the loops which lead to (4.7). However, the situation is much simplified by the fact that the divergencies are of order $1/\varepsilon$, so not much cancellation is needed to get an integrable rest.

The reader may find it instructive to look back at the derivation of (4.7) under this viewpoint. The original divergency of the first summand in (4.3) and $\dfrac{d}{d\varepsilon}\kappa_1$ is of order $\varepsilon^{-3/2}$, and it would have been easy to get a cancellation with a rest of order $\varepsilon^{-3/2+\delta}$ for some $\delta>0$, or even with $\delta=1/2$. The difficulty there is to get this with $\delta>1/2$.

$$\int_0^{1-\varepsilon} ds\,E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})J_{0,s;s,s+\varepsilon}\Psi) \tag{4.8}$$

$$\leqq \int_0^{1-\varepsilon} ds\int_0^s du\int_s^{s+\varepsilon} dv\,j^\varepsilon(v-u)E(e^{-\bar{J}_{0,u}-\bar{J}_{u,s}-\bar{J}_{s+\varepsilon,1}}\delta(\omega_u-\omega_v)\delta(\omega_s-\omega_{s+\varepsilon})$$

$$\times\Psi_{0,u}\Psi_{u,s}\Psi_{s+\varepsilon,1})(1+O(\varepsilon))$$

$$=\int_0^{1-\varepsilon} ds\int_0^s du\int_s^{s+\varepsilon} dv\,j^\varepsilon(v-u)\,\|\bar{g}_{0,u}*(\bar{g}_{u,s}p_{s,v}p_{v,s+\varepsilon})*\bar{g}_{s+\varepsilon,1}\|_1(1+O(\varepsilon))$$

where for $0\leqq s<t\leqq 1$, $\bar{g}_{s,t}(x)=E(\exp(-\bar{J}_{0,t-s})(\Psi_{s,t}\circ\theta_s)\delta(x-\omega_{t-s}))$.

Obviously, $\bar{g}_{s,t} \leqq c g_{t-s}$, and using Proposition 3.1, one calculates that the right hand side in (4.8) is bounded above by c/ε. Therefore, the term with $O(\sqrt{\varepsilon})$ in (4.7) is harmless, giving a summand of order $\varepsilon^{-1/2}$, which can be incorporated in $i(\varepsilon)$. We also remark that we can restrict the domain of integration in the first summand of (4.7), to keep s away from the boundary and u (in the integration in $J_{0,s;s,s+\varepsilon}$) close to s. More precisely, for arbitrary $0 < \alpha < \beta < 1$, we have:

$$\int_0^{1-\varepsilon} ds\, E(e^{-\tilde{J}^s_{0,1}} \delta(\omega_s - \omega_{s+\varepsilon}) J_{0,s;s,s+\varepsilon} \Psi)$$

$$(4.9) \qquad = \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv\, E(e^{-\bar{J}^s_{0,1}} \delta(\omega_s - \omega_{s+\varepsilon}) \delta(\omega_u - \omega_v) \Psi) + i(\varepsilon)$$

which follows in exactly the same way as in (4.8). We now want to keep any further interaction at distance from the $u, v, s, s+\varepsilon$ in the above expression. To this end we replace $\tilde{J}^s_{0,1}$ by a quantity where all the interactions with a short time interval before s and behind $s+\varepsilon$ are dropped. This will help to extract the divergency out of the expectation. It should perhaps be remarked that here the argument is quite brute. As indicated before, we can proceed in such a way because not much cancellation is needed. We assume $\beta > \frac{1}{2} > \alpha$ and choose $\gamma > \frac{1}{2}$ with $\gamma < \beta$. Then

$$\tilde{J}^s_{0,1} \geqq \bar{J}_{0,s-\varepsilon^\gamma} + \bar{J}_{s+\varepsilon^\gamma,1} + \lambda J_{0,s-\varepsilon^\gamma;s+\varepsilon^\gamma,1} - O(\varepsilon^\gamma \kappa_1(\varepsilon)) = \check{J}^s_{0,1} - O(\varepsilon^\gamma \kappa_1(\varepsilon)), \text{ say}.$$

Remark that $\varepsilon^\gamma \kappa_1(\varepsilon) \sim \varepsilon^{\gamma - 1/2}$, so this part can be incorporated in the $i(\varepsilon)$ – part in (4.9). Therefore, the expression on the left hand side of (4.9) is

$$\leqq \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv\, E(e^{-\check{J}^s_{0,1}} \delta(\omega_s - \omega_{s+\varepsilon}) \delta(\omega_u - \omega_v) \times \Psi_{0,s-\varepsilon^\gamma} \Psi_{s+\varepsilon^\gamma,1}) + i(\varepsilon).$$

Keeping the positions at the times $s - \varepsilon^\gamma$ and $s + \varepsilon^\gamma$ fixed, the divergency can be extracted easily. Proceeding in this way, to integrand in the above expression is:

$$(4.10) \quad = \int dx \int dy E(e^{-\check{J}^s_{0,1}} \delta(\omega_s - \omega_{s+\varepsilon}) \delta(\omega_u - \omega_v) \Psi_{0,s-\varepsilon^\gamma} \Psi_{s+\varepsilon^\gamma,1} \delta(x - \omega_{s-\varepsilon^\gamma}) \times \delta(y - \omega_{s+\varepsilon^\gamma}))$$

$$= \int dx \int dy E(e^{-\check{J}^s_{0,1}} \delta(x - \omega_{s-\varepsilon^\gamma}) \delta(y - \omega_{s+\varepsilon^\gamma}) \Psi_{0,s-\varepsilon^\gamma} \Psi_{s+\varepsilon^\gamma,1}) q_{u,s,v}(y - x),$$

where $q_{u,s,v} = p_{2\varepsilon^\gamma - \varepsilon - (s-u)} * (p_{s-u} p_{v-s} p_{s+\varepsilon-v})$.

$$(4.11) \qquad q_{u,s,v}(x) = p_{2\varepsilon^\gamma}(x) \| p_{s-u} p_{v-s} p_{s+\varepsilon-v} \|_1 + \tilde{q}_{u,s,v}(x), \text{ where}$$

$$\tilde{q}_{u,s,v}(x) = \int dz [p_{2\varepsilon^\gamma - \varepsilon - (s-u)}(z) - p_{2\varepsilon^\gamma}(x)] p_{s-u}(x-z) p_{v-s}(x-z) p_{s+\varepsilon-v}(x-z)$$

$$= \tau [p_{2\varepsilon^\gamma - \varepsilon - (s-u) + \sigma}(z) - p_{2\varepsilon^\gamma}(x)]$$

with $\tau = c[(s-u)(v-s) + (s-u)(s+\varepsilon-v) + (v-s)(s+\varepsilon-v)]^{-3/2}$, $\sigma = (s-u)(v-s)(s+\varepsilon-v)\tau^{3/2}$. The first summand on the right hand side of (4.11) now looks nice: if we implement this into (4.10), the factor $p_{2\varepsilon^\gamma}(x)$ "glues" the loose ends x, y in the expectation together, and the second factor is our counterterm. So we are essentially only left with proving that the contribution of $\tilde{q}_{u,s,v}$ is negligible.

Proceeding in this way, we get for the left hand side of (4.9) the estimate

$$\leqq \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds\, E(e^{-\check{J}^s_{0,1}}\Psi_{0,s-\varepsilon^\gamma}\Psi_{s+\varepsilon^\gamma,1}) \int_{s-\varepsilon^\beta}^{s} du \int_{s}^{s+\varepsilon} dv \,\|p_{s-u}p_{v-s}p_{s+\varepsilon-v}\|_1 + R + i(\varepsilon)$$

$$\leqq -\frac{1}{2}\frac{d}{d\varepsilon}\kappa_2(\varepsilon) \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds E(e^{-\check{J}^s_{0,1}}\Psi) + R + i(\varepsilon), \text{ where}$$

$$R \leqq \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_{s}^{s+\varepsilon} dv \int dx \int dy E(e^{-\check{J}^s_{0,1}}\Psi_{0,s-\varepsilon^\gamma}\Psi_{s+\varepsilon^\gamma,1}$$

$$\times \delta(x-\omega_{s-\varepsilon^\gamma})\delta(y-\omega_{s+\varepsilon^\gamma}))|\tilde{q}_{u,s,v}(y-x)|$$

which by $\check{J}^s_{0,1} \geqq \bar{J}_{0,s-\varepsilon^\gamma} + \bar{J}_{s+\varepsilon^\gamma}$ and (1.3a) is

$$\leqq c \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_{s}^{s+\varepsilon} dv \,\|\tilde{q}_{u,s,v}\|_1 = i(\varepsilon)\,.$$

The last estimate follows from the same kind of arguments as in the bound for C_1 in Sect. 3. In fact, from (3.3), we get $\|\tilde{q}_{u,s,v}\|_1 \leqq c\tau\varepsilon^{-\gamma}\varepsilon^\beta$ and $\int_{s-\varepsilon^\beta}^{s} du \int_{s}^{s+\varepsilon} dv\tau \leqq c/\varepsilon$.

Therefore, we get

$$\int_0^{1-\varepsilon} ds E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})J_{0,s;s,s+\varepsilon}\Psi) \leqq -\frac{1}{2}\frac{d}{d\varepsilon}\kappa_2(\varepsilon) \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds E(e^{-\check{J}^s_{0,1}}\Psi) + i(\varepsilon)$$

which by using the same kind of arguments again (splitting $\bar{J}_{0,1}$ into $\check{J}^s_{0,1}$ and a rest) is $\leqq -\frac{1}{2}\frac{d}{d\varepsilon}\kappa_2(\varepsilon)\rho(\varepsilon) + i(\varepsilon)$. The same estimate holds when $J_{0,s;s,s+\varepsilon}$ is replaced by $J_{s,s+\varepsilon;s+\varepsilon,1}$. Implementing this into (4.7) gives $\frac{d}{d\varepsilon}\rho(\varepsilon) \geqq -i(\varepsilon)$ which is the desired bound for (4.1).

It only remains to prove (4.6) which had been left open. Differentiating $E(\exp(-\tilde{J}_{0,v})\Psi)$ with respect to v yields for $v \geqq \varepsilon$:

$$\frac{d}{dv}E(\exp(-\bar{J}_{0,v})\Psi) = \lambda \int_0^{v-\varepsilon} ds E(\delta(\omega_s-\omega_v)\exp(-\bar{J}_{0,v})\Psi) \tag{4.12}$$

$$+ (-\lambda\kappa_1(\varepsilon) + \lambda^2\kappa_2(\varepsilon))E(\exp(-\bar{J}_{0,v})\Psi)\,.$$

Remark that this gives the very crude and insufficient bound

$$\left|\frac{d}{dv}E(\exp(-\bar{J}_{0,v})\Psi)\right| \leqq c\varepsilon^{-1/2}\,. \tag{4.13}$$

Indeed, for the second part on the right hand side of (4.12), this is clear and the first part is

$$\leqq c \int_0^{v-\varepsilon} ds E(\delta(\omega_s-\omega_v)\exp(-\bar{J}_{0,s}-\bar{J}_{s,v})) \tag{4.14}$$

$$= c \int_0^{v-\varepsilon} ds \|g_s\|_1 g_{v-s}(0) = O(\varepsilon^{-1/2})\,,$$

by Proposition 3.1. To get a slightly better bound, we need some cancellation in (4.12). We split first the integral in the first summand on the right hand side of (4.12) in the part where $s \leqq v - \varepsilon^{3/4}$ and where $s > v - \varepsilon^{3/4}$. By the same kind of estimate as in (4.14), we get for $v \geqq \varepsilon^{3/4}$:

$$\int_0^{v-\varepsilon^{3/4}} ds E(\delta(\omega_s - \omega_v)\bar{J}_{0,v}\Psi) = O(\varepsilon^{-3/8}) . \tag{4.15}$$

$$\begin{aligned}
\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds E(\delta(\omega_s - \omega_v)e^{-\bar{J}_{0,v}}\Psi) &\leqq \int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds E(e^{-\bar{J}_{0,s}}\Psi_{0,s})p_{v-s}(0)(1 + O(\varepsilon^{1/4})) \\
&\leqq \int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds E(e^{-\bar{J}_{0,s}}\Psi)p_{v-s}(0)(1 + O(\varepsilon^{1/4})) \\
&\leqq E(e^{-\bar{J}_{0,v}}\Psi) \int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds p_{v-s}(0) + O(\varepsilon^{-1/4}), \text{ by (4.13)} \\
&= E(\exp(-\bar{J}_{0,v})\Psi)\kappa_1(\varepsilon) + O(\varepsilon^{-1/4}) .
\end{aligned}$$

To have an estimate in the other direction, we perform the same kind of analysis as in Sect. 3.

$$\begin{aligned}
&\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds E(\delta(\omega_s - \omega_v)\exp(-\bar{J}_{0,v})\Psi) \\
&\geqq \int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds E(\exp(-\bar{J}_{0,s})\Psi_{0,s})g_{v-s}(0)(1 - O(\varepsilon^{1/4})) \\
&- c \int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds \int_0^s du \int_s^v dt j^\varepsilon(t-u) E(\delta(\omega_s - \omega_v)\delta(\omega_t - \omega_u) \\
&\qquad \times \exp(-\bar{J}_{0,u} - \bar{J}_{u,s} - \bar{J}_{s,t} - \bar{J}_{t,v})) .
\end{aligned}$$

The first summand on the right hand side is, again by (4.14) and Proposition 3.1 $\geqq E(\exp(-\bar{J}_{0,v})\Psi)\,\kappa_1(\varepsilon) - O(\varepsilon^{-1/4})$ and the second is in absolute value

$$\leqq \int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds \int_0^s du \int_s^v dt j^\varepsilon(t-u) \|g_s\|_1 \|g_{s-u}g_{t-s}g_{v-t}\|_1 = O(|\log \varepsilon|) .$$

Therefore, we get

$$\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} ds E(\delta(\omega_s - \omega_v)\exp(-\bar{J}_{0,v})\Psi) \geqq E(\exp(-\bar{J}_{0,v})\Psi)\kappa_1(\varepsilon) - O(\varepsilon^{-1/4}) . \tag{4.16}$$

(4.12), (4.14), (4.15) and (4.16) prove $\left|\frac{d}{dv} E(\exp(-\bar{J}_{0,v})\Psi)\right| \leqq \varepsilon^{-3/8}$, which proves (4.6).

Appendix. Proof of Proposition 2.1

We restrict to the discussion of $Y_{T,\lambda}^{\varepsilon,a}$, the other cases being similar.

(A1) Lemma. *If* $0 \leqq \gamma < \frac{1}{2}$, $m \in \mathbb{N}$ *then*

(a) $\int_{\mathbb{R}^{3m}} du_1 du_2 \ldots du_m \prod_{j=1}^m |u_j|^\gamma \left(1 + \left|\sum_{s=1}^j u_s\right|\right)^{-4} < \infty$, *where* $u_1, \ldots, u_m \in \mathbb{R}^3$

(b) $\int_{\mathbb{R}^{3m}} du_1 du_2 \ldots du_m \prod_{j=1}^{m} \left(1 + \left|\sum_{s=1}^{j} u_s\right|\right)^{-4+2\gamma} < \infty$.

Proof. If $a \in \mathbb{R}^3$, then

$$\int_{\mathbb{R}^3} \frac{|u|^\gamma}{(1+|a+u|)^{4-\gamma}} du = \int_{\mathbb{R}^3} \frac{|x-a|^\gamma}{(1+|x|)^{4-\gamma}} dx$$

$$\leqq 2^\gamma \int_{\mathbb{R}^3} \frac{(1+|a|)^\gamma + (1+|x|)^\gamma}{(1+|x|)^{4-\gamma}} dx \leqq c(\gamma)(1+|a|)^\gamma .$$

Therefore, we get

$$\int_{\mathbb{R}^3} du_k \prod_{j=1}^{k} |u_j|^\gamma \prod_{j=1}^{k-1} \left[1 + \left|\sum_{s=1}^{j} u_s\right|\right]^{-4} \left[1 + \left|\sum_{s=1}^{k} u_s\right|\right]^{-4+\gamma}$$

$$\leqq c(\gamma) \sum_{j=1}^{k-1} |u_j|^\gamma \prod_{j=1}^{k-2} \left[1 + \left|\sum_{s=1}^{j} u_s\right|\right]^{-4} \left[1 + \left|\sum_{s=1}^{k-1} u_s\right|\right]^{-4+\gamma} .$$

From this, (a) follows and (b) is similar.

We now start with the proof of (2.1) and use modifications of arguments in Rosen [5].

$$Y_{T,x}^{\varepsilon,a} = \sqrt{T} \int_0^{1-\varepsilon/T} ds \int_{s+\varepsilon/T}^{1} dt p_{a/T}(\omega(t) - \omega(s) - (t-s)\omega(1) + (t-s)x/\sqrt{T}) .$$

We see that it suffices to prove the proposition for a fixed T, for simplicity, say for $T = 1$, and for $\varepsilon \geqq \varepsilon_0$ for an arbitrary but fixed ε_0. Let $\Delta_\varepsilon = \{(s,t) \in [0,1]: 0 \leqq s, s+\varepsilon \leqq t \leqq 1\}$. If $S = (s,t) \in \Delta_\varepsilon$, we write $X_S = \omega(t) - \omega(s) - (t-s)\omega(1)$. We will consider rectangles of the form $R = [s,t] \times [u,v]$, where $0 \leqq s < t < u < v \leqq 1$. We define $\delta(R) = \max((t-s),(v-u))$, $g(R) = u - t$. Clearly, for any $\varepsilon_0 > 0$ and $\delta_0 > 0$, we can cover Δ_ε with finitely many rectangles R satisfying $\delta(R) \leqq \delta_0$, $g(R) \geqq \varepsilon_0/2$. If R is such a rectangle, we write

$$\tilde{Y}_{R,x,y}^{\varepsilon,a} = \iint_{\Delta_\varepsilon \cap R} ds\, dt\, p_a(X_S + (t-s)x - y),\ x, y \in \mathbb{R}^3;\ \varepsilon, a > 0 .$$

It suffices to prove that a continuous version of $\tilde{Y}_{R,x,y}^{\varepsilon,a}$ exists for rectangles with $g(R) \geqq \varepsilon_0/2$ and $\delta(R)$ arbitrarily small. All what we have to prove is the following result:

(A.2) Lemma. *If $n \in \mathbb{N}$, $\varepsilon_0 > 0$, then there exists $\delta_0 > 0$ such that for a rectangle R with $\delta(R) \leqq \delta_0$, $g(R) \geqq \varepsilon_0/2$, there exists $C > 0$ with*

$$E[\tilde{Y}_{R,x,y}^{\varepsilon,a} - \tilde{Y}_{R,x',y'}^{\varepsilon',a'}]^n \leqq C[|x-x'|^{n/4} + |y-y'|^{n/4} + |\varepsilon-\varepsilon'|^{n/8} + |a-a'|^{n/8}] .$$

Proof. We first keep $\varepsilon = \varepsilon'$ and $a = a'$. A simple calculation gives

$$E(\tilde{Y}_{R,x,y}^{\varepsilon,a} - \tilde{Y}_{R,x',y'}^{\varepsilon,a})^n = (2\pi)^{-3n} \int_{\mathbb{R}^{3n}} d\underline{k} \int_{(\Delta_\varepsilon \cap R)^n} d\underline{s}\, d\underline{t} \prod_{j=1}^{n} \{\exp(-ik_j[(t_j - s_j)x' - y'])$$

$$\text{(A.3)} \quad - \exp(-ik_j[(t_j - s_j)x - y])\} \exp\left[-a \sum_{j=1}^{n} |k_j|^2/2 - \frac{1}{2} \operatorname{Var}\left(\sum_{j=1}^{k} k_j X_{S_j}\right)\right]$$

where $\underline{k} = (k_1, \ldots, k_n)$, $k_i \in \mathbb{R}^3$, $\underline{s} = (s_1, \ldots, s_n)$, $\underline{t} = (t_1, \ldots, t_n)$, $S_j = (s_j, t_j)$. The above expression is in absolute value

$$\text{(A.4)} \qquad \leqq C(\gamma, n)(|x - x'|^{n/4} + |y - y'|^{n/4}) \int_{\mathbb{R}^{3n}} d\underline{k} \int_{R^n} d\underline{s}\, d\underline{t}$$

$$\times \prod_{j=1}^{n} |k_j|^{\frac{1}{4}} \exp\left(-\frac{1}{2} \operatorname{var}\left[\sum_{j=1}^{n} k_j X_{S_j}\right]\right).$$

Remark that this bound does not depend on a. We split the integration with respect to $\underline{s}, \underline{t}$ according to the relative ordering of the s_j, t_j. It suffices to consider the cases

$$a \leqq s_1 < s_2 < \ldots < s_n \leqq b < c \leqq t_{\pi(1)} < t_{\pi(2)} < \ldots < t_{\pi(n)} \leqq c$$

where π is a permutation of $1, \ldots, n$ and $R = [a, b] \times [c, d]$. We denote by $R_{n,\pi}$ the set of $2n$-tuples with $(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n)$ satisfying this condition. We assume $\delta(R) \leqq \frac{1}{4}$ and distinguish two cases:

$$\text{(I)} \quad d - a \leqq \tfrac{3}{4}, \quad \text{and (II)} \quad d - a > \tfrac{3}{4}.$$

In case (I), the Brownian bridge looks like the ordinary Brownian motion (at least if $\delta(R)$ is small). In case (II) the situation is like if one would have two independent Brownian motions (also if $\delta(R)$ is small, so $g(R)$ is large).

We first consider the case (I).

We have

$$\sum_{j=1}^{n} k_j X_{S_j} = \sum_{j=1}^{n} k_{\pi(j)} \beta(t_{\pi(j)}) - \sum_{j=1}^{n} k_j \beta(s_j)$$

where $\beta(t) = \omega(t) - t\omega(t)$. Substituting t_j' for $t_{\pi(j)}$ and dropping the dash afterwards, we have

$$\sum_{j=1}^{n} k_j X_{S_j} = \left(\sum_{s=1}^{n} k_s\right)(\beta(t_1) - \beta(s_n)) + \sum_{j=1}^{n-1} \left(\sum_{s=1}^{j} k_s\right)(\beta(s_{j+1}) - \beta(s_j))$$

$$+ \sum_{j=2}^{n} \left(\sum_{s=j}^{n} k_{\pi(s)}\right)(\beta(t_j) - \beta(t_{j-1}))$$

with $a \leqq s_1 < s_2 < \ldots < s_n \leqq b < c \leqq t_1 < t_2 < \ldots < t_n \leqq d$. β is a three dimensional process, but the three components are independent. We write one component of the above expression as $\sum_{j=0}^{2n-2} \lambda_j \xi_j$ where ξ_0 is the component of $\beta(t_1) - \beta(s_n)$ and ξ_i is the component of $\beta(s_{i+1}) - \beta(s_i)$ for $1 \leqq i \leqq n-1$ and of $\beta(t_{i-n+2}) - \beta(t_{i-n+1})$ for $n \leqq i \leqq 2n-2$. If ξ_j is the component of $\beta(u) - \beta(v)$, we write Δ_j for $u - v$. Then

$$\operatorname{var}\left(\sum_{j=0}^{2n-2} \lambda_j \xi_j\right) = \sum_j \lambda_j^2 (\Delta_j - \Delta_j^2) - \sum_{j \neq k} \lambda_j \lambda_k \Delta_j \Delta_k .$$

Remark that $\Delta_0 \leqq 3/4$ and $\Delta_j \leqq \delta(R)$ for $j \geqq 1$. Then, one easily checks that if $\delta(R) \leqq \dfrac{1}{100n}$, one has for $0 \leqq j \leqq 2n-2$, $\Delta_j - \Delta_j^2 \geqq \frac{1}{5}\Delta_j + \sum_{j \neq k} \Delta_j \Delta_k$. From this,

one obtains by the Schwartz inequality $\operatorname{var}(\sum_j \lambda_j \xi_j) \geqq \frac{1}{5}\sum_j \lambda_j^2 \Delta_j$. Therefore

$$\int_{R_{\pi,n}} d\underline{s}\, d\underline{t} \exp\left(-\frac{1}{2}\operatorname{var}\left[\sum_{j=1}^{n} k_j X_{s_j}\right]\right) \leqq \prod_{m=1}^{n-1} \int_0^\infty du \exp\left(-\frac{u}{5}\left|\sum_{j=1}^{m} k_j\right|^2\right)$$

$$\times \prod_{m=2}^{n} \int_0^\infty du \exp\left(-\frac{u}{5}\left|\sum_{j=m}^{n} k_{\pi(j)}\right|^2\right) \int_{\varepsilon_0/2}^\infty du \exp\left(-\frac{u}{5}\left|\sum_{j=1}^{n} k_j\right|^2\right)$$

$$\leqq \prod_{m=1}^{n}\left(1+\left|\sum_{j=1}^{m} k_j\right|\right)^{-2} \prod_{m=1}^{n}\left(1+\left|\sum_{j=m}^{n} k_{\pi(j)}\right|\right)^{-2}.$$

Combining this with (A.3) and (A.4) gives

$$E(\tilde{Y}^{\varepsilon,a}_{R,x,y} - \tilde{Y}^{\varepsilon,a}_{R,x',y'})^n$$

$$\leqq C(n)\sum_\pi (|x-x'|^{n/4} + |y-y'|^{n/4}) \int_{\mathbb{R}^{3n}} d\underline{k} \prod_{j=1}^{n} |k_j|^{1/4} \prod_{m=1}^{n}\left(1+\left|\sum_{j=1}^{m} k_j\right|\right)^{-2}$$

$$\times\left(1+\left|\sum_{j=m}^{n} k_{\pi(j)}\right|\right)^{-2}$$

$$\leqq C(n)(|x-x'|^{n/4} + |y-y'|^{n/4}) \int_{\mathbb{R}^{3n}} d\underline{k} \prod_{j=1}^{n} |k_j|^{1/4} \prod_{m=1}^{n}\left(1+\left|\sum_{j=1}^{m} k_j\right|\right)^{-4}$$

$$\leqq C(n)(|x-x'|^{n/4} + |y-y'|^{n/4}), \text{ by (A.1.a)}$$

The estimate of $E(\tilde{Y}^{\varepsilon,a}_{R,x,y} - \tilde{Y}^{\varepsilon,a'}_{R,x,y})^n$ is similar. In fact, we just have to estimate $|\exp(-a|k|^2) - \exp(-a'|k|^2)| \leqq |a-a'|^{1/8}|k|^{1/4}$ and argue as before.

Next, we estimate $E(\tilde{Y}^{\varepsilon',a}_{R,x,y} - \tilde{Y}^{\varepsilon,a}_{R,x,y})^n$ still in the case (I) for $\varepsilon_0 \leqq \varepsilon < \varepsilon'$. Clearly, $Y^\varepsilon \geqq Y^{\varepsilon'}$. If $D = \Delta_\varepsilon \backslash \Delta_{\varepsilon'}$, then we have

$$E(\tilde{Y}^{\varepsilon,a}_{R,x,y} - \tilde{Y}^{\varepsilon',a}_{R,x,y})^n$$

$$= (2\pi)^{-3n} \int_{\mathbb{R}^{3n}} d\underline{k} \int_{(R\cap D)^n} d\underline{s}\, d\underline{t} \prod_{j=1}^{n} \exp(-ik_j[(t_j - s_j)x - y])$$

$$\times \exp\left[-a\sum_{j=1}^{n} |k_j|^2/2 - \frac{1}{2}\operatorname{Var}\left(\sum_{j=1}^{k} k_j X_{s_j}\right)\right]$$

$$\leqq C(n)\sum_\pi |R\cap D|^{n/q} \int_{\mathbb{R}^{3n}} d\underline{k} \left\{\int_{R^n} d\underline{s}\, d\underline{t} \exp\left[-(p/2)\operatorname{var}\left[\sum_{j=1}^{n} k_j X_{s_j}\right]\right]\right\}^{1/p}$$

when $1/p + 1/q = 1$, $|R\cap D|$ denoting the Lebesgue measure of $R\cap D$

$$\leqq C(n)|R\cap D|^{n/q} \int_{\mathbb{R}^{3n}} d\underline{k} \prod_{m=1}^{n}\left(1+\left|\sum_{j=1}^{m} k_j\right|\right)^{-4/p} \leqq C(n,p)|R\cap D|^{n/q}$$

if $4/p > 3$, i.e. if $1/q < 1/4$ by (A.1.b). Therefore, we have proved (A.2) in case (I).

Everything up to now was for the case (I). The arguments in case (II) are essentially the same with only minor modifications. We again have to estimate $\operatorname{var}[\sum_{j=1}^n k_j X_{s_j}]$ from below on $a \leqq s_1 < s_2 < \ldots < s_n \leqq b < c \leqq t_{\pi(1)} < t_{\pi(2)} < \ldots < t_{\pi(n)} \leqq d$. Putting now $(1-t_1') = t_{\pi(n)}$, $(1-t_2') = t_{\pi(n-1)}, \ldots, (1-t_n') = t_{\pi(1)}$

and putting $\beta'(t') = \beta(1-t')$, we have to look at

$$\sum_{m=1}^{n}\left(\sum_{j=m}^{n} k_{\pi(j)}\right)(\beta'(t'_j)-\beta'(t'_{j-1}))-\sum_{m=1}^{n}\left(\sum_{j=m}^{n} k_j\right)(\beta(s_j)-\beta(s_{j-1}))\,.$$

Arguing now as in the case (I), one easily sees that the covariance matrix of a component of $(\beta'(t'_1), \beta'(t'_2)-\beta'(t'_1), \ldots, \beta'(t'_n)-\beta'(t'_{n-1}), \beta(s_1), \beta(s_2)-\beta(s_1), \ldots, \beta(s_n)-\beta(s_{n-1}))$ dominates a diagonal matrix with entries $\frac{1}{5}t'_1$, $\frac{1}{5}(t'_2-t'_1), \ldots, \frac{1}{5}s_1, \ldots, \frac{1}{5}(s_n-s_{n-1})$, if $\delta(R)$ is small enough. Performing the integration of $\exp(-\frac{1}{2}\mathrm{var}[\sum_{j=1}^{n} k_j X_{s_j}])$ with respect to $\underline{s}$ and $\underline{t}$ gives an upper bound

$$c\left(1+\left|\sum_{j=1}^{m} k_j\right|\right)^{-2}\left(1+\left|\sum_{j=m}^{n} k_{\pi(j)}\right|\right)^{-2}.$$

The rest of the argument is the same as before.

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References

1. Bovier, A., Felder, G., Fröhlich, J.: On the critical properties of the Edwards and self-avoiding walk model of polymer chains. Nucl. Phys. B **230**, 119–147 (1984)
2. Brydges, D.C., Fröhlich, J., Sokal, A.: A new proof of the existence and nontriviality of the continuum ϕ_2^4 and ϕ_3^4 quantum field theories. Commun. Math. Phys. **91**, 141 (1983)
3. Dynkin, E.B.: Regularised self-intersection local times of planar Brownian motion. Ann. Probab. **16**, 58–74 (1988)
4. Kusuoka, S.: On the path property of Edwards' model for long polymer chains in three dimensions. In: Albeverio, S. (ed.) Proc. Bielefeld Conf. in infinite dimensional analysis and stochastic processes. (Pitman Res. Notes Math. Sci., vol. 124, pp. 48–65) Harlow: Longman and New York: Wiley 1985
5. Rosen, J.: A local time approach to the self-intersections of Brownian paths in space. Commun. Math. Phys. **88**, 327–338 (1983)
6. Varadhan, S.R.S.: Appendix to "Euclidian quantum field theory" by K. Symanzik. In: Jost, R. (ed.) Local quantum theory. New York London: Academic Press 1969
7. Westwater, J.: On Edwards' model for polymer chains. Commun. Math. Phys. **72**, 131–174 (1980)
8. Westwater, J.: On Edwards' model for polymer chains. III. Borel summability. Commun. Math. Phys. **84**, 459–470 (1982)
9. Yor, M.: Complements aux formules de Tanaka-Rosen. In: Azéma, J., Yor, M. (eds.) Seminaire de Probabilites XIX. (Lect. Notes Math., vol. 1123, pp. 332–349) Berlin Heidelberg New York: Springer 1985.
10. Zhou, Z.H.: The intersection local time for the Westwater process. Probab. Theory Relat. Fields **91**, 375 (1992)