FIRST-ORDER MODELING

This chapter opens the last part of the monograph, which is more specifically dedicated to an engineering audience. It begins with a general presentation and discussion of prediction methods for turbulent flows, based on statistical — or Reynolds — averaged Navier-Stokes equations (RANS). Then, the incidence of density changes and the incorporation of variable-density and compressibility effects in first-order closure models are analyzed with respect to (i) "modifications" to incompressible schemes and (ii) introduction of additional "specific contributions" to non-constant density flows. At last, some zero-, one-, two- and three-equation models are reviewed.

10.1. Introduction

The ultimate objective of this third part of the monograph is to discuss tractable models that have been developed to calculate quantities of interest and practical relevance in turbulent flows of variable density fluid. Turbulence is considered here as a moderate-to-high Reynolds number flow, produced by unsteady, non-linearly interacting instabilities generating a large dynamic range of 3-D velocity and vorticity fluctuations in space and time. It is assumed that the motion is governed by the Navier-Stokes equations, along with the continuity equation, the first law of thermodynamics and the equation of state, as detailed in Chapter 4. For practical applications, several predictive methods[1] are available at the present time:

- Large Eddy Simulations (LES), in which the motion equations are solved for a *filtered* velocity field, which is representative of the large scale turbulent motion. Hence, a model is required for the smaller-scale motions which are not represented;
- Probability Density Function methods (PDF), in which a model transport equation is solved for a probability density function of the fluctuating velocity and other single or joined scalar fluctuations;
- Single point modeling of Reynolds Averaged Navier-Stokes equations, the so called RANS approach, in which closure schemes are introduced to provide a set of statistical equations governing one point moments.

Only the last approach will be addressed hereafter.

[1] At present, Direct Numerical Simulation (DNS) is not considered as a predictive method for industrial applications.

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The conventional hierarchy of such turbulence closure approaches is well known. It is mainly based upon a simple division between *first*, *second* and other higher order closure levels. In first-order models, turbulence or Reynolds stresses are directly coupled with the mean flow, using an eddy-viscosity concept or more generally linear or non-linear constitutive schemes. Now, since turbulence is not a fluid property but depends on the motion itself, the eddy-viscosity is to be prescribed as a function of some flow characteristics. This can be achieved algebraically, yielding zero-equation models, or by solving one or more additional transport equations, producing one-equation, two-equation... turbulence models.
In second-order models[2], transport equations are solved for the Reynolds stress tensor and all other second-order moments, yielding Reynolds Stress (transport) Equations models (RSE).

10.2. Synopsis of one-point turbulence modeling status

10.2.1. CONSTANT DENSITY FLOWS

Historically, since the late 60's, two groups have mainly contributed to the development of one-point modeling under the leadership of B.E. Launder, now at UMIST and J.L. Lumley, now at Cornell University. Since then, a wide amount of literature has been published on the topic by many other authors. Up to date reviews of first and second-order closure schemes for *incompressible* turbulent flows can be found in Schiestel [421], Chen & Jaw [89], Piquet [366], Pope [370], and Chassaing [84].

It is commonly agreed that eddy-viscosity models:

- perform quite satisfactorily in quasi-parallel, equilibrium, 2-D, wall-attached and free flows;
- are relatively numerically 'robust'.

On the other hand, the generic problems associated with linear eddy-viscosity models have been known for many years and include the inability to:

- track rapid, inviscidly induced mean flow alterations;
- predict counter-gradient fluxes;
- predict negative production zones;
- generate turbulence induced secondary flows.

With RSE modeling, such flaws are not present and significant improvements have been obtained, concerning the representation of:

[2]Algebraic stress models (ASM) should also be mentioned. They implicitly determine the local Reynolds stresses as a function of the turbulence kinetic energy, its dissipation rate, and mean velocity gradients. Due to the approximations they involved, they are simpler but less general and accurate than RSE. They are not considered here.

- pressure-strain correlations;
- low-Reynolds number effects;
- anisotropy in the near-wall region.

Nevertheless, some problems are still present with this type of models. They concern:

- the dissipation (or length-scale) equation(s);
- the gradient diffusion (turbulent transport) assumptions.

In constant density flows, some of the dominant physical processes of the turbulent regime can be captured with statistical, single point, closure models. However, such a modeling approach is neither sufficiently developed nor intrinsically adapted to account for all turbulence features, such as coherent structures and free flow intermittency, or energy cascade and small-scale intermittency.
In simple free shear layers, for instance, many observations reveal the presence of large (coherent) structures which are specific to each type of flow (plane or round jet, wakes, mixing layers). Such evidence, in turn, suggests that the energy-containing lower wave number portion of the turbulence spectrum should be flow-dependent. Hence the problem of turbulence modeling cannot simply be reduced to how reflecting modifications to a common basic state, mainly according to Kolmogorov's ideas. As suggested by Bushnell [65], "this is one of the root causes of the *variable* constants required thus far for all modeling approaches".
Another incidence of such large scale motions on the physics of turbulence is the irregular and intermittent behavior at the edge of free flows, with direct consequences on the mixing process, entrainment and expansion. Many proposals have been made to account for free flow intermittency, but at the present time, none of them is actually involved in practical predicting tools for engineering applications.

10.2.2. VARIABLE DENSITY FLOWS

An increasing number of what is considered as variable density effects has been now identified from experimental investigations, theoretical analysis such as stability analysis and Rapid Distortion Theory, or direct numerical simulations. As briefly presented in Chapter 2, the influence of variable density and compressibility on a turbulent flow is manifold and may results in:

- mean temperature/concentration effects;
- mean flow bulk dilatation/compression effects;
- additional baroclinic torque generation/destruction;
- additional instability mechanisms;

– alterations of turbulent eddies interactions and energy cascade;
– shock-induced modifications;
– dilatational contributions to turbulence;
– variation of the physical coefficients and bulk viscosity effects.

During 1975-85 decade, an important effort was dedicated by Ha Minh-Chassaing and co-workers, [80], [87], [193], [195], [85], [194], [473], [197], [332], [199] to the extension of constant density first and second-order closure schemes to variable density turbulent flows, in low and high speed fluid motions. The contributions due to Launder's and Lumley's groups in low-speed modeling for scalar fields were also extended to heterogeneous flows, considering the equations governing concentration or temperature correlations.
At that time and using density weighted averages, the main question, as pointed out by Janicka and Lumley [232] in 1980, was to know to what extent "model assumptions which are developed for constant density flows can be adopted for closure of density-weighted moments for variable density flows."

Later on, a serious attempt to extend RSE closure to super/hypersonic flows was promoted by Speziale-Sarkar *et al.* [443] at NASA Langley. Since then, there have been many proposals to model *new* compressibility effects in high-speed flows, by (i) adding "corrections" to the model "constants" via terms depending on a turbulence Mach number, (ii) introducing "extra-compressibility terms" accounting for dilatational effects (compressible or dilatational dissipation, pressure dilatation correlation). Both procedures are reviewed in the present chapter, as far as they deal with first-order closure schemes.

A second point emerged, giving rise to a slightly different discussion in variable and constant density situations. It directly addresses the choice of the closure level. For "theoretical" reasons, which will become clear in the next chapter, second-order level appears to be particularly suitable in modeling variable density flows. Thus, when reviewing first-order closure schemes, it is worth keeping in mind the following statement by Fulachier *et al.* [173] in 1989 "The improvements obtained by a second-order modeling are so evident, and the ability of this type of modeling to predict the effects of density differences in complex flows so clear, that it is to be recommended that such model be used even for industrial purposes."

10.3. The first-order modeling issue

When deriving closure schemes for the mean motion equations of variable density fluids, one is faced with a three-fold challenge:

- selecting the intrinsic mechanisms which are actually responsible for the dominant variable density effects that are observed in a given flow configuration;
- identifying, in the open set of equations, the corresponding terms that can be handled at a given closure level;
- deriving proper closure schemes to such terms, capturing the correct underlying physics.

Although present in the modeling issue for constant density fluid flows, the challenge is even more crucial in variable density situations. Let us now give some examples.

Identification of intrinsic mechanisms. The search for the explanation of the reduction in the mixing layer growth rate through compressibility can be considered as an illustrative example of the difficulty in identifying intrinsic mechanisms of turbulence in modeling variable density fluid motions.

After the discovery by Passot and Pouquet [359] in 1987, from numerical simulations, of the distinct possibility of random shock-like structures to appear in the flow domain, even when the turbulence or r.m.s. Mach number M_t is subsonic, the presence of "shocklets" was taken as one of a plausible explanation of the phenomenon. However, by the same time, a new controversy about the actual mechanism responsible for the Mach number stabilizing effect to be introduced in turbulence modeling was opened.

– In 1990, Zeman [495] proposed the dilatation dissipation concept to account for such an effect.

– In 1989, to improve the modeling of shock wave/boundary layer interaction in flows over compression ramps, Grasso & Speziale [187] argued that the pressure-dilatation correlation[3] $\overline{p'\partial u'_j/\partial x_j}$ and the turbulent mass flux coupled with the mean pressure gradient $(\overline{\rho' u'_i}/\overline{\rho})(\partial\overline{P}/\partial x_i)$ play an important role in this situation.

– In 1990, Nichols [346], addressing the same flow configuration, suggested that a turbulent "velocity-density" dissipation term $\overline{U}_i\,\overline{\rho' u'_j}(\partial\overline{U}_i/\partial x_j + \partial\overline{U}_j/\partial x_i)$ has to be introduced in the modeling of compressibility effects in that case.

A similar lack of consensus can also observed in the choice of the additional functions which are used as arguments in the various closure schemes accounting for density/compressibility effects. The example of the $(k\ \text{-}\ \epsilon)$ model is particularly illustrative in this respect. Several proposals have been made to extend the incompressible version of this model to compressible situations, generally adopting the turbulence Mach number $M_t = \sqrt{2\overline{k}}/\overline{c}$,

[3] f' denotes a Reynolds fluctuation.

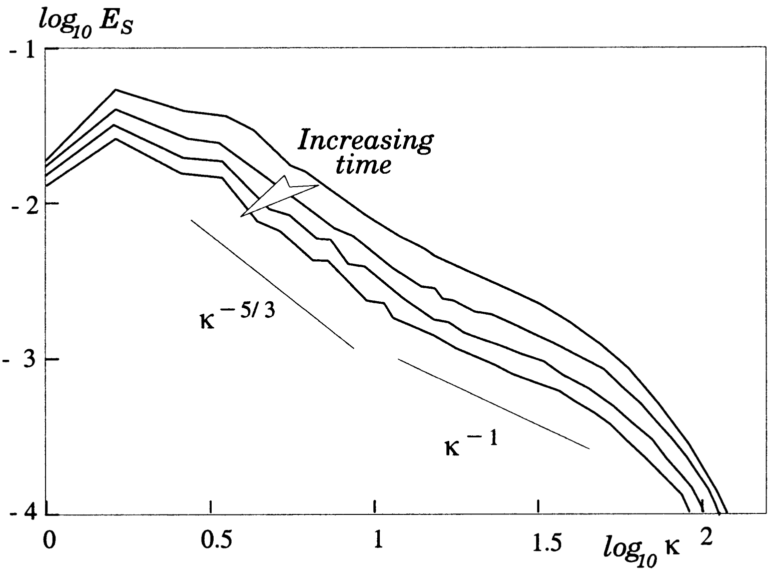

Figure 10.1. Spectrum of the solenoidal velocity fluctuations in a decaying homogeneous isotropic supersonic turbulence, adapted from Pouquet [371].

where $\bar{c}$ is the mean speed of sound, as the parameter accounting for compressibility effects. Some of them are reviewed in the present chapter. However such straightforward extensions have revealed to be not entirely satisfactory, and the need for additional functions has been investigated. Various candidates have been suggested and will be examined in §10.12.

The underlying physics. In modeling constant density fluid turbulence, single point closure of mean transport equations in fully developed, high Reynolds number flows may be addressed first. Far from the walls or in free turbulent flows, this leads to consider that turbulence can reach some equilibrium state, depending on the boundary conditions, but with only a limited memory of its past state. Accordingly, the energetic process is statistically depicted as obeying the Kolmogorov cascade, and large scale organization effects, if present, are not explicitly taken into consideration. To transpose "incompressible" closure schemes to variable density fluid turbulence, it is (*generally implicitly*) assumed that compressibility effects do not radically change the physics, so that most of the incompressible closure schemes can be adapted to compressible turbulence.
Although not always justified, this procedure will be adopted in the following presentation, in which, from a "practical" modeling point of view, compressibility effects are considered as (i) modifying existing incompressible schemes and (ii) introducing new additional terms into the incompres-

sible closure formulation.
However, it is worth recalling that this attitude corresponds to a reduction in the physics. Let us give an illustration of this point. Direct *3-D* numerical simulations of supersonic homogeneous and isotropic turbulence by Pouquet [371] (see Fig.10.1) reveal a double scaling of the spectrum E_S of the *solenoidal* part of the velocity fluctuations: (i) as $\kappa^{-5/3}$ for the energetic wave number range $\kappa < \kappa_t$, where κ_t is the Taylor wave number, and (ii) as κ^{-1}, for the small scale structures $\kappa_t < \kappa < \kappa_\eta$, where κ_η is the Kolmogorov wave number. To the author's knowledge, such a feature has not yet been included in turbulence modeling.

10.4. The open set of equations and the closure issue

Regarding the instantaneous equations governing various types of turbulent motion of a variable density fluid, several approximate models to the full variable-density Navier-Stokes equations can be derived. This question has been addressed in Chapter 3, where such approximate models have been presented, depending on various physical assumptions concerning the dominant mechanisms driving density variations.
With respect to the general objective of the monograph, closure schemes[4] that have been developed for such simplified situations will not be considered here. Hence, due to the wide variety of variable density turbulent flows, the full set of Navier-Stokes equations is adopted (see Chapter 4).

The non-linearities present in these equations generate unknown moments in any *finite* set of *averaged* equations deduced from the instantaneous ones. When deriving "modeled equations", one basically aims at producing a closed set of equations by recovering only that part of information, lost from the averaging procedure, which is required at a given level of statistical description. As far as *first*-order closure is concerned, this level of description simply refers to the "*mean motion equations*". These equations have been detailed in Chapter 5, adopting either conventional or density

[4]Such models are largely restricted to the situation for which they have been derived. For example, Boussinesq's approximations, or more generally weak compressible assumptions, yield simplification of the pressure role and make the modeling issue relatively easily tractable as an extension from the isovolume/incompressible regime. As a direct consequence, the density variance can be readily deduced from the temperature or concentration variance, as solution to a modeled transport equation for a *quasi-passive* contaminant.
On the other hand, in compressible turbulent flows, even within the limit of linear Kovasznay's modes decomposition (see Chapter 3), density fluctuations are included in both acoustic ($p' \neq 0$, $\rho' \neq 0$, and $s' = \omega' = 0$) and entropy ($s' \neq 0$, $\rho' \neq 0$, and $\omega' = p' = 0$) modes. Therefore, in compressible boundary layers and channel flows, for instance, the density variance is not a good parameter to describe compressibility effects accurately.

weighted averages viz., respectively:

$$F = \overline{F} + f' = \widetilde{F} + f'', \qquad \text{where} \qquad \widetilde{F} = \overline{\rho F}/\overline{\rho}$$

and noticing that: $\overline{f'} = 0, \quad \widetilde{f''} = 0, \quad \widetilde{f'} \neq 0, \quad \overline{f''} \neq 0,$

and $$\overline{F} - \widetilde{F} \equiv f'' - f' = -\overline{\rho' f''}/\overline{\rho} = -\overline{\rho' f'}/\overline{\rho} = -\overline{\rho f'}/\overline{\rho}\,.$$

Adopting density-weighted or Favre's averages[5], and considering a binary mixture of perfect gases (see Chapter 5), the equations governing the mean motion are:

Mean continuity :
$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{U}_k)}{\partial x_k} = 0\,, \tag{10.1}$$

Mean momentum :
$$\frac{\partial(\overline{\rho}\widetilde{U}_i)}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{U}_i\widetilde{U}_j)}{\partial x_j} = \overline{\rho}\widetilde{F}_i - \frac{\partial \overline{P}}{\partial x_i} - \frac{\partial(\overline{\rho}\,\widetilde{u''_i u''_j})}{\partial x_j} + \frac{\partial \overline{\tau}_{ij}}{\partial x_j}\,, \tag{10.2}$$

Mean internal energy :
$$\frac{\partial(\overline{\rho}\widetilde{e})}{\partial t} + \frac{\partial(\overline{\rho}\,\widetilde{e}\,\widetilde{U}_j)}{\partial x_j} = \overline{P}\frac{\partial \widetilde{U}_i}{\partial x_i} - \overline{P\frac{\partial u''_i}{\partial x_i}} + \overline{\tau}_{ij}\frac{\partial \widetilde{U}_i}{\partial x_j} + \overline{\tau_{ij}\frac{\partial u''_i}{\partial x_i}} + \frac{\partial \overline{q}_i}{\partial x_i} - \frac{\partial(\overline{\rho}\,\widetilde{e''u''_j})}{\partial x_j}\,, \tag{10.3}$$

Mean mass fraction :
$$\frac{\partial(\overline{\rho}\widetilde{C})}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{C}\widetilde{U}_j)}{\partial x_j} = -\frac{\partial(\overline{\rho}\,\widetilde{\gamma''u''_j})}{\partial x_j} + \frac{\partial \overline{q}_{m_j}}{\partial x_j}\,. \tag{10.4}$$

In the previous equations, τ_{ij} stands for the viscous stress tensor, q_j and q_{m_j} for the molecular heat and mass fluxes respectively.

With respect to the previous equations, the closure issue is concerned with two types of terms:

[5]The equations governing the *mean* properties of variable density fluid motions have been derived in Chapter 5. As compared with the incompressible, constant density situation, new non-linearities due to density variations introduce additional correlations. These correlations can be handled in different ways. Two of them have been more specifically discussed, with reference to binary regrouping (density weighted or Favre's averaging) and ternary regrouping (conventional or Reynolds averaging). The two formulations differ by the way the extra correlation terms due to density variation, the so called d.f.c. (density fluctuation correlations), are handled. This formal distinction may have a direct incidence on the expression of the closure schemes.
Favre averaged equations bear great *formal* resemblance to those governing incompressible flows, since most d.f.c.'s are explicitly "eliminated" from the formulation, but are implicitly present in mean mass-weighted operators.

– (i) turbulent diffusion (or transport) terms, viz. $\overline{\rho}\,\widetilde{\alpha'' u_j''} \equiv \overline{\rho\alpha'' u_j''}$;
– (ii) correlations with the Favrian velocity fluctuation u_i''.

The modeling of the former can be sought from formal extensions of the corresponding constant density fluxes, where, for instance, the Reynolds stress tensor $\rho_0\overline{u_i' u_j'}$ is simply changed to $\overline{\rho u_i' u_j'}$ or $\overline{\rho u_i'' u_j''}$.
The latter introduce new *dilatational* or *compressible* contributions that are specific to variable density fluid motions, as we shall see now.

10.4.1. PRESSURE-DILATATION CORRELATION

In eq.(10.3), the coupling term between the instantaneous pressure and the Favrian dilatation fluctuation reads:

$$\overline{P\frac{\partial u_i''}{\partial x_i}} = \overline{P}\frac{\partial \overline{u_i''}}{\partial x_i} + \overline{p'\frac{\partial u_i''}{\partial x_i}} \, .$$

The last term in the right-hand-side introduces the pressure-dilatation correlation Π_d, which can be equally written as (see Chapters 5 and 6):

$$\Pi_d \equiv \overline{p'\vartheta'} = \overline{p'\frac{\partial u_i''}{\partial x_i}} = \overline{p'\frac{\partial u_i'}{\partial x_i}} \tag{10.5}$$

This term is specific to non-isovolume *fluctuating* motions.

10.4.2. DILATATION DISSIPATION

Similarly, the coupling term between the viscous stress and the Favrian dilatation fluctuation in eq.(10.3) reads:

$$\overline{\tau_{ij}\frac{\partial u_i''}{\partial x_i}} = \overline{\tau}_{ij}\frac{\partial \overline{u_i''}}{\partial x_i} + \overline{\tau_{ij}'\frac{\partial u_i''}{\partial x_i}} \, ,$$

where τ_{ij}' denotes the fluctuation *centered* on the mean value $\overline{\tau}_{ij}$.
Now, as shown in Chapter 6-§6.4.4, the last term in the right-hand-side expression can be rewritten as (neglecting viscosity fluctuations):

$$\overline{\rho}\,\overline{\epsilon} \equiv \overline{\tau_{ij}'\frac{\partial u_i''}{\partial x_i}} = \overline{\rho}\,\overline{\epsilon}_s + \overline{\rho}\,\overline{\epsilon}_d + \overline{\rho}\,\overline{\epsilon}_{nh} \, . \tag{10.6}$$

Here, $\overline{\epsilon}_s$, $\overline{\epsilon}_d$ and $\overline{\epsilon}_{nh}$ denote respectively the solenoidal, dilatational and non-homogeneous contributions to the dissipation rate:

$$\overline{\epsilon}_s = 2\frac{\overline{\mu}}{\overline{\rho}}\overline{\omega_{ij}'\omega_{ij}'} \, , \quad \overline{\epsilon}_d = \frac{4}{3}\frac{\overline{\mu}}{\overline{\rho}}\overline{\vartheta'^2} \, , \quad \overline{\epsilon}_{nh} = 2\frac{\overline{\mu}}{\overline{\rho}}\left(\frac{\partial^2(\overline{u_i' u_j'})}{\partial x_i \partial x_j} - 2\frac{\partial(\overline{\vartheta' u_j'})}{\partial x_j}\right) \, , \tag{10.7}$$

where $\omega'_{ij} = (\partial u'_i/\partial x_j - \partial u'_j/\partial x_i)/2$.

10.4.3. THE CLOSURE PROBLEM

To summarize, the closure problem of the open set of equations governing the mean motion of a variable density fluid requires to derive closure schemes for different types of terms which:

- are formally analogous to those considered in constant density flows, e.g., the "diffusion" terms, say $\overline{\rho}\,\widetilde{\alpha'' u''_j}$;
- directly originate from density fluctuations, such as the turbulent mass flux $\overline{u''_i}$;
- are specific to density fluctuations, namely Π_d and $\overline{\epsilon}_d$.

Some closure schemes for each type of terms are separately reviewed in the following sections.

10.5. Turbulent momentum transport modeling

10.5.1. EDDY-VISCOSITY REPRESENTATION OF REYNOLDS STRESSES

A widely used representation of the Reynolds stresses in incompressible flows is based on the so-called Boussinesq scheme, introducing an eddy or turbulent viscosity μ_t:

$$-\rho_0\overline{u'_iu'_j} + \frac{2}{3}\rho_0\overline{k}\delta_{ij} = 2\mu_t\overline{S}_{ij} \equiv \mu_t\left(\frac{\partial\overline{U}_i}{\partial x_j} + \frac{\partial\overline{U}_j}{\partial x_i}\right), \qquad (10.8)$$

or

$$-\rho_0\, b_{ij} = 2\mu_t\overline{k}\,\overline{S}_{ij}\,, \qquad (10.9)$$

where $b_{ij} = (\overline{u'_iu'_j}/\overline{k}) - \frac{2}{3}\delta_{ij}$ is the Reynolds stress anisotropy tensor. ρ_0 denotes the *constant* density of the *incompressible* fluid in an *isovolume* motion and $\overline{k} \equiv \frac{1}{2}\overline{u'_iu'_i}$ is the turbulence kinetic energy.

Apart from the general validity of such a linear relation (see §10.5.4), transposition of eq.(10.8) in variable density flows raises at least three specific questions:

- What accounts for the "flux", i.e., the compressible equivalent to the incompressible Reynolds stress tensor?
- What accounts for the "forces" driving the flux, i.e., the mean strain rate tensor in isovolume/incompressible flows?
- Are the "incompressible" eddy-viscosity expressions acceptable in variable density fluid situations?

Some answers to these questions will be now examined.

Direct transpositions

A straightforward transposition of eq.(10.8) to variable density flows yields the following expressions, based on Reynolds and Favre decompositions respectively:

$$-\overline{\rho}\,\overline{u_i'u_j'} + \frac{2}{3}\overline{\rho}\,\overline{k}\delta_{ij} = 2\mu_t(\overline{S}_{ij} - \frac{1}{3}\frac{\partial \overline{U}_l}{\partial x_l}\delta_{ij}), \tag{10.10}$$

and

$$-\overline{\rho}\,\widetilde{u_i''u_j''} + \frac{2}{3}\,\overline{\rho}\,\widetilde{k}\delta_{ij} = 2\mu_t(\widetilde{S}_{ij} - \frac{1}{3}\frac{\partial \widetilde{U}_l}{\partial x_l}\delta_{ij}), \tag{10.11}$$

where $\overline{k} = \frac{1}{2}\overline{u_l'u_l'}$ and $\widetilde{k} = \frac{1}{2}\widetilde{u_l''u_l''}$ in eqs.(10.10) and (10.11), respectively. The previous relations are not equivalent, as it can be seen by converting Favre averages into Reynolds averages, according to the simple analytical relationship given in Chapter 5 (see Table 5.2). When Favre averages are used, eq.(10.11) is the most widely adopted expression, beginning with Jones and Launder [238], Jones [237], in 1979 and Ha Minh *et al.* [195] in 1981.

Nevertheless, the following alternate formulations have been proposed.

Wilcox-Rubesin (1980). A more complex relation was introduced in 1980 by Wilcox and Rubesin [485] as part of a (k-ω^2) model, where $\omega = \epsilon/k$ is the dissipation rate per unit kinetic energy. It reads

$$-\overline{\rho}\,\widetilde{u_i''u_j''} + \frac{2}{3}\,\overline{\rho}\,\widetilde{k}\delta_{ij} = 2\mu_T(\widetilde{S}_{ij} - \frac{1}{3}\widetilde{S}\delta_{ij}) + \frac{8}{9}\widetilde{k}\frac{\widetilde{S}_{im}\widetilde{R}_{mj} + \widetilde{S}_{jm}\widetilde{R}_{mi}}{\beta^*\omega^2 + 2\widetilde{S}_{nm}\widetilde{S}_{nm}}, \tag{10.12}$$

where $\beta^* = 0.09$ is a model coefficient.

Dussauge-Quine (1988). Dussauge and Quine [141] used the following eddy-viscosity in a (k - ϵ) model for the turbulent shear stress

$$-\overline{\rho}\widetilde{u''v''} = \frac{C_\mu}{[1 - \frac{3}{2}\beta C(2)\, M^2]^2}\,\overline{\rho}\frac{\widetilde{k}^2}{\overline{\epsilon}},$$

with $C_\mu = 0.09$, $C(2) = 1.5$. M denotes the local Mach number, and

$$\beta = \frac{\alpha(\gamma - 1)}{C_1 - 1 + P_{rod}/\overline{\epsilon}},$$

where $\gamma = C_p/C_v = 1.4$ for air, and $C_1 = 1.5$. P_{rod} denotes the production rate in the turbulence kinetic energy equation. Several values of the last model coefficient have been tested, ranging between -1.35 and -0.8.

Applied to the calculation of a supersonic shear layer, the model gives encouraging results, since it predicts a decrease of the spreading rate as the Mach number increases, in contrast with the usual (k - ϵ) model which appears practically insensitive to such an effect.

Taulbee-VanOsdol (1991). Extra terms due to turbulent fluxes are included in the model adopted by Taulbee and VanOsdol [454] since

$$-\overline{\rho}\,\widetilde{u_i''u_j''} + \frac{2}{3}\,\overline{\rho}\,\widetilde{k}\delta_{ij} = \mu_t\left(\frac{\partial\widetilde{U}_i}{\partial x_j} + \frac{\partial\widetilde{U}_j}{\partial x_i} - \frac{2}{3}\frac{\partial\widetilde{U}_l}{\partial x_l}\delta_{ij}\right) - \overline{\rho}\,\overline{u_i''}\,\overline{u_j''} + \frac{1}{3}\overline{\rho}\,\overline{u_l''}\,\overline{u_l''}\delta_{ij}\ . \tag{10.13}$$

This scheme was introduced in a (k-ϵ-$\overline{\rho'^2}$-$\overline{u''}_i$) model to predict the adiabatic-wall boundary layer and the compressible free shear layer. As noticed by the authors, "the extra terms appearing in the equation are probably not too important" in the first situation. Indeed, according to Kistler's data, $\overline{u''}/\overline{U}$ in this case is about -0.0011 and -0.0028 for $M_\infty = 1.7$ and 4.7, respectively.

10.5.2. EDDY-VISCOSITY EXPRESSIONS

In contrast with the molecular viscosity which can be considered as a macroscopic physical property of the *fluid*, the eddy-viscosity is tightly linked with the properties of the *flow* field. Hence, the next step in the modeling procedure based on such a concept is to prescribe the eddy-viscosity, and more generally, all turbulence transport coefficients, such as the thermal and mass diffusivities. This can be achieved from different ways, associated with:

- *zero-equation* models, based on algebraic expressions of the eddy-viscosity;
- *one-equation* models, in which the eddy-viscosity is directly derived from a modeled transport equation, or deduced from another transportable quantity;
- *two-equation* models, where two distinct characteristics are introduced as transportable functions governing eddy-viscosity variations.

Mixing-length models. When restricted to two-dimensional boundary layers flows, the eddy-viscosity is obtained from a *mixing length*, as given by:

$$\overline{\mu}_t = \overline{\rho}\,\ell_m^2 \left|\frac{\partial\overline{U}}{\partial y}\right|\ . \tag{10.14}$$

The *mixing length* ℓ_m is to be prescribed algebraically as a local function of position in the flow field. Some expressions of ℓ_m are recalled in section 10.9.

Turbulence-kinetic-energy models. From simple dimensional considerations, the eddy-viscosity can be written as:

$$\overline{\mu}_t = \overline{\rho}\, u_{ref} \times l\,, \tag{10.15}$$

where u_{ref} — resp. l —, is a characteristic velocity scale — resp. length scale —, of the fluctuating turbulent motion.
Adopting $u_{ref} \propto \sqrt{k}$ and $l \propto \ell_m$, we reduce eq.(10.15) to

$$\overline{\mu}_t = C\overline{\rho}\,\sqrt{\overline{k}}\,\ell_m\,, \tag{10.16}$$

where C stands for a nondimensional constant.
To be adopted as a closure scheme, eq.(10.16) requires that the local amount of turbulence kinetic energy is known. This can be achieved by solving a model transport equation for this function, introducing the one-equation $(\overline{k})$ turbulence model. Closure schemes involved in the modeling of this equation in variable density situation are discussed in section 10.10.2.

Two-equation models. The next step in prescribing the eddy-viscosity from eq.(10.15) consists in solving a second modeled transport equation for the characteristic length scale l, producing a two-equation (k- l) turbulence model. In actual fact, this idea can be generalized to obtain the eddy-viscosity from any dimensionnally correct group of two independent functions, based on any combination of suitable powers of characteristic velocity and length scales. Formally, this can be written as:

$$\overline{\mu}_t \propto \overline{\rho}\,\overline{k}^m \zeta^n\,. \tag{10.17}$$

Some examples of the additional transportable function ζ, which has been used in predicting variable density fluid motions are given in table 10.1. Additional references can be found in Wilcox [484], pages 459 & 460.

Since 1975, the constant density version of the two-equation (k - ϵ) model has become one of the most popular in this category and is widely adopted in predicting turbulent flows for industrial applications. It has also been extended to predicting variable density and compressible fluid motions and will be detailed in section 10.11.1.

10.5.3. COMPRESSIBILITY EFFECTS ON THE EDDY-VISCOSITY CONCEPT

In most proposals to extending eq.(10.15) to variable density and/or compressible situations, Batchelor's scaling $\overline{\epsilon} \propto \overline{k}^{3/2}/\ell$, is adopted for the dissipation rate. Thus, it is assumed that only *one* characteristic length scale

TABLE 10.1. Some examples of two-equation ($k - \zeta$) models

Author - Ref.	Year	Additional function ζ
Kolmogorov [256]	1942	
Wilcox [483]	1988	$\overline{k}^{1/2} / l (\equiv \omega)$
Saffman [403]	1970	
Spalding [441]	1972	$\overline{k}/l^2 \ (\equiv \omega^2)$
Wilcox & Traci [486]	1976	
Davidov [123]	1961	
Harlow & Nakayama [209]	1967	$\overline{k}^{3/2}/l \ (\equiv \overline{\epsilon})$
Jones & Launder [238]	1972	
Cousteix, Saint-Martin, Messing, Bézard, Aupoix [108]	1997	$\overline{\epsilon}/\overline{k}^{1/2} \ (\equiv \varphi)$

$l \propto \ell$ accounts for *both* turbulent transport, say l, and energy transfer by fluctuating motions, say ℓ. In high turbulence Reynolds number flows, it results, for instance, that:

$$\nu_t = C_\mu \frac{\overline{k}^2}{\overline{\epsilon}} \, . \qquad (10.18)$$

Directly adopting eq.(10.18) in variable density and/or compressible fluid motions is questionable for, at least, the following reasons:

a) A possible consequence of compressibility effects is that the model coefficient C_μ (0.09, in incompressible flows) could change with the turbulence Mach number $M_t = \sqrt{\overline{k}}/c$. Such modifications have been actually proposed, as reviewed by Nichols [346].

b) As recalled by Ristorcelli [394], even in constant density, high Reynolds number flows, the ratio l/ℓ slightly depends on the type of flow, so that the coefficient in eq.(10.18), should be flow dependent.

c) In the context of compressible turbulent flows and even restricting Batchelor's scaling to the solenoidal dissipation, the choice of a *single* characteristic length scale is even more questionable. This can be seen from different results obtained, for instance, in compressed turbulence (Zeman & Coleman [499]), in shock-turbulence interaction (Jamme [231]) or in a compressible mixing layer (Freund *et al.* [163]).

For the latter situation, the characteristic length scale of the energy containing eddies ℓ exhibits a dependence on the convective Mach number which is different from the transverse large-eddy length scale l. It seems then plausible that compressible mean flow interacts directly with the large eddies which govern the production of turbulence kinetic energy and the expansion rate. On the other hand, turbulent dissipation occurs mainly at high frequency, i.e., involves small-scale fluctuations that are less influenced by the mean flow and associated compressibility effects. Thus, a possible way to handle this specific action of compressibility could consist in adopting a multiple-scale model, as proposed by Liou *et al.* [300] in 1995.

In 1997, using such a two-scale method, Yoshizawa et al. [492] reached the conclusion that compressibility effects on Boussinesq's scheme were two-fold. One consequence should be a modified expression of the eddy-viscosity, the other comes from the deviation of the Reynolds stress from a turbulent viscosity representation, which is written using the *Lagrange* derivative of the mean velocity and the *spatial* derivatives of the mean density and internal energy. The Reynolds stress representation proposed by these authors reads:

$$\overline{u_i'u_j'} - \frac{2}{3}\overline{k}\delta_{ij} = -2\nu_{t_C}(\overline{S}_{ij} - \frac{1}{3}S_{nn}\delta_{ij}) \; + \; \text{non linear correction} \; + \; \text{mean density gradient contribution} \; + \; \text{mean internal energy contribution}\,, \tag{10.19}$$

where $\overline{S}_{ij} = (\partial\overline{U}_j/\partial x_i + \partial\overline{U}_i/\partial x_j)/2$.
The contributions due to mean density and internal-energy gradients are supposed to increase near a shock, where these gradients become large. The non-linear correction includes $\overline{S}_{ij}$-$\overline{S}_{ij}$ and $\overline{S}_{ij}$-$\overline{R}_{ij}$ parts (see next section). The compressible correction to the eddy-viscosity ν_{t_C} is written as follows:

$$\nu_{t_C} = \frac{1}{[1 + A(I_\rho/M_t)]^{3/2}}\,\nu_t\,, \tag{10.20}$$

where $\nu_t = C_\mu\overline{k}^2/\overline{\epsilon}$ is the usual expression for the incompressible eddy-viscosity, with $C_\mu = 0.09$. Hence, from eq.(10.20), compressibility effects result from the influence of the turbulence density intensity $I_\rho = \sqrt{\overline{\rho'^2}}/\overline{\rho}$ and the turbulence Mach number M_t. Two values of the model coefficient have been investigated ($A=5$ and $A=10$). The differences observed with these values on the prediction of the spreading rate of a compressible shear layer cannot be significantly corroborated, due to the scattering of the experimental data.

10.5.4. NON-LINEAR CONSTITUTIVE RELATIONSHIP

Even in incompressible flows, a linear expression such as eq.(10.8) has no firm basis, neither mathematically nor physically. In particular, it results from eq.(10.9) that the principal axes of the Reynolds stresses are systematically aligned with those of the mean rate of strain tensor, a point which is in disagreement with experimental evidence in homogeneous shear flows, for instance. From a theoretical point of view, such linear expressions can be considered as crude approximations to a more general constitutive expression of the turbulent stresses, see, for instance, Lumley [304].

As a first step toward non-linear eddy-viscosity models in incompressible fluid motions, the following second-order expansion, Lumley (1970) [304], Saffman (1976) [404] can be derived

$$-\overline{u_i u_j} + \frac{2}{3}\overline{k}\,\delta_{ij} = 2\nu_t \overline{S}_{ij} - D\frac{\overline{k}^3}{\overline{\epsilon}^2}\left(\overline{S}_{ik}\overline{R}_{kj} + \overline{S}_{jk}\overline{R}_{ki}\right)$$

where $$\overline{R}_{kj} = \frac{1}{2}\left(\frac{\partial \overline{U}_k}{\partial x_j} - \frac{\partial \overline{U}_j}{\partial x_k}\right)$$

and one additional constant D is introduced.
This relation was adapted in 1980 by Wilcox and Rubesin [485] as part of a $(k\text{ - }\omega^2)$ model for compressible flows, yielding:

$$-\overline{\rho}\,\widetilde{u_i'' u_j''} + \frac{2}{3}\,\overline{\rho}\,\widetilde{k}\delta_{ij} = 2\mu_t(\widetilde{S}_{ij} - \frac{1}{3}\frac{\partial \widetilde{U}_l}{\partial x_l}\delta_{ij}) + \frac{8}{9}\widetilde{k}\frac{\widetilde{S}_{im}\widetilde{R}_{mj} + \widetilde{S}_{jm}\widetilde{R}_{mi}}{\beta^*\omega^2 + 2\widetilde{S}_{nm}\widetilde{S}_{nm}},$$

where $\beta^* = 0.09$ is a model coefficient.
With this scheme, the model improves the prediction of the anisotropy of the normal stresses in the logarithmic region of a compressible boundary layer, with a fair restitution of pressure gradient effects.

The promising idea behind the derivation of more general non-linear eddy-viscosity models in incompressible fluid motions is the expected capability to capture turbulence effects that are directly linked with Reynolds stress anisotropy in complex situations, such as secondary flows in ducts or two-component behaviour of turbulence near a solid surface. It is hoped that predictions based on such types of closure could be as good as those obtained with second-order models and cheaper, in terms of computing resources. Several non-linear constitutive relations have been proposed. The first one, by Pope [369] in 1975, has been extended to three dimensional flows and non-inertial reference frames by Gatski & Speziale [175], in 1993. Various other formulations have been proposed by Shih *et al.* [426] in 1993, Craft *et al.* [111] in 1996, Wang [478] in 1997, in addition to proposals derived from the renormalization group theory, see, for instance, Rubinstein & Barton [400]. They are briefly reviewed in Chassaing [84].

A comparative study of various linear and non linear eddy-viscosity schemes in two- and three-equation models was published in 2000 by Barakos and Drikakis [29]. These authors aim at assessing the capability and accuracy of the models in predicting transonic flows featuring shock/boundary-layer interaction and separation. The general conclusion is balanced: as compared with those obtained from *linear* eddy-viscosity models, the predictions are improved when using *non-linear* models. However the computed results are still far from the experimental data. For the flow over a bump for example (ONERA Case C, as provided by Délery at the 1980-81 Stanford Conference), the maximum of the turbulent shear stress profile is not predicted at the same location as that given by the measurements, and its value is underpredicted by about 40% to 50%.

Since it has not yet been demonstrated that compressible anisotropy effects can be accounted for by formal transposition of non-linear incompressible expressions, and owing to the fact that, even in incompressible flows, the implementation of some non-linear eddy-viscosity formulations may become cumbersome, this type of closure will not be discussed in more detail here.

10.6. Turbulent heat/mass transport modeling

10.6.1. GRADIENT DIFFUSION ASSUMPTION

For heat and mass turbulent fluxes, the counterpart of the gradient diffusion assumption for turbulent momentum transport are:

$$-\overline{\rho}\widetilde{f''u''_j} = \overline{\rho}\,\Gamma_t \frac{\partial \widetilde{F}}{\partial x_j} \quad \text{or} \quad -\overline{\rho f'u'}_j = \overline{\rho}\,\Gamma_t \frac{\partial \overline{F}}{\partial x_j}\,, \tag{10.21}$$

where f' (resp. f'') denotes temperature or mass-fraction fluctuations with respect to conventional (resp. density-weighted) averages.
Eq.(10.21), can be viewed as a direct transposition of Fourier's and Fick's laws for *molecular* heat and mass *transfer* to *turbulent transport*, where $\Gamma_t(L^2T^{-1})$ stands for a heat or mass *turbulence* diffusivity. Hence, the ratio of the thermal ($\mathrm{a_t}$) and mass ($\mathcal{D}_t$) turbulence diffusivity to the turbulence kinematic eddy-viscosity ν_t results in turbulence Prandtl and Schmidt numbers:

$$\sigma_t = \frac{\nu_t}{\mathrm{a_t}} = \frac{\nu_t}{\mathcal{D}_t}\,. \tag{10.22}$$

In addition to the validity of the gradient diffusion assumption, eq.(10.22) raises the question of the value of the turbulence Prandtl-Schmidt numbers in compressible and variable density flows. As reported by Cousteix & Aupoix [107], no systematic effects of Mach number in boundary layers flows have been observed on the turbulence Prandtl number, which is rather flow-dependent: 0.8$\sim$0.9 (flat plate boundary layer), 0.7 (jet) and 0.5 (wake).

10.6.2. ALTERNATIVE TO GRADIENT DIFFUSION SCHEMES

Even in incompressible fluid motions, there exists several classes of turbulent flows where simple eddy-viscosity and gradient diffusion schemes are not appropriate. The reason can be found from at least two arguments, in terms of (i) time scale ratio and (ii) level of anisotropy, Pope [370].

- (i) As pointed out by Corrsin [103], the gradient diffusion scheme requires that the time behaviour of the diffusing motions should be locally and instantaneously dominated by the time scale imposed by the mean velocity gradient, say $(\partial \overline{U}/\partial y)^{-1}$ in a thin shear layer. This can be presumably achieved by the small-scale turbulent eddies but not by the large scale turbulent motions;
- (ii) A linear relationship between stress and strain tensors can be mathematically derived when assuming isotropic conditions. Hence, the deviatoric part of the Reynolds stress tensor cannot be determined from the local mean strain rates in flows where high anisotropy levels are developed, due to curvature, secondary flows or swirl, for instance.

Examples of local and non-local formulations, accounting for curvature and large scale structure effects (intermittency, convective transport) are reviewed in Chassaing [84].

Some of the previous flaws can be obviously transposed to variable density flows. In low Mach number flows, a severe limitation of such simple gradient-type approximations, due to large scale motions, can be expected in fire flows where very large density gradients are present.

In 1999, Shimomura [428] derived a new model for thermally driven turbulent flow, by applying a two-scale direct-interaction approximation to the Rehm-Baum [379] equations governing a low Mach number flow (see Chapter 3). The model is theoretically deduced by considering the fluctuating fields without perturbing the buoyant term and assuming an isobaric evolution for the thermal pressure ($\rho T = C^{te}$), both in space and time. The result, to be used as a turbulence closure in a first-order model, is as follows:

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right) - \frac{2}{3}\left(\overline{k} + \nu_t \frac{\partial \overline{U}_l}{\partial x_l}\right) \delta_{ij}\,, \tag{10.23}$$

$$-\overline{\theta' u_j'} = \mathrm{a_t}\left(\frac{\partial \overline{\mathrm{T}}}{\partial \mathrm{x_j}}\right) + \eta_0 \frac{\partial \overline{\mathrm{P}}}{\partial \mathrm{x_j}}\,, \tag{10.24}$$

with:

$$\nu_t = 0.12\frac{\overline{k}^2}{\overline{\epsilon}},\ \mathrm{a_t} = 0.14\frac{\overline{\mathrm{k}}^2}{\overline{\epsilon}}\left(1 + 0.27\frac{\overline{\mathrm{k}}}{\overline{\epsilon}}\frac{\mathrm{Q_s}}{\rho_0 \mathrm{C_P T_0}}\right) \text{ and } \eta_0 = 0.32\frac{1}{\mathrm{T_0}}\frac{\overline{\theta'^2}}{\overline{\epsilon}_\theta}\,.$$

Here, Q_s is the heat source and $\overline{\epsilon}_\theta = 2\lambda\overline{(\partial\theta'/\partial x_i)^2}$ the turbulent dissipation rate of the thermal variance $\overline{\theta'^2}$.

As compared with the usual gradient type expressions — eqs.(10.10) and (10.11) —, it appears that the main modification concerns the turbulent heat flux, due to the presence in eq.(10.24) of the mean pressure gradient term. Also noticeable is the change in the turbulence thermal diffusivity a_t, as a function of the heat source when $Q_s \neq 0$. Without external heat release, the equivalent turbulence Prandtl number is about 0.86.
These closure schemes were applied to the prediction of the natural turbulent convection along a heated vertical plate. The model prediction of the longitudinal heat flux profile $\overline{u'\theta'}$ is in better agreement with the experimental data.

10.7. Modeling d.f.c terms

According to the classification introduced in §10.4.3, we turn now to the second point to be addressed in first-order closure of variable density turbulent flows, viz. the modeling of first-order d.f.c. terms $\overline{\rho' f'} \equiv \overline{\rho f'} = -\overline{\rho}\,\overline{f''}$. Particular attention is given to the turbulent mass flux $\overline{u_i''}$ in relation with Favre's averaged formulation of the mean motion and turbulence kinetic energy equations.

Linearized approximations
Approximate expressions of the turbulent mass flux $\overline{\rho u_i'}$ can be easily obtained when considering *separately* one of the following three situations:

(a) perfect compressible gas (homogeneous composition);
(b) isobaric evolution (homogeneous composition);
(c) isothermal, ideal mixing of two non-reactive species.

The corresponding equations of state are respectively:

$$(a)\ \frac{1}{\rho} = R\frac{T}{P}, \qquad (b)\ P = C^t \times \rho^n, \qquad (c)\ \frac{1}{\rho} = \frac{C}{\rho_1} + \frac{1-C}{\rho_2}. \qquad (10.25)$$

In eq.10.25(b), n is the polytropic coefficient ($n = 0$: isobaric evolution, $n = 1$: isothermal evolution, $n = C_p/C_v$: isentropic evolution).
In eq.10.25(c), C denotes the mass fraction of one of the two species and ρ_1 and ρ_2 are the densities of each pure species.
First-order approximate expressions of density fluctuations can be deduced from the previous expressions, assuming small fluctuations, viz., $\rho'/\overline{\rho} \ll 1$, $p'/\overline{P} \ll 1$, $\theta'/\overline{\theta} \ll 1$ and $\gamma'/\overline{\gamma} \ll 1$, where ρ', p', θ' and γ' are the fluctuations of density, pressure, temperature, and mass-fraction corresponding to the Reynolds averages $\overline{\rho}$, $\overline{P}$, $\overline{T}$ and $\overline{C}$, respectively. They read:

$$(a)\ \frac{\rho'}{\overline{\rho}} \simeq \frac{p'}{\overline{P}} - \frac{\theta'}{\overline{T}} \qquad (b)\ n\frac{\rho'}{\overline{\rho}} \simeq \frac{p'}{\overline{P}} \qquad (c)\ \frac{\rho'}{\overline{\rho}^2} \simeq \frac{\rho_1 - \rho_2}{\rho_1\rho_2}\gamma' \quad (10.26)$$

Thus the turbulent mass fluxes are linearly linked with pressure-velocity, temperature-velocity and mass-fraction-velocity correlations:

$$(a)\,\frac{\overline{\rho' u_i'}}{\overline{\rho}}=\frac{\overline{p' u_i'}}{\overline{P}}-\frac{\overline{\theta' u_i'}}{\overline{T}}\,,\quad (b)\,n\frac{\overline{\rho' u_i'}}{\overline{\rho}}=\frac{\overline{p' u_i'}}{\overline{P}}\,,\quad (c)\,\frac{\overline{\rho' u_i'}}{\overline{\rho}}=\frac{\rho_1-\rho_2}{\rho_1\rho_2}\overline{\rho}\,\overline{\gamma' u_i'}\,. \quad (10.27)$$

Hence, for a non isothermal evolution ($n \neq 1$) of a perfect gas, it can be deduced that:

$$\frac{\overline{\rho' u_i'}}{\overline{\rho}}=\frac{1}{n-1}\frac{\overline{\theta' u_i'}}{\overline{T}}\,. \quad (10.28)$$

The approximate expression 10.27(c) was used in 1987 by Shih *et al.* [427] in second-order modeling of a variable density mixing layer.
In compressible flows, and with the aid of additional assumptions, eq.(10.28) can be used to get the expected scheme, as first proposed by Rubesin.

Rubesin (1976). The simplification introduced by Rubesin [397] in 1976, consists in assuming that the constant total-temperature condition ($C_pT + U_jU_j/2 = C^t$) also applies to the turbulent field. Hence, to first-order, one can assume that $C_p\,\theta' + \overline{U}_i u_i' \simeq 0$, from which, in *2-D* thin shear layer flows ($\overline{U}_1(\equiv \overline{U}) >> \overline{U}_2$), it results that

$$\frac{\overline{\rho' u'}}{\overline{\rho}} \simeq -\frac{\overline{U}}{(n-1)C_p\overline{T}}\,\overline{u'^2} \simeq -\frac{\overline{U}}{(n-1)C_p\overline{T}}\,\overline{k}\,. \quad (10.29)$$

This expression, with with $n = 1.2$ as recommended by Rubesin, leads to a systematically negative mass flux. This is not the case with eq.(10.28).

Galmes-Dussauge-Dekeyser (1983). In a modified version of the Jones-Launder (k - ϵ) model, Galmes *et al.* [174] suggested to account for compressibility effects in the prediction of supersonic boundary layers with non-zero pressure gradient, by including the mean pressure term $\overline{u''}(\partial\overline{P}/\partial x)$. The closure scheme adopted for the turbulent mass flux $\overline{\rho' u'}/\overline{\rho}$ is based on the strong Reynolds analogy[6]. For a *2-D* boundary layer, it reads:

$$\frac{\overline{\rho' u'}}{\overline{\rho}} = R_{\rho u}(\gamma-1)M^2\overline{\rho}\frac{\widetilde{u''^2}}{\widetilde{U}}\,,$$

where M is the Mach number. The value adopted for the correlation coefficient between density and velocity fluctuations $R_{\rho u}$ is 0.8.
It can be noticed that the previous expression is not galilean invariant. Finally, it is to be added that in [174], the eddy-viscosity is also modified by

[6] The model is only valid in boundary layers over adiabatic walls, not too far from equilibrium.

the previous compressibility term, as mentioned in §10.5.3. The expression is:

$$\mu_t = C_\mu \frac{\overline{\rho}\widetilde{k}^2}{\overline{\epsilon}(1 - A_\rho/\overline{\epsilon})} .$$

Vandromme-Ha Minh (1985). A direct transposition of eq.(10.29) using Favre's averages was extensively adopted by Ha Minh and Vandromme [332], [471] to predict various types of compressible flows, including shock-boundary layer interaction. It reads:

$$\overline{u_i''} = \frac{\widetilde{U}_j}{(n-1)C_p\widetilde{T}} \widetilde{u_i''u_j''} . \qquad (10.30)$$

Dussauge-Quine (1988). A slightly different expression was used by Dussauge and Quine [141] in the prediction of *2-D* supersonic mixing layers:

$$\overline{\rho' u_i'} = C(i)(\gamma - 1)M^2 \overline{\rho} \frac{\widetilde{u_1''u_i''}}{\widetilde{U}} ,$$

where M denotes the local Mach number, and subscript 1 refers to the direction of mean advection. The value of the model constant $C(i)$ is adjusted to the flux component: $C(1) = 0.8$ and $C(2) = 1.5$.

Basically, most of the previous schemes are directly derived from the utilization of the strong Reynolds analogy of Morkovin (see Chapter 4, section 4.4.2), yielding the generic form $\overline{u''} \propto \overline{u''^2}/\widetilde{U}$. According to the DNS data of Zeman [496], such an expression is of wrong sign for the response of an initially isotropic turbulence to a normal shock. It results in gross errors in the turbulence kinetic energy balance (gain, instead of loss of energy), which yields Zeman to "strongly suggest that the application of SRA be avoided in modeling turbulent boundary layers in the presence of shocks."

Gradient diffusion hypothesis

As reviewed by Purwanto [376] in 1994, various proposals suggested to model d.f.c. terms $\overline{\rho' u_i'}$, $\overline{\rho'\theta'}$,..., using a generalized gradient diffusion expression of the form:

$$-\overline{\rho' u_i'} = D_t \frac{\partial \overline{\rho}}{\partial x_i} , \qquad (10.31)$$

where D_t denotes a turbulence diffusivity.

A closure scheme of this type was adopted by Milinazzo & Saffman [331] in 1976 to model the dominant turbulent mass flux ($\overline{\rho' v'}$) in the mean continuity equation of an inhomogeneous, two-dimensional mixing layer. It reads

$$-\overline{\rho' v'} = \frac{\mu_t}{\sigma_\rho} \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial y} \, .$$

The value of turbulence Prandtl-Schmidt mass diffusion number is ranging between 0.25 and 1.

Such gradient diffusion schemes were also introduced very early in combustion applications[7] (see Kent and Bilger [245] in 1977 and Janicka and Kollmann [233] in 1979, for instance). This explains why the validity of such gradient diffusion schemes was first questioned in premixed flames. In this case, measurements showed that the local heat release drives the turbulent heat flux in a direction opposite to that predicted by eq.(10.31). In 1982, the measurements in a nonpremixed flame by Driscoll *et al.* [134] exhibit clear evidence of counter-gradient diffusion of density for the axial turbulent flux component, so that the gradient diffusion assumption is seldom obeyed for $\overline{\rho' u'}$. On the contrary, the radial flux $\overline{\rho' v'}$ was found to follow a gradient diffusion relation throughout the flow field.

Some years later, the gradient diffusion relation for turbulent mass flux was also questioned in non-reactive, binary gas jets. In 1989, the results of Zhu *et al.* [503] validated a gradient diffusion relation

$$\overline{\rho' u'} = -D_t \frac{\partial \overline{\rho}}{\partial x} \, , \tag{10.32}$$

only for a downstream distance $x/D_0 \geq 6$ in a premixed helium/air round jet, with 50% of helium and $D_0 = 9.5\,\mathrm{mm}$. In the near field of the jet, eq.(10.32) is not valid over a substantial part of the flow, except near the core of the jet. Now considering the radial flux component, the corresponding gradient relation

$$\overline{\rho' v'} = -D_t \frac{\partial \overline{\rho}}{\partial y} \, , \tag{10.33}$$

only applies for $x/D_0 \geq 5.1$. But, unlike the axial flux, eq.(10.33) is not valid inside the potential core of the jet, but is true over the rest of the flow field.

[7] In 1975, Lockwood and Naguib [302] proposed a $(k - \epsilon)$ model, based on conventional averaged equations, to predict free turbulent diffusion flames in a round jet. Although not *explicitly* associated with the turbulent mass flux, the modeled equation of the turbulence kinetic energy $\overline{k}$ incorporates a density gradient term, to account for "the generation of $\overline{k}$ due to the density fluctuations". In a *2*-D-thin-shear-layer situation, it reads

$$C_{omp} = C_\rho \, \overline{\gamma'^2} \, \frac{\mu_t}{\sigma_\rho} \frac{\partial \overline{\rho}}{\partial x} \, ,$$

where γ' is the fluctuation of the mixture fraction, C_ρ a model constant and σ_ρ the turbulence Schmidt number.

In 1990, So *et al.* [436] extended the measurements to the self-preserving region of a helium/air, isothermal round jet with the same exit diameter D_0. The exit Reynolds number was 4,300 and the jet-to-ambient fluid density ratio 0.64. In this case, mean quantities, turbulent mass fluxes and second-order turbulent correlations achieve self-preservation at a downstream location of about 24 D_0. For $13.8 \leq x/D_0 \leq 24.46$, the axial turbulent mass flux $\overline{\rho' u'}$ and the mean density gradients have opposite signs only in the central region of the jet ($r \leq 0.8\delta$, where δ is the half-velocity width of the jet). Thus, according to the authors, a gradient diffusion relation does not apply in the outer region of the jet, as a consequence of the physics of the mixing which is mainly controlled by large-scale motions in this region. Finally, as shown in Chapter 6, limitations to gradient diffusion formulations can be pointed out analytically in free, isobaric jets.

Gradient diffusion expressions for the turbulent mass flux were also introduced in various first and second-order models to predict compressible flows, Taulbee & VanOsdol [454] in 1991, Sarkar & Lakshmanan [413] in 1991, for instance. In the latter reference, similar gradient diffusion schemes are introduced for both mass and heat turbulent fluxes:

$$\overline{\rho' u_i'} = -\frac{\nu_t}{\sigma_\rho}\frac{\partial \overline{\rho}}{\partial x_i} \quad \text{and} \quad \overline{\theta' u_i'} = -\frac{\nu_t}{\sigma_T}\frac{\partial \overline{T}}{\partial x_i} \, ,$$

where $\nu_t = 0.09\overline{k}^2/\overline{\epsilon}$. The values of the turbulent Schmidt and Prandtl numbers are $\sigma_\rho = \sigma_T = 0.7$.
An original contribution was proposed by Rubesin [398] in 1990. Within the *2-D* thin shear layer approximations it reads:

$$\overline{u''} = \frac{\gamma - 1}{n - 1} c_e \frac{\widetilde{k}}{\widetilde{\epsilon}} \frac{1}{\widetilde{a}^2} \frac{\overline{\rho u'' v''}}{\overline{\rho}} \frac{\partial \widetilde{h}}{\partial y} \, . \tag{10.34}$$

In eq.(10.34), γ is the isentropic coefficient, $\widetilde{h}$ and $\widetilde{a}$ are the enthalpy and speed of sound of the mean flow ($\widetilde{a}^2 = (\gamma - 1)\widetilde{h}$), $\widetilde{k}$ and $\widetilde{\epsilon}$ the turbulence kinetic energy and its dissipation rate in a Favrian formulation. The value of the model constant is $c_e = 0.35$.
This scheme was developed to improve a previous proposal by Rubesin in 1976 —see eq.(10.29) —, in the prediction of super and hypersonic boundary layers with non-adiabatic conditions. In particular, it can be noticed that in a flow where the Reynolds stress has a given sign, the sign of $\overline{u''}$ is different below or above the point where $\widetilde{h}$ is extremum.

Transport equation of the turbulent mass flux
An open transport equation for the turbulent mass flux $\overline{\rho' u_i'} \equiv -\overline{\rho}\, \overline{u''}_i$ can be obtained by manipulation of the equations governing velocity and density

fluctuations (see Chapter 5-§5). Further simplification to the results given in that chapter can be obtained, by assuming, for instance, a linearized form of $1/\rho$ (Liou & Shih [299]):

$$\frac{1}{\rho} = \frac{1}{\overline{\rho}}(1+\frac{\rho'}{\overline{\rho}})^{-1} = \frac{1}{\overline{\rho}}[1-\frac{\rho'}{\overline{\rho}}+(\frac{\rho'}{\overline{\rho}})^2+...] \simeq \frac{1}{\overline{\rho}}-\frac{\rho'}{\overline{\rho}^2}+\mathcal{O}(\frac{\rho'^2}{\overline{\rho}^3})\,.$$

Jones (1979). Jones [237] was probably the first author to propose a modeled transport equation for the turbulent mass, based on the following general form in high turbulence-Reynolds-number flows:

$$\overline{\rho}(\frac{\partial\overline{u''_i}}{\partial t}+\widetilde{U}_j\frac{\partial\overline{u''_i}}{\partial x_j}) = \text{Production} \;+\; \text{Diffusion} \;-\; \text{Dissipation}\,. \quad (10.35)$$

Zeman (1991). Dealing with the response of an initially isotropic turbulence to a normal shock, Zeman [496] suggested the following transport equation:

$$\frac{D\overline{\rho' u'}_i}{Dt} \approx -\frac{\overline{\rho' u'}_i}{\tau_a} - \overline{u''_i u''_j}\frac{\partial\overline{\rho}}{\partial x_j} - \overline{\rho' u'}_j\frac{\partial\widetilde{U}_i}{\partial x_j}\,.$$

The first term in the right-hand-side drives the turbulent mass flux to relax to zero after the shock, on the fast acoustic time scale $\tau_a = 0.4M_t\tau_t$, where $\tau_t = \widetilde{k}/\overline{\epsilon}_s$ is the vortical turbulence time scale based on the solenoidal dissipation (see Chapter 6).
Within the shock, this equation yields a negative normal component of the turbulent mass flux ($\overline{\rho' u'}_1 < 0$), so that the mean pressure coupling term (d) in eq.(10.52) damps turbulence, as expected from the Rayleigh-Taylor analogy (Chapter 2).

Taulbee and VanOsdol (1991). Taulbee and VanOsdol [454] proposed the following closure schemes to the modeling of eq.(10.35):

$$\text{Production} = -\overline{\rho}\,\overline{u''_j}\,\frac{\partial\widetilde{U}_i}{\partial x_j}+\frac{1}{\overline{\rho}}(\overline{\rho u''_i u''_j}-\overline{\rho' u''_i u''_j})\frac{\partial\overline{p}}{\partial x_j}\,, \quad (10.36)$$

$$\text{Diffusion} = \frac{\partial}{\partial x_j}[\mu_t(\frac{\partial\overline{u''_i}}{\partial x_j}+\frac{\partial\overline{u''_j}}{\partial x_i}-\frac{2}{3}\frac{\partial\overline{u''_m}}{\partial x_m}\delta_{ij})+\frac{1}{3}\overline{\rho' u'^2_m}\delta_{ij}]\,, \quad (10.37)$$

$$\text{Dissipation} = C_{u_2}\,\overline{\rho}\,\frac{\overline{\epsilon}}{\widetilde{k}}\,\overline{u''_i}\,. \quad (10.38)$$

The production term in eq.(10.36) includes two contributions involving mean velocity and density gradients. They are both "exact", i.e., directly

deduced from transportable quantities, when adopting the eddy-viscosity formulation of the authors (see eq.(10.13). Due to the various assumptions introduced during its derivation, the dissipation scheme (10.38) is restricted to low Mach number situations where $\sqrt{\overline{\rho'^2}}/\overline{\rho} \ll 1$ and the density pressure-gradient correlations are negligible. The model constant is $C_{u_2} = 5.3$.
The cross-stream averaged mass fluctuating velocity profiles $(\overline{v''})$ calculated with this model in a flat plate boundary layer can be compared with those obtained from a simple gradient hypothesis:

$$\overline{\rho}\,\overline{v''} = \nu_t \frac{\partial \overline{\rho}}{\partial y} \quad \text{with} \quad \nu_t = C_\mu \frac{\overline{k}^2}{\overline{\epsilon}} \, .$$

The same shape of profiles results from both formulations, the profiles deduced from a simple gradient hypothesis being systematically lower than those obtained with the previous model. The departure between the two computed profiles is all the more important than the free stream Mach number is high.

10.8. Modeling density effects

We consider now the final question introduced in §10.4.3. It is concerned with the modeling of the terms that are specific to variable density fluid turbulence, namely the *pressure-dilatation correlation* and the *dilatational* or *compressible dissipation*. Some of the various schemes, that have been derived so as to be used in first-order closure models, are reviewed in this section, beginning with the pressure-dilatation correlation:

$$\Pi_d = \overline{p'\vartheta'} = \overline{p'\frac{\partial u_i''}{\partial x_i}} \equiv \overline{p'\frac{\partial u_i'}{\partial x_i}} \, .$$

10.8.1. PRESSURE-DILATATION CORRELATION

Review of some closure schemes

Viegas-Horstman (1978). Several expressions were developed by Rubesin and co-workers at NASA-Ames for the compressibility contribution in a turbulence kinetic energy modeled equation. As an example, we just mention here the proposal by Viegas and Horstman [476] in 1978:

$$\overline{p'\vartheta'} = \xi \overline{\rho} \frac{\widetilde{k}}{\gamma} M^2 \frac{\partial \widetilde{U}_i}{\partial x_i} \, , \tag{10.39}$$

where M is the local Mach number and $\xi = 0.73$ a model constant.
The derivation of this scheme is based on the same kind of assumptions as

those discussed in section 10.7 (Rubesin's scheme, 1976).

Horstman (1987). Various inclusions of compressibility effects into the Jones-Launder model were examined by Horstman [218]. In the $\widetilde{k}$ equation, the following scheme for the pressure-dilatation correlation was used

$$\overline{p'\vartheta'} = C\frac{n}{\gamma}(\frac{\gamma-1}{n-1})^2\overline{\rho}\widetilde{k}\,M^2\frac{\partial\widetilde{U}_i}{\partial x_i},$$

where M is the local Mach number. The model constants are $n = 1.2$ and $C = 0.12$.
A similar modification (with a factor $0.3\,\widetilde{k}/\overline{\epsilon}$) was introduced into the dissipation equation.

Zeman (1991). In homogeneous, shocklets free, decaying turbulence or shear driven turbulence, Zeman [497] suggested that the pressure variance equation reduces to

$$\overline{p'\vartheta'} = -\frac{1}{2\overline{\rho}\,a^2}\frac{D\overline{p'^2}}{Dt},$$

where a stands for the local speed of sound and D/DT is the material derivative with respect to the mean motion. In other words, according to this relation, the pressure-dilatation correlation is proportional to the rate of change of potential energy $\overline{p'^2}$ due to compression work.
Hence, the closure issue is focused on the pressure variance equation, as discussed in a following section, see eq.(10.57).

Aupoix *et al.* (1990). Using direct numerical simulations for homogeneous, compressible turbulence, Aupoix *et al.* [24] proposed to model the evolution of the pressure-dilatation term from the following transport equation:

$$\frac{\mathrm{d}\overline{p'\vartheta'}}{\mathrm{dt}} = -C_1\overline{\rho}\frac{1}{\tau_a}M_t^2\frac{\mathrm{d}\overline{k}}{\mathrm{dt}} - C_2\frac{1}{\tau_a}\overline{p'\vartheta'},$$

with,

$$M_t = \frac{\sqrt{2\overline{k}}}{a},\ \tau_a = \frac{\overline{k}^{3/2}}{\overline{\epsilon}\,a},\ C_1 = 0.25,\ C_2 = 0.2\,.$$

This model was only validated in homogeneous sheared compressible flows.

Durbin-Zeman (1992). Applying rapid distortion theory to the analysis of compressed turbulence, Durbin and Zeman [138] suggested that the pressure-dilatation term makes a rapid contribution which is proportional to $\overline{p'^2}\times\partial\widetilde{U}_i/\partial x_i$. Hence, a new expression has been proposed, in which a rapid part, accounting for the rapid compression contribution, is added to the slow relaxation term in eq.(10.57):

$$2\overline{\rho}\, a^2\, \overline{p'\vartheta'} = \frac{\overline{p'^2} - p_e^2}{\tau_a} + C_d\, \overline{p'}^2\, \frac{\partial \widetilde{U}_i}{\partial x_i}\,. \tag{10.40}$$

Here, p_e^2 is the equilibrium pressure variance and τ_a an acoustic relaxation time scale (see §10.12.2). The model constant is $C_d = (5-3\gamma)/6$ as determined from RDT, so that $C_d = 0$ for mono-atomic gases and 0.066 for air.

Zeman-Coleman (1991). As pointed out by Zeman and Coleman [499], the isotropic model — eq.(10.40) — is entirely inadequate for 1D compression, since it does not distinguish between spheric (*3-D*) and anisotropic (directional) compression. These authors suggested to add a new contribution, solely due to compression anisotropy:

$$\overline{p'\vartheta'} = \frac{1}{2\overline{\rho}a^2}\left(\frac{\overline{p'^2} - p_e^2}{\tau_a} + C_d\overline{p'^2}\frac{\partial \widetilde{U}_i}{\partial x_i}\right) - C_A\frac{\sqrt{\overline{p'^2}}}{\overline{P}M_t^2}\widetilde{k}\tau S_{ij}^* S_{ij}^*\,, \tag{10.41}$$

where $S_{ij}^* = \frac{1}{2}(\partial\widetilde{U}_i/\partial x_j + \partial\widetilde{U}_j/\partial x_i - \frac{2}{3}\partial\widetilde{U}_l/\partial x_l\delta_{ij})$ is the trace-free deformation tensor. The *vortical* turbulence time scale $\tau = 2\widetilde{k}/\overline{\epsilon}_s$ is based on the *solenoidal* dissipation rate $\overline{\epsilon}_s$.

Sarkar (1992). The analysis of Sarkar [408] is concerned with the evolution of compressible flow statistics for time intervals larger than the acoustic time scale. Hence, the compressible part of the pressure-dilatation correlation $\overline{p'^C\vartheta'}$ can be neglected, as compared with the incompressible one, $\overline{p'^I\vartheta'}$. Here, p'^I is the *incompressible pressure* associated with the solenoidal part of the velocity vector. In such situations, the pressure-dilatation correlation can be formally written as

$$\overline{p'\vartheta'} \simeq \overline{p'^I\vartheta'} = \overline{p'^S\vartheta'} + \overline{p'^R\vartheta'}\,.$$

As inferred from the previous relation, the closure scheme of $\overline{p'^I\vartheta'}$ is made of two additive terms, as it is usual in pressure-strain modeling for constant density flows. Thus the incompressible pressure contribution is split into the so-called *slow* and *rapid* parts.

– The slow part is modeled as

$$\overline{p'^S\vartheta'} = \alpha_3\overline{\rho}\epsilon_s M_t^2\,,$$

where ϵ_s is the solenoidal dissipation associated with isovolume velocity fluctuations, and $M_t = \sqrt{2\widetilde{k}}/\overline{c}$ is the turbulence Mach number.

– The rapid part — associated with p'^R — that reacts instantaneously to a change in the mean velocity gradient, can be approximated by the algebraic relation

$$\overline{p'^R\vartheta'} = 2\,\overline{\rho}\frac{\partial \overline{U}_m}{\partial x_n}A_{mn} + 2\,\overline{\rho}\frac{\partial \overline{U}_m}{\partial x_m}A_{nn}\ ,$$

provided that an equilibrium scaling exists in the flow. Hence, additional compressible correlations do not dominate the incompressible terms. As shown by Sarkar, $A_{mm} = q_C^2 \equiv \overline{u_i'^C u_i'^C}$, where $u_i'^C$ is the dilatational part of the fluctuating velocity. Introducing

$$A_{mn} = \frac{1}{3}q_C^2\,\delta_{mn} + B_{mn}\ ,$$

the deviatoric tensor B_{mn}, modeled by the leading-order term in a Taylor-series expansion around the isotropic state, is taken as proportional to the anisotropy tensor $a_{ij} = \overline{u_i'u_j'} - (2\overline{k}\delta_{ij}/3)$. Assuming that, for $M_t < 0.5$, the anisotropic part of A_{mn} varies as $M_t q^2$, the final scheme for the rapid part is:

$$\overline{p'^R\vartheta'} = \frac{8}{3}\alpha_4 M_t^2\overline{\rho}\frac{\partial \overline{U}_m}{\partial x_m}\overline{k} + \alpha_2 M_t\overline{\rho}\frac{\partial \overline{U}_m}{\partial x_n}a_{mn}\overline{k}\ .$$

The final form of the pressure-dilatation model is

$$\overline{p'\vartheta'} = \alpha_2 M_t\overline{\rho}\frac{\partial \overline{U}_m}{\partial x_n}a_{mn}\overline{k} + \alpha_3 M_t^2\overline{\rho}\epsilon_s + \frac{8}{3}\alpha_4 M_t^2\overline{\rho}\frac{\partial \overline{U}_m}{\partial x_m}\overline{k}\ . \qquad (10.42)$$

Ristorcelli (1997). The analysis of Ristorcelli [394] treats turbulence in which compressible effects are generated by the turbulent motions, under several hypothesis and, in particular,

- (i) a compact-source assumption: a turbulent eddy is small with respect to the length scale of its acoustic radiation;
- (ii) a compact flow assumption: the size of the turbulent field, D is small or in the order of the acoustic scale; $D/\lambda \leq 1$, where λ is the characteristic length scale of the propagation of pressure and density fluctuations.

In this regard, homogeneous compressible DNS, as treated by Sarkar *et al.* [416], for example, is not a compact flow, since it is concerned with turbulence of scale ℓ irradiated by an *infinite* external acoustic field generated by turbulence whose statistics are the same as those of the local turbulent region.
According to Ristorcelli's assumptions, the pressure-dilatation is found to

be a non-equilibrium phenomena. It is modeled as the sum of a slow and rapid part and includes the substantial derivative following a mean fluid particle of a relative time scale based on the non-dimensional strain and rotation rates of the mean motion. Restricted to low-M_t^2 situations, it scales as

$$\overline{p'\vartheta'} \propto M_S^2[\frac{P_{rod}}{\overline{\epsilon}_s} - 1] \,. \qquad (10.43)$$

Here P_{rod} stands for the production rate of the turbulence kinetic energy, $\overline{\epsilon}_s$ is the *solenoidal* dissipation (as modeled in next section) and $S = (S_{ij}S_{ij})^2$ the trace of the square of the mean strain rate matrix $S_{ij} = (\partial\overline{U}_i/\partial x_j + \partial\overline{U}_j/\partial x_i)/2$.
Hence, from eq.(10.43), conditions for the pressure-dilatation correlation to be important, are, that the square of the strain (gradient) Mach number $M_S = M_t S\overline{k}/\overline{\epsilon}_s$ is large and turbulence far from the energetic equilibrium $P_{rod} \neq \overline{\epsilon}_s$. Now, depending upon the departure from energetic equilibrium, eq.(10.43) shows that the pressure-dilatation can be either positive or negative, and therefore the gradient Mach number can equally have a stabilizing or destabilizing effect.

Hamba (1999). The following scheme, due to Hamba [203], is part of the modeling of the pressure variance equation (see §10.12.2). It reads

$$\overline{p'\vartheta'} = -(1 - C_{pd3}\chi_p) \times [C_{pd1}M_t^2 \frac{D(\overline{\rho}\overline{k})}{Dt} + C_{pd2}\gamma\overline{\rho}M_t^2\overline{k}\,\frac{\partial\overline{U}_i}{\partial x_i}] \,. \qquad (10.44)$$

In this expression, D/Dt stands for the material derivative following the mean motion and χ_p is a non-dimensional pressure variance corresponding to the ratio of potential to kinetic energy for weak fluctuations

$$\chi_p = \frac{\overline{p'^2}}{2\overline{\rho}^2\overline{c}^2\overline{k}} \,. \qquad (10.45)$$

To achieve good agreement between model predictions and direct numerical simulations, the constants are set to $C_{pd1} = 1.2$ and $C_{pd3} = 6$. The value of C_{pd2} needs not to be prescribed in homogeneous shear flows where the mean velocity divergence is zero.

Discussion
Dealing with compressible homogeneous turbulence at a moderate Mach number, Sarkar *et al.* [416] suggested in 1991 that, for the purpose of turbulence modeling, the effect of the pressure-dilatation correlation can be absorbed in the model of the compressible dissipation (see the next section). This statement is based on DNS of isotropic turbulence which

indicates that the average of $\overline{p'\vartheta'}$ over its oscillations is significantly smaller than the compressible dissipation.

Also shown by direct numerical simulation results, the pressure-dilatation contribution to the evolution of turbulence kinetic energy is more important in homogeneous shear compressible flows than in decaying turbulence. In this case, the major contributor to the pressure-dilatation comes from the *incompressible pressure* p'^I associated with the solenoidal velocity.

To some extent, this explains why the model derived by Sarkar [408] in 1992 — with $\alpha_2 = 0.15$ and $\alpha_3 = 0.2$ in eq.(10.42) —, can be considered as a reasonable approximation for weakly inhomogeneous flows, such as compressible shear layer and flat plate boundary layer, far from the wall.

Although the pressure-dilatation term $\overline{p'\vartheta'}$ may be negligible in the $\overline{k}$ equation for homogeneous compressible shear flow, it cannot be inferred that there is no need for an appropriate closure scheme, for at least two reasons:

- The model derived by Zeman and Coleman [499], with $C_A = 0.0008$ for the closure parameter in eq.(10.41), is capable of replicating the important feature of kinetic to pressure energy transfer, specific to 1D compression and the resulting different amplification rates of $\overline{p'^2}$ for low and high Mach number situations;
- The pressure-dilatation correlation plays an important role in the pressure variance equation (see §10.12.2), so that the modeling of this term is to be addressed when deriving a closure expression to this equation. In compressible homogeneous shear flow, for instance, the DNS results of Hamba [203] in 1999 show that the pressure-dilatation correlation is the dominant term in the pressure variance equation.
  As far as Hamba's scheme is concerned — eq.(10.44) —, it can be observed that
  - if χ_p is neglected and approximating $D(\overline{\rho}\overline{k})/Dt$ by $\rho(P_{rod} - \epsilon)$, then eq.(10.44) is almost the same as the model derived by Sarkar [408] in 1992;
  - compressibility effects are represented by the factor $(1 - C_{pd3}\chi_p)$, which amounts to 0.88 and 0.58 for M_t equal to 0.1 and 0.3, respectively, thus reducing the pressure-dilatation growth rate when increasing the turbulent Mach number.

Finally, in both Ristorcelli's and Hamba's schemes, the pressure-dilatation correlation is related to the non-equilibrium or unsteady properties of the turbulent field. This is particularly apparent with the material derivative of the turbulence kinetic energy in Hamba's scheme (eq.(10.44)) and a bit more intricate in the *almost* algebraic expression of Ristorcelli (see eq. (87) in [394]). As pointed out by the latter, the model produces predictions which are consistent with the study by Simone *et al.* [431] where the observed behavior is related to the anisotropy component b_{12} and leads to negligible contributions in a quasi-equilibrium situation, as it is the case in channel flows and boundary layers without strong pressure gradients.

10.8.2. DILATATION DISSIPATION

Review of some models

We turn now to the modeling of the last term which is specific to variable density turbulent flows, viz. the dilatation or compressible dissipation $\bar{\epsilon}_d$, eq.(10.7).

Zeman (1990). According to Zeman [495], the dilatation dissipation is considered as proportional to the solenoidal dissipation

$$\bar{\epsilon}_d = c_d\, F(M_t, K)\, \bar{\epsilon}_s \ . \tag{10.46}$$

The adjustable model constant is $c_d = 0.75$. The scalar function F depends on the rms or turbulence Mach number $M_t = \sqrt{2\tilde{k}}/\bar{c}$, where $\bar{c} = \sqrt{\gamma R\tilde{T}}$, and on the probability distribution function of the turbulence, through the kurtosis of the p.d.f. of the velocity fluctuation, $K = \overline{u''^4}/(\overline{u''^2})^2$ (ranging from 4 to 20). The function $F(M_t, K)$, of order unity, is found to increase rapidly for $0 < M_t < 2$ and reaches a quasi constant level for $M_t > 2$ (close to unity, for $K = 6$).
This closure was introduced in a second-order model to predict compressible mixing layers. It appeared able to predict the reduction of layer growth rates as a function of the convective Mach number in agreement with the experimental data.

Sarkar-Erlebacher-Hussaini-Kreiss (1991). Sarkar *et al.* [416] come to a very similar expression, but postulate a fundamentally different basis. The model is simply

$$\bar{\epsilon}_d = \alpha_1 M_t^2\, \bar{\epsilon}_s \, , \tag{10.47}$$

where the model constant is $\alpha_1 = 1$, based on direct numerical simulations of the decay of isotropic compressible turbulence at a Reynolds number based on the Taylor micro-scale $R_\lambda = 15$. The turbulence Mach number is $M_t = \sqrt{2\bar{k}}/\bar{c}$, where $\bar{c}$ is the local mean speed of sound.

Wilcox (1994). As shown in Wilcox [484], page 185, the dilatation-dissipation concept can be introduced in a (k - ω) model (ω is the Kolmogorov frequency of the energy bearing eddies $\omega = \sqrt{\bar{k}}/l \equiv \bar{\epsilon}/(C_\mu \bar{k})$). It merely consists in letting the closure coefficients β and β^* of the model vary with the turbulence Mach number, thus departing from their incompressible values β_0 and β_0^* as

$$\beta^* = \beta_0^*[1 + \xi^* F(M_t)] \qquad \text{and} \qquad \beta = \beta_0 - \beta_0^* \xi^* F(M_t)\,,$$

$$\text{with} \quad F(M_t) = (M_t^2 - M_{t_0}^2)\mathcal{H}(M_t - M_{t_0})\,, \quad \xi^* = 3/2\,, \qquad M_{t_0} = 1/4\,,$$

where $\mathcal{H}(x)$ is the Heaviside step function.

Ristorcelli (1997). Within the same frame of analysis as that previously mentioned in deriving the pressure-dilatation closure scheme, Ristorcelli [394] succeeded in producing a representation for the effects of the compressible (or dilatational) dissipation $\bar{\epsilon}_d$ as a sum of slow ($\bar{\epsilon}_d^s$) and rapid ($\bar{\epsilon}_d^r$) portions.

The ratio $\bar{\epsilon}_d/\bar{\epsilon}_s$ is found to be a function of the turbulence Reynolds number $R_t = 4\bar{k}^2/9\nu\bar{\epsilon}$, scaling as

$$\frac{\bar{\epsilon}_d^s}{\bar{\epsilon}_s} \propto \frac{M_t^4}{R_t} \qquad \text{and} \qquad \frac{\bar{\epsilon}_d^r}{\bar{\epsilon}_s} \propto \frac{M_t^2 M_S^2}{R_t}\,, \tag{10.48}$$

for high-R_t and low-M_t^2 non-equilibrium flows, respectively. Here, M_t and M_S are the turbulence and strain (gradient) Mach numbers, respectively. An important feature of the previous model is the dependence of the dilatation dissipation on the viscosity: for fixed M_t, $\bar{\epsilon}_d$ vanishes at a sufficiently high turbulence Reynolds number.

Discussion

Prediction of the reduction in the spreading rate of a compressible mixing layer can be taken as one of the most challenging issues to turbulence modeling of compressible effects in such flows. Indeed, most "classical" extensions of incompressible models, performed without incorporating dilatational terms, fail to predict the reduction growth rate when the convective Mach number increases. This can be observed in Figure 10.2 for first-order (k - ω) and second-order (R_{ij}) closure models as well. The (k - ϵ) model generates predictions similar to those of (k - ω), if not worse.

In this figure, the spreading rate of the compressible mixing layer C_δ is defined as

$$C_\delta = \frac{\mathrm{d}\delta}{\mathrm{d}x}\left(\frac{U_1 + U_2}{U_1 - U_2}\right).$$

It is normalized by its incompressible value C_{δ_0}. The shear layer thickness $\delta(x)$ is the distance between the points where the mean velocity is, respectively, $U_2 + 0.1(U_1 - U_2)$ and $U_2 + 0.9(U_1 - U_2)$. The ratio C_δ/C_{δ_0} is plotted against the convective Mach number

$$M_c = \frac{U_1 - U_2}{c_1 + c_2}\,,$$

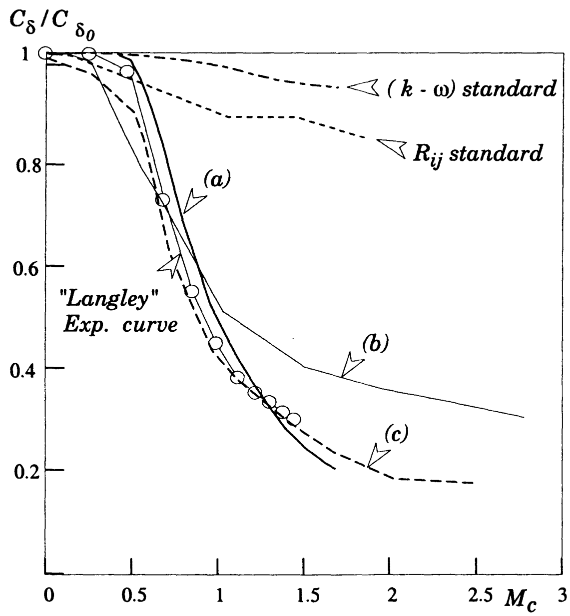

Figure 10.2. Prediction of the spreading rate of a compressible mixing layer according to (a) Wilcox [484], (b) Sarkar *et al.* [416] and (c) Zeman [495].

where c_1 and c_2 stand for the speed of sound in the two incident free streams.

As shown Fig.10.2, a better agreement with experimental data is obtained when incorporating compressible or dilatation effects in the closure schemes. Except the first-order model of Wilcox [484], all other closure schemes, as derived by Zeman [495], eq.(10.46) and Sarkar *et al.* [416], eq.(10.47), have been introduced in second-order models to predict the reduction of the mixing layer growth rate as a function of the convective Mach number (see, for instance, Sarkar and Lakshmanan [413] in 1991 for the latter).

These examples can be considered as representative of a "second generation"[8] of compressible models [495], [415], [416], [484], in which it was presumed that *explicit* dilatational terms could account for compressibility effects, namely the pressure-dilatation correlation and the dilatation dissipation.
Later literature on the topic, Sarkar [409] in 1995, Vreman *et al.* [477] in 1996, Hamba [203] in 1999, *inter alia*, pointed out important *implicit* effects of compressibility, associated with structural changes of the turbulent field.

[8]The "first generation" only involves simple adjustments (turbulence Mach number parameterization) and formal extensions of "incompressible" closure schemes.

Moreover, even though the importance of dilatational terms is difficult to assess, *a priori* from theoretical analysis, or check, *a posteriori* from experimental data, further developments — Ristorcelli [394] and Simone *et al.* [431] for example —, support the conclusion that compressible dissipation is negligible in most compressible flows at a moderate turbulence Mach number (see the M_t^2 and M_t^4 scalings in eqs.(10.48), for instance).

Hence, one can reasonably question the capability of this early generation of compressible models (based on specific modeling of explicit dilatational terms) in capturing the actual mechanism responsible for predicted compressibility effects and suspect whether an effect attributed to compressible dissipation could not actually represent an alteration in production, as recently suggested by Guézengar et al. [191].

10.9. Examples of zero-equation models

After having analyzed *separate* closure schemes accounting for variable density or compressibility effects, this section introduces a second part of the chapter which is devoted to the presentation of some *complete* models that have been used to predict such types of turbulent flows. The review does not aims at being exhaustive, but simply illustrative of some salient contributions.

Owing to the great number of algebraic expressions of eddy-viscosity, the presentation is limited here to those which have been used in predicting compressible turbulent flows for practical applications. For a more detailed review, the reader is referred to Vandromme [469] or Cousteix & Aupoix [107], for instance.

Michel-Quémard-Durant (1969). In the mixing length scheme proposed by Michel *et al.* [327], the eddy-viscosity in a turbulent boundary layer is given by:

$$\mu_t = \overline{\rho} F^2 \, \ell_m^2 \, |\frac{\partial \overline{U}}{\partial y}| \, .$$

The mixing length ℓ_m is taken as a function of the distance to the wall y

$$\frac{\ell_m}{\delta} = 0.085 \, tanh(\frac{\chi}{0.085} \frac{y}{\delta}) \, , \quad \text{with} \quad \chi = 0.41 \, .$$

Here δ is the conventional boundary layer thickness. F is a modified version of the Van Driest damping function:

$$F = 1 - exp[-\frac{\ell_m \sqrt{\overline{\rho}\, \tau}}{26 \, \chi \, \mu}] \, ,$$

where $\tau = \mu\frac{\partial \overline{U}}{\partial y} - \overline{\rho u'v'}$ is the total shear stress. The counterpart of this scheme for the turbulent heat flux is:

$$-\overline{\rho v'\theta'} = -\frac{1}{\sigma_t}\overline{\rho}F^2\,\ell_m^2\frac{\partial \overline{U}}{\partial y}\frac{\partial \overline{T}}{\partial y}\,,$$

where $\sigma_t = 0,89$ is the turbulence Prandtl number.
This model is able to reproduce quite satisfactorily compressibility effects in supersonic turbulent boundary layers not too far from equilibrium. It was also included in a integral method by Michel *et al.* [326] to predict satisfactorily adiabatic boundary layers.

Cebeci-Smith-Mosinskis (1970). The Cebeci-Smith-Mosinskis (C.S.M.) scheme [74] is an extension to compressible flows of Cebeci's model. Several adaptations of this kind have been proposed. For a rather complete inventory, the reader is referred to the book by Cebeci and Smith [73].
In the C.S.M. model, the eddy-viscosity is derived from a two-layer formulation:

$$Inner\ region: \quad 0 \le y \le y_c \quad \mu_{ti} = \overline{\rho}\,\kappa^2\,y^2(1-e^{-y/A})^2\,\frac{\partial \overline{U}}{\partial y}\,,$$

$$Outer\ region: \quad y_c \le y \le \delta \quad \mu_{to} = \alpha\,\overline{\rho}\,U_E\,\delta_1/\Gamma\,,$$

where y_c is the distance from the wall where the two expressions are matched $(\mu_{to} = \mu_{ti})$.
In the first expression, $\kappa = 0.41$, $\alpha = 0.0168$, $\delta_1 = \int_0^\infty(1-\overline{U}/U_E)dy$. Γ is the intermittency correction factor:

$$\Gamma = [1 + 5.5(\frac{y}{\delta})^6]^{-1}\,.$$

In the inner region, the damping coefficient A is defined as

$$A = A^+\frac{\nu}{N}(\frac{\tau_w}{\rho_w})^{1/2}(\frac{\overline{\rho}}{\rho_w})^{-1/2}\,,$$

where $A^+ = 26$ and N is given by the following expression, accounting for pressure gradient and wall blowing (transpiration velocity V_w):

$$N = \frac{\mu}{\mu_e}(\frac{\rho_e}{\rho_w})^2\frac{P^+}{V_w^+}[1 - exp(11.8\frac{\mu_w}{\mu}V_w^+)] + exp(11.8\frac{\mu_w}{\mu}V_w^+)\,,$$

with

$$P^+ = (\frac{\nu_e U_e}{u_\tau^3}\frac{\partial U_e}{\partial x}) \qquad V_w^+ = \frac{V_w}{u_\tau} \quad u_\tau^2 = \frac{\tau_w}{\rho_w}\,.$$

In these formulae, subscripts w and e refer to wall and external flow conditions respectively.

Baldwin-Lomax (1978). One of the major drawbacks of the C.S.M model is revealed in separated flows, where δ_1 is not well defined. This is not the case with the Baldwin-Lomax (B.L.) scheme, where an implicit matching is adopted between two eddy-viscosity values ν_{ti} and ν_{to}. The local eddy-viscosity is taken as the lowest value between the two previous ones, respectively defined by:

$$Inner\,region: \ \nu_{ti} = l_m^2 \Omega \quad where \ \ l_m = \chi y \left(1 - e^{y^+/A^+}\right) \ \text{ and } \ \Omega = \left\|\vec{\Omega}\right\| ,$$

$$Outer\,region: \ \nu_{to} = \alpha C_{cp} F \widetilde{I}_K \left(x, y\right) ,$$

$$\text{where} \quad F = \min\left[y_{max} f_{max}; C_w y_{max} U_{dif}^2 / f_{max}\right] , \ \ f_{max} = \frac{1}{\chi} \sup_y \left[l_m \Omega\right]$$
$$\text{and } \widetilde{I}_K = \left[1 + 5,5 \left(C_K y / y_{max}\right)^6\right]^{-1} .$$

$\vec{\Omega}$ is the mean vorticity vector and y_{max} the distance from the wall where $[l_m \Omega]\,(y)$ is maximum. U_{dif} stands for the mean velocity difference over y_{max}.
The model constants and parameters are given in table 10.2.

TABLE 10.2. Model constants of the Baldwin-Lomax eddy-viscosity model.

Constant	χ	α	A^+	C_{cp}	C_ω	C_K
Value	0.41	0.0168	26	1.6	0.25	0.3

Discussion
Algebraic eddy-viscosity models were primarily derived for closing the averaged Navier-Stokes equations under thin shear layer approximations and were mainly dedicated to wall bounded flows. They are easy to use, cheap, and produce rather accurate results in boundary layers that are not too far from equilibrium, even at high Mach numbers, Cousteix & Aupoix [107]. Mean flow predictions of attached supersonic boundary layers with C.B.S. and B.L. models are generally found in reasonably good agreement with experimental data.

10.10. One-equation models

The following table give some examples of one-equation models that have been proposed for predicting incompressible flows. As far as applications to variable density and compressible flows are concerned, only two of them will be detailed here, based on modeled transport equations for (i) the eddy-viscosity and (ii) the turbulence kinetic energy.

TABLE 10.3. Examples of one-equation incompressible turbulence models

Transportable function	Author	Ref.	Year
Turbulence kinetic energy	Prandtl	[372]	1945
	Emmons	[148]	1954
Turbulence shear stress	Bradshaw & Ferriss	[53]	1972
Eddy viscosity	Nee & Kovasznay	[345]	1968
	Secundov	[424]	1971
	Baldwin & Barth	[28]	1990
	Spalart & Allmaras	[440]	1992
	Gulyaev, Kozlov & Secundov	[192]	1993
	Durbin, Mansour & Yang	[139]	1994

10.10.1. EDDY-VISCOSITY TRANSPORT EQUATION

The first model based on an eddy-viscosity transport equation was proposed by Nee & Kovasznay [345] in 1968. After an eclipse of more than twenty years, this type of closure is again in use for practical applications, thanks to the proposals of Baldwin & Barth [28], and more recently, Spalart & Allmaras [440]. In the latter models, the eddy-viscosity is taken as:

$$\nu_t = \alpha \hat{\nu} \,, \tag{10.49}$$

where α is a model coefficient and $\hat{\nu}$ an effective eddy-viscosity, which is governed by a model transport equation of the general form[9]:

$$\frac{\partial \hat{\nu}}{\partial t} + \overline{U}_j \frac{\partial \hat{\nu}}{\partial x_j} = D_{iff} + \text{Source} - \text{Sink} \,. \tag{10.50}$$

[9] The Spalart-Allmaras model can predict a laminar solution. The complete version of this model includes a "trip" term to initiate transition in a smooth manner. For sake of shortness, the detailed expressions of this term are not given in Table 10.4.

TABLE 10.4. Details of Baldwin-Barth and Spalart-Allmaras eddy-viscosity transport equation models, in reference to eqs.(10.49) and (10.50).

Term	Baldwin-Barth	Spalart-Allmaras
Diffusion	$(\nu + \frac{\nu_t}{\sigma})\frac{\partial^2 \hat{\nu}}{\partial x_j \partial x_j}$	$\frac{1}{\sigma}\frac{\partial}{\partial x_j}[(\nu + \hat{\nu})\frac{\partial \hat{\nu}}{\partial x_j}]$ $+$ $\frac{C_{b2}}{\sigma}\frac{\partial \hat{\nu}}{\partial x_j}\frac{\partial \hat{\nu}}{\partial x_j}$
Source	$(C_{\epsilon 2} f_2 - C_{\epsilon 1})\sqrt{\hat{\nu} P}$, where $f_2 = \frac{C_{\epsilon 1}}{C_{\epsilon 2}} + (1 - \frac{C_{\epsilon 1}}{C_{\epsilon 2}})(\frac{1}{\chi y^+} + D_1 D_2)\times$ $[\sqrt{D_1 D_2} + \frac{y^+}{\sqrt{D_1 D_2}}(\frac{1}{A_0^+} e^{-\frac{y^+}{A_0^+} D_2} + \frac{1}{A_2^+} e^{-\frac{y^+}{A_2^+} D_1})]$ and $P = \nu_t[(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i})\frac{\partial \overline{U}_i}{\partial x_j} - \frac{2}{3}\frac{\partial \overline{U}_k}{\partial x_k}\frac{\partial \overline{U}_l}{\partial x_l}]$	$C_{b1}\hat{S}\hat{\nu}$, where $\hat{S} = (2\overline{\Omega}_{ij}\overline{\Omega}_{ij})^{1/2} + \gamma\frac{\hat{\nu}}{\chi^2 y^2}$ $\gamma = 1 - \frac{\zeta}{1+\alpha\zeta}$ and $\zeta = \hat{\nu}/\nu$
Sink	$\frac{1}{\sigma}\frac{\partial \nu_t}{\partial x_j}\frac{\partial \hat{\nu}}{\partial x_j}$	$C_{w1} f_w (\frac{\hat{\nu}}{y})^2$, where $f_w = g\,(\frac{1+C_{w3}^6}{g^6 + C_{w3}^6})^{1/6}$ $g = r + C_{w2}(r^6 - r)$ $r = \hat{\nu}/\hat{S}\chi^2 y^2)$
α	$C_\mu D_1 D_2$, where $D_1 = 1 - e^{-y^+/A_0^+}$, *and* $D_2 = 1 - e^{-y^+/A_2^+}$	$\frac{\zeta^3}{\zeta^3 + C_{v1}^3}$

TABLE 10.5. Model constants.

Model	Constants
Baldwin & Barth	$C_\mu = 0.09$ $C_{\epsilon 1} = 1.2$ $C_{\epsilon 2} = 2$ $A_0^+ = 26$ $A_2^+ = 10$ $\sigma = 0.7$ $\chi = 0.41$
Spalart & Allmaras	$C_{b1} = 0.1355$ $C_{b2} = 0.622$ $C_{v1} = 7.1$ $C_{w1} = 3.2391$ $C_{w2} = 0.3$ $C_{w3} = 2$ $\sigma = 2/3$ $\chi = 0.41$

The one-equation model of Spalart & Allmaras has been used by Wong [487] to predict supersonic separated flows in an axisymmetric configuration that were not validated by the authors of the model [440]. According to Wong, the model is able to predict relaminarization but cannot be trusted to predict accurately the onset and duration of transition. Nevertheless, this corresponds to an improved situation as compared with the results obtained from an algebraic eddy-viscosity model (Baldwin-Lomax) which cannot predict relaminarization without *ad hoc* tuning of the constants.

10.10.2. TURBULENCE KINETIC ENERGY MODEL EQUATION

A widely adopted expression of the eddy-viscosity in incompressible turbulence reads:

$$\nu_t \propto \overline{\rho}\hat{u}\, l\,, \tag{10.51}$$

where $\hat{u}$ and l are characteristic velocity and length scales of the turbulent motion, respectively.

Assuming that l can be prescribed algebraically, the characteristic turbulence velocity is generally taken as $\hat{u} \propto \sqrt{\widetilde{k}}$ or $\sqrt{\overline{k}}$ (it is recalled that $\widetilde{k} \equiv \widetilde{u''_i u''_i}/2$ — resp. $\overline{k} \equiv \overline{u'_i u'_i}/2$ — denotes the kinematic turbulence kinetic energy, according to density weighted or classical averages). This function is governed by an exact transport equation which reads, when Favre's decomposition is adopted (see Chapter 5):

$$\underbrace{\overline{\rho}\left(\frac{\partial \widetilde{k}}{\partial t}+\widetilde{U}_j\frac{\partial \widetilde{k}}{\partial x_j}\right)}_{(a)} = \underbrace{-\overline{\rho}\widetilde{u''_i u''_j}\frac{\partial \widetilde{U}_i}{\partial x_j}}_{(b)} \underbrace{-\frac{\partial}{\partial x_j}\left[\frac{1}{2}\overline{(\rho u''_i u''_i u''_j)} + \overline{p' u''_j}\right]}_{(c)} \underbrace{-\overline{u''_i}\frac{\partial \overline{P}}{\partial x_i}}_{(d)} + \underbrace{\overline{p'\frac{\partial u''_i}{\partial x_i}}}_{(e)}$$

$$+\underbrace{\frac{\partial \overline{(\tau_{ij} u''_i)}}{\partial x_j}}_{(g)} - \underbrace{\overline{\tau_{ij}\frac{\partial u''_i}{\partial x_j}}}_{(h)}\,. \tag{10.52}$$

In eq.(10.52):

(a) is the mean material variation of the turbulence kinetic energy, based on a *transport* formulation. Making use of the mean continuity equation, it can be easily transformed into a *conservative* expression:

$$\frac{\partial \overline{\rho}\widetilde{k}}{\partial t} + \frac{\partial (\overline{\rho}\widetilde{k}\widetilde{U}_j)}{\partial x_j} \equiv \frac{\partial (\overline{\rho}\widetilde{k})}{\partial t} + \widetilde{U}_j \frac{\partial (\overline{\rho}\widetilde{k})}{\partial x_j} + \overline{\rho}\widetilde{k}\,\frac{\partial \widetilde{U}_j}{\partial x_j}\,;$$

(b) is the mean shear production;

(c) is the turbulent transport or diffusion of turbulence kinetic energy, including pressure effects;

(d) corresponds to an energy transfer by coupling of the turbulent mass flux with the mean pressure field. It will be briefly called the mean pressure work, and is specific to variable density flows *and* Favre formulation;

(e) is the pressure-dilatation correlation, which can also be written as $\overline{p'(\partial u'_i/\partial x_i)}$. It is specific to non-solenoidal velocity fluctuations.

The last two terms in eq.(10.52) are accounting for molecular viscosity effects, since τ_{ij} is the *instantaneous* viscous stress tensor, viz. for a Newton-Stokes behavior:

$$\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu\frac{\partial U_l}{\partial x_l}\delta_{ij} .$$

Thus:
(g) is the work done by external viscous forces in the fluctuating motion;
(h) is the Favre-averaged dissipation rate.

Now, as far as the *modeled* expression is concerned, the generic form of the turbulence kinetic energy equation can be taken as:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\overline{\rho}\widetilde{k}) = P^{(k)}_{rod} + D^{(k)}_{iff} - \overline{\epsilon}_s + A_\rho , \tag{10.53}$$

where $\mathrm{D}/\mathrm{D}t$ denotes the material derivative with reference to the mean flow.
In eq.(10.53), $P^{(k)}_{rod}$, $D^{(k)}_{iff}$, $\overline{\epsilon}_s$ are the production, diffusion and dissipation terms, the closure schemes of which are 'similar' to those developed for incompressible or constant density flows (direct extension from the incompressible situation). Hence, A_ρ accounts for all compressible or variable density contributions. Its modeling requires specific closure schemes being derived for:
- the turbulent mass flux;
- the pressure-dilatation correlation[10];
- the dilatation dissipation.

Some of the corresponding closure schemes, dedicated to the modeling of A_ρ, have been already discussed in the previous sections. Therefore, we are simply concerned here with closure schemes for the production, diffusion and solenoidal dissipation terms.

Production. When direct transpositions of the linear eddy-viscosity concept are adopted in variable density fluid motions, — eqs.(10.10) and (10.11) —, the production term is obtained explicitly:

$$P^{(k)}_{rod} = -\overline{\rho}\,\widetilde{u''_i u''_j}\frac{\partial \widetilde{U}_i}{\partial x_j} = 2\mu_t\widetilde{S}_{ij}\frac{\partial \widetilde{U}_i}{\partial x_j} - \frac{2}{3}\mu_t\left(\frac{\partial \widetilde{U}_l}{\partial x_l}\right)^2 - \frac{2}{3}\overline{\rho}\widetilde{k}\frac{\partial \widetilde{U}_l}{\partial x_l} .$$

[10]By regrouping all pressure contributions in eq.(10.52), only one pressure term is obtained, $u''_i\frac{\partial P}{\partial x_i}$. In Aupoix *et al.* [24], an exact transport equation is derived for this term, assuming a perfect gas equation of state. However, as pointed out by the authors, a term-by-term closure of this equation cannot be applied, since some of them go to infinity and only the difference remains finite. Thus the splitting of the pressure terms, as given in eq.(10.52) is generally the one adopted in the modeling process.

Diffusion. A widely adopted closure scheme for the diffusion term in eq.(10.52) is based on a generalized gradient-type expression:

$$D_{iff}^{(k)} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_k}\frac{\partial \widetilde{k}}{\partial x_j}\right),$$

where the turbulent transport coefficient (or diffusivity) of $\widetilde{k}$ is a model constant, taken as a turbulence Prandtl-Schmidt number σ_k.
This expression was introduced, for instance, in 1972 by Jones and Launder [238], in a $(k - \epsilon)$ model, with $\sigma_k = 1$.

Solenoidal dissipation. In the modeling of the energy transport equation, the solenoidal or incompressible contribution to the dissipation ($\overline{\epsilon}_s$) is simply deduced from $\widetilde{k}$ and l. Assuming that Batchelor's incompressible scaling still applies in variable density flows, the expression reads:

$$\overline{\epsilon}_s \propto \frac{\widetilde{k}^{3/2}}{l},$$

introducing only one model constant.

10.11. Two-equation models

In two-equation models for incompressible fluid motions, the characteristic length scale l in eq.(10.51) is obtained from any product of a given power of the turbulence kinetic energy with another transportable quantity of suitable dimension.
Some examples of such available combinations that have been used in variable density flows are reported in Table 10.1. Only three of them will be discussed here.

10.11.1. THE (κ - ϵ) MODEL

In 1979, Jones [237] suggested that the eddy-viscosity can be obtained from the turbulence kinetic energy and the dissipation rate, according to a direct extension of the usual incompressible relation (Jones & Launder [238], 1972). Using Favre averages, this scheme reads

$$\mu_t = C_\mu \overline{\rho}\frac{\widetilde{k}^2}{\overline{\epsilon}}, \tag{10.54}$$

where $C_\mu = 0.09$ is a model constant, and $\overline{\epsilon}$ the dissipation rate, obtained from a model transport equation.

As reviewed by Vandromme and Ha Minh [472], the common form of the transport equation for the dissipation rate in high Reynolds number compressible flows can be written as (see also Rubesin [398]):

$$\frac{\partial(\bar{\rho}\bar{\epsilon})}{\partial t} + \frac{\partial(\bar{\rho}\bar{\epsilon}\widetilde{U}_j)}{\partial x_j} = -C_{\epsilon 1}\frac{\bar{\epsilon}}{\bar{\tilde{k}}}\bar{\rho}\widetilde{u''_i u''_j}\frac{\partial \widetilde{U}_i}{\partial x_j} + D_\epsilon - C_{\epsilon 2}\bar{\rho}\frac{\bar{\epsilon}^2}{\bar{\tilde{k}}}$$

$$+\underbrace{C_{\epsilon 3}\frac{\bar{\epsilon}}{\bar{\tilde{k}}}\overline{p'\frac{\partial u''_i}{\partial x_i}}}_{(d)} - \underbrace{C_{\epsilon 4}\frac{\bar{\epsilon}}{\bar{\tilde{k}}}\overline{u''_i}\frac{\partial \bar{P}}{\partial x_i}}_{(e)} - \underbrace{C_{\epsilon 5}\bar{\rho}\bar{\epsilon}\frac{\partial \widetilde{U}_i}{\partial x_i}}_{(f)} \quad . \tag{10.55}$$

In the constant density situation, eq.(10.55) reduces to the first line, where the right-hand-side terms are the shear production, the diffusion and the dissipation. The second line is thus accounting for compressible effects. It is made of three additional terms: (d) and (e) are simply the counterparts for the pressure-dilatation and mean pressure contributions, already present in the turbulence kinetic energy equation. The last one (f) is introduced to accounting for the dependence of turbulence length scale on passing through a shock wave. The constant model is $C_{\epsilon 5} = 1/3$ in isotropic turbulence and unity otherwise.

When dilatation (compressible) dissipation schemes are introduced, this equation is adopted with adapted values of the constants to calculate the solenoidal part of the dissipation rate ($\bar{\epsilon}_s$). We shall now just give two examples.

Ha Minh *et al.* (1981-91). A simplified version of eq.(10.55) was intensively used by Ha Minh and co-workers [195], [199], [280], [198], [196], [471], [470], for the conventional dissipation rate, as part of (k - ϵ) model and second-order closure models to predict wall bounded flows, including shock-boundary layer interaction.
In this case, the low Reynolds number corrections of Jones-Launder [239] are adopted, in addition to the following assumptions, [195]:

- The gradient diffusion is simply $D_\epsilon = \frac{\partial}{\partial x_j}(\frac{\mu_t}{\sigma_\epsilon}\frac{\partial \bar{\epsilon}}{\partial x_j})$ with $\sigma_\epsilon = 1.3$;
- When present[11], compressibility correction is limited to the mean pressure gradient term, adopting a Rubesin-derived expression for the turbulent mass flux — eq.(10.30) —, with $C_{\epsilon 4} = 1$.

When applied to the shock-induced boundary layer separation, one of the most striking results found by Vandromme and Ha Minh (see [471], for instance), is the great sensitivity of the prediction to the value of $C_{\epsilon 1}$. In the transonic flow over a bump (Délery & Le Diuzet [130]), the "lambda"

[11] In [280], no compressibility terms are added.

shape of the shock wave is not observed with $C_{\epsilon 1} = 1.28$, while it is with $C_{\epsilon 1} = 1.57$. However, even in this case, the model is not able to produce the correct level of turbulence kinetic energy.

Taulbee-VanOsdol (1991). In Taulbee and VanOsdol [454], the usual (incompressible) form of eq.(10.55) is used to calculate the *solenoidal* dissipation, in addition to a closure model for the dilatation-dissipation. Coupled transport equations of the density variance and turbulent mass flux $(\overline{u_i''})$ are also included in the model.
For *2-D* thin shear layers, the modeled dissipation equation reads

$$\overline{\rho}\widetilde{U}\frac{\partial\overline{\epsilon}}{\partial x} + \overline{\rho}\widetilde{V}\frac{\partial\overline{\epsilon}}{\partial y} = \frac{\partial}{\partial y}(\frac{\mu_t}{\sigma_\epsilon}\frac{\partial\overline{\epsilon}}{\partial y}) + C_{\epsilon 1}\frac{\overline{\epsilon}}{\widetilde{k}}\mu_t(\frac{\partial\widetilde{U}}{\partial y})^2 - C_{\epsilon 2}\overline{\rho}\frac{\overline{\epsilon}^2}{\widetilde{k}} .$$

The "standard" values of the "incompressible" constants are used ($C_{\epsilon 1} = 1.44$, $C_{\epsilon 2} = 1.92$, $\sigma_\epsilon = 1.3$).
With this model, the prediction of the growth rate of a compressible mixing layer is in reasonable agreement with the experimental data.

Discussion. Most proposals extending incompressible versions of the (k - ϵ) model to variable density and/or compressible situations, consist in adding compressibility corrections to the baseline incompressible formulation of the model. Some of them have been reviewed in section 10.8. With such (k - ϵ) closure types, it is worth noticing that, from computational grounds, specific schemes accounting for variable density effects in the dissipation equation cannot be introduced without a similar counterpart into the turbulence kinetic energy equation and vice versa.
Although not discussed here, the same kind of variable-density or compressible extension can also be applied to other types of two-equation models, such as the (k - ω) model proposed by Wilcox & Traci [486], or the (k - ϕ) model by Cousteix et al. [108] However, in all cases, Batchelor's scaling $\overline{\epsilon} \propto \overline{k}^{3/2}/\ell$, is adopted, assuming that only *one* characteristic length scale $l \propto \ell$ is accounting for *both* turbulent transport and energy transfer by fluctuating motions.
As discussed in §10.5.3, such an assumption could be at least questionable in variable density and/or compressible turbulent flows.

10.12. Three-equation models

As it has been seen in the previous section, different modifications of the incompressible versions of two-equation models have been proposed, (i) including changes in the incompressible schemes, and (ii) introducing new additional terms, specific to density variations.

With the aim of modeling latter contributions, one can presume that it could be more suitable to account for variable density and compressibility effects by adding, at least, one more additional transportable function, yielding a three-equation model.

Several proposals have been made for such extra transport equations. In 1979, Chen and Chen [91] derived a modeled transport equation for the temperature variance in order to predict vertical buoyant jets. The pressure variance was introduced by Zeman [497] in 1991 for predicting homogeneous compressible shear flows, and revisited by Hamba [203] in 1999. Three-equation models (k - ϵ - $\overline{\rho'^2}$) using a density variance equation, have been developed in order to predict compressible mixing layers by Lejeune *et al.* [287], [286] in 1996 and 1997, Yoshizawa *et al.* [492] in 1997, and separated flows by Duranti & Pittaluga [137] in 2000. In Taulbee and VanOsdoll [454], the density variance is also obtained from a modeled transport equation, the closure of which is achieved with a model transport equation for the turbulent mass fluxes.

10.12.1. DENSITY VARIANCE EQUATION

By direct manipulation of the open set of equations, an exact transport equation can be derived for the density variance $\overline{\rho'^2}$ (see Chapter 4, eq.(4.32) and also Taulbee and VanOsdoll [454], for instance). Some models that have been derived for closing this equation are now presented.

Taulbee-VanOsdol (1991). The modeled form of the density variance transport equation proposed by Taulbee and VanOsdol [454] in *2-D* thin shear layers is

$$\widetilde{U}\frac{\partial\overline{\rho'^2}}{\partial x}+\widetilde{V}\frac{\partial\overline{\rho'^2}}{\partial y}=-2\overline{\rho'^2}\frac{\partial\widetilde{U}_j}{\partial x_j}+2\overline{\rho}\,\overline{v''}\frac{\partial\overline{\rho}}{\partial y}+\frac{\partial}{\partial y}(\frac{\mu_t}{\sigma_\rho}\frac{\partial}{\partial y}(\frac{\overline{\rho'^2}}{\overline{\rho}}))-C_\rho\frac{\overline{\epsilon}}{\widetilde{k}}\frac{\overline{\rho'^2}}{\rho}, \tag{10.56}$$

where the values of the constants $\sigma_\rho = 1$ and $C_\rho = 5.3$ are chosen by trial and error to get reasonable agreement with experimental data.
In reference [454], the compressible flat plate boundary layer over an adiabatic wall is calculated with the low-Reynolds version of the (k - ϵ) model of Chien [94], *without compressibility corrections.* The density fluctuations variance is obtained from eq.(10.56), where the spanwise component $\overline{v''}$ of the turbulent mass fluxis calculated from a modeled transport equation, based on closure schemes (10.35), (10.36) and (10.37). Thus k and ϵ equations can be solved independently from $\overline{\rho'^2}$, $\overline{u''}$ and $\overline{v''}$ equations.

The turbulence density intensity $(\overline{\rho'^2})^{1/2}/\overline{\rho}$ and the longitudinal turbulent mass flux normalized by the local mean velocity $\overline{u''}/\overline{U}$ are given in table 10.6. The predicted values are compared with the experimental data of

Kistler and the results obtained from Morkovin's strong Reynolds analogy (see Chapter 4, section 4.4.2):

$$I_\rho \equiv \frac{\sqrt{\overline{\rho'^2}}}{\overline{\rho}} = (\gamma - 1)M^2 \frac{\sqrt{\overline{u'^2}}}{\overline{U}} \quad \text{and} \quad \frac{\overline{u''}}{\overline{U}} = -(\gamma - 1)M^2 \frac{\overline{u'^2}}{\overline{U}^2} \,.$$

It is seen that the overall agreement between predictions and experiment is reasonable for the two values of the Mach number.
Finally, for this flow configuration, it should be added that according to the authors, the dominant production in the model equation (10.37), comes from the mean density gradient, so that a good modeling of the turbulent normal mass flux $\overline{v''}$ is crucial, and cannot be achieved with gradient type diffusion schemes.

TABLE 10.6. Comparison of turbulence density intensity and longitudinal turbulent mass flux in a flat plate boundary layer over an adiabatic wall, at $y/\delta = 0.5$, from Taulbee and VanOsdol [454].

	I_ρ		$\overline{u''}/\overline{U}$	
	$M_\infty = 1.7$	$M_\infty = 4.7$	$M_\infty = 1.7$	$M_\infty = 4.7$
Morkovin's Analogy	0.035	0.131	- 0.0016	- 0.0124
Model [454]	0.028	0.120	- 0.0008	- 0.0028
Kistler's Exp.	0.027	0.115	- 0.0011	- 0.0028

Lejeune-Kourta-Chassaing (1996). To predict high-speed turbulent mixing layers, up to a convective Mach number of 0.8, the following model transport equation for the density variance was derived by Lejeune *et al.* [287]:

$$\frac{D\overline{\rho'^2}}{Dt} = -\overline{\rho'^2}\frac{\partial \widetilde{U}_j}{\partial x_j} + 2\frac{\nu_t}{\sigma_\rho}\left(\frac{\partial \overline{\rho}}{\partial x_j}\right)^2 + \frac{\partial}{\partial x_j}\left(\mu_t \frac{\partial}{\partial x_j}\left(\frac{\overline{\rho'^2}}{\overline{\rho}}\right)\right) - 2\frac{\overline{\rho}^2}{\gamma \overline{P}}\overline{p'\frac{\partial u'_j}{\partial x_j}} \,,$$

where $\gamma = C_p/C_v$.
The scheme for the pressure-dilatation term is

$$\overline{p'\frac{\partial u'_j}{\partial x_j}} = \alpha \frac{\overline{\rho'^2}}{\overline{\rho}^2}\frac{\gamma \overline{P}}{M_t}\frac{\overline{\epsilon}}{\overline{k}} \,,$$

where M_t is the turbulence Mach number.
With $\alpha = -0.05$, $\sigma_\rho = 0.7$, the model predicts turbulence kinetic energy

profiles which are in good agreement with the measurements, but exhibits a too loose dependence of the mixing layer growth rate on the convective Mach number.

10.12.2. PRESSURE VARIANCE EQUATION

Zeman (1991). The phenomenological closure of the pressure variance equation proposed by Zeman [497] in homogeneous compressible shear flows is based on the heuristic argument that, in the absence of forcing, the pressure fluctuations (i) tend to relax to an equilibrium value depending on the turbulence Mach number, $p_e(M_t)$, and (ii) at a rate set by an acoustic timescale $\tau_a \propto L/a$ over an eddy of size L. Thus:

$$-2\overline{\rho}\, a^2\, \overline{p'\vartheta'} = \frac{D\overline{p'^2}}{Dt} = -\frac{\overline{p'^2} - p_e^2}{\tau_a} \,. \tag{10.57}$$

The acoustic time scale is taken as $\tau_a = 0.2\,\tau M_t$, where the turbulence time scale τ is related to vortical turbulence since $\tau = 2\widetilde{k}/\epsilon_s$, where ϵ_s denotes the solenoidal dissipation. Finally it is inferred from DNS data and theory that the equilibrium pressure variance can be taken as

$$p_e^2 = 2(\frac{M_t^2 + M_t^4}{1 + M_t^2 + M_t^4})\,\overline{\rho}^2\,\widetilde{k}\,a^2 \,.$$

As shown by Zeman and Coleman [499], the model mechanism of relaxation to equilibrium is supported by direct numerical simulation data of homogeneous shear turbulence at moderate shear rates.

Hamba (1999). Restricting the analysis to homogeneous flows, the pressure variance equation reduces to, Hamba [203]:

$$\frac{D\overline{p'^2}}{Dt} = -2\gamma\overline{P}\,\overline{p'\vartheta'} - \epsilon_p + Res\,, \tag{10.58}$$

where $\vartheta' = \partial u_i'/\partial x_i$, ϵ_p is the pressure-variance dissipation and Res stands for residual terms which can be considered as negligible.
At sufficiently high turbulence Mach number, the dissipation ϵ_p does not depend on the viscosity ν, and can be modeled as

$$\epsilon_p = C_{\epsilon p1}(\gamma - 1)\frac{\overline{p'^2}}{\overline{k}P_{rod}}\epsilon\,, \tag{10.59}$$

where $P_{rod} = -\overline{u_i'u_j'}\partial\overline{U}_i/\partial x_j$ is the production rate of turbulence kinetic energy. The value of the model parameter $C_{\epsilon p1}$, which depends on the

turbulence Mach number M_t, is $C_{\epsilon p1} = 1.3$ for $M_t = 0.3$.
Although the pressure-dilatation term $\overline{p'\vartheta'}$ may be negligible in the $\overline{k}$ equation, it plays an important (actually dominant) role in the $\overline{p'^2}$ equation. The scheme for the pressure-dilatation correlation is as given in eq.(10.44).

10.13. Closure models for buoyant flows

Many problems involving buoyant modification to turbulence occur in natural environment where, to the exception of fire flows situations, the density change is never more than a few percent of the mean value. The present part of the monograph is not dedicated to the modeling of such types of flow configurations. Readers with direct interest in buoyancy affected turbulent flows may refer to Launder [274] or Craft [109], *inter alia.*

The following example is merely given to illustrate the way body-force influence was included in early modeling of transport (k - ϵ) equations and scalar variance equations (see e.g. Gibson & Launder [181] in 1976, Chen & Chen [91] in 1979).

Chen and Chen (1979). These authors [91] adapted the standard incompressible version of the (k - ϵ) model to predict vertical buoyant jets. They derived a three-equation (k - ϵ- $\overline{\theta'^2}$) model, adding a transport equation for the temperature variance $\overline{\theta'^2}$. With the thin shear layer approximations, the main modifications to the standard (k - ϵ) closure concern the expressions of the significant Reynolds stresses and heat flux components, which become:

$$\text{Reynolds stresses :}\quad -\overline{u'v'} = \frac{1-c_0}{c_1}\frac{\overline{v^2}}{\overline{k}}\left(1+\frac{\overline{k}\,g\;\partial\overline{T}/\partial y}{c_h\,\overline{\epsilon}\,T_a\;\partial\overline{U}/\partial y}\right)\frac{\overline{k}^2}{\overline{\epsilon}}\frac{\partial\overline{U}}{\partial y}\,,$$

$$\overline{v'^2} = c_2\,\overline{k}\,,$$

$$\text{Turb. heat fluxes :}\quad -\overline{u'\theta'} = \frac{\overline{k}}{c_h\,\overline{\epsilon}}\left[\overline{u'v'}\frac{\partial\overline{T}}{\partial y}+\overline{u'\theta'}(1-c_{h1})\frac{\partial\overline{U}}{\partial y}+g(1-c_{h1})\frac{\overline{\theta'^2}}{T_a}\right],$$

$$-\overline{v'\theta'} = \frac{1}{c_h}\frac{\overline{v^2}}{\overline{k}}\frac{\overline{k}^2}{\overline{\epsilon}}\frac{\partial\overline{T}}{\partial y}\,.$$

The previous expressions are obtained from the exact transport equations of the second-order correlations $\overline{u_i'u_j'}$ and $\overline{u_i'\theta'}$ when neglecting convection and diffusion terms. T_a is the ambient temperature, g, the gravitational constant, and the model constants are $c_0 = 0.55$, $c_1 = 2.2$, $c_2 = 0.53$, $c_h = 3.2$ and $c_{h1} = 0.5$. The incompressible closure of $\overline{k}$ and $\overline{\epsilon}$ equations is unchanged, except for the production terms, in which the additional buoyant contributions are respectively added:

$$P^k_{rod} = -\overline{u'v'}\frac{\partial\overline{U}}{\partial y} + g\,\frac{\overline{u'\theta'}}{T_a}\qquad\text{and}\qquad P^\epsilon_{rod} = C_{\epsilon 1}\frac{\overline{\epsilon}}{\overline{k}}P^k_{rod}\,.$$

10.14. Final discussion and concluding remarks

Before coming to scientific conclusions, it is worth keeping in mind the following two general points:

- Turbulence modeling of variable density flows addresses a wide range of situations, due to the various origins of density variations. Scientific faithfulness should require the review to be exhaustive, which is not the case of the present one. Progress in turbulence modeling has developed in a way far different from a logical, gradual, systematic approach. Historical faithfulness results in an impressionistic picture, which could make turbulence modeling look like a black art for non-specialists, as could probably appear from the present review;
- This chapter is restricted to reviewing some of the aspects of the modeling of density variations and compressible effects in low and high speed flows in first-order closure models. This choice is not motivated on scientifically argued grounds and simply results from pedagogical considerations: second-order modeling is addressed in next chapter. Therefore, it would be unwise to conclude from this review that first-order is an appropriate closure level for capturing the compressible effects that are now documented.

Within the previous frame, the following scientific conclusions can be drawn:

• In low-speed, non-reactive fluid motions, density variations arise from changes in temperature and/or composition, which can produce high density-intensity levels ($\sqrt{\overline{\rho'^2}}/\overline{\rho}$) in turbulent flows. As far as the modeling of statistical averaged equations is concerned, one is faced with the specific closure of turbulent mass flux and, more generally, correlations with density fluctuation (d.f.c.). Such terms, which have no equivalence in constant density flows, are present even when using density-weighted averages. Free shear flows can be predicted by using direct extensions of closure schemes derived for incompressible and buoyant flows, provided a suitable closure for such d.f.c. is adopted. In free jets, it has been demonstrated that a gradient-type diffusion closure for d.f.c. is not necessarily appropriate, depending upon the signs of mean velocity and density gradients.

• In non-separated high-speed boundary layers, up to supersonic free stream Mach numbers $M_\infty < 3$ to 5, and channel flows, the effects of density and pressure fluctuations under adiabatic conditions at the wall are small. Hence, compressibility effects are mostly due to *mean* density and temperature variations, and mean velocity profiles can be recovered from that in incompressible turbulence, using the Van Driest transformation [220]. To some extent, this explains why direct extensions of incompressible closure schemes to that of density-weighted moments for variable density flows ("first" generation), do not yield irrelevant predictions, as demonstrated by Ha Minh and co-workers [195], [199], [198], [196].

• In eddy-viscosity models, structural changes of the turbulence field associated with modifications in the Reynolds stress anisotropy tensor cannot be introduced explicitly into closure schemes, as it is the case with second-order modeling (see next chapter). Nevertheless, some consequences of such dominant effects can be indirectly and subtly accounted for in first-order closure models. This is basically the reason for improving the non-equilibrium schemes derived for the pressure dilatation correlation, Ristorcelli [394], Hamba [203], as part of the closure of the pressure variance transport equation.

• Even within the limit of Kovasznay's modes decomposition, density fluctuations are included in both acoustic ($p' \neq 0$, $\rho' \neq 0$, and $s' = \omega' = 0$) and entropy ($s' \neq 0$, $\rho' \neq 0$, and $\omega' = p' = 0$) modes. With nonadiabatic walls, density fluctuations can exist with little compressible turbulence effects [221]. Therefore, for boundary layers and channel flows, the density variance is not a suitable choice of an independent function to accurately account for compressibility effects in a (k - ϵ) model, for instance.

• In high-speed free shear flows, direct extensions of incompressible models, similar to those adopted in predicting wall bounded flows, fail to predict the strong compressibility effects observed in such flows. In a "second" generation of new compressible models, specific dilatational terms (pressure-dilatation correlation, compressible or dilatation dissipation) were presumed to account for such effects.

• This statement was not confirmed by later literature on the topic. In 1995, Sarkar [409] showed that the reduced growth rate of turbulence kinetic energy in homogeneous shear flow was primarily due to the reduced level of turbulence production, as a consequence of a change in the anisotropy of the Reynolds stress due to compressibility. In 1996, Vreman *et al.* [477] showed that reduced pressure fluctuations are responsible for the reduction in the growth rate of mixing layers via the pressure-strain term. In 1999, Hamba [203] confirmed the conclusion that, in compressible homogeneous shear flow, the anisotropy of the Reynolds stress, which reduces the turbulence production, is primarily due to the decrease in the pressure-strain term $\Pi_{12}(=\overline{p'\partial u_1'/\partial x_2})$ which, in turn, results from the reduced level of pressure fluctuations.

Since dilatational corrections for pressure and dissipation terms in this second generation of compressible models were introduced in second-order closure models, Zeman [495], Sarkar and Lakshmanan [413], the observed improvements are hardly separable from those resulting from other modifications.

• The observations quoted in the previous item are based on DNS studies. They are not yet entirely confirmed by experimental evidence, due to technical difficulty in the measurements. For example, there is no clear indication of changes in the Reynolds stress anisotropy in [146] or [77], as opposed to [186].