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Optimal Strategy at Roulette

GERALD JOHN SMITH

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1. This paper solves the problem posed by DUBINS and SAVAGE in [1] of finding the optimal betting strategy at roulette where one's object is to maximize the chance of reaching a certain fixed level of fortune, given a certain lesser fortune level to begin with.

Roulette is defined as a game where the gambler may bet any amount of money, not exceeding in total that which he has available to him at that instant, on any of p numbers, perhaps betting on several numbers simultaneously, and receiving as a payoff an amount q times the money bet on the single winning number that comes up, less the total of all the money bet for that turn of the wheel, where p and q are positive integers with $p > q$. It is assumed that on any turn of the wheel each number has a probability $1/p$ of being the winning number.

DUBINS and SAVAGE in [1], p. 119–121, solved this problem with the additional restriction that bets made on several numbers simultaneously were allowable only if identical amounts were bet on each of the numbers. With such a restriction the game is called uniform roulette, and the optimal strategy is described as follows, supposing without loss of generality that the object is to reach a fortune 1 starting with a fortune x in $(0, 1)$. If a person's fortune is less than $1/q$, he bets his whole fortune on a single number. If his fortune x is greater than $1/q$ and less than 1 he bets an amount $(1-x)/q\left(1-\frac{1}{q}\right)$ on a single number. Note that this strategy consists of betting the largest amount possible on a single number which will not give him a fortune greater than 1 should he win the bet. This strategy is called the bold strategy.

Still unsolved is the problem of generalized uniform roulette, played as follows. The gambler selects each betting turn an amount to be bet x, not exceeding the amount of money then available to him, and a number $\lambda \in [1, q]$. He then wins that bet with probability $q/\lambda p$ and wins an amount $(\lambda - 1)x$ if he wins. If he loses, he just loses his bet x. Note that if λ is restricted to integral values in $[1, q]$ the game then becomes ordinary uniform roulette.

If λ is allowed to assume values arbitrarily close to zero (and positive) then the game is solved in [1], p. 176–182.

2. It will be shown that the bold strategy is not only optimal for uniform roulette but also optimal for roulette with no restrictions on the relative sizes of bets on various numbers. The two main theorems of the proof are theorems 6 and 7.

Before the proof is given, some preliminary definitions are needed. It is assumed that the roulette wheel has p numbers, each having a chance $1/p$ of being the winning number, and that the payment to the gambler is q times the amount bet on the winning number, less the total amount bet for that turn of the wheel, where p and q are positive integers with $p > q$.

Definition 1. X is an allowable random variable if X assumes at most a finite number of values $t_1, t_2, \ldots, t_n$, all in $[0, 1]$, and such that there exist positive integers $r_1, r_2, \ldots, r_n$ with Prob $\{X = t_i\} = r_i/p$, all i. The ordered set

$$(t_1, t_2, \ldots, t_n)\,, \quad \text{with} \quad t_1 < t_2 < \cdots < t_n\,,$$

is called the set of attainable values of X. The (ordered) sequence $\{r_1, r_2, \ldots, r_n\}$ is called the sequence of weights of X.

Henceforth all random variables considered in this paper will be allowable, and the word allowable will be dropped.

Definition 2. φ is the utility function, defined on $[0, 1]$, when the bold strategy is played, i. e. $\varphi(x)$ is the probability that the gambler will reach fortune 1 if he starts with fortune x and plays the bold strategy.

The properties of φ are described in [1], chapter 6. φ is the utility function for roulette with bets restricted to one number, and also for uniform roulette. It is a strictly increasing, continuous singular function with $\varphi(0) = 0$ and $\varphi(1) = 1$.

Definition 3. If X is a random variable, $V(X) = \sum_{t_i} \varphi(t_i)$ Prob $\{X = t_i\}$, where the sum is taken over the set of attainable values of X.

Definition 4. A number $x \in [0, 1]$ is said to generate the random variable X if either

1. Prob $\{X = x\} = 1$, or

2. if $(t_1, t_2, \ldots, t_n)$, $n \geqq 2$, are the attainable values of X and $\{m_1, m_2, \ldots, m_n\}$ their weights, then there exist numbers $a_2, a_3, \ldots, a_n$ with

$$0 \leqq a_2 < a_3 < \cdots < a_n \leqq 1$$

such that

$$t_i - x = q a_i - \sum_{j=2}^{n} m_j a_j, \qquad i = 2, 3, \ldots, n \tag{1}$$

and

$$x - t_1 = \sum_{j=2}^{n} m_j a_j\,. \tag{2}$$

It is easily checked that in case 2 X is the distribution of fortunes arising if the gambler starts at x and on one turn of the wheel makes bets of amount a_i on m_i numbers for all $i = 2, 3, \ldots, n$, such that the gambler cannot attain values less than 0 or greater than 1. Thus the random variables generated by x give the possible distributions resulting from one turn of the wheel starting at x, and all allowable bets, with the exception of bets where money is placed on every number or bets which may result in a fortune greater than 1.

Suppose, starting at fortune x, a bet is made with money placed on every number, say a_i on the number i for $i = 1, 2, \ldots, p$, with $m = \min a_i > 0$. This bet generate a random variable X giving the distribution of fortunes resulting from this bet. Now suppose, starting at x, a bet is made with $(a_i - m)$ bet on the number i for $i = 1, 2, \ldots, p$. It is clear, if Y represents the random variable giving the distribution of fortunes resulting from this new bet, that $Y = X + (p - q)m > X$ everywhere. Hence it is clear that X is a completely inefficient bet compared to Y, and so no optimal strategy need use bets where money is placed on every number. Therefore such bets will be ignored in the rest of the paper and it will be assumed all random variables are generated by bets where money is not placed on every number.

In a similar manner it may be easily shown that bets which may result in fortunes greater than 1 are inefficient, and will also be ignored.

Theorem 1. *There is exactly one generating point for any random variable.*

Proof. Let X have attainable values $(t_1, t_2, \ldots, t_n)$ and weights $\{m_1, m_2, \ldots, m_n\}$. If $n = 1$ the theorem is trivial, so assume $n \geqq 2$.

Set

$$x = t_1 + \frac{1}{q}\sum_{i=2}^{n} m_i(t_i - t_1)$$

and

$$a_i = \frac{1}{q}(t_i - t_1), \qquad i = 2, 3, \ldots, n.$$

These x and a_i are as required since

$$x - \sum_{i=2}^{n} m_i a_i = t_1 + \frac{1}{q}\left[\sum_{i=2}^{n} m_i (t_i - t_1) - \sum_{i=2}^{n} m_i(t_i - t_1)\right] = t_1$$

and

$$q a_i - \sum_{j=2}^{n} m_j a_j = t_i - t_1 - \frac{1}{q}\sum_{j=2}^{n} m_j(t_j - t_1) = t_i - x, \qquad i = 2, 3, \ldots, n.$$

But x and the a_i are uniquely determined, for substitution of (2) in (1) gives $qa_i - x + t_1 = t_i - x$, or

$$a_i = \frac{1}{q}(t_i - t_1).$$

x is then determined by substituting for the a_i in (2). Theorem 1 allows the following definition.

Definition 5. X and Y are related if both are generated from the same x.

Thus this definition allows the set of random variables to be partitioned into a set equivalence classes.

Definition 6. A basic random variable is a random variable having two attainable values t_1, t_2 with $t_1 < t_2$,

$$\text{Prob}\{X = t_1\} = \frac{p-1}{p},$$

$$\text{Prob}\{X = t_2\} = \frac{1}{p},$$

and either $t_1 = 0$, or $t_2 = 1$.

Theorem 2. *For any random variable not identically* 0 *or* 1 *there exists exactly one basic random variable related to it.*

Proof. Clearly the x generating the random variable lies in $(0, 1)$. Hence from theorem 1 it suffices to show that any $x \in (0, 1)$ generates exactly one basic random variable.

Case 1. $$0 < x < \frac{1}{q}.$$

In this case it is impossible for the basic X to have $t_2 = 1$, for

$$t_2 - x = (q - 1)\,a_2 \quad \text{and} \quad x - t_1 = a_2$$

would imply $1 = qx - qt_1 + t_1$ which implies $t_1 < 0$. Hence $t_1 = 0$ and $t_2 = qx$ gives the unique, basic X generated by x.

Case 2. $$\frac{1}{q} < x < 1.$$

In this case t_1 cannot equal 0 for this would similarly imply $t_2 > 1$. Hence $t_2 = 1$ and $t_1 = (qx - 1)/(q - 1)$, determining X.

Theorem 3. *If $Q(x) \geqq 0$ for $x \in [0, 1)$, $Q(1) \geqq 1$, and if for all X generated by x*

$$\frac{1}{p} \sum_{t_i} m_i\, Q(t_i) \leqq Q(x),$$

where the t_i and m_i are the attainable values and the weights of X, then $Q(x) \geqq U(x)$ for all $x \in [0, 1]$ where U is the utility function giving the probability of attaining fortune 1 starting at fortune x.

Proof. This is theorem 1 on page 28 of [1].

Theorem 4. *Let X be the basic random variable generated by $x \in (0, 1)$ having attainable values t_1 and t_2 and weights $p - 1$ and 1. Then*

$$\varphi(x) = \frac{p-1}{p}\,\varphi(t_1) + \frac{1}{p}\,\varphi(t_2).$$

Proof. This follows from the definition of φ as the utility in roulette when bets are restricted to only one number and the fact that bold play, i. e. the strategy of making bets which generate basic random variables, is optimal in that restricted game.

Definition 7. The order of X is p times $\text{Prob}\{x = t_1\}$, where t_1 is the lowest attainable value of X.

Theorem 5. *For each random variable X of order $\leqq (p - q)$ there exists a random variable Y such that X is related to Y, X has smaller order than Y, and* $\text{Prob}\{Y > z\} \geqq \text{Prob}\{X > z\}$ *for all $z \in [0, 1]$.*

Proof. Let X have order $k \leqq (p - q)$, attainable values $(t_1, t_2, \ldots, t_n)$ and weights $\{m_1, m_2, \ldots, m_n\}$. Let $a_2, a_3, \ldots, a_n$ be the corresponding a_i's of definition 4. The generator of X thus equals

$$x = t_1 + \sum_{j=2}^{n} m_j\, a_m.$$

Let Y be defined by having attainable values $(t_1', t_2', \ldots, t_{n-1}')$ and weights $\{m_1', m_2', \ldots, m_{n-1}'\}$ where the t_i''s, m_i''s and the corresponding a_i''s of definition 4 are defined by

$$a_i' = a_{i+1} - a_2, \qquad i = 2, 3, \ldots, n - 1,$$
$$m_1' = m_1 + m_2,$$

$$m_i' = m_{i+1}, \qquad i = 2, 3, \ldots, n-1,$$
$$t_1' = x - \sum_{j=2}^{n-1} m_i' a_i',$$
$$t_i' = q a_i' + t_1', \qquad i = 2, 3, \ldots, n-1.$$

From the construction of Y and definition 4 it is clear that x generates Y, and hence X and Y are related.

$$t_1 < t_1' \text{ since} \sum_{j=2}^{n-1} m_j' a_j' < \sum_{j=2}^{n} m_j a_j,$$

and

$$t_2 \leqq t_1'$$

since

$$\begin{aligned} t_2 &= q a_2 + t_1 = q a_2 + x - \sum_{j=2}^{n} m_j a_j \\ &= q a_2 + x - m_2 a_2 - \sum_{j=2}^{n-1} m_j' a_j' - a_2 \sum_{j=2}^{n-1} \\ &= t_1' + a_2 \left(q - m_2 - \sum_{j=3}^{n} m_j \right) = t_1' + a_2 (q - p + m_1) \\ &\leqq t_1' \end{aligned}$$

since the order of X is m_1.

Also

$$t_i \leqq t_{i-1}', \qquad i = 3, 4, \ldots, n$$

since

$$\begin{aligned} t_i &= q a_i + t_1 = q (a_{i-1}' + a_2) + x - \sum_{j=2}^{n} m_j a_m \\ &= q (a_{i-1}' + a_2) + x - m_2 a_2 - \sum_{j=2}^{n-1} m_j' a_j' - a_2 \sum_{j=2}^{n-1} m_j' \\ &= t_1' + q a_{i-1}' + a_2 \left(q - m_2 - \sum_{j=2}^{n-1} m_j' \right) \\ &= t_{i-1}' + a_2 (q - p + m_1) \leqq t_{i-1}'. \end{aligned}$$

From the definition of the m_1' it is clear that the order of Y is greater than that of X and that

$$\operatorname{Prob}\{Y > z\} \geqq \operatorname{Prob}\{X > z\}, \quad \text{all} \quad z \in [0, 1].$$

Theorem 6. *If for each basic random variable X, $V(X) \geqq V(Y)$ for all Y of order $> (p - q)$ related to X, then the bold strategy is an optimal betting strategy.*

Proof. The fact that the inequality need only be satisfied for all Y of order $> (p - q)$ related to X follows from theorem 5 and the remarks following definition 4. Theorem 6 then reduces to theorem 3 with Q set equal to V and theorem 4. Since the bold strategy gives payoff φ, this is then the utility and the bold strategy is optimal.

Definition 8. If the order of X with attainable values $(t_1, t_2, \ldots, t_n)$ is greater than $p - q + 2$ then X^* is the set of random variables Y such that Y is related to X and either

1. $(a, b, t_2, \ldots, t_n)$ are the *non*-ordered attainable values of Y with

$$\begin{aligned} \mathrm{Prob}\{X = t_i\} = \mathrm{Prob}\{Y = t_i\}, \qquad & i = 2, 3, \ldots, n; \\ a < t_1 < b; \quad a, b \neq t_i, \qquad & i = 1, 2, \ldots, n; \quad \text{and} \\ \mathrm{Prob}\{Y = b\} = \frac{1}{p} \quad \text{or} & \end{aligned}$$

2. $(a, t_2, \ldots, t_n)$ are the (ordered) attainable values of Y, and for a fixed $j \in (2, 3, \ldots, n)$

$$\mathrm{Prob}\{Y = t_i\} = \mathrm{Prob}\{X = t_i\} + \frac{\delta_{ij}}{p}, \qquad i = 2, 3, \ldots, n \text{ and } a < t_1.$$

The next theorem is preceded by six lemmas.

Lemma 1. *If X has order $k \geqq p - q + 2$, is not identically* 0 *or* 1, *and $Y \in X^*$, then either*

1. $$\frac{t_1 - a}{b - a} = \frac{1}{k - p + q} \qquad \textit{or}$$

2. $$\frac{t_1 - a}{t_j - a} = \frac{1}{k - p + q}$$

respectively, corresponding to the two cases of definition 8.

Conversely, if X has order $m_1 \geqq p - q + 2$, attainable values $(t_1, t_2, \ldots, t_n)$, and weights $\{m_1, m_2, \ldots, m_n\}$, and Y has either

1. *non-ordered attainable values $(a, b, t_2, t_3, \ldots, t_n)$ and corresponding weights $\{m_1 - 1, 1, m_2, m_3, \ldots, m_n\}$ such that $a < t_1 < b$ and*

$$\frac{t_1 - a}{b - a} = \frac{1}{m_1 - p + q} \qquad \textit{or}$$

2. *there exists $j \in (2, 3, \ldots, n)$ such that Y has attainable values $(a, t_2, t_3, \ldots, t_n)$ with $a < t_2$, and weights $\{m_1 - 1, m_2', m_3', \ldots, m_n'\}$ such that $m_i' = m_i + \delta_{ij}$,*

$$j = 2, 3, \ldots, n, \quad a < t_1, \quad \textit{and} \quad \frac{t_1 - a}{t_j - a} = \frac{1}{m_1 - p + q};$$

then $Y \in X^*$.

Proof. Let X have attainable values $(t_1, t_2, \ldots, t_n)$ and weights $\{m_1, m_2, \ldots, m_n\}$, with $m_1 = k$. Let x generate X. Then there exist $a_2, a_3, \ldots, a_n$ all > 0 such that

$$t_i - x = q a_i - \sum_{i=2}^{n} m_i a_i, \qquad i = 2, 3, \ldots, n$$

and

$$x - t_i = \sum_{i=2}^{n} m_i a_i.$$

From these two equations follow

$$a_i = \frac{1}{q}(t_i - t_1), \qquad i = 2, 3, \ldots, n$$

and

$$x = t_1 + \frac{1}{q} \sum_{j=2}^{n} m_j (t_j - t_1).$$

Case 1. Let x' generate $Y \in X^*$ by case 1 of definition 8. Then

$$x' = a + \frac{1}{q} \sum_{i=2}^{n} m_i (t_i - a) + \frac{1}{q} (b - a).$$

$x' = x$ if and only if

$$a + \frac{1}{q} \sum_{i=2}^{n} m_i (t_i - a) + \frac{1}{q} (b - a) = t_1 + \frac{1}{q} \sum_{i=2}^{n} m_i (t_i - t_1).$$

Since $\sum_{i=2}^{n} m_i = p - k$, this is equivalent to

$$a - t_1 + \frac{1}{q} (b - a) = \frac{a - t_1}{q} (p - k);$$

$$(t_1 - a) \left(\frac{p - k}{q} \right) + \frac{b - a}{q} = t_1 - a;$$

$$\frac{t_1 - a}{b - a} \left(\frac{p - k}{q} - 1 \right) + \frac{1}{q} = 0;$$

$$\frac{t_1 - a}{b - a} = \frac{1}{k - p + q}.$$

Also, if this last equation holds for Y, then $x = x'$ and hence Y is related to X and $Y \in X^*$.

Case 2. The analysis is analogous with t_j taking the place of b.

Lemma 2. *For any Y of order k with $p - q + 1 \leqq k \leqq p - 1$ there exists an X of order $k + 1$ such that $Y \in X^*$.*

Proof. Let Y have attainable values $(t_1, t_2, \ldots, t_n)$ and weights $\{m_1, m_2, \ldots, m_n\}$. If $m_2 = 1$, then set

$$t_1' = \frac{t_2 - t_1}{k + 1 - p + q} + t_1,$$
$$t_i' = t_{i+1}, \qquad i = 2, 3, \ldots, n - 1,$$
$$m_1' = m_1 + m_2 = k + 1,$$

and

$$m_i' = m_{i+1}, \qquad i = 2, 3, \ldots, n - 1.$$

Then lemma 1 clearly shows $Y \in X^*$.

Similarly if $m_2 > 1$,

$$t_1' = \frac{t_2 - t_1}{k + 1 - p + q} + t_1,$$
$$t_i' = t_i, \qquad i = 2, 3, \ldots, n - 1,$$
$$m_1' = k + 1,$$
$$m_2' = m_2 - 1,$$

and

$$m_i' = m_i, \qquad i = 3, 4, \ldots, n - 1$$

clearly determine an X such that $Y \in X^*$.

Lemma 3. *For all a, b, t with $0 \leqq a < b \leqq 1$ and $t \in [0, 1]$,*

$$\varphi(t) \leqq \frac{\varphi(a + t[b-a]) - \varphi(a)}{\varphi(b) - \varphi(a)}.$$

Proof. The fact that φ is the utility of a casino is proven in section 3, p. 101 of [1]. Theorem 1, p. 64 of [1] then shows the desired equation holds for all casino functions. (See explanation at the bottom of p. 69.)

Lemma 4. *Suppose X has order k with $p - q + 2 \leqq k \leqq p - 1$, $Y \in X^*$, and $V(Y) > V(X)$. Then $V(Z) > V(W)$ where W is the basic random variable with $t_1 = 1/(k - p + q)$, $t_2 = 1$, and Z is the random variable of order $k - 1$ with $(0, 1/(k - p + q), 1)$ as its attainable values and $\{k - 1, p - k - 1, 2\}$ as its set of weights. (If $k = p - 1$, then Z has only two attainable values.)*

Proof. Suppose $Y \in X^*$, X having order $k \geqq p - q + 2$ and $V(Y) > V(X)$. With a, b, and t_1 as in definition 8 we have, if case 1 of the definition applies,

$$V(Y) - V(X) = \frac{k-1}{p}\varphi(a) + \frac{1}{p}\varphi(b) - \frac{k}{p}\varphi(t_1) > 0;$$
$$(k-1)\varphi(a) + \varphi(b) > k\varphi(t_1).$$

Since $(t_1 - a)/(b - a) = 1/(k - p + q)$, lemma 3 gives

$$\varphi\left(\frac{1}{k-p+q}\right) \leqq \frac{\varphi(t_1) - \varphi(a)}{\varphi(b) - \varphi(a)};$$

$$\begin{aligned}
V(Z) - V(W) &= \frac{1}{p} - \frac{k}{p}\varphi\left(\frac{1}{k-p+q}\right) \geqq \frac{1}{p} - \frac{k}{p}\frac{\varphi(t_1) - \varphi(a)}{\varphi(b) - \varphi(a)} \\
&= \frac{1}{p[\varphi(b) - \varphi(a)]}(\varphi(b) - \varphi(a) - k\varphi(t_1) + k\varphi(a)) \\
&> \frac{1}{p[\varphi(b) - \varphi(a)]}(\varphi(b) - \varphi(a) - [k-1]\varphi(a) - \varphi(b) + k\varphi(a)) = 0.
\end{aligned}$$

If case 2 of definition 8 applies, the above analysis with b replaced by t_j again shows that $V(Z) - V(W) > 0$.

Lemma 5. *If X is a basic random variable then for no $Y \in X^*$ is $V(Y) > V(X)$. If X' is of order $p - 1$ and related to the basic random variable X, then $V(X') \leqq V(X)$.*

Proof. If such a $Y \in X^*$ existed, then by lemma 4 $V(Z) > V(W)$ where W has values $(1/(q-1), 1)$ and weights $\{p - 1, 1\}$ and Z has values $(0, 1)$ and weights $(p - 2, 2)$. But this W arises from starting at $x = 2/q$ and betting $\left[\frac{1}{q} - \frac{1}{q(q-1)}\right]$ on one number, while Z arises by starting at $x = 2/q$ and betting $1/q$ on each of two numbers. Since W arises by the bold strategy, $V(W) = \varphi(x)$ and since the bold strategy has been shown in [1], p. 119, to be optimal in uniform roulette, $V(Z) \leqq \varphi(x)$, and hence $V(Z) \leqq V(W)$, giving a contradiction.

The second part of the lemma follows from the optimality of the bold strategy for roulette with bets restricted to one number.

The following lemma is the heart of theorem 7. The proof is by induction and uses lemma 4 in an important way.

Lemma 6. *If X is a random variable of order k with $p - q + 2 \leqq k \leqq p - 1$, then for no $Y \in X^*$ is $V(Y) > V(X)$.*

Proof. Let $k \geqq p - q + 2$ be the largest k for which there exists X of order k and $Y \in X^*$ with $V(Y) > V(X)$. Then by lemma 4, $V(Z) > V(W)$ where W is the basic random variable with $t_1 = 1/(k - p + q)$, $t_2 = 1$, and Z is the random variable of order $k - 1$ with $(0, 1/(k - p + q), 1)$ as its attainable values and $\{k - 1, p - k - 1, 2\}$ as its set of weights, and Z and W are related.

Let $Z_1, Z_2, \ldots, Z_{p-k-1}$ be defined inductively as follows. Z_i, $i = 1, 2, \ldots, p - k - 2$, has $(s_i, 1/(k - p + q), 1)$ as its set of attainable values where

$$s_1 = \frac{1}{(k-p+q)^2}$$

and

$$s_{i+1} = \frac{\dfrac{1}{k-p+q} - s_i}{k+i-p+q} + s_i, \qquad i = 1, 2, \ldots, p - k - 3,$$

and has $\{k - 1 + i, p - k - i - 1, 2\}$ as its set of weights.

$Z_i \in Z_{i+1}^*$, $i = 1, 2, \ldots, p - k - 3$ from lemma 1 since

$$\frac{s_{i+1} - s_i}{\dfrac{1}{k-p+q} - s_i} = \frac{1}{k+i-p+q}$$

and Z_{i+1} is of order $k + i$. Similarly $Z \in Z_1^*$. Let Z_{p-k-1} be defined by having attainable values $(s_{p-k-1}, 1)$ and weights $\{p - 2, 2\}$, where

$$s_{p-k-1} = \frac{\dfrac{1}{k-p+q} - s_{p-k-2}}{q-2} + s_{p-k-2}\,.$$

As above, $Z_{p-k-2} \in Z_{p-k-1}^*$. Also, since Z, W, and all the Z_i are related, definitions 4 and 8 imply that $Z_{p-k-1} \in W^*$.

By hypothesis $V(Z_1) \leqq V(Z_2) \leqq V(Z_3) \leqq \cdots \leqq V(Z_{p-k-1}) \leqq V(W)$.

$$V(Z) - V(W) = \frac{1}{p} - \frac{k}{p}\varphi\left(\frac{1}{k-p+q}\right) > 0,$$

$$V(Z) - V(Z_1) = \frac{1}{p}\varphi\left(\frac{1}{k-p+q}\right) - \frac{k}{p}\varphi(s_1)\,.$$

By lemma 3

$$\varphi(s_1) = \varphi\left(\frac{1}{(k-p+q)^2}\right) \geqq \varphi^2\left(\frac{1}{k-p+q}\right).$$

Therefore

$$\begin{aligned} V(Z) - V(Z_1) &\leqq \frac{1}{p}\varphi\left(\frac{1}{k-p+q}\right) - \frac{k}{p}\varphi^2\left(\frac{1}{k-p+q}\right) \\ &= \varphi\left(\frac{1}{k-p+q}\right)\left[\frac{1}{p} - \frac{k}{p}\varphi\left(\frac{1}{k-p+q}\right)\right] < V(Z) - V(W)\,. \end{aligned}$$

Therefore $V(W) < V(Z_1)$, giving a contradiction.

Theorem 7. *For each random variable X, $V(Y) \geqq V(X)$ where Y is the basic random variable related to X.*

Proof. For X of order p, i.e. constant X, theorem 4 gives the result.

For X of order $p - 1$ lemma 5 gives the result.

For X of order k with $p - q + 1 \leqq k \leqq p - 2$, lemma 2 gives the existence of a sequence of random variables X_i, $i = 1, 2, \ldots, p - k - 1$ such that $X \in X_1^*$, $X_i \in X_{i+1}^*$, $i = 1, 2, \ldots, p - k - 2$, and each X_i is of order $k + i$. Lemma 6 then shows $V(X) \leqq V(X_1) \leqq V(X_2) \leqq \cdots \leqq V(X_{p-k-1})$, and the result follows from the fact already noted for random variables of order $p - 1$ that $V(X_{p-k-1}) \leqq V(Y)$ where Y is the basic random variable related to X and all the X_i.

For X of order less than $p - q + 1$ the result follows from successive applications of theorem 5 together with the fact that the result holds for X of order at least equal to $p - q + 1$.

The fact that φ is the utility function of roulette now follows from theorem 7, theorem 3 with $Q = \varphi$, definition 3, and the remarks following definition 4. Thus the bold strategy is optimal at roulette.

Reference

1. Dubins, L. E., and L. J. Savage: How to gamble if you must. New York: McGraw-Hill 1965.

Gerald John Smith
2419 Durant Avenue (Apt. 27)
Berkeley, California, USA