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Itô Excursion Theory Via Resolvents

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1. Introduction

Let $(X_t^\partial)_{t \geqq 0}$ be a strong Markov process with state space F, which is killed at the instant τ of first reaching some distinguished state $a \in F$. *What are the strong Markov processes X on F which behave like X^∂ up to the time τ?* In a profound study, Itô [6] answered this question in terms of the local time of the process at a and the Poisson point process of excursions. Formulated in terms of the resolvent R_λ of X, behaving "like X^∂ up to time τ" is expressed as

$$R_\lambda f(x) = R_\lambda^\partial f(x) + E^x(e^{-\lambda\tau}) R_\lambda f(a) \qquad (x \in F) \tag{1}$$

where R_λ^∂ is the resolvent of the killed process X^∂, and f is a bounded measurable function. This identity, which is obvious from the strong Markov property, plainly imposes certain constraints on $R_\lambda f(a)$ by way of the resolvent identity. Using *only* the resolvent identity, we are able to show from (1) that $R_\lambda f(a)$ must have the form

$$R_\lambda f(a) = \frac{n_\lambda f + \gamma f(a)}{\delta + \lambda n_\lambda 1 + \lambda \gamma} \tag{2}$$

where γ, δ are non-negative reals, and $\{n_\lambda; \lambda > 0\}$ are a family of non-negative measures on $F \setminus \{a\}$ such that for λ, $\mu > 0$, distinct,

$$n_\lambda R_\mu^\partial = \frac{n_\lambda - n_\mu}{\mu - \lambda}. \tag{3}$$

These formulae will come as no great surprise to those who are familiar with excursion theory, in terms of which (2) and (3) have natural interpretations. However, their potential does not yet appear to have been generally realised, and it is the purpose of this paper to emphasise their usefulness by presenting excursion explanations of two well-known results originally proved by completely different means.

Formulae (2) and (3) are implicit in Itô's excursion solution to the original problem, phrased in terms of resolvents at (1). Thus it should not be surprising that one can derive (2) and (3) (which, in a sense, *are* Itô excursion theory!) from (1), since the resolvent contains all there is to know about the process. Yet there *is* a surprise, of a different kind. When I showed the first draft of this paper to David Williams, he made several valuable suggestions, one of which was to look up Reuter [12]; in this paper, I found essentially the same proof as is presented here (although only for the case of Markov chains), established more than twenty years earlier, and *more than ten years before Itô's paper*! Reuter's fine results (also proved independently by Neveu) deserve to be better known, so perhaps I may be forgiven for reworking them here in a more general context.

These matters occupy Section 2 of the paper. In Section 3, the characterisation of Feller Brownian motions (strong Markov processes in $\mathbb{R}^+$ which behave like Brownian motion when in $(0,\infty)$), due originally to Feller [1], is proved briskly with the aid of (2) and (3). See Itô-McKean [7] for further discussion. As a further application, in Sect. 4 we show briefly how to represent the Laplace transforms $(n_\lambda)_{\lambda>0}$ of the excursion law (for that is what they are) as integrals over a suitable (Martin type) compactification of $F\setminus\{a\}$, whose boundary points correspond to the different ways in which excursions can exit from a continuously. The part of the integral representation over $F\setminus\{a\}$ corresponds to excursions which jump out of a. Another of David William's suggestions was to consult Neveu [10], where again I found that the chain experts had got there first, and again a long time before excursion theory had been invented! However, Neveu's construction is valid in a much more general setting than the chains which he dealt with; we here prove the result in this greater generality, and indicate its connections with the integral representation of excursion measures in the final section of Itô's paper [6]. As a zoological footnote, we illustrate the construction in the case of the Brownian hedgehog. Finally in Sect. 5 we show how David Williams' path decompositions [16] can be explained in terms of excursions, and we characterise the excursion law of drifting Brownian motion.

We shall work throughout with Ray processes, and in several places lean heavily on their properties. This is an unfortunate technical necessity, but many of our results hold an intuitive content which is meaningful in a "strong Markov" context even though we may be compelled to resort to the Ray setting to state them accurately!

Acknowledgement. Those who have read this far will realise how much I have benefited from David Williams' advice, for which I am very grateful.

2. Excursion Theory from Resolvents

Let $F_\partial \equiv F\cup\{\partial\}$ be the (compact metric) state space of a Ray process $(X_t^\partial)_{t\geqq 0}$ with Ray resolvent $(R_\lambda^\partial)_{\lambda>0}$ (see Getoor [3] or Williams [18] for definitions and the chief properties). The state ∂ is a graveyard to which X^∂ is sent when it approaches some (fixed) distinguished state $a \in F$;

$$R_\lambda^\partial(a, \{\partial\}) = R_\lambda^\partial(\partial, \{\partial\}) = \lambda^{-1}.$$

For a clean statement of results (and for technical reasons) we assume that

$$R_\lambda^\partial(x, \{\partial\}) = \lambda^{-1} \Leftrightarrow x = a \quad \text{ or } x = \partial. \tag{4}$$

Intuitively, only at a is the process certain to be killed. We also assume to avoid triviality that for some $x \in F$,

$$P^x(\text{for some } t > 0, X_{t-}^\partial = a) > 0.$$

In his historic paper [6], Itô considers the following question. Let $(X_t)_{t \geqq 0}$ be a Ray process on F_∂ with (honest) resolvent $(R_\lambda)_{\lambda > 0}$, and define the stopping times

$$\tau_+ \equiv \inf\{t \geqq 0; X_t = a\}, \qquad \tau_- \equiv \inf\{t > 0; X_{t-} = a\}$$
$$\tau \equiv \tau_+ \wedge \tau_-.$$

If we define the killed process X' by

$$\begin{aligned} X_t' &= X_t \qquad 0 \leqq t < \tau \\ &= \partial \qquad \tau \leqq t, \end{aligned}$$

Itô asks the question, „*What are the processes X such that X' has the same law as the given process X^∂?*" Itô's answer, in terms of local time at a and the Poisson point process of excursions from a, is of a theoretical interest and practical importance at least as great as his famous change of variables formula for stochastic integrals. The approach we shall adopt here is much more elementary, based on the resolvent identity, though the insight into sample paths is absent. Recall that the credit here is due to Reuter.

Remarks. Typically, we will be given a Ray process X^0 on a compact metric space F_0 with a distinguished state a, and the process X^∂ will be X^0 killed on first approach to a, just as we defined X' from X. However, the process X^∂ so defined need not be Ray; we have to take the Ray-Knight compactification of $F_e \equiv \{x \in F_0;\ x \text{ is not a branch point of } X^0\}$ given by the resolvent of X^∂. See Williams [18] III.57 for more details. Since our resolvent R_λ^∂ will usually arise in this way, we have taken a and ∂ as distinct points, which is not otherwise strictly necessary.

To exclude trivialities, we shall henceforth assume $R_\lambda \not\equiv R_\lambda^\partial$, i.e. $R_\lambda(a, F) \neq 0$.

Now define for each $\lambda > 0$

$$\psi_\lambda(x) \equiv E^x[e^{-\lambda\xi}] \qquad (x \in F)$$

where $\xi \equiv \inf\{t > 0; X_t^\partial = \partial\}$. To avoid clumsiness, we shall make the *notational convention* that *any bounded measurable* $f\colon F_\partial \to \mathbb{R}$ *vanishes at* ∂, with the sole exception of multiples of $I_{\{\partial\}}$! Thus we have the concise and useful identity

$$\psi_\lambda = \lambda R_\lambda^\partial I_{\{\partial\}} = 1 - \lambda R_\lambda^\partial 1. \tag{5}$$

Notice particularly that, since R^∂_λ is Ray, R^∂_λ: $C(F_\partial)\to C(F_\partial)$ and so $\psi_\lambda\in C(F_\partial)$. Also, $\psi_\lambda(a)=1$, and, by our assumption (4), $\psi_\lambda(x)<1$ for $x\in F\setminus\{a\}$.

With this notation, we have the following obvious fundamental result.

Lemma 1. *The resolvent R_λ of X satisfies*

$$\begin{aligned} R_\lambda f(x) &= R^\partial_\lambda f(x)+\psi_\lambda(x)R_\lambda f(a) && (x\in F) \qquad (6)\\ R_\lambda I_{\{\partial\}}(x) &= \psi_\lambda(x)R_\lambda I_{\{\partial\}}(a) && (x\in F)\\ &= \lambda^{-1} && (x=\partial)\end{aligned}$$

where $f\in C(F_\partial)$ (vanishing at ∂ by convention).

Remarks. The second statement above follows from the first and the assumption that R_λ is honest; $\lambda R_\lambda 1+\lambda R_\lambda I_{\{\partial\}}\equiv 1$. The third statement is just another way of saying ∂ is a graveyard.

Proof. In view of the remarks, we only need prove (6). Two cases arise, according as a is a branch point or not. We suppose a is a branch point; the other case is even easier and is left as an exercise. Since a is a branch point, $P^\mu(X_t\neq a\,\forall t)=1$ for any initial law μ, so the stopping time τ is equal to τ_-, and there exist stopping times $\tau_n\uparrow\tau_-$, $\tau_n<\tau_-$ for all n. Thus for $f\in C(F_\partial)$, $R_\lambda f\in C(F_\partial)$ and

$$\begin{aligned} R_\lambda f(x) &\equiv E^x\int_0^\infty e^{-\lambda t}f(X_t)\,dt\\ &=E^x\left[\int_0^{\tau_n} e^{-\lambda t}f(X_t)\,dt+e^{-\lambda\tau_n}\int_0^\infty e^{-\lambda s}f(X_{s+\tau_n})\,ds\right]\\ &=E^x\left[\int_0^{\tau_n} e^{-\lambda t}f(X_t)\,dt+e^{-\lambda\tau_n}R_\lambda f(X_{\tau_n})\right]\end{aligned}$$

by the strong Markov property of X;

$$=E^x\left[\int_0^{\tau_n} e^{-\lambda t}f(X^\partial_t)\,dt\right]+E^x[e^{-\lambda\tau_n}R_\lambda f(X_{\tau_n})]$$

by the hypothesis on X;

$$\begin{aligned}&\to E^x\left[\int_0^{\tau} e^{-\lambda t}f(X^\partial_t)\,dt\right]+E^x[e^{-\lambda\tau}R_\lambda f(a)]\\ &=R^\partial_\lambda f(x)+\psi_\lambda(x)R_\lambda f(a)\end{aligned}$$

as $n\to\infty$.

Thus Lemma 1 reduces the study of the resolvent $(R_\lambda)_{\lambda>0}$ to the study of $R_\lambda(a,\cdot)_{\lambda>0}$. The following result gives a detailed description of this family of measures.

Theorem 1. *There exist non-negative γ, δ, and non-negative measures $(n_\lambda)_{\lambda>0}$ concentrated on $F\setminus\{a\}$ such that for all bounded measurable f: $F_\partial\to\mathbb{R}$, for $\lambda>0$*

$$R_\lambda f(a)=\frac{n_\lambda f+\gamma f(a)}{\delta+\lambda n_\lambda 1+\lambda\gamma} \qquad (7)$$

and for $\lambda \neq \mu$, $\lambda, \mu > 0$,

$$n_\lambda R_\mu^\partial = \frac{n_\lambda - n_\mu}{\mu - \lambda}. \tag{8}$$

Proof. Fix $\beta > 0$, and define for $\lambda > 0$, bounded measurable $f\colon F \to \mathbb{R}$

$$m_\lambda f \equiv R_\beta f(a) + (\beta - \lambda) R_\beta R_\lambda^\partial f(a).$$

Using (6) to re-express $R_\lambda^\partial f$, together with the resolvent equation, we have

$$m_\lambda f = R_\lambda f(a)[1 + (\lambda - \beta) R_\beta \psi_\lambda(a)]. \tag{9}$$

Now $\beta R_\beta \psi_\lambda(a) \leqq \beta R_\beta 1(a) \leqq 1$, with equality holding throughout only if ψ_λ is almost everywhere equal to 1 (with respect to $R_\beta(a, \cdot)$). But if this happens then $\lambda R_\beta \psi_\lambda(a) = \lambda R_\beta 1(a) > 0$, so, either way,

$$1 + (\lambda - \beta) R_\beta \psi_\lambda(a) > 0,$$

and the right side of (9) defines a non-zero non-negative continuous linear form on $C(F)$. Thus m_λ is a non-negative measure on F. Reworking the term in brackets on the right of (9),

$$1 + (\lambda - \beta) R_\beta \psi_\lambda(a) \equiv 1 + (\lambda - \beta) R_\beta (1 - \lambda R_\lambda^\partial 1)(a),$$

by (5);

$$= 1 - \beta R_\beta 1(a) + \lambda m_\lambda 1$$

from the definition of m_λ. Writing $\delta \equiv 1 - \beta R_\beta 1(a) \geqq 0$, (9) rearranges to give

$$R_\lambda f(a) = \frac{m_\lambda f}{\delta + \lambda m_\lambda 1}. \tag{10}$$

Next, a few calculations with the resolvent equation show that

$$m_\lambda R_\mu^\partial = \frac{m_\lambda - m_\mu}{\mu - \lambda} \tag{11}$$

for $\lambda, \mu > 0$ distinct, and, since $R_\mu^\partial I_{\{a\}} = 0$, it follows that $m_\lambda I_{\{a\}}$ is the same for all $\lambda > 0$. Let the common value be γ, which is non-negative, since m_λ is. If we now define

$$n_\lambda = m_\lambda|_{F \setminus \{a\}}, \tag{12}$$

then

$$R_\lambda f(a) = \frac{n_\lambda f + \gamma f(a)}{\delta + \lambda n_\lambda 1 + \lambda \gamma}$$

which is (7). Finally, to establish (8), notice that $R_\lambda^\partial f(a) = 0$ for all $f \in C(F)$, so that $m_\lambda R_\mu^\partial f = n_\lambda R_\mu^\partial f$, and also $m_\lambda f - m_\mu f = n_\lambda f - n_\mu f$ as m_λ and m_μ both put mass γ on a. Thus (8) follows from (11).

Remarks. (i) Equation (8) looks like the resolvent equation, and in fact it is. If n is the Itô excursion law on the space U of excursions from a (see Itô [6] for definitions), then the measures $(n^t)_{t>0}$ defined by

$$n^t(A) \equiv n(\{w \in U;\, w(t) \in A,\ t < \xi(w)\}) \tag{13}$$

(where A is a Borel subset of $F \setminus \{a\}$, and $\xi(w)$ is the lifetime of excursion w), give an entrance law for the killed semigroup P_t^∂ whose resolvent is R_λ^∂;

$$n^t P_s^\partial = n^{t+s} \qquad (t, s > 0). \tag{14}$$

Laplace transforming yields (8), where

$$n_\lambda \equiv \int_0^\infty dt\, e^{-\lambda t} n^t. \tag{15}$$

Conversely, given a family $(n_\lambda)_{\lambda>0}$ such that (8) holds, does there exist a family $(n^t)_{t>0}$ satisfying (14) such that (15) is valid? This is a more difficult point, but Getoor and Sharpe [4] show that the answer is that there does exist such a family.

(ii) As we see from (7) on taking $f = I_{\{a\}}$, the Lebesgue measure of the time spent at a by the process X is zero iff $\gamma = 0$. Thus γ can be interpreted as the *stickiness* of a. Likewise, the resolvent R_λ restricted to $C(F)$ is honest iff $\delta = 0$, so δ can be viewed as the rate of *killing* in a. Itô concentrates on the case $\delta = 0$, so our result may be considered a slight extension of his. It is not perhaps obvious that killing at a can only occur in this way. Reuter is only concerned with the case $\gamma = 0$, in the Markov chain setting.

(iii) The converse to Theorem 1 is of some interest; given non-negative δ, γ, and $(n_\lambda)_{\lambda>0}$ satisfying (8), does there exist a strong Markov process with resolvent given by (7)? Itô's approach synthesises such a process from a U-valued Poisson point process; the approach used here also gives the existence of such a process. We have the following result.

Theorem 2. *With our notation as before, suppose $(n_\lambda)_{\lambda>0}$ is a family of non-negative measures on $F \setminus \{a\}$ such that* (8) *holds, suppose δ, γ are non-negative reals, and, for each $\lambda > 0$, suppose U_λ: $C(F_\partial) \to C(F_\partial)$ is defined by*

$$U_\lambda f(x) = R_\lambda^\partial f(x) + \psi_\lambda(x) \cdot U_\lambda f(a) \qquad (x \in F, f \in C(F_\partial)) \tag{16}$$

$$U_\lambda f(a) = \frac{n_\lambda f + \gamma f(a)}{\delta + \lambda n_\lambda 1 + \lambda \gamma} \tag{17}$$

$$\begin{aligned} U_\lambda I_{\{\partial\}}(x) &= \psi_\lambda(x)\, U_\lambda I_{\{\partial\}}(a) && (x \in F) \\ &= \lambda^{-1} && (x = \partial) \end{aligned}$$

and

$$U_\lambda I_{\{\partial\}}(a) = \frac{\lambda^{-1}\delta}{\delta + \lambda n_\lambda 1 + \lambda \gamma}.$$

Then $(U_\lambda)_{\lambda>0}$ is an honest Ray resolvent on $C(F_\partial)$; in particular, there exists a right continuous strong Markov process with left limits, taking values in F_∂, with resolvent $(U_\lambda)_{\lambda>0}$.

Proof. Spelling it out, we have to check (i) $U_\lambda C(F_\partial) \subseteq C(F_\partial)$, (ii) $0 \leqq f \leqq 1 \Rightarrow 0 \leqq \lambda U_\lambda f \leqq 1$, (iii) $\lambda U_\lambda 1 = 1$, (iv) $U_\lambda - U_\mu + (\lambda - \mu) U_\lambda U_\mu = 0$ for all λ, $\mu > 0$ (v) if

$CSM^\alpha \equiv \{f \in C(F_\partial);\ 0 \leqq \lambda U_{\lambda+\alpha} f \leqq f \text{ for all } \lambda \geqq 0\}$, then $\bigcup_\alpha CSM^\alpha$ separates points of F_∂.

Since R_λ^∂ is a Ray resolvent itself, (i) follows from the definition of U_λ. The positivity of U_λ is immediate, (iii) follows by inspection, giving (ii) as well, which leaves only (iv) and (v).

Using the identity

$$(\lambda-\mu) R_\lambda^\partial \psi_\mu = \psi_\mu - \psi_\lambda, \tag{18}$$

it is easy to prove that U_λ satisfies the resolvent equation iff

$$(\lambda-\mu) U_\lambda U_\mu f(a) = U_\mu f(a) - U_\lambda f(a) \tag{19}$$

for all $\lambda, \mu > 0$, $f \in C(F)$, and, with the aid of the identity

$$(\mu-\lambda) n_\mu \psi_\lambda = \mu n_\mu 1 - \lambda n_\lambda 1, \tag{20}$$

(19) follows quite easily from (16), (8).

Thus there remains only (v), the Ray condition. We can easily separate ∂ from any point of F, since for each $0 \leqq \theta \leqq 1$ and each $\alpha > 0$, $\psi_\alpha + \theta I_{\{\partial\}} \in CSM^\alpha$; and we can choose θ as needed. This also allows us to separate a from any point of $F \setminus \{a\}$, since $\psi_\alpha(x) = 1$ iff $x = a$ by assumption (4).

Finally suppose $\xi_1 \neq \xi_2$ are two distinct points of $F \setminus \{a\}$. Since R_λ^∂ is a Ray resolvent, for some $\alpha > 0$, there exists $f_0 \in C(F)$ such that for $\lambda > 0$

$$0 \leqq \lambda R_{\lambda+\alpha}^\partial f_0 \leqq f_0,$$

and $f_0(\xi_1) \neq f_0(\xi_2)$. Unfortunately, f_0 need not be anywhere near α-supermedian for U_λ; for example, if U_λ is the resolvent of reflecting Brownian motion on $\mathbb{R}$, then the function $x \to \sqrt{x}$ is α-supermedian for the resolvent of Brownian motion killed when it hits 0, but not for U_λ. However, the function $R_\alpha^\partial 1$ is also continuous α-supermedian for R_λ^∂, so for any positive θ,

$$f = (\theta R_\alpha^\partial 1) \wedge f_0$$

is again continuous α-supermedian. Moreover, for large enough θ, f separates ξ_1 and ξ_2, since, by assumption (4), $R_\alpha^\partial 1$ vanishes only at a. Now a few calculations show that the function $g \equiv f + c\psi_\alpha$ is in CSM^α if for all $\lambda > 0$,

$$c \geqq \lambda U_{\lambda+\alpha} g(a),$$

equivalently, using (17), (20)

$$c(\delta + \alpha n_\alpha 1 + \gamma^\alpha) \geqq \lambda n_{\lambda+\alpha} f + \lambda \gamma f(a) \tag{21}$$

for all $\lambda > 0$. But, by construction, $f(a) = 0$, and $f \leqq \theta R_\alpha^\partial 1$, so

$$\lambda n_{\lambda+\alpha} f \leqq \theta(n_\alpha 1 - n_{\lambda+\alpha} 1) \leqq \theta n_\alpha 1.$$

Thus by taking c large enough, (21) will be valid, implying $g = f + c\psi_\alpha$ is in CSM^α, and g separates ξ_1, ξ_2 for some choice of c. Theorem 2 is proved.

Finally, we record a small result which will be needed later, and whose content comes as no surprise. Recall that F_e denotes the set of extreme points of R^∂_λ, while F_{br} will denote $F\setminus F_e$, the set of branch points.

Lemma 2. (i) *For all* $\lambda>0$, $n_\lambda(F_{br})=0$. (ii) *As* $\lambda\uparrow\infty$, $n_\lambda 1$ *decreases to zero.*

Proof. From the resolvent identity

$$n_\lambda R^\partial_\mu f=\frac{n_\lambda f-n_\mu f}{\mu-\lambda}$$

it is plain that if $f\geqq 0$, then $n_\lambda f$ decreases in λ. Now take f to be I_{F_e}. Since $R^\partial_\mu(x,F_{br})=0$ for all x, for all $\mu>0$, it follows that $R^\partial_\mu I_{F_e}=R^\partial_\mu 1$, so multiplying the above equation by μ and letting $\mu\to\infty$ yields

$$\begin{aligned}\lim_{\mu\to\infty} n_\lambda(\mu R^\partial_\mu 1)&=n_\lambda(F_e)-\lim_{\mu\to\infty} n_\mu(F_e)\\ &\leqq n_\lambda(F_e).\end{aligned}$$

Now $\lim\limits_{\mu\to\infty}\mu R^\partial_\mu 1(x)=P^x(\tau>0)\equiv f_0(x)$, say; $f_0(x)$ is 1 for $x\in F_e$, and positive for all $x\in F\setminus\{a\}$, by assumption (4). Thus the above inequality yields

$$n_\lambda(F_e)\geqq n_\lambda f_0\geqq n_\lambda(F_e)$$

whence immediately $n_\lambda(F_{br})=0$, $\lim n_\mu(F_e)=0$, completing the proof.

3. Feller Brownian Motions

To demonstrate the usefulness of the excursion-resolvent identities (1)-(3) we apply them here to a problem first treated by Feller [1] using differential equations theory, and subsequently studied by Itô-McKean [7]. The problem is to find all the (right continuous left limits) strong Markov processes on $[0,\infty)$ which behave like Brownian motion when in $(0,\infty)$. As formulated by Feller, if $\mathcal{G}$ is the generator of a Feller semigroup on $C(\mathbb{R}^+)$ with domain $\mathcal{D}(\mathcal{G})$, if

$$D=\mathcal{D}(\mathcal{G})\cap C^2([0,\infty)),$$

and if $\mathcal{G}|_D=\dfrac{1}{2}\dfrac{d^2}{dx^2}$, the problem is to characterise all the possible subspaces D. Feller's answer is that there are non negative reals p_1, p_2, p_3, and a σ-finite measure p_4 on $(0,\infty)$ such that

$$p_1+p_2+p_3+\int_{(0,\infty)} p_4(dx)(1-e^{-x})=1$$

(a normalisation condition), and that D is the subspace

$$D=\{f\in C^2;\, p_1 f(0)-p_2 f'(0)+\tfrac{1}{2}p_3 f''(0)=\int p_4(dx)\,[f(x)-f(0)]\}.$$

See Itô-McKean [8] p. 186. We shall call the quadruple (p_1,p_2,p_3,p_4) the *characteristics* of the Feller Brownian motion whose generator is $\mathcal{G}$. The intuitive content of this result is not immediately apparent; nor is its method of proof by differential equations.

On the other hand, if we take the approach adopted in Sect. 2, both the meaning and the proof are much clearer. Taking the closed interval $[0, \infty]$ as F, the compact metric statespace of the killed process, we have the explicit representation of the resolvent of Brownian motion killed on hitting zero;

$$\begin{aligned} R_\lambda^\partial(x, dy)/dy &= \theta^{-1}\{e^{-\theta|x-y|} - e^{-\theta|x+y|}\} && (0<x, y<\infty) \\ R_\lambda^\partial(x, \{\partial\}) &= \lambda^{-1} e^{-\theta x} && (x \in [0, \infty]) \\ R_\lambda^\partial(\infty, \{\infty\}) &= \lambda^{-1}, \end{aligned} \tag{22}$$

where $\theta \equiv \sqrt{2\lambda}$, and, with τ the first hitting time of zero,

$$\psi_\lambda(x) \equiv E^x e^{-\lambda\tau} = e^{-\theta x} \qquad (0 \leqq x \leqq \infty) \tag{23}$$

These formulae are well known; see Itô-McKean [8] for more details. Thus if we ask for the most general Ray resolvent on F_∂ of the form

$$R_\lambda f(x) = R_\lambda^\partial f(x) + \psi_\lambda(x) R_\lambda f(0) \qquad (0 \leqq x \leqq \infty),$$

then Theorem 1 supplies the answer in terms of non-negative δ, γ and a family $(n_\lambda)_{\lambda>0}$ of measures on $(0, \infty]$ such that

$$(\mu-\lambda) n_\lambda R_\mu^\partial = n_\lambda - n_\mu,$$

so we have only to characterise such families $(n_\lambda)_{\lambda>0}$. Rearranging (8) gives

$$\lambda n_\lambda \left((1-\psi_1) \frac{R_\mu^\partial f}{1-\psi_1}\right) = \frac{\lambda}{\lambda-\mu}(n_\mu - n_\lambda) f$$

for any $f \in C(F)$. Now if we let $\lambda \to \infty$, the right side tends to $n_\mu f$, by the fact (Lemma 2) that $n_\lambda f \to 0$. As for the left side, if k_λ is the measure with density $\lambda(1-\psi_1)$ with respect to n_λ,

$$k_\lambda 1 \equiv \lambda n_\lambda(1-\psi_1) = \lambda n_\lambda R_1^\partial 1 = \frac{\lambda}{\lambda-1}(n_1 1 - n_\lambda 1)$$

which remains bounded as $\lambda \to \infty$, so the k_λ are bounded measures on $[0, \infty]$, and if $\beta \equiv \sqrt{2\mu}$,

$$\lim_{x \downarrow 0} \frac{R_\mu^\partial f(x)}{1-\psi_1(x)} = \sqrt{2} \int_0^\infty e^{-\beta y} f(y)\, dy$$

$$\lim_{x \uparrow \infty} \frac{R_\mu^\partial f(x)}{1-\psi_1(x)} = \mu^{-1} f(\infty)$$

so that $(1-\psi_1)^{-1} R_\mu^\partial f$ extends to a continuous function on $[0, \infty]$. By passing to a subsequence if need be, we may assume that the measures k_λ converge weakly to k, and hence that n_μ has the representation

$$n_\mu f = \Delta k(0) \sqrt{2} \int_0^\infty e^{-\beta y} f(y)\, dy + \int_{(0,\infty]} k(dx) \frac{R_\mu^\partial f(x)}{1-\psi_1(x)}. \tag{24}$$

Writing $p_4(dx)\equiv(1-\psi_1(x))^{-1}\,k(dx)$, and $p_2\equiv\Delta k(0)/\sqrt{2}$, we see that p_4 satisfies the integrability condition

$$\int_{(0,\infty]} p_4(dx)\,(1-e^{-x})<\infty \tag{25}$$

and

$$n_\mu f=2p_2\int_0^\infty e^{-\beta y} f(y)\,dy+\int_{(0,\infty]} p_4(dx)\,R_\mu^\partial f(x). \tag{26}$$

Conversely, if p_4 is a measure satisfying (25) and $p_2\geqq 0$, then if n_μ is defined by (26), the Eq. (8) is easily shown to be satisfied. By writing $\delta=p_1$ and $\gamma=p_3$ in Theorem 1, we have, using the existence result Theorem 2, the following.

Theorem 3. *If* $(R_\lambda)_{\lambda>0}$ *is a Ray resolvent on* F_∂ *satisfying*

$$R_\lambda f(x)=R_\lambda^\partial f(x)+\psi_\lambda(x)\,R_\lambda f(0) \tag{27}$$

for all $\lambda>0$, $f\in C(F)$ *and* $x\in[0,\infty]$ *where* $R_\lambda^\partial, \psi_\lambda$ *are as at* (22), (23) *then there exist non-negative* p_1, p_2, p_3 *and a non-negative measure* p_4 *on* $(0,\infty]$ *such that*

$$\int_{(0,\infty]} p_4(dx)\,(1-e^{-x})<\infty \tag{28}$$

and such that for $\lambda>0$, $f\in C(F)$,

$$R_\lambda f(0)=\frac{2p_2\int_0^\infty e^{-\theta x} f(x)\,dx+p_3 f(0)+\int_{(0,\infty]} p_4(dx)\,R_\lambda^\partial f(x)}{p_1+p_2\sqrt{2\lambda}+\lambda p_3+\int_{(0,\infty]} p_4(dx)\,(1-e^{-\theta x})} \tag{29}$$

where $\theta\equiv\sqrt{2\lambda}$.

Conversely, given $p_1, p_2, p_3\geqq 0$, p_4 satisfying (28), the Eqs. (27), (29) define a Ray resolvent $(R_\lambda)_{\lambda>0}$ on $[0,\infty]$.

Remarks. (i) The state ∞ is included because it is part of the Ray-Knight compactification, but it is not essential; we can restrict f in (29) to continuous functions of compact support in $[0,\infty)$ and then any atom at infinity of p_4 in the numerator vanishes, and any atom at infinity in the denominator can be absorbed into p_1. The state ∞ is just an alternative graveyard!

(ii) For any $f\in C(\mathbb{R}^+)$, it easily verified that $R_\lambda f\in C^2(\mathbb{R}^+)$ and that the boundary condition in the definition of D is satisfied by $R_\lambda f$, and, conversely, if $h\in D$, then $(\lambda-\mathscr{G})\,h$ is in $C(\mathbb{R}^+)$ and $R_\lambda(\lambda-\mathscr{G})\,h=h$. Thus

$$R_\lambda(C(\mathbb{R}^+))=D,$$

recovering Feller's result.

(iii) Taking the special case $p_1=1$, $p_2=p_3=p_4=0$ we get the resolvent of Brownian motion killed at zero; as in the general case, p_1 measures *killing at* 0. Taking $p_2=1$, $p_1=p_3=p_4=0$ gives the resolvent of reflecting Brownian motion; p_2 measures the tendency for X to *exit zero continuously*. This is not surprising in view of the way that p_2 arose, as the limiting mass on zero of $\lambda\,n_\lambda(1-\psi_1)$; the limiting mass on $x>0$ of $\lambda\,n_\lambda(1-\psi_1)$ measures the rate in local time at zero with which there appear excursions from zero starting at x. Indeed p_4 is the *Lévy measure of jumps into* $(0, \infty)$ *of* x. The constant p_3, as in the general case, measures *the stickiness of zero;* taking $p_3=1$, $p_1=p_2=p_4=0$ gives the resolvent of Brownian motion absorbed at zero, for example.

(iv) We can easily deduce the Brownian excursion law from the preceding, by considering the case $p_2=1$, $p_1=p_3=p_4=0$ of reflecting Brownian motion. Indeed, by (24), we have for some $c>0$

$$\begin{aligned} n_\lambda f &= c\int_0^\infty e^{-\sqrt{2\lambda}y} f(y)\,dy \\ &= c\int_0^\infty f(y)\,dy)\int_0^\infty \frac{y\,e^{-y^2/2t}}{\sqrt{2\pi t^3}} e^{-\lambda t}\,dt \end{aligned}$$

by the familiar Brownian first passage time density Laplace transform. But if n is the excursion measure on $U=\{$continuous $\rho\colon \mathbb{R}^+\to\mathbb{R}^+$ such that for some $\xi>0$, $\rho^{-1}((0,\infty))=(0,\xi)\}$, the Brownian excursion space, then

$$n_\lambda f=\int_U n(d\rho)\int_0^\xi e^{-\lambda t} f(\rho_t)\,dt$$

whence on comparing with the above

$$n(\{\rho\in U;\, \xi(\rho)>t, \rho_t\in dy\})=c\cdot\frac{y\,e^{-y^2/2t}}{\sqrt{2\pi t^3}}\,dy, \tag{30}$$

as is given, for example, in Ikeda-Watanabe [5] Ch. 4.3.

For another characterisation of the Brownian excursion law, see Williams [18] II.67 and Rogers [13].

4. Integral Representation of the Excursion Law

We saw in the last section how the excursions of a Feller Brownian motion decomposed according to the starting point; an excursion starting in $(0,\infty)$ developed like Brownian motion killed at zero, but an excursion starting from 0 developed according to the Brownian excursion law. Here we shall extend this result to the general setting; as before, an excursion starting away from a develops according to the semigroup P_t^∂, but now an excursion starting at a can exit a in many different ways. We construct a compactification of $F\setminus\{a\}$ whose boundary points correspond to the different ways of exiting a, and we perform the construction in the case of the "Brownian hedgehog" (David Williams' terminology - Walsh [15] calls it a "roundhouse singularity") which perfectly illustrates the theory.

Fix $\beta>0$, and suppose $(n_\lambda)_{\lambda>0}$ is a family measures on $F\setminus\{a\}$ satisfying the resolvent Eq. (8). If $v\equiv n_\beta$, it is immediate that for $\lambda>0$,

$$0\leqq \lambda\, v R^\partial_{\lambda+\beta}=v-n_{\lambda+\beta}\leqq v,$$

so that v is β-supermedian. The converse is also true; we have the following.

Proposition 1. *There is a* $1-1$ *correspondence between* β*-supermedian measures* ν *and families* $(n_\lambda; \lambda>0)$ *of measures satisfying the resolvent Eq.* (8). *The correspondence is given by*

$$\nu = n_\beta,\ n_\lambda = \nu + (\beta-\lambda)\,\nu R_\lambda^\partial. \tag{31}$$

Proof. If ν is β-supermedian, then $n_\lambda = \nu + (\beta-\lambda)\,\nu R_\lambda^\partial$ is always non-negative, and satisfies (8) as is easily verified.

So our study is the study of β-supermedian measures. Notice also that by Lemma 2, $n_\beta(F_{br})=0$, so for any β-supermedian measure ν

$$\nu = \lim_{\lambda\to\infty} \nu(\lambda R_\lambda^\partial)$$

so ν is β-excessive. Let $\mathscr{M}$ denote the set of all β-supermedian measures.

We turn now to the construction of the compactification. If $(f_n)_{n\geqq 0}$ is some dense sequence of continuous functions on $F\setminus\{a\}$, with $f_0\equiv 1$, and if $(g_n)_{n\geqq 0}$ is some enumeration of $\{f_n, \lambda R_{\lambda+\beta}^\partial f_n;\ n\in\mathbb{Z}^+, \lambda\in\mathbb{Q}^+\}$, with $g_0\equiv 1$, then the linear map $\Phi\colon \mathscr{M}\to\mathbb{R}^{\mathbb{Z}^+}$ defined by

$$\Phi\colon \nu\to(\nu g_0, \nu g_1, \ldots)$$

is $1-1$, and $\nu_n\Rightarrow\nu$ iff $\Phi(\nu_n)\to\Phi(\nu)$. Since the weak limit of elements of $\mathscr{M}$ is again in $\mathscr{M}$, $\Phi(\mathscr{M})$ is a closed convex cone in the locally convex metrisable topological vector space $\mathbb{R}^{\mathbb{Z}^+}$ and the section

$$S\equiv\{x\in\Phi(\mathscr{M});\ x_0=1\}$$

is a compact convex metrisable subset. We may thus use Choquet's theorem on integral representation of elements of S; in order to assert that the representing measure is unique, we need the following result.

Lemma 3. *$\mathscr{M}$ is a lattice in its natural ordering.*

Proof. If ν^1, ν^2 are two elements of $\mathscr{M}$, with corresponding $(n_\lambda^1)_{\lambda>0}$, $(n_\lambda^2)_{\lambda>0}$, by Getoor-Sharpe [4] there exist entrance laws $(\nu_t^i)_{t>0}$, $i=1,2$, such that

$$\nu^i = \int_0^\infty e^{-\beta t}\,\nu_t^i\,dt.$$

If ν_t^i has density h_t^i with respect to $(\nu_t^1+\nu_t^2)$, following Neveu, we define for bounded continuous f

$$\mu_t(f)\equiv\lim_{s\downarrow 0}\int(\nu_s^1+\nu_s^2)(dx)\,(h_s^1(x)\vee h_s^2(x))\,P_{t-s}^\partial f(x).$$

It is easy to show that this is an increasing limit, so is well defined, and that μ_t is a measure. Moreover, $(\mu_t)_{t>0}$ is an entrance law, and

$$(\mu_t-\nu_t^1)(f)=\lim_{s\downarrow 0}\int(\nu_s^1+\nu_s^2)(dx)\,(h_s^2-h_s^1)^+(x)\,P_{t-s}^\partial f(x)$$

is again an entrance law, so $\mu\geqq\nu^1,\nu^2$ in the natural order on $\mathscr{M}$. Likewise, if $\bar\mu\geqq\nu_1,\nu_2$, then it is easy to show $\bar\mu\geqq\mu$, so that $\mu=\nu_1\vee\nu_2$. We define $\nu_1\wedge\nu_2$ similarly.

Remark. It is quite possible to prove this result directly from the Laplace transforms of the entrance laws without recourse to Getoor and Sharpe's result, but it is not possible to see what is going on that way!

If we now denote by $\mathcal{M}_0$ the subset of $\mathcal{M}$ consisting of all these β-supermedian measures of the form mR^{∂}_{β}, where m is a measure on $F \setminus \{a\}$, then from (31) we have for $\nu \in \mathcal{M}$

$$\begin{aligned}\nu &= n_{\lambda+\beta} + \lambda\, \nu R^{\partial}_{\lambda+\beta} \\ &= n_{\lambda+\beta} + \lambda\, n_{\lambda+\beta} R^{\partial}_{\beta}\end{aligned}$$

and since, by Lemma 2, $n_{\lambda+\beta} \to 0$ as $\lambda \to \infty$, it follows that ν is a weak limit of elements of $\mathcal{M}_0$; equivalently, $\Phi(\mathcal{M}) = \overline{\Phi(\mathcal{M}_0)}$.

Moreover, defining a map $\Psi: F \setminus \{a\} \to \mathcal{M}_0$ by

$$\Psi: x \to \kappa(x, \cdot) \equiv R^{\partial}_{\beta}(x, \cdot)/R^{\partial}_{\beta} 1(x), \tag{32}$$

it is clear that (i) the subspace generated by $\Psi(F \setminus \{a\})$ is dense in $\mathcal{M}_0$ (ii) Ψ is $1-1$ on F_e, the set of extreme points in F. (The second is because if $\nu = mR^{\partial}_{\beta}$, we recover m by $mP^{\partial}_0 = \lim\limits_{\lambda \to \infty} \lambda(\nu - \lambda\, \nu R^{\partial}_{\lambda+\beta})$, and if m is concentrated on F_e, then $m = mP^{\partial}_0$). The kernel κ is the *Martin kernel* for the problem. To summarise, we have the following.

Theorem 4. *The mapping $i \equiv \Phi \circ \Psi: F \setminus \{a\} \to S$ is a continuous map of $F \setminus \{a\}$ into the compact metric space S, whose range S_0 is dense in S. If $B \equiv S \setminus S_0$, then for each $\nu \in \mathcal{M}$, there exists a unique measure m on the extreme points of B, and a measure m' on F_e such that*

$$n = \int_B m(d\xi)\, \kappa(\xi, \cdot) + \int_{F_e} m'(dx)\, R^{\partial}_{\beta}(x, \cdot). \tag{33}$$

Proof. We show (i) if $x \in F \setminus \{a\}$, then $i(x)$ is extreme in S iff $x \in F_e$ (ii) the integral over extreme points in S may be pulled back to an integral over F_e, whence the result is immediate.

(i) Clearly if $x \in F_{br}$, $i(x)$ is not extreme in S. Conversely, suppose $x \in F_e$. Now $\nu \in \mathcal{M}$ is represented as μR^{∂}_{β} for some measure μ on F_e iff $\lambda\, n_\lambda \Rightarrow \mu$ as $\lambda \to \infty$. Indeed,

$$\begin{aligned}\nu - \lambda\, n_\lambda R^{\partial}_{\beta} &= \nu - \lambda(\nu + (\beta - \lambda) R^{\partial}_{\lambda}) R^{\partial}_{\beta} \\ &= \nu - \lambda\, \nu R^{\partial}_{\lambda} \\ &\to 0 \quad \text{as } \lambda \to \infty\end{aligned}$$

since, as we have already remarked, ν is β-excessive; if $\lambda\, n_\lambda \Rightarrow \mu$, this implies $\nu = \mu R^{\partial}_{\beta}$, and the converse implication is immediate. So if $i(x)$ were not extreme in S, we could find $(n^i_\lambda)_{\lambda > 0}$, $i = 1, 2$, such that

$$i(x) = \Phi(\nu^1) + \Phi(\nu^2).$$

Thus

$$\lim_{\lambda \to \infty} \lambda(n^1_\lambda + n^2_\lambda) = \delta_x.$$

Since $\lambda\, n^i_\lambda 1$ increases with λ, this implies $\lim\limits_{\lambda \to \infty} \lambda\, n^i_\lambda = c_i\, \delta_x$, so $\Phi(\nu_i) = c_i \cdot i(x)$ and $i(x)$ *is* extreme, a contradiction.

(ii) The unique Choquet integral representation is an integral over extreme points of B and over $i(F_e)$, the set of extreme points of S_0. But i is a continuous $1-1$ map on F_e, and F_e is a G_δ in a compact metric space, therefore it is a Polish space. By Lusin's theorem (see Getoor [3] Theorem 8.7), i^{-1} is measurable, justifying the form (33) of the integral representation.

Remarks. The Martin-type compactification and integral representation result using Choquet's theorem is similar to Neveu's construction of the entrance space [10]. In the final section of [6], Itô also considers integral representation of excursion laws; his approach is to make a probability measure out of the excursion measure by conditioning the excursions to last longer than an independent exponential random variable, and then he gets the different types of continuous exit from a by taking a regular conditional distribution for the path of the process given $\mathscr{F}_{0+}$. A closely related approach is to define a new process on $F\setminus\{a\}$ by killing X^∂ at rate β, but conditioning it not hit a before the killing time. The resolvent $(R^\beta_\lambda)_{\lambda>0}$ of this process is defined by

$$R^\beta_\lambda f(x)=\frac{R^\partial_{\beta+\lambda}((1-\psi_\beta)f)(x)}{1-\psi_\beta(x)}\qquad (x\in F\setminus\{a\}).$$

It is not too hard to prove that *the Ray-Knight compactification of F_e under $(R^\beta_\lambda)_{\lambda>0}$ is the same as the Martin compactification of F_e given in Theorem* 4, thus linking the two approaches. We shall not go into the details, "whereof a little more than a little is by much too much".

To conclude this section, let us find the excursion boundary for the "Brownian hedgehog". The habitat of this endearing process is $\mathbb{R}^2$. It is a descendent of Brownian motion in that if B_t is Brownian motion on $\mathbb{R}$, with local time at zero L_t, the Brownian hedgehog is

$$(|B_t|\cos L_t,\, |B_t|\sin L_t).$$

One interesting feature of this process is that it is a continuous Markov process which is not strong Markov. However, we shall harden our hearts and kill the Brownian hedgehog when it first reaches zero, up to which time it *is* strong Markov, moving radially like Brownian motion. The uncompactified state space of the killed process is $\mathbb{R}^2\setminus\{0\}$ which it will be helpful to consider in polar coordinates as $(0,\infty)\times T$, where T is the unit circle. Thus the resolvent of the killed process is specified by

$$R^\partial_\lambda f(r,\theta)=h(\theta)\int_0^\infty \frac{dx}{\sqrt{2\lambda}}\left(e^{-\sqrt{2\lambda}|x-r|}-e^{-\sqrt{2\lambda}(x+r)}\right)g(x),$$

where $f(r,\theta)\equiv g(r)\,h(\theta)$ is of separable variable type, and

$$\psi_\lambda(r,\theta)=\exp(-\sqrt{2\lambda}\,r).$$

It is not hard to prove that the Ray-Knight compactification of $\mathbb{R}^2\setminus\{0\}$ under this resolvent is the one point compactification of $\mathbb{R}^2$.

To obtain the excursion boundary though, we must take the Martin compactification induced by the Martin kernel (32). As we saw earlier, the sequence $(x_n)_{n\geqq 1}\equiv((r_n,\theta_n))_{n\geqq 1}$ in $(0,\infty)\times T$ is Cauchy in this topology iff

$$\frac{R_\lambda^\partial f(x_n)}{1-\psi_\beta(x_n)} \quad \text{is convergent for each } f\in C_K(\mathbb{R}^2),\ \lambda>0,$$

and for this it is sufficient to check convergence on f of separable variable form, $f(r,\theta)=g(r)\,h(\theta)$. We have the explicit expression

$$\frac{R_\lambda^\partial f(x_n)}{1-\psi_\beta(x_n)}=\frac{h(\theta_n)\left\{e^{-\alpha r_n}\int\limits_0^{r_n}\sinh\alpha x\,g(x)\,dx+\sinh\alpha r_n\int\limits_{r_n}^{\infty}e^{-\alpha x}g(x)\,dx\right\}}{\frac{1}{2}\alpha(1-e^{-\sqrt{2\beta}\,r_n})}$$

where $\alpha\equiv\sqrt{2\lambda}$, which is convergent for all bounded continuous h, g iff *either* $r_n\to\infty$ *or* $r_n\to r\in[0,\infty)$, and $\theta_n\to\theta$ in T. Thus the Ray-Knight compactification is homeomorphic to the closed unit disc, with the embedding given by

$$(0,\infty)\times T\ni(r,\theta)\to(1+r)^{-1}\,e^{i\theta},$$

and the most general excursion law $(n_\lambda)_{\lambda>0}$ has the representation

$$\begin{aligned} n_\lambda f=&\int\limits_T m_b(d\theta)\int\limits_0^\infty e^{-\sqrt{2\lambda}\,r}f(r,\theta)\,dr\\ &+\int\limits_{\mathbb{R}^2\setminus\{0\}} m(dx)\,R_\lambda^\partial f(x), \end{aligned}$$

made up of a contribution from excursions starting in $\mathbb{R}^2\setminus\{0\}$, and a mixture of the laws of excursions "kicked out" of zero in a given direction.

5. Williams' Path Decompositions

In an influential paper [16], Williams proved a decomposition of the path of an upward drifting Brownian motion at its minimum. This result has since been proved in various forms by numerous others (see, for example, Pitman [11], Jeulin [9], Rogers-Pitman [14], Ikeda-Watanabe [5]) and generalised in many directions. It may seem superfluous to provide yet another proof of what must be one of the most proven results of the past decade, but, as the excursion viewpoint so well illustrates the themes of the earlier sections and provides a conceptually simple approach to Williams' result, we include it.

Let $(X_t)_{t\geqq 0}$ be Brownian motion on $\mathbb{R}$, with drift $c\in\mathbb{R}$, started at 0. Define

$$L_t\equiv\sup\{-X_s;\,s\leqq t\}.$$

The state 0 is regular for $\{0\}$ for the non-negative strong Markov process $Y_t\equiv X_t+L_t$, and, to within multiples, *the local time of Y at zero is L*. We can see this either by appeal to a result of Fristedt ([2], Corollary 9.9) on spectrally positive Lévy processes, or prove it directly from the fact that for $c=0$,

$$L_t = \text{a.s.} \lim_{\varepsilon \downarrow 0} \varepsilon\, n_t(\varepsilon), \tag{34}$$

where $n_t(\varepsilon)$ is the number of downcrossings of $(0, \varepsilon)$ completed by Y before t (see Williams [17] for a short proof of this). This displays L as a continuous homogeneous additive functional of Y, at least in the case $c=0$, and the case $c \neq 0$ follows immediately by the equivalence of the laws of Brownian motion with different drifts.

The path decomposition of Williams will follow from the explicit characterisation of the excursion law of Y which we shall shortly establish. Let us define $U \equiv \{\text{continuous } \rho: \mathbb{R}^+ \to \mathbb{R}^+ \text{ s.t. } \rho^{-1}((0,\infty)) = (0,\xi) \text{ for some } 0<\xi \leqq \infty\}$, the excursion space for Y, and let us define

$$U_\infty \equiv \{\rho \in U;\ \text{the lifetime } \xi = \xi(\rho) = \infty\},$$

the subspace of excursions which never return to zero, with $U_0 \equiv U \setminus U_\infty$. The excursion law on U will be denoted by n^c (where c is the drift of X), and, as before, for $\lambda > 0$ we define for bounded measurable f

$$n_\lambda^c f \equiv \int_U n^c(d\rho) \int_0^\xi e^{-\lambda t} f(\rho_t)\, dt.$$

We now characterise the excursion law n^c, just as we found the excursion law n^0 at the end of Sect. 3, beginning by finding R_λ^∂. As is well known, if we kill Y when it reaches zero, the resulting process has transition density

$$p_t^c(x,y) \equiv e^{c(y-x) - \frac{1}{2}c^2 t} \left[e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t} \right] (2\pi t)^{-\frac{1}{2}}. \tag{35}$$

Indeed, the case $c=0$ is just the familiar reflection principle, the case $c \neq 0$ follows from it by the Cameron-Martin formula. Hence the resolvent has transition density

$$r_\lambda^c(x,y) = \alpha^{-1} e^{c(y-x)} \left[e^{-\alpha|x-y|} - e^{-\alpha(x+y)} \right], \tag{36}$$

where $\lambda > 0$, $\alpha \equiv (c^2 + 2\lambda)^{\frac{1}{2}}$, and the λ-excessive function ψ_λ has the explicit form

$$\psi_\lambda(x) = \exp(-(c+\alpha)\,x). \tag{37}$$

We can thus write down explicitly the excursion Martin kernel (32), and we find that the excursion compactification of $(0,\infty)$ is simply $[0,\infty]$, with the boundary point 0 corresponding to continuous exits from 0. Since excursions of Y can only leave 0 continuously, it follows from (33) and (36) that

$$n_\lambda^c(dy)/dy = 2 e^{(c-\alpha)y}, \tag{38}$$

at least to within some multiplicative constant. We shall later prove that the normalisation we have chosen here makes L the local time at zero. Assuming this for the moment, we examine n^c in the case $c>0$. In this case, there is a possibility that an excursion may drift off to ∞ before returning to zero. Since the excursion evolves in $(0,\infty)$ like the process X, and since

$$P(X_t = 0 \text{ for some } t>0 \mid X_0 = x > 0) = e^{-2cx}, \tag{39}$$

we have

$$n^c(U_\infty) = \int_0^\infty \lambda\, n_\lambda^c(dy)(1-e^{-2cy}) \\ = 2c, \tag{40}$$

using (38). Unravelling the Laplace transform (38) yields

$$n^c(\{\rho;\ \xi(\rho)>t,\ \rho_t \in dy\}) = \frac{2y}{\sqrt{2\pi t^3}} \exp[-(y-ct)^2/2t]\, dy \tag{41}$$

whence immediately from (39),

$$n^c(\{\rho;\ \rho_t \in dy,\ \xi=\infty\}) = \frac{2y}{\sqrt{2\pi t^3}} e^{-(y-ct)^2/2t}(1-e^{-2cy})\, dy, \tag{42}$$

and the excursion process on U_∞ envolves according to the transition semigroup whose density is

$$p_t^{*c}(x,y) \equiv (1-e^{-2cx})^{-1}\, p_t^c(x,y)\,(1-e^{-2cy}). \tag{43}$$

Put another way, if $\mathbb{P}^c$ is the probability on U_∞ which is the law of the diffusion started at zero with generator

$$\mathscr{G} \equiv \frac{1}{2}\frac{d^2}{dx^2} + c \coth cx \frac{d}{dx}$$

then n^c *restricted to* U_∞ *is* $2c\,\mathbb{P}^c$ (which is exactly what it must be from Williams' result. See Rogers-Pitman for the interpretation of this process as the modulus of a drifting three-dimensional Brownian motion).

As for the excursion law on U_0, we have directly from (39) and (41) that

$$n^c(\{\rho;\ \rho_t\, dy,\ t<\xi(\rho)<\infty\}) = \frac{2y}{\sqrt{2\pi t^3}} e^{-(y-ct)^2/2t} e^{-2cy}\, dy \\ = \frac{2y}{\sqrt{2\pi t^3}} e^{-(y+ct)^2/2t}\, dy$$

and the excursion process on U_0 under n^c evolves according to the transition density

$$e^{2cx} p_t^c(x,y)\, e^{-2cy} \equiv p_t^{-c}(x,y).$$

To summarise, *for* $c>0$, $n^c = n^{-c}$ *on* U_0, *and* $n^c = 2c\,\mathbb{P}^c$ *on* U_∞. From this, we deduce William's path decomposition.

Theorem (Williams). *Take three independent random elements* $\{X'_t;\ t\geqq 0\}$ *a Brownian motion on* $\mathbb{R}$ *with drift* $-c<0$, $X'_0=0$, $\{R_t;\ t\geqq 0\}$ *a process with law* $\mathbb{P}^c$, *and* L_∞ *a random variable with* $P(L_\infty>t)=e^{-2ct}$. *Then the process*

$$\begin{aligned} X''_t &= X'_t && (0\leqq t\leqq \tau) \\ &= X'_\tau + R_{t-\tau} && (\tau \leqq t) \end{aligned}$$

is Brownian motion with drift c, *where* $\tau \equiv \inf\{t;\ X'_t = -L_\infty\}$.

Proof. As we have already seen, $L_t' \equiv \sup\{-X_s'; s \leqq t\}$ is the local time at zero of $Y_t' \equiv X_t' + L_t'$, and, referred to this local time, the Poisson point process of excursions of Y' has characteristic measure n^{-c}. Thus the Poisson point process of excursions of Y'' has characteristic measure n^{-c} up to an independent exponential $(2c)$ time, at which time there is an excursion in U_∞ with law $\mathbb{P}^c$, and after that the point process dies. This is precisely the behaviour of the point process of excursions of Y, so, since we can reconstruct X from the path of Y, the law of X'' is the law of X.

To conclude, we verify our earlier claim that the normalisation chosen at (38) renders L the local time at zero of Y. Our approach may not be the quickest method of proving this, but we also deduce a decomposition of the drifting Brownian excursion analogous to that of Williams [18] II.67. For $0<x<b$,

$$P(Y \text{ hits } b \text{ before } 0 \mid Y_0 = x) = (1-e^{-2cx})(1-e^{-2cb})^{-1},$$

whence, if T is an exponential (λ) r.v. independent of the excursion process,

$$\begin{aligned} n^c(\{\rho; \rho \text{ hits } b \text{ after } T\}) &= \int_0^b \lambda\, n_\lambda^c(dx)(1-e^{-2cx})(1-e^{-2cb})^{-1} \\ &\quad + \int_b^\infty \lambda\, n_\lambda^c(dx) \to \frac{2c}{1-e^{-2cb}} \quad \text{as } \lambda \to \infty \end{aligned}$$

That is,

$$n^c(\{\rho; \sup_t \rho_t > b\}) = 2c(1-e^{-2cb})^{-1}, \tag{44}$$

so if the claim made is correct, if $\sigma \equiv \inf\{t; Y_t > b\}$, then L_σ will be exponential $(2c/(1-e^{-2cb}))$. However, it is easy to show using Itô's formula that for each $\theta > 0$

$$(2c\theta^{-1} + 1 - e^{-2cY_t})\, e^{-\theta L_t} \quad \text{is a martingale,}$$

bounded on $[0, \sigma]$, whence by applying the optional sampling theorem at σ, we have

$$E e^{-\theta L_\sigma} = \frac{2c}{2c + \theta(1-e^{-2cb})}$$

verifying the claim.

Finally, suppose $c>0$, let $(R_t)_{t \geqq 0}$, $(R_t')_{t \geqq 0}$ be two independent processes with law $\mathbb{P}^c$, and for each $x>0$, let $\tau_x \equiv \inf\{t; R_t = x\}$, $\tau_x' \equiv \inf\{t; R_t' = x\}$. Define the process Z^x by

$$\begin{aligned} Z_t^x &= R_t & &(0 \leqq t \leqq \tau_x) \\ &= x - R_{t-\tau_x}' & &(\tau_x \leqq t \leqq \tau_x + \tau_x') \\ &= 0 & &(\tau_x + \tau_x' \leqq t). \end{aligned}$$

Let $n^c(x, \cdot)$ be the law of Z^x, a probability measure on U_0. The following characterisation of n^c is proved in a method analogous to that used in Rogers [13] to prove the case $c=0$. We leave the proof as an exercise.

Theorem. (i) *If $c>0$, then*

$$n^{-c}(\{\rho; \sup_t \rho_t \in dx\}) = \frac{4c^2 e^{2cx}}{(e^{2cx}-1)^2} dx$$

and

$$n^{-c}(\cdot)=\int_0^\infty dx \frac{4c^2 e^{2cx}}{(e^{2cx}-1)^2} n^c(x,\cdot)$$

(ii) *If* $c>0$, *then*

$$n^c(\cdot)=n^{-c}(\cdot)+2c\mathbb{P}^c.$$

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