5 Mechanical systems on Riemannian manifolds

5.1 The generalized Newton law

Let $(Q, \langle , \rangle)$ be a Riemannian manifold, $q = q(t)$ be a C^2-curve on Q and ∇ be the Levi-Civita connection associated to the given Riemannian metric $\langle , \rangle$. The acceleration of $q(t)$ is the covariant derivative of the velocity field $\dot{q} = \dot{q}(t)$, that is,

$$\textbf{acceleration of} \quad q(t) \stackrel{def}{=} \frac{D\dot{q}}{dt}. \tag{5.1}$$

If V is any (local) vector field extending $\dot{q} = \dot{q}(t)$, we also write, for simplicity, $\frac{D\dot{q}}{dt} = \nabla_{\dot{q}}\dot{q} = \nabla_{\dot{q}}V$. When $\dot{q}(t) \neq 0$, there exists such a V in a neighborhood of $q(t)$.

In local coordinates $(\Omega; q_1, \ldots, q_n)$ of Q, the functions $g_{ij} = \langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \rangle$ and the Γ_{ij}^k given by $\nabla_{\frac{\partial}{\partial q_j}} \frac{\partial}{\partial q_i} = \sum_{k=1}^n \Gamma_{ji}^k \frac{\partial}{\partial q_k}$, are well known C^1-functions on Ω and the expressions 3.20 give each Γ_{ij}^k as a function of the $g_{ij}(q_1, \ldots, q_n)$ and their derivatives, hence as a function of $q_1, \ldots, q_n$. If $(q_i, \dot{q}_i)$ are the corresponding natural coordinates of TQ on $\tau^{-1}(\Omega)$ (recall that $\tau : TQ \to Q$ is the natural projection), one can write:

$$\dot{q} = \sum_{i=1}^n \dot{q}_i \frac{\partial}{\partial q_i} \tag{5.2}$$

and so, we have along $q = q(t)$ (see 3.7):

$$\frac{D\dot{q}}{dt} = \sum_{k=1}^n \left[\ddot{q}_k + \sum_{i,j} \dot{q}_i \dot{q}_j \Gamma_{ij}^k \right] \frac{\partial}{\partial q_k} \tag{5.3}$$

The **kinetic energy** associated to the Riemannian metric $\langle , \rangle$ is the C^k-function $K : TQ \to \mathbb{R}$ given by $K(v_p) = \frac{1}{2}\langle v_p, v_p \rangle$.

As we will see in some examples, the masses appear in the definition of the metric $\langle , \rangle$; the **Legendre transformation** (see Appendix A) or **mass operator** μ is a diffeomorphism from TQ onto T^*Q,

$$\mu : TQ \to T^*Q \tag{5.4}$$

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given by $\mu(v_p)(.) = \langle v_p, . \rangle$ for all $v_p \in TQ$. TQ is also called the phase space of velocities and T^*Q is called the phase space of momenta. Since $\langle , \rangle_p$ is non degenerate, we see easily that μ takes the fiber T_pQ onto the fiber $T_p{}^*Q$ and μ identifies, diffeomorphically, TQ with T^*Q. A **field of (external) forces** is a C^1-differentiable map

$$\mathcal{F} : TQ \to T^*Q \tag{5.5}$$

that sends the fiber T_pQ into the fiber $T^*{}_pQ$, for all $p \in Q$.

We remark that, by definition, $\mathcal{F}$ is not necessarily surjective but sends fibers into fibers. When $\mathcal{F}(v_p)$ is constant (for all $p \in Q$ and $v_p \in T_pQ$) the field of forces is said to be **positional**. As an example of a positional field of forces one defines

$$\mathcal{F}_U(v_p) = -dU(p) \qquad \forall v_p \in T_pQ, p \in Q,$$

where $U : Q \to \mathbb{R}$, the **potential energy**, is a given C^2-differentiable function. In that case one says that $\mathcal{F}_U$ is a **conservative** field of forces. It is clear that $\mathcal{F}_U$ is a positional field of forces. The map $\mu^{-1} \circ \mathcal{F}_U : TQ \to TQ$ defines, in this case, a vector field $\mathcal{X}$ on the manifold Q:

$$\mathcal{X} : p \in Q \longmapsto \mu^{-1} \circ \mathcal{F}_U(v_p) \in T_pQ,$$

that does not depend on $v_p \in T_pQ$, but on U and $p \in Q$, only. In fact $\mathcal{X}$ is equal to $-grad\ \ U$ (**- gradient** of U); take $w_p \in T_pQ$ and so:

$$\begin{aligned}\langle \mathcal{X}(p), w_p \rangle &= \langle \mu^{-1}\mathcal{F}_U(v_p), w_p \rangle = \mu(\mu^{-1}\mathcal{F}_U(v_p))(w_p) \\ &= \mathcal{F}_U(v_p)(w_p) = -dU(p)(w_p), \quad \text{that is } \mathcal{X}(p) = -(grad\ \ U)(p).\end{aligned}$$

Exercise 5.1.1. Show that in local coordinates we have

$$\mu(\frac{D\dot{q}}{dt}) = \sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \right) dq_j. \tag{5.6}$$

A **mechanical system on a Riemannian manifold** $(Q, \langle , \rangle)$ is a triplet $(Q, \langle , \rangle, \mathcal{F})$ where $\mathcal{F}$ is an (external) field of forces. The manifold Q is said to be the **configuration space** and the corresponding **generalized Newton law** is the relation

$$\mu(\frac{D\dot{q}}{dt}) = \mathcal{F}(\dot{q}). \tag{5.7}$$

A **motion** $q = q(t)$ is a C^2-curve, with values on Q, that satisfies the Newton law (5.7). A **conservative mechanical system** is a triplet $(Q, \langle , \rangle, \mathcal{F} = -dU)$ where $U : Q \to \mathbb{R}$ is its potential energy. The function $E_m = K + U \circ \tau$ is the **mechanical energy**.

Proposition 5.1.2. *(Conservation of energy) In any conservative mechanical system* $(Q, <, >, -dU)$ *the mechanical energy* $E_m = K + U \circ \tau$ *is constant along a given motion* $q = q(t)$.

Proof:

$$\tfrac{d}{dt}E_m(\dot q) = \tfrac{d}{dt}\left[K(\dot q) + U\circ\tau(\dot q)\right] = \tfrac{d}{dt}\left[\tfrac{1}{2}\langle \dot q, \dot q\rangle + U(q)\right] =$$

$$= \langle (\tfrac{D\dot q}{dt}), \dot q\rangle + (dU(q))\dot q = \langle \mu^{-1}[-dU(q)], \dot q\rangle + (dU(q))\dot q$$

$$= -(dU(q))\dot q + (dU(q))\dot q = 0.$$

■

5.2 The Jacobi Riemannian metric

Let $(Q, \langle,\rangle, -dU)$ be a conservative mechanical system on a Riemannian manifold $(Q, \langle,\rangle)$ and U be a C^2-potential energy. Let $v_p \in TQ$ be a critical point of the mechanical energy $E_m = K + U\circ\tau : TQ \to \mathbb{R}$, that is, $dE_m(v_p) = 0$. In local coordinates we have $v_p = (q_i, \dot q_i)$ and $E_m(v_p) = \frac{1}{2}\sum_{ij} g_{ij}(p)\dot q_i \dot q_j + U(q_1(p), \ldots, q_n(p))$, so

$$dE_m(q_i, \dot q_i) = \sum_{k=1}^{n}\left[\frac{1}{2}\sum_{ij}\frac{\partial g_{ij}}{\partial q_k}\dot q_i \dot q_j + \frac{\partial U}{\partial q_k}\right] dq_k + \sum_{k=1}^{n}\left[\sum_i g_{ik}\dot q_i\right] d\dot q_k = 0$$

and that implies the following equations:

$$\sum_i g_{ik}\dot q_i = 0, \qquad k = 1, \ldots, n, \tag{5.8}$$

$$\left[\frac{1}{2}\sum_{ij}\frac{\partial g_{ij}}{\partial q_k}\dot q_i \dot q_j + \frac{\partial U}{\partial q_k}\right] = 0, \qquad k = 1, \ldots, n. \tag{5.9}$$

By (5.8) and (5.9), and since $det(g_{ij}) \neq 0$, $v_p \in TQ$ is a critical point of E_m if, and only if:

$$\dot q_i = 0, \quad i = 1, \ldots, n, \qquad \text{and} \quad \frac{\partial U}{\partial q_k}(p) = 0.$$

This means that v_p is a **critical point** of E_m if, and only if, $p \in Q$ is a critical point of U and $v_p = 0_p \in T_pQ$.

Let $h \in \mathbb{R}$ be a (not necessarily regular) value of the mechanical energy E_m with $E_m^{-1}(h) \neq \emptyset$ and consider the open set of Q:

$$Q_h = \{p \in Q \mid U(p) < h\}. \tag{5.10}$$

On the manifold Q_h one can define the so called **Jacobi metric** g_h **associated to** $\langle,\rangle$; for each $p \in Q_h$ define $g_h(p)$ by

$$g_h(p)(u_p, v_p) \stackrel{def}{=} 2(h - U(p))\langle u_p, v_p\rangle, \tag{5.11}$$

Since $(h - U(p)) > 0$ for $p \in Q_h$, one sees that g_h is a Riemannian metric on Q_h.

Proposition 5.2.1. *(Jacobi) The motions of a conservative mechanical system $(Q, \langle,\rangle, -dU)$ with mechanical energy h are, up to reparametrization, geodesics of the open manifold Q_h with the Jacobi metric associated to $\langle,\rangle$.*

Before proving the 5.2.1 one goes to show the following (see [54]):

Proposition 5.2.2. *Let $(Q, \langle,\rangle)$ be a Riemannian manifold, $\rho : Q \to \mathbb{R}$ to be a C^2 function and grad ρ denote a vector field on Q, the gradient corresponding to $\langle,\rangle$ of the function ρ. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections associated to $\langle,\rangle$ and $e^{2\rho}\langle,\rangle$, respectively. Then, for all $X, Y \in \mathcal{X}(Q)$ we have:*

$$\tilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y\rangle grad\rho \tag{5.12}$$

Proof: By the definition of $\tilde{\nabla}$ and making $\ll, \gg = e^{2\rho}\langle,\rangle$, formula (5.19) gives

$$\begin{aligned}2 \ll \tilde{\nabla}_X Y, Z \gg &= Y \ll X, Z \gg + X \ll Z, Y \gg - Z \ll X, Y \gg \\ &- \ll [Y, Z], X \gg - \ll [X, Z], Y \gg - \ll [Y, X], Z \gg .\end{aligned}$$

On the other hand we have

$$\begin{aligned}Y \ll X, Z \gg &= Y(e^{2\rho}\langle X, Z\rangle) = e^{2\rho} Y\langle X, Z\rangle + \langle X, Z\rangle Y(e^{2\rho}) = \\ &= e^{2\rho}[Y\langle X, Z\rangle + \langle X, Z\rangle Y(2\rho)],\end{aligned}$$

so,

$$\begin{aligned}2 \ll \tilde{\nabla}_X Y, Z \gg = e^{2\rho}\{&Y\langle X, Z\rangle + \langle X, Z\rangle Y(2\rho) + X\langle Z, Y\rangle + \\ &+ \langle Z, Y\rangle X(2\rho) - Z\langle X, Y\rangle - \langle X, Y\rangle Z(2\rho) \\ &- \langle [Y, Z], X\rangle - \langle [X, Z], Y\rangle - \langle [Y, X], Z\rangle\}.\end{aligned}$$

From (3.19) one obtains

$$\begin{aligned}2 \ll \tilde{\nabla}_X Y, Z \gg &= 2e^{2\rho} < \nabla_X Y, Z > + e^{2\rho}\{\langle X, Z\rangle Y(2\rho) \\ &+ \langle Z, Y\rangle X(2\rho) - \langle X, Y\rangle Z(2\rho)\} \\ &= 2 \ll \nabla_X Y, Z \gg + \ll X, Z \gg Y(2\rho) \\ &+ \ll Z, Y \gg X(2\rho) - \ll X, Y \gg Z(2\rho).\end{aligned}$$

Since $Y(2\rho) = 2Y(\rho) = 2d\rho(Y)$ we have

$$\begin{aligned}\ll \tilde{\nabla}_X Y, Z \gg &= \ll \nabla_X Y, Z \gg + \ll X, Z \gg d\rho(Y) \\ &+ \ll Z, Y \gg d\rho(X) - \ll X, Y \gg d\rho(Z).\end{aligned}$$

The definition of $grad\ \rho$ gives

$$d\rho(Z) = \langle grad\ \rho, Z\rangle$$

for all Z, thus

$$\begin{aligned}\langle\tilde{\nabla}_X Y, Z\rangle &= \langle\nabla_X Y, Z\rangle + \langle X, Z\rangle d\rho(Y) + \langle Z, Y\rangle d\rho(X)\\ &\quad - \langle X, Y\rangle\langle grad\ \rho, Z\rangle \quad \text{for all} \quad Z.\end{aligned}$$

So, one obtains (5.12). ∎

Proof: (of 5.2.1) One defines $\rho : Q_h \to \mathbb{R}$ by the equality $e^{2\rho} = 2(h - U)$ so $e^{2\rho}d\rho = -dU$ and then

$$e^{2\rho} grad\ \rho = -grad\ U \quad \text{with respect to} \quad \langle,\rangle, \tag{5.13}$$

that is

$$2(h - U)d\rho = -dU. \tag{5.14}$$

Let $\gamma = \gamma(t)$ be a motion of $(Q, \langle,\rangle, -dU)$ with mechanical energy h and contained in Q_h. By (5.7) we have

$$\nabla_{\dot\gamma}\dot\gamma = -(grad\ U)(\gamma(t)). \tag{5.15}$$

As

$$2K(\dot\gamma) = \langle\dot\gamma, \dot\gamma\rangle = 2(h - U(\gamma(t)) = e^{2\rho(\gamma(t))},$$

that implies $\dot\gamma(t) \neq 0$ for all t in the maximal interval of γ.

Using (5.12), (5.15), (5.13) and (5.14) one can write

$$\begin{aligned}\tilde{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\dot\gamma + 2d\rho(\dot\gamma)\dot\gamma - \langle\dot\gamma, \dot\gamma\rangle grad\ \rho\\ &= -(grad\ U)(\gamma(t)) + 2d\rho(\dot\gamma)\dot\gamma - e^{2\rho(\gamma(t))} grad\ \rho, \quad \text{so}\end{aligned}$$

$$\tilde{\nabla}_{\dot\gamma}\dot\gamma = 2d\rho(\dot\gamma)\dot\gamma. \tag{5.16}$$

Let s and $\tilde{s}$ be the arc lengths in $\langle,\rangle$ and $\ll,\gg$ respectively. Call $\mu(s) = \gamma(t(s))$ and $c(\tilde{s}) = \mu(s(\tilde{s}))$. So $c(\tilde{s}) = \gamma(t(s(\tilde{s})))$ and $c^{'}(\tilde{s}) = \frac{dc(\tilde{s})}{d\tilde{s}} = \dot\gamma(t(s(\tilde{s})))\frac{dt}{d\tilde{s}}(s(\tilde{s})) = \dot\gamma(t(s(\tilde{s}))).\frac{dt(s)}{ds}.\frac{ds(\tilde{s})}{d\tilde{s}}$. But

$$(\frac{dt(s)}{ds})^2 = (\frac{ds(t)}{dt})^{-2} = \langle\dot\gamma, \dot\gamma\rangle^{-1} = e^{-2\rho(\gamma(t(s)))}$$

and then

$$\frac{dt(s)}{ds} = e^{-\rho(\gamma(t(s)))}. \tag{5.17}$$

Analogously

$$\begin{aligned}\left(\frac{ds(\tilde{s})}{d\tilde{s}}\right)^2 &= \left(\frac{d\tilde{s}(s)}{ds}\right)^{-2} =\ll \mu'(s), \mu'(s) \gg^{-1}\\ &= \ll \dot{\gamma}(t(s))\frac{dt(s)}{ds}, \dot{\gamma}(t(s))\frac{dt(s)}{ds} \gg^{-1}\\ &= \left(\frac{dt(s)}{ds}\right)^{-2} \ll \dot{\gamma}(t(s)), \dot{\gamma}(t(s)) \gg^{-1}\end{aligned}$$

that gives

$$\begin{aligned}\left(\frac{ds(\tilde{s})}{d\tilde{s}}\right) \cdot \left(\frac{dt(s)}{ds}\right) &= \ll \dot{\gamma}(t(s)), \dot{\gamma}(t(s)) \gg^{-1/2}\\ &= e^{-\rho(\gamma(t(s)))}\langle \dot{\gamma}(t(s)), \dot{\gamma}(t(s))\rangle^{-1/2}\end{aligned}$$

then $(\frac{ds(\tilde{s})}{d\tilde{s}}).(\frac{dt(s)}{ds}) = e^{-2\rho(\gamma(t(s)))}$ and

$$c'(\tilde{s}) = \dot{\gamma}(t(s(\tilde{s}))).e^{-2\rho(\gamma(t(s)))}. \tag{5.18}$$

Now compute $\tilde{\nabla}_{c'(\tilde{s})}c'(\tilde{s})$ using (5.18) and obtain

$$\begin{aligned}\tilde{\nabla}_{c'}c' &= \tilde{\nabla}_{e^{-2\rho}\dot{\gamma}}\left(e^{-2\rho}\dot{\gamma}\right) = e^{-2\rho}\tilde{\nabla}_{\dot{\gamma}}\left(e^{-2\rho}\dot{\gamma}\right) = e^{-2\rho}[e^{-2\rho}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} + d(e^{-2\rho})(\dot{\gamma})]\dot{\gamma}\\ &= e^{-4\rho}[\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} - 2d\rho(\dot{\gamma})\dot{\gamma}];\end{aligned}$$

from (5.16) we get $\tilde{\nabla}_{c'}c' = 0$, so $c(\tilde{s}) = \gamma(t(s(\tilde{s})))$ is a geodesic in the Jacobi metric. ∎

5.3 Mechanical systems as second order vector fields

Let $(Q, \langle,\rangle, \mathcal{F})$ be a mechanical system on the Riemannian manifold $(Q, \langle,\rangle)$ and $q(t)$ a motion, that is, a solution of the generalized Newton law $(\frac{D\dot{q}}{dt}) = \mu^{-1}(\mathcal{F}(\dot{q}))$.

In local coordinates we have (see (5.3)):

$$\sum_{k=1}^{n}\left[\ddot{q}_k + \sum_{ij}\Gamma^k_{ij}\dot{q}_i\dot{q}_j\right]\frac{\partial}{\partial q_k} = \sum_{k=1}^{n} f_k(q,\dot{q})\frac{\partial}{\partial q_k}$$

where the $f_k(q,\dot{q})$ are the components of $\mu^{-1}(\mathcal{F}(\dot{q}))$, that is, the Newton law is locally equivalent to the 2nd order system of ordinary differential equations:

$$\ddot{q}_k = -\sum_{i,j}\Gamma^k_{ij}\dot{q}_i\dot{q}_j + f_k(q,\dot{q}), \quad k = 1, \ldots, n,$$

or, to the first order system of ordinary differential equations:

$$\begin{cases} \dot{q}_k = v_k \\ \dot{v}_k = -\sum_{i,j} \Gamma_{ij}^k(q) v_i v_j + f_k(q, \dot{q}), \end{cases} \tag{5.19}$$

$k = 1, \ldots, n$.

Using (5.6) we also have

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \sum_{k=1}^{n} g_{jk} f_k(q, \dot{q}), \quad j = 1, \ldots, n \tag{5.20}$$

which are called the **Lagrange equations** for the system (the free external forces case can be seen in Appendix A taking K as the Lagrangian function).

This way, in natural coordinates $(q, \dot{q}) = (q, v)$ of TQ we have, well defined, the vector-field

$$E : (q, v) \longmapsto ((q, v), (\dot{q}, \dot{v}))$$

where the $(\dot{q}, \dot{v})$ are given by (5.19). The map above is a vector field E on TQ,

$$E : v_p \in TQ \longmapsto E(v_p) \in T(TQ).$$

The tangent space TQ is called the **phase space** and the vector field E defined on TQ is said to be a **second order vector field** because the first equation (see (5.19)) is $\dot{q} = v$. This is equivalent to say that any trajectory of $E = E(v_p)$ is the derivative of its projection on Q. In the special case where $\mathcal{F} = 0$, the vector field E reduces to the geodesic flow S of $\langle , \rangle$, (see (4.21)), given locally by

$$S : (q, v) \longmapsto ((q, v), (v, \gamma))$$

where $\gamma = (\gamma_1, \ldots, \gamma_k)$ is given by $\gamma_k = -\sum_{i,j} \Gamma_{ij}^k v_i v_j$.

In order to write an explicit expression for $E = E(v_p)$, let us introduce the concept of **vertical lifting operator** . It is an operator denoted by C_{v_p} associated to an element $v_p \in T_pQ$. C_{v_p} is a map

$$C_{v_p} : T_pQ \longrightarrow T_{v_p}(TQ)$$

defined by

$$C_{v_p}(w_p) = \frac{d}{ds}(v_p + s w_p) \mid_{s=0} . \tag{5.21}$$

C_{v_p} takes $w_p \in T_pQ$ into a tangent vector of $T(TQ)$ at the point $v_p \in TQ$. This tangent vector $C_{v_p}(w_p)$ is **vertical**, that is, is tangent at the point v_p to the fiber T_pQ since the curve $s \mapsto v_p + s w_p$ passes through v_p at $s = 0$ and has values on T_pQ for all s. In local coordinates, if $v_p = (q_i, v_i)$ and $w_p = (q_i, w_i)$, we have

$$C_{v_p} : (q_i, w_i) \longmapsto ((q_i, v_i), (0, w_i))$$

because the curve $v_p + s w_p$ is given, in local coordinates by $v_p + s w_p = (q_i, v_i + s w_i)$ and its tangent vector at $s = 0$ is written as $((q_i, v_i), (0, w_i))$.

The map C_{v_p} is linear and injective so is an isomorphism of T_pQ onto its image

$$C_{v_p}(T_pQ) = T_{v_p}(\tau^{-1}(p)).$$

So, the vector field $E = E(v_p)$ is given, in local coordinates, by the expression

$$E(v_p) = E((q_i, v_i)) = ((q_i, v_i), (v_i, \gamma_i + f_i))$$

where $\gamma_i = -\sum_{r,s} \Gamma^i_{rs} v_r v_s$ and the (f_i) are defined by

$$\mu^{-1}(\mathcal{F}(v_p)) = \sum_{i=1}^{n} f_i \frac{\partial}{\partial q_i}(p).$$

Then

$$E(v_p) = ((q_i, v_i), (v_i, \gamma_i)) + ((q_i, v_i), (0, f_i)), \quad \text{or}$$

$$E(v_p) = S(v_p) + C_{v_p}(\mu^{-1}(\mathcal{F}(v_p))). \tag{5.22}$$

Proposition 5.3.1. *The second order vector field $E = E(v_p)$ defined on TQ and associated to the generalized Newton law of the mechanical system $(Q, \langle,\rangle, \mathcal{F})$ is given by the expression* (5.22) *where $S = S(v_p)$ is the geodesic flow of $\langle,\rangle$. The trajectories of E are the derivatives of the motions satisfying $\mu(\frac{D\dot{q}}{dt}) = \mathcal{F}(\dot{q})$. When $\mathcal{F}(v_p) = -dU(p)$, and h is a regular value of E_m, the manifold $E_m^{-1}(h)$ is invariant under the flow of the vector field $E = E(v_p)$.*

5.4 Mechanical systems with holonomic constraints

Let $\mathcal{F} : TQ \to T^*Q$ be a C^1-field of external forces acting on a Riemannian manifold $(Q, \langle,\rangle)$.

A **holonomic constraint** is a submanifold $N \subset Q$ such that $dim \ N < dim \ Q$. A C^2-curve $q : I \subset \mathbb{R} \to Q$ is said to be **compatible** with N if $q(t) \in N$ for all $t \in I$. In order to obtain motions compatible with N we have to introduce a **field of reactive forces** $\mathcal{R} : TN \longrightarrow T^*Q$ depending on $Q, \langle,\rangle, N$ and $\mathcal{F}$ only, and to consider the **generalized Newton law**

$$\mu(\frac{D\dot{q}}{dt}) = (\mathcal{F} + \mathcal{R})(\dot{q}). \tag{5.23}$$

The constraint N is said to be **perfect** (with respect to reactive forces) or to satisfy **d'Alembert principle** if, for a given $\mathcal{F}$, the field of reactive forces $\mathcal{R}$ is such that $\mu^{-1}\mathcal{R}(v_q)$ is orthogonal to T_qN for all $v_q \in TN$. Here orthogonality is understood with respect to $\langle,\rangle$, μ is the mass operator and ∇ is the Levi-Civita connection associated to the Riemannian structure $(Q, \langle,\rangle)$. Using the decomposition $v_q = {v_q}^T + {v_q}^\perp$ for all $q \in N$ and $v_q \in T_qQ$, that is

$$T_qQ = T_qN \oplus (T_qN)^\perp, \quad q \in N,$$

one obtains from (5.23), assuming $\dot{q} \neq 0$, the following relations:

$$(\nabla_{\dot{q}}\dot{q})^T - [\mu^{-1}(\mathcal{F}(\dot{q}))]^T = 0 \tag{5.24}$$

$$\mu^{-1}(\mathcal{R})(\dot{q}) = (\nabla_{\dot{q}}\dot{q})^{\perp} - [\mu^{-1}(\mathcal{F}(\dot{q}))]^{\perp}. \tag{5.25}$$

Denoting by D the Levi-Civita connection associated to the Riemannian metric $\ll,\gg$ induced by $\langle,\rangle$ on N, Exercise 5.4.1 shows that if N is perfect, the C^2 solution curves compatible with N are precisely the motions of the mechanical system (without constraints) $(N, \ll,\gg, \mathcal{F}_N)$ where $\mathcal{F}_N(v_q) = \mu_N[(\mu^{-1}\mathcal{F}(v_q))^T]$, $v_q \in T_qN$, μ_N being the mass operator of $(N, \ll,\gg)$.

In fact, since $D_{\dot{q}}\dot{q} = (\nabla_{\dot{q}}\dot{q})^T$ (by Exercise 5.4.1) one obtains from (5.24) that

$$\mu_N(D_{\dot{q}}\dot{q}) = \mathcal{F}_N(\dot{q}) = \mu_N([\mu^{-1}\mathcal{F}(\dot{q})]^T) \tag{5.26}$$

which is the generalized Newton law corresponding to $(N, \ll,\gg, \mathcal{F}_N)$.

Also, from (5.25) we see that

$$\mu^{-1}(\mathcal{R})(\dot{q}) = \nabla_{\dot{q}}\dot{q} - (\nabla_{\dot{q}}\dot{q})^T - [\mu^{-1}\mathcal{F}(\dot{q})]^{\perp},$$

that is,

$$\mu^{-1}(\mathcal{R})(\dot{q}) = \nabla_{\dot{q}}\dot{q} - D_{\dot{q}}\dot{q} - [\mu^{-1}\mathcal{F}(\dot{q})]^{\perp}. \tag{5.27}$$

If X, Y are local vector fields on N and $\bar{X}, \bar{Y}$ be local extensions to Q, we have

$$B(X, Y) = \nabla_{\bar{X}}\bar{Y} - D_XY \tag{5.28}$$

where B is bilinear and symmetric with $B(X, Y)(q)$ depending only on $X(q)$ and $Y(q)$; B is called the **second fundamental form of the embedding** $i : N \to Q$ (see [17]) So, from (5.27) and (5.28) we can write $\mu^{-1}(\mathcal{R})(\dot{q}) = B(\dot{q}, \dot{q}) - [\mu^{-1}\mathcal{F}(\dot{q})]^{\perp}$, suggesting that

$$\mathcal{R}(v_q) = \mu[B(v_q, v_q) - [\mu^{-1}\mathcal{F}(v_q)]^{\perp}] \in T_q^*Q \tag{5.29}$$

for all $q \in N$ and $v_q \in T_qN$. The last expression gives the way to compute the **reactive force** introduced in (5.23) when the constraint is perfect.

Using (5.6) for $\mu_N(D_{\dot{q}}\dot{q})$ with $\dot{q} \neq 0$, in local coordinates of N, and also (5.26), we obtain the so-called **Lagrange equations** for obtaining the motions compatible with the perfect constraints without computing the reaction force of the constraints.

Exercise 5.4.1. Let N be a submanifold of a Riemannian manifold $(Q, \langle,\rangle)$ with Levi-Civita connection ∇. For any pair of vector fields X, Y on N we define D_XY as the vector field on N that at the point $p \in N$ is equal to $(D_XY)(p) = [(\nabla_{\bar{X}}\bar{Y})(p)]^T$ where $\bar{X}, \bar{Y}$ are local vector fields that extend X and Y in a neighborhood of $p \in Q$, respectively, $[(\nabla_{\bar{X}}\bar{Y})(p)]^T$ being the orthogonal projection of $(\nabla_{\bar{X}}\bar{Y})(p)$ onto T_pN, under $\langle,\rangle$. Show that $(D_XY)(p)$ does not depend on the chosen extensions and that

$$D : \mathcal{X}(N) \times \mathcal{X}(N) \to \mathcal{X}(N)$$

has the properties of an affine connection. Verify also that D is symmetric and compatible with the pseudo-Riemannian metric $\ll, \gg$ induced by $\langle , \rangle$ on N. So, D is the Levi-Civita connection associated to the pseudo-Riemannian manifold $(N, \ll, \gg)$.

5.5 Some classical examples

The study of a system of particles with or without constraints starts, in classical analytical mechanics, with the consideration of a manifold of configurations Q endowed, in general, with two metrics, $(,)$ and $\langle , \rangle$; the first one is called the **spatial metric** and the second is the one corresponding to the kinetic energy that defines the mass operator $\mu : TQ \to T^*Q$. With the two metrics one introduces the **tensor** of inertia $I : \mathcal{X}(Q) \to \mathcal{X}(Q)$ characterized by the relation

$$(I(X), Z) = \langle X, Z \rangle \tag{5.30}$$

for all $X, Z \in \mathcal{X}(Q)$. It is clear that:

i) I is non degenerate with respect to $(,)$ so I^{-1} exists.
ii) I is symmetric with respect to $(,)$, since:

$$(I(X), Z) = \langle X, Z \rangle = \langle Z, X \rangle = (I(Z), X) = (X, I(Z)).$$

iii) I is symmetric with respect to $\langle , \rangle$. In fact,

$$(I(I(X)), Z) = \langle I(X), Z \rangle \quad \text{and}$$

$$(I(I(X)), Z) = (I(X), I(Z)) = (I(I(Z)), X) = \langle I(Z), X \rangle$$

iv) I^{-1} is symmetric with respect to $\langle , \rangle$ and $(,)$:

$$\langle I^{-1}(X), Z \rangle = (X, Z) = (X, I(I^{-1}(Z))) = (I(I^{-1}(Z)), X) = \langle I^{-1}(Z), X \rangle$$

and

$$\begin{aligned}(I^{-1}(X), Z) &= (I^{-1}(X), I(I^{-1}(Z))) = \langle I^{-1}(X), I^{-1}(Z) \rangle \\ &= (I(I^{-1}(X)), I^{-1}(Z)) = (X, I^{-1}(Z)).\end{aligned}$$

v) Assume $(,)$ and $\langle , \rangle$ are positive definite. Then I and I^{-1} are positive definite with respect to the metrics:

$$(I(X), X) = \langle X, X \rangle;$$

$$\langle I(X), X \rangle = \langle I(X), I^{-1}(I(X)) \rangle = (I(X), I(X));$$

$$\langle I^{-1}(X), X \rangle = (X, X);$$

$$(I^{-1}(X), X) = (I^{-1}(X), I(I^{-1}(X))) = \langle I^{-1}(X), I^{-1}(X) \rangle.$$

In the applications, the usual forces are given by a map $F : TQ \to TQ$ which is fiber preserving, that is, $F(T_pQ) \subset T_pQ$ for all $p \in Q$; the notion of **work** is introduced using the spatial metric. So, the **work of** $F(v_p)$ **along** w_p is defined as $(F(v_p), w_p)$. To obtain the external field of forces $\mathcal{F} : TQ \to T^*Q$ from F we write

$$\mathcal{F} \stackrel{def}{=} \mu I^{-1} F \tag{5.31}$$

and, then, the generalized Newton law can be written under one of the two equivalent forms:

$$\frac{D\dot{q}}{dt} = I^{-1}F(\dot{q}) \quad \text{ or } \quad I(\frac{D\dot{q}}{dt}) = F(\dot{q})$$

(In (5.31), as in the last formulae, I is considered as a fiber preserving map $I : TQ \longrightarrow TQ$.)

Example 5.5.1. **The system of n mass points**

Let k be a three dimensional oriented Euclidean vector space also considered as affine space associated to itself. A pair (q_i, m_i) such that $q_i \in k$ and $m_i > 0$ is said to be a **mass point** and m_i is the **mass** of point $q_i, i = 1, \ldots, n$. To give n mass points is to consider $q = (q_1, \ldots, q_n) \in k^n$ and $(m_1, \ldots, m_n) \in {\mathbb{R}_+}^n$.

Assume that at each point $q_i \in k$ acts an **external force** ${f_i}^{ext} = {f_i}^{ext}(q, \dot{q}) \in k$ and $(n-1)$ **internal forces** $f_{ij} \in k, j \in \{1, \ldots, n\}\backslash\{i\}$, due to the action of q_j on q_i. The laws, in classical mechanics, determining the motions $q_i(t)$ of the mass points (q_i, m_i) are the following:

I - **Newton laws:**

$$m_i \ddot{q}_i = f_i \stackrel{def}{=} ({f_i}^{ext} + \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij}), \qquad i = 1, \ldots, n.$$

II - **Principle of action and reaction:**

f_{ij} and $(q_i - q_j)$ are linearly dependent and $f_{ij} = -f_{ji}$.

The two laws above imply the following:

(a) $\sum_{i=1}^{n} m_i \ddot{q}_i = \sum_{i=1}^{n} {f_i}^{ext}$

(b) $\sum_{i=1}^{n} m_i \ddot{q}_i \times (q_i - c) = \sum_{i=1}^{n} {f_i}^{ext} \times (q_i - c)$ for any $c \in k$.
(here $\times$ means the usual vector product in k).

In fact, case (a) is trivial. Using Newton's law one proves case (b) under the hypothesis $c = 0$, provided that $\sum_{i,j} f_{ij} \times q_i = 0$; but since $f_{ij} \times (q_i - q_j) = 0$, we have

$$\sum_{i,j} f_{ij} \times q_i = \sum_{i,j} f_{ij} \times q_j = -\sum_{i,j} f_{ji} \times q_j = -\sum_{i,j} f_{ij} \times q_i = 0.$$

The case (b) for arbitrary $c \in k$ follows from case (a) and from case (b) with $c = 0$.

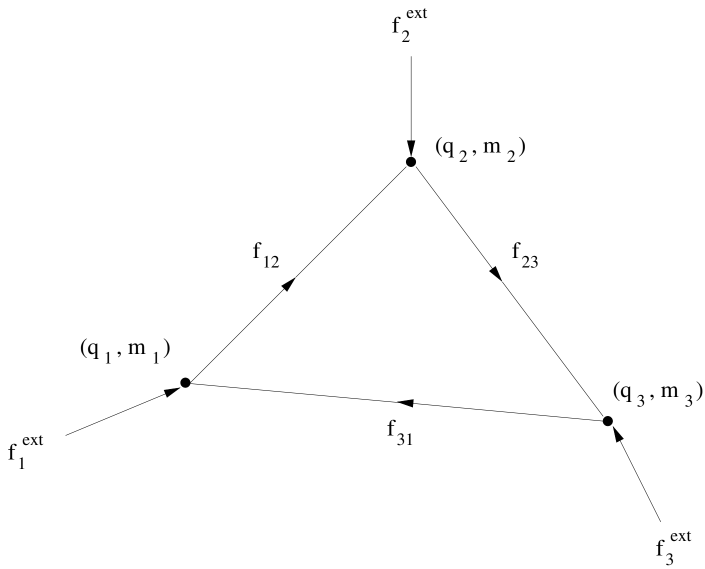

Fig. 5.1. System of $n = 3$ mass points.

The **kinetic energy of a motion** is $K = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{q}_i, \dot{q}_i)$ where $(,)$ is the inner product of k. The manifold $Q = k^n$ is the configuration space that can be endowed with two Riemannian metrics: $(u, v) = (u_1, v_1) + \ldots + (u_n, v_n)$, the **spatial metric**, and $\langle u, v\rangle = m_1(u_1, v_1) + \ldots + m_n(u_n, v_n)$, the metric corresponding to the kinetic energy, where the masses appear.

The Levi-Civita connection ∇ associated to $\langle , \rangle$ has the g_{ij} as constant functions, so the Christoffel symbols are all zero (see 3.2.6) and then

$$\frac{D\dot{q}}{dt} = \ddot{q} = (\ddot{q}_1, \ldots, \ddot{q}_n).$$

The mass operator $\mu : Tk^n \to T^*k^n$ is defined by $\mu(w_x)(.) = \langle w_x, .\rangle$ for all $w_x \in T_x k^n \cong k^n$. If the usual forces are given by $F : Tk^n \to Tk^n$ with $F = (f_1, \ldots, f_n)$, one defines $\mathcal{F} : Tk^n \to T^*k^n$, the field of external forces, using the formula $\mathcal{F} = \mu I^{-1} F$ where I is given by (5.30). Then one can write:

$$\begin{aligned}\mathcal{F}(v_x)u_x &= (\mu I^{-1}F)(v_x)u_x = \langle I^{-1}F(v_x), u_x\rangle \\ &= (I \circ I^{-1}F(v_x), u_x) = (F(v_x), u_x),\end{aligned}$$

so,

$$\mathcal{F}(v_x)u_x = \sum_{i=1}^{n}(f_i(v_x), {u_x}^i), \quad \text{where} \quad u_x = ({u_x}^1, \ldots, {u_x}^n). \tag{5.32}$$

Then $\mathcal{F}(v_x)u_x$ is the total work of the external forces $f_i(v_x)$ along ${u_x}^i$.

From the generalized Newton law (5.7) we have

$$\mathcal{F}(\dot{q})u_x = \mu(\frac{D\dot{q}}{dt})u_x = \mu(\ddot{q})u_x = \langle \ddot{q}, u_x \rangle = \sum_{i=1}^{n} (m_i \ddot{q}_i, {u_x}^i)$$

and (5.32) implies $\mathcal{F}(\dot{q})u_x = \sum_{i=1}^{n}(f_i(q,\dot{q}), {u_x}^i)$; so, since u_x is arbitrary in k^n one obtains the classical Newton's law:

$$m_i \ddot{q}_i = f_i(q, \dot{q}), \qquad i = 1, \ldots, n,$$

and conversely.

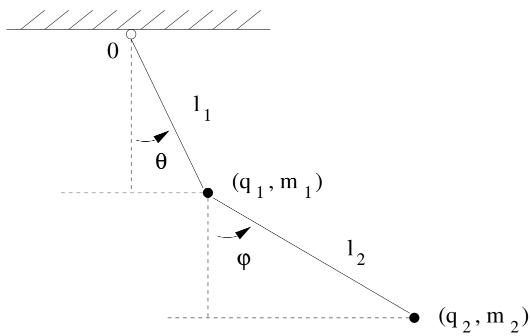

Fig. 5.2. Planar double pendulum.

Example 5.5.2. - **The planar double pendulum** One may consider two mass points (q_1, m_1) and (q_2, m_2), $q_i \in \mathbb{R}^2, i = 1, 2$, in the configuration space $Q = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ and a holonomic constraint N defined by the conditions:

$$| q_1 - 0 |^2 = {\ell_1}^2 \tag{5.33}$$
$$| q_2 - q_1 |^2 = {\ell_2}^2, \tag{5.34}$$

where $0 \in \mathbb{R}^2$ is the origin. If $a, b \in \mathbb{R}^2$, $a.b$ denotes the usual inner product of $\mathbb{R}^2$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ vectors in $\mathbb{R}^4$, that is, $u_i, v_i, \in \mathbb{R}^2, i = 1, 2$.

The spatial metric in $\mathbb{R}^4$ is given by

$$(u, v) = u_1.v_1 + u_2.v_2,$$

and

$$\langle u, v \rangle = m_1 u_1.v_1 + m_2 u_2.v_2$$

is the metric corresponding to the kinetic energy

$$K(\dot{q}) = \frac{1}{2}[m_1 \dot{q}_1.\dot{q}_1 + m_2 \dot{q}_2.\dot{q}_2], \quad \dot{q} = (\dot{q}_1, \dot{q}_2) \in \mathbb{R}^4.$$

The Levi-Civita connection ∇ associated to the metric $\langle , \rangle$ gives the acceleration of $q(t) = (q_1(t), q_2(t)) \in \mathbb{R}^4$ with Christoffel symbols equal to zero:

$$\frac{D\dot{q}}{dt} = \ddot{q} = (\ddot{q}_1, \ddot{q}_2). \tag{5.35}$$

The usual external forces acting on q_1 and q_2 are

$$F_1 = (0, m_1 g) \quad \text{and} \quad F_2 = (0, m_2 g),$$

respectively. As in the previous 5.5.1, one defines the field of external forces

$$\mathcal{F} : T(\mathbb{R}^2 \times \mathbb{R}^2) \to T^*(\mathbb{R}^2 \times \mathbb{R}^2)$$

using the total work of the physical external forces:

$$\mathcal{F}(\dot{q})(u_1, u_2) = (F_1(\dot{q}), u_1) + (F_2(\dot{q}), u_2) \tag{5.36}$$

where $F_i(\dot{q}) = F_i = (0, m_i g), \quad i = 1, 2.$

Assuming that the submanifold N defined by (5.33) and (5.34) is a perfect constraint, that is, satisfies the d'Alembert principle, we have by (5.23) that for any C^2 curve compatible with N,

$$\mathcal{R}(\dot{q}) = \mu(\frac{D\dot{q}}{dt}) - \mathcal{F}(\dot{q}), \quad \mathcal{R}(\dot{q}) \in T^*_{q(t)}Q,$$

is such that the vector $\mu^{-1}(\mathcal{R}(\dot{q}))$ is, at the point $q(t) \in N$, orthogonal to $T_{q(t)}N$ with respect to the metric $\langle , \rangle$, for all t; that is,

$$\langle \mu^{-1}\mathcal{R}(\dot{q}), (v_1, v_2) \rangle = 0 \tag{5.37}$$

for all $(v_1, v_2) \in T_{q(t)}N$. But $(v_1, v_2) \in T_{q(t)}N$ means that v_1 and v_2 in $\mathbb{R}^2$ have to satisfy:

$$v_1.(q_1 - 0) = 0 \tag{5.38}$$

$$(v_2 - v_1).(q_2 - q_1) = 0 \tag{5.39}$$

where (5.38) and (5.39) were obtained by differentiation, with respect to time, of (5.33) and (5.34), respectively. If one denotes

$$I\mu^{-1}\mathcal{R}(\dot{q}) \overset{def}{=} (R_1(\dot{q}), R_2(\dot{q})), \tag{5.40}$$

condition (5.37) and the definitions (5.30) and (5.40) give

$$\begin{aligned} 0 &= \langle \mu^{-1}\mathcal{R}(\dot{q}), (v_1, v_2) \rangle = (I\mu^{-1}\mathcal{R}(\dot{q}), (v_1, v_2)) \\ &= ((R_1(\dot{q}), R_2(\dot{q})), (v_1, v_2)) = (R_1(\dot{q})).v_1 + (R_2(\dot{q})).v_2 \end{aligned}$$

so, $R_1(\dot{q})$ and $R_2(\dot{q})$ defined in (5.40) satisfy

$$(R_1(\dot{q})).v_1 + (R_2(\dot{q})).v_2 = 0 \tag{5.41}$$

for all v_1, v_2 in $\mathbb{R}^2$ that verify (5.38) and (5.39).

From (5.35), and the definition of μ we obtain

$$\begin{aligned}\mu(\frac{D\dot{q}}{dt})(u_1, u_2) &= \langle \frac{D\dot{q}}{dt}, (u_1, u_2)\rangle = \langle(\ddot{q}_1, \ddot{q}_2), (u_1, u_2)\rangle \\ &= m_1\ddot{q}_1.u_1 + m_2\ddot{q}_2.u_2. \end{aligned} \tag{5.42}$$

From (5.23), (5.36), (5.40) and (5.42) we have

$$\begin{aligned} m_1\ddot{q}_1.u_1 + m_2\ddot{q}_2.u_2 &= (F_1(\dot{q})).u_1 + (F_2(\dot{q})).u_2 + \mathcal{R}(\dot{q})(u_1, u_2) \\ &= (F_1(\dot{q})).u_1 + (F_2(\dot{q})).u_2 + (R_1(\dot{q})).u_1 + (R_2(\dot{q})).u_2; \end{aligned}$$

in fact,

$$\begin{aligned} \mathcal{R}(\dot{q})(u_1, u_2) &= \mu I^{-1}(R_1(\dot{q}), R_2(\dot{q}))(u_1, u_2) \\ &= \langle I^{-1}(R_1(\dot{q}), R_2(\dot{q})), (u_1, u_2)\rangle \\ &= ((R_1(\dot{q}), R_2(\dot{q})), (u_1, u_2)) \\ &= (R_1(\dot{q})).u_1 + (R_2(\dot{q})).u_2 \quad , \end{aligned}$$

and then

$$m_1\ddot{q}_1.u_1 + m_2\ddot{q}_i u_2 = (F_1(\dot{q}) + R_1(\dot{q})).u_1 + (F_2(\dot{q}) + R_2(\dot{q})).u_2 \quad ;$$

since $(u_1, u_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ is arbitrary (see (5.24)) we have

$$\begin{aligned} m_1\ddot{q}_1 &= F_1(\dot{q}) + R_1(\dot{q}) \\ m_2\ddot{q}_2 &= F_2(\dot{q}) + R_2(\dot{q}). \end{aligned} \tag{5.43}$$

Equations (5.43) are the classical Newton law for two mass points; $R_1(\dot{q}), R_2(\dot{q})$ are the constraint's reactions that have to satisfy (5.41) for all (v_1, v_2) such that (5.38) and (5.39) hold, that is, "the virtual work of the reactive forces is equal to zero (classical d'Alembert principle)".

One can also show that (5.41) for all (v_1, v_2), under the hypotheses that (5.38) and (5.39) hold, is equivalent to

$$\begin{aligned} R_2(\dot{q}) &= \rho(q_2 - q_1) \\ R_1(\dot{q}) + R_2(\dot{q}) &= \alpha(q_1 - 0), \qquad (\rho, \alpha \in \mathbb{R}). \end{aligned}$$

Let us derive now the **Lagrange equations** (5.20) corresponding to the generalized Newton law (5.26) for the planar double pendulum. From (5.36) the field of external forces is given by

$$\mathcal{F}(\dot{q})(u_1, u_2) = (F_1(\dot{q}), u_1) + (F_2(\dot{q}), u_2) = (m_1 u_1^y + m_2 u_2^y)g$$

provided that $u_1 = (u_1^x, u_1^y)$ and $u_2 = (u_2^x, u_2^y)$.
The function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$U(q_1, q_2) = -m_1 g y_1 - m_2 g y_2,$$

where $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$, are such that $\mathcal{F}(v_p) = -dU(p), v_p \in T_p\mathbb{R}^4$. So, $\mathcal{F}$ is a conservative field of forces. The manifold N is a torus with coordinates (φ, θ), so, the potential energy U and the kinetic energy K restricted to N are $\bar{U}$ and $\bar{K}$ respectively:

$$\begin{aligned} \bar{U} &= -m_1 g \ell_1 \cos\theta - m_2 g(\ell_1 \cos\theta + \ell_2 \cos\varphi) \\ \bar{K} &= \frac{1}{2}[m_1(\dot{q}_1, \dot{q}_1) + m_2(\dot{q}_2, \dot{q}_2)] = \frac{1}{2}\sum_{i=1}^{2} m_i(\dot{x}_i^2 + \dot{y}_i^2) \end{aligned}$$

where $\dot{q}_1 = (\dot{x}_1, \dot{y}_1)$ and $\dot{q}_2 = (\dot{x}_2, \dot{y}_2)$ for $x_1 = \ell_1 \sin\theta$, $y_1 = \ell_1 \cos\theta$, $x_2 = \ell_1 \sin\theta + \ell_2 \sin\varphi$, $y_2 = \ell_1 \cos\theta + \ell_2 \cos\varphi$. Then $\dot{x}_1 = \ell_1\dot{\theta}\cos\theta$, $\dot{y}_1 = -\ell_1\dot{\theta}\sin\theta$, $\dot{x}_2 = \ell_1\dot{\theta}\cos\theta + \ell_2\dot{\varphi}\cos\varphi$, $\dot{y}_2 = -\ell_1\dot{\theta}\sin\theta - \ell_2\dot{\varphi}\sin\varphi$ and consequently:

$$\frac{\partial \bar{U}}{\partial \theta} = (m_1 + m_2)g\ell_1 \sin\theta, \qquad \frac{\partial \bar{U}}{\partial \varphi} = m_2 g \ell_2 \sin\varphi;$$

$$\begin{aligned} \frac{\partial \bar{K}}{\partial \theta} &= m_1\dot{x}_1\frac{\partial \dot{x}_1}{\partial \theta} + m_1\dot{y}_1\frac{\partial \dot{y}_1}{\partial \theta} + m_2\dot{x}_2\frac{\partial \dot{x}_2}{\partial \theta} + m_2\dot{y}_2\frac{\partial \dot{y}_2}{\partial \theta} \\ &= m_1\ell_1\dot{\theta}\cos\theta(-\ell_1\dot{\theta}\sin\theta) + m_1\ell_1\dot{\theta}\sin\theta(\ell_1\dot{\theta}\cos\theta) \\ &+ m_2(\ell_1\dot{\theta}\cos\theta + \ell_2\dot{\varphi}\cos\varphi)(-\ell_1\dot{\theta}\sin\theta) \\ &+ m_2(\ell_1\dot{\theta}\sin\theta + \ell_2\dot{\varphi}\sin\varphi)\ell_1\dot{\theta}\cos\theta, \end{aligned}$$

i.e.,

$$\frac{\partial \bar{K}}{\partial \theta} = m_2\ell_1\ell_2\dot{\varphi}\dot{\theta}\sin(\varphi - \theta);$$

$$\begin{aligned} \frac{\partial \bar{K}}{\partial \varphi} &= m_2\dot{x}_2\frac{\partial \dot{x}_2}{\partial \varphi} + m_2\dot{y}_2\frac{\partial \dot{y}_2}{\partial \varphi} \\ &= m_2(\ell_1\dot{\theta}\cos\theta + \ell_2\dot{\varphi}\cos\varphi)(-\ell_2\dot{\varphi}\sin\varphi) \\ &+ m_2(\ell_1\dot{\theta}\sin\theta + \ell_2\dot{\varphi}\sin\varphi)\ell_2\dot{\varphi}\cos\varphi, \end{aligned}$$

i.e.,

$$\frac{\partial \bar{K}}{\partial \varphi} = m_2\ell_1\ell_2\dot{\varphi}\dot{\theta}\sin(\theta - \varphi);$$

$$\frac{\partial \bar{K}}{\partial \dot{\theta}} = m_1 {\ell_1}^2 \dot{\theta} \cos^2 \theta + m_1 {\ell_1}^2 \dot{\theta} \sin^2 \theta + m_2 (\ell_1 \dot{\theta} \cos \theta + \ell_2 \dot{\varphi} \cos \varphi) \ell_1 \cos \theta$$
$$+ m_2 (\ell_1 \dot{\theta} \sin \theta + \ell \dot{\varphi} \sin \varphi) \ell_1 \sin \theta,$$

i.e.,

$$\frac{\partial \bar{K}}{\partial \dot{\theta}} = m_1 {\ell_1}^2 \dot{\theta} + m_2 \ell_1^2 \dot{\theta} + m_2 \ell_1 \ell_2 \dot{\varphi} \cos(\theta - \varphi);$$

$$\frac{\partial \bar{K}}{\partial \dot{\varphi}} = m_2 (\ell_1 \dot{\theta} \cos \theta + \ell_2 \dot{\varphi} \cos \varphi) \ell_2 \cos \varphi$$
$$+ m_2 (\ell_1 \dot{\theta} \sin \theta + \ell_2 \dot{\varphi} \sin \varphi) \ell_2 \sin \varphi,$$

i.e.,

$$\frac{\partial \bar{K}}{\partial \dot{\varphi}} = m_2 {\ell_2}^2 \dot{\varphi} + m_2 \ell_1 \ell_2 \dot{\theta} \cos(\theta - \varphi).$$

The two Lagrange's equations are

$$\frac{d}{dt} \frac{\partial \bar{K}}{\partial \dot{\theta}} - \frac{\partial \bar{K}}{\partial \theta} = - \frac{\partial \bar{U}}{\partial \theta}, \qquad \frac{d}{dt} \frac{\partial \bar{K}}{\partial \dot{\varphi}} - \frac{\partial \bar{K}}{\partial \varphi} = - \frac{\partial \bar{U}}{\partial \varphi},$$

i.e.

$$\frac{d}{dt} [m_1 {\ell_1}^2 \dot{\theta} + m_2 {\ell_1}^2 \dot{\theta} + m_2 \ell_1 \ell_2 \dot{\varphi} \cos(\theta - \varphi)] - m_2 \ell_1 \ell_2 \dot{\varphi} \dot{\theta} \sin(\theta - \varphi)$$
$$= -(m_1 + m_2) g \ell_1 \sin \theta$$

$$\frac{d}{dt} [m_2 {\ell_2}^2 \dot{\varphi} + m_2 \ell_1 \ell_2 \dot{\theta} \cos(\theta - \varphi)] - m_2 \ell_1 \ell_2 \dot{\varphi} \dot{\theta} \sin(\theta - \varphi)$$
$$= -m_2 g \ell_2 \sin \varphi.$$

.

These two equations determine a second order system of ordinary differential equations on the torus of coordinates (θ, φ):

$$\begin{aligned} (m_1 + m_2) \ell_1^2 \ddot{\theta} &+ m_2 \ell_1 \ell_2 [\ddot{\varphi} \cos(\theta - \varphi) - \dot{\varphi}(\dot{\theta} - \dot{\varphi}) \sin(\theta - \varphi)] - \\ &- m_2 \ell_1 \ell_2 \dot{\varphi} \dot{\theta} \sin(\theta - \varphi) + \\ &+ (m_1 + m_2) g \ell_1 \sin \theta = 0, \end{aligned} \tag{5.44}$$

$$\begin{aligned} m_2 \ell_2^2 \ddot{\varphi} &+ m_2 \ell_1 \ell_2 [\ddot{\theta} \cos(\theta - \varphi) - \dot{\theta}(\dot{\theta} - \dot{\varphi}) \sin(\theta - \varphi)] - \\ &- m_2 \ell_1 \ell_2 \dot{\varphi} \dot{\theta} \sin(\theta - \varphi) + m_2 g \ell_2 \sin \varphi = 0. \end{aligned} \tag{5.45}$$

One can compute $\ddot{\theta}$ and $\ddot{\varphi}$ in (5.44) and (5.45) and get a system of two ordinary differential equations in the normal form; in fact the matrix

$$\begin{bmatrix} (m_1+m_2)\ell_1^2 & m_2\ell_1\ell_2\cos(\theta-\varphi) \\ \\ m_2\ell_1\ell_2\cos(\theta-\varphi) & m_2\ell_2^2 \end{bmatrix}$$

is positive definite, with determinant equal to

$$m_1 m_2 \ell_1^2 \ell_2^2 + m_2^2 \ell_1^2 \ell_2^2 \sin^2(\theta-\varphi) > 0.$$

The mechanical energy $E_m = \bar{K} + \bar{U}$ is a first integral of system (5.44), (5.45) (see 5.1.2) expressed as:

$$E_m = \frac{1}{2}(m_1+m_2)\ell_1^2\dot{\theta}^2 + \frac{1}{2}m_2\ell_2^2\dot{\varphi}^2 + m_2\ell_1\ell_2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi) - \\ - (m_1+m_2)g\ell_1\cos\theta - m_2 g\ell_2\cos\varphi.$$

The critical points are the zero vectors $0_p \in T_pN$ such that $d\bar{U}(p) = 0$, that is, $\frac{\partial \bar{U}}{\partial \theta}(p) = \frac{\partial \bar{U}}{\partial \varphi}(p) = 0$, or, equivalently, $p = (\theta, \varphi)$ such that $\sin\theta = \sin\varphi = 0$; so, one has 4 critical configurations on the torus N:

$$p_1 = (0,0), \quad p_2 = (0,\pi), \quad p_3 = (-\pi, 0) \text{ and } p_4 = (\pi,\pi).$$

5.6 The dynamics of rigid bodies

Let K and k be two oriented Euclidean vector spaces also considered as affine spaces associated to K and k, respectively. Assume that both spaces have dimension 3 so, each one has well defined the vector product operation (denoted by $\times$) corresponding to the inner product $(,)$.

An **isometry** $M : K \to k$ is a distance preserving map, that is, $\|X-Y\| = \|MX - MY\|$ for all $X, Y \in K$. The induced map $M^* : K \to k$ is defined by: ($0 \in K$ is the zero vector)

$$M^*X = M(X) - M(0), \quad \text{for all} \quad X \in K \tag{5.46}$$

Proposition 5.6.1. *Let M^* be the induced map of an isometry M. Then one has the following:*

1. *M^* is modulus preserving.*
2. *M^* preserves inner products and is linear.*
3. *M^* is a bijection, so M is an affine (bijective) transformation.*
4. *The inverse of M is an isometry.*
5. *If M^* is orientation preserving then M^* preserves vector product.*

Proof:

1. $\|M^*X\| = \|M(X) - M(0)\| = \|X - 0\| = \|X\|$.
2. One has

$$(M^*X, M^*Y) = \frac{1}{2}(\|M^*X\|^2 + \|M^*Y\|^2 - \|M^*X - M^*Y\|^2)$$
$$= \frac{1}{2}(\|X\|^2 + \|Y\|^2 - \|X - Y\|^2) = (X, Y).$$

So, M^* preserves inner product. Moreover M^* is linear: for any $\alpha \in \mathbb{R}$ and $X \in K$ we have

$$\begin{aligned} &\|M^*(\alpha X) - \alpha M^*X\|^2 \\ &= \|M^*(\alpha X)\|^2 + \alpha^2\|M^*X\|^2 - 2(M^*(\alpha x), \alpha M^*X) \\ &= \|\alpha X\|^2 + \alpha^2\|X\|^2 - 2\alpha(M^*(\alpha X), M^*X) \\ &= 2\alpha^2\|X\|^2 - 2\alpha(\alpha X, X) = 0; \end{aligned}$$

and

$$\begin{aligned} &\|M^*(X - Y) - (M^*X - M^*Y)\|^2 \\ &= \|M^*(X - Y)\|^2 + \|M^*X - M^*Y\|^2 - 2(M^*(X - Y), M^*X - M^*Y) \\ &= \|X - Y\|^2 + \|X - Y\|^2 - 2(X - Y, X) + 2(X - Y, Y) = 0. \end{aligned}$$

3. Since M^* is linear, it is enough to prove that M^* is an injection; but if $M^*X = 0$ ($0 \in k$ is the zero vector) one has $\|M^*X\| = \|X\| = 0$, so $X = 0$ and M^* has an inverse $(M^*)^{-1}$.
4. The map $N : k \to K$ defined by

$$N(x) = (M^*)^{-1}(x - M(0)) \quad \text{for all} \quad x \in k, \tag{5.47}$$

is the inverse of M since by (5.46) and (5.47) we have:

$$M(N(x)) = M(0) + M^*(N(x)) = M(0) + (x - M(0)) = x$$

But (5.47) gives $N(0) = -(M^*)^{-1}(M(0))$, so,

$$N(x) = (M^*)^{-1}x - (M^*)^{-1}(M(0)) = N(0) + (M^*)^{-1}x \tag{5.48}$$

and N is an isometry with $N^* = (M^*)^{-1}$ as induced map. In fact (5.48) shows that $N^* = (M^*)^{-1}$ and (5.47) implies:

$$\begin{aligned} \|N(x) - N(y)\| &= \|(M^*)^{-1}x - (M^*)^{-1}y\| \\ &= \|M^*(M^*)^{-1}x - M^*(M^*)^{-1}y\| = \|x - y\|, \end{aligned}$$

so N preserves distances.

■

Exercise 5.6.2. Prove property 5. in Proposition 5.6.1.

An isometry $M : K \to k$ is said to be a **proper isometry** if its induced map $M^* : K \to k$ is orientation preserving.

A **rigid motion of** K **relative to** k is a C^2 curve

$$M : t \longmapsto M_t$$

where M_t is a proper isometry. If, moreover, $M_t(0) = 0$ for all t, then M is said to be a **rotation**.

Proposition 5.6.3. *Any rigid motion M of K relative to k is such that M_t has a unique decomposition $M_t = T_t \circ R_t$ where $R_t = {M_t}^* : K \to k$ defines a rotation and $T_t : k \to k$ is given by $T_t x = x + r(t)$, that is, T_t is a translation in k, for each t.*

Proof: From (5.46) we have:

$$\begin{aligned} M_t(X) &= {M_t}^* X + M_t(0) = R_t X + M_t(0) \\ &= T_t(R_t X) = (T_t \circ R_t)X \end{aligned}$$

where $T_t(x) \overset{def}{=} x + r(t)$ for all $x \in k$, $\quad r(t) \overset{def}{=} M_t(0)$. If $M_t = \bar{T}_t \circ \bar{R}_t$ is another decomposition such that $\bar{T}_t(x) = x + \bar{r}(t)$ for all $x \in k$ and $\bar{R}_t 0 = o$ then $\bar{T}_t(\bar{R}_t X) = T_t({M_t}^* X)$ or $\bar{R}_t X + \bar{r}(t) = M_t^* X + r(t)$ for all $X \in K$; in particular for $X = 0$ one gets $r(t) = \bar{r}(t)$ and consequently $\bar{R}_t = {M_t}^*$. ∎

A rigid motion M is said to be **translational** if in the (unique) decomposition $M_t = T_t \circ {M_t}^*$, the linear isometry ${M_t}^*$ does not depend on t, that is, ${M_t}^* = {M_{t_o}}^*$ for some t_o. In that case we have $M_t(X) = {M_{t_o}}^* X + r(t)$.

We will derive now, the expression that describes the kinematics of a rigid motion M of a (**moving**) system K with respect to a (**stationary**) system k, that is, for t in some interval I of the real line, $M_t : K \to k$ is the corresponding proper isometry. Let us denote by $Q(t) \in K$ a moving C^2 radius vector also defined in I and let $q(t) = M_t(Q(t))$ be the radius vector, in k, corresponding to the action of M_t on the moving point $Q(t)$. Let us denote by $r(t) \in k$ the vector $r(t) = M_t(0)$.

Taking into account that $M_t(X) = {M_t}^* X + M_t(0)$ for all $X \in K$ one obtains:

$$q(t) = M_t(Q(t)) = M_t^* Q(t) + r(t). \tag{5.49}$$

By differentiating (5.49) with respect to time one has

$$\dot{q}(t) = \dot{M}_t^* Q(t) + M_t^* \dot{Q}(t) + \dot{r}(t). \tag{5.50}$$

Special cases:

a) If the rigid motion M is **translational**, that is, ${M_t}^* = {M_{t_o}}^*$ for all t, one obtains from (5.50) that

$$\dot{q}(t) = M_{t_o}{}^*\dot{Q}(t) + \dot{r}(t) \tag{5.51}$$

and so, the **absolute velocity** $\dot{q}(t)$ is equal to the sum of the **relative velocity** $M_{to}{}^*\dot{Q}(t)$ with the **velocity** $\dot{r}(t)$ (of the origin 0) **of the moving system** K.

b) If the rigid motion M is a **rotation** of the moving system K with respect to the stationary system k, that is, if $r(t) = 0$ for all t, one obtains from (5.49):

$$q(t) = M_t{}^*Q(t) \quad \text{and} \quad \dot{q}(t) = \dot{M_t}^*Q(t) + M_t{}^*\dot{Q}(t). \tag{5.52}$$

If, moreover, $Q(t) = \xi =$ constant, (5.52) shows that

$$q(t) = M_t{}^*\xi \qquad \text{for all} \quad t \tag{5.53}$$

and the motion of $q(t)$ is called a **transferred rotation of** ξ.

Exercise 5.6.4. Assume it is given a skew-symmetric linear operator $A : V \to V$ acting on an oriented 3-dimensional Euclidean vector space V. Prove that there exists a unique vector $\omega \in V$ such that $Ay = \omega \times y$ for all $y \in V$, and also that $\omega = 0$ if and only if $A = 0$. We use to denote simply $A = \omega\times$.

Let us consider the induced linear map $M_t{}^*$ associated to a rigid motion $M : t \to M_t$ of K with respect to k. One can construct two linear operators (with C^1 dependence on time):

$$\dot{M}_t^*(M_t{}^*)^{-1} : k \to k \qquad \text{and} \qquad (M_t{}^*)^{-1}\dot{M}_t^* : K \to K.$$

From Proposition 5.6.1 (2. and 3.) $M_t{}^*$ is a linear isometry:

$$(M_t^*X, M_t^*Y) = (X, Y), \quad \text{for all} \quad X, Y \in K. \tag{5.54}$$

By differentiating (5.54) with respect to time we obtain

$$(\dot{M}_t^*X, M_t{}^*Y) + (M_t{}^*X, \dot{M}_t^*Y) = 0, \quad \text{for all} \quad X, Y \in K. \tag{5.55}$$

Since $(M_t{}^*)^{-1}$ is also a linear isometry one gets from (5.55) that

$$((M_t^*)^{-1}\dot{M}_t^*X, Y) + (X, (M_t^*)^{-1}\dot{M}_t^*Y)) = 0, \quad \text{for all} \quad X, Y \in K \tag{5.56}$$

and also

$$(\dot{M}_t^*(M_t{}^*)^{-1}x, y) + (x, \dot{M}_t^*(M_t{}^*)^{-1}y) = 0 \quad \text{for all} \quad x, y \in k, \tag{5.57}$$

where $x = M_t^*X$ and $y = M_t{}^*Y$ are arbitrary in k. Then (5.56) and (5.57) show that $(M_t^*)^{-1}\dot{M}_t^*$ and $\dot{M}_t^*(M_t^*)^{-1}$ are skew-symmetric linear operators acting on K and k, respectively. Using the result of Exercise 5.6.4 above one can state the following:

Proposition 5.6.5. *Let $M : t \to M_t$ be a rigid motion of K with respect to k and ${M_t}^*$ its induced linear isometry. Then there exist unique vectors $\Omega(t) \in K$ and $\omega(t) \in k$ such that $({M_t}^*)^{-1}\dot{M}_t^* = \Omega(t)\times$ and $\dot{M}_t^*({M_t}^*)^{-1} = \omega(t)\times$. Moreover $\omega(t) = {M_t}^*\Omega(t)$.*

Proof: We only need to prove that $\omega(t) = {M_t}^*\Omega(t)$. But from the definition of $\Omega(t)$ we know that

$$({M_t}^*)^{-1}\dot{M}_t^* Y = \Omega(t) \times Y \quad \text{for all} \quad Y \in K;$$

so, making $Y = ({M_t}^*)^{-1}y$, one obtains

$$({M_t}^*)^{-1}\dot{M}_t^*({M_t}^*)^{-1}y = \Omega(t) \times ({M_t}^*)^{-1}y,$$

and then

$$\dot{M}_t^*(M_t)^{-1}y = {M_t}^*[\Omega(t) \times ({M_t}^*)^{-1}y].$$

The last expression and Proposition 5.6.1 (5.) show that

$$\dot{M}_t^* {M_t}^{-1}y = [{M_t}^*\Omega(t)] \times y \quad \text{for all} \quad y \in k,$$

thus the definition and the uniqueness of $\omega(t)$ enable us to conclude the result. ∎

We will now give the interpretation of $\omega(t)$ and $\Omega(t)$ when we are dealing with the special cases considered above. We start with a rotation M ($r(t) = 0$ for all t) such that $Q(t) = \xi =$ constant, that is, the motion of $q(t)$ is a transferred rotation of $\xi \in K$. We have the following result:

Proposition 5.6.6. *If $q(t)$ is a transferred rotation of ξ, to each time t for which $\dot{M}_t^* \neq 0$ there corresponds an axis of rotation, that is, a line in k through the origin whose points have zero velocity at that time. Each point out of the axis of rotation has velocity orthogonal to the axis with the modulus proportional to the distance from the point to the mentioned axis; if, otherwise, we have $\dot{M}_t^* = 0$, then all the points in k have zero velocity at this time t.*

Proof: By (5.53) we have

$$\dot{q}(t) = \dot{M}_t^*\xi. \tag{5.58}$$

If $\dot{M}_t^* = 0$, (5.58) shows that $\dot{q}(t) = 0$. Assume otherwise $\dot{M}_t^* \neq 0$; in this last case (5.53) and (5.58) imply that

$$\dot{q}(t) = \dot{M}_t^*({M_t}^*)^{-1}q(t). \tag{5.59}$$

One sees that the skew-symmetric linear operator $\dot{M}_t^*({M_t}^*)^{-1} : k \to k$ is non zero: in fact $\dot{M}_t^*({M_t}^*)^{-1} = 0$ implies $\dot{M}_t^* = 0$ (contradiction). From Proposition 5.6.5 there exists a unique non zero vector $\omega(t) \in k$ such that

$$\dot{M}_t^*(M_t{}^*)^{-1} = \omega(t)\times; \tag{5.60}$$

then equations (5.59) and (5.60) imply that

$$\dot{q}(t) = \omega(t) \times q(t). \tag{5.61}$$

The **instantaneous axis of rotation** at the time t is the line in k through the origin and direction $\rho\omega(t), \rho \in \mathbb{R}$, and (5.61) shows that $|\dot{q}(t)| = |\omega(t)|\ |q(t)| \sin\theta$ where $|q(t)|\ \sin\theta$ is the distance from $q(t)$ to the axis of rotation . ■

Another case to be considered is a general rotation ($r(t) = 0$ for all t); so equations (5.52) imply

$$\dot{q}(t) = \dot{M}_t^*(M_t{}^*)^{-1}q(t) + M_t{}^*\dot{Q}(t) \tag{5.62}$$

and using Proposition 5.6.5 there exists a unique $\omega(t) \in k$ so that equation (5.62) can be written

$$\dot{q}(t) = \omega(t) \times q(t) + M_t^*\dot{Q}(t). \tag{5.63}$$

So, for a rotation M, the **absolute velocity** $\dot{q}(t)$ is equal to the sum of the **relative velocity** $M_t{}^*\dot{Q}(t)$ and the **transferred velocity of rotation** $\omega(t) \times q(t)$.

The **dynamics of mass points in a non-inertial frame** can be studied by assuming that k is an inertial and that K is a non-inertial coordinate system subjected to a rigid motion $M : t \to M_t$. From (5.50) we know that $\dot{q}(t) = \dot{M}_t^*Q(t) + M_t{}^*\dot{Q}(t) + \dot{r}(t)$. Let us suppose also that the motion of the point $q \in k$ with mass $m > 0$ satisfies the Newton's equation

$$m\ddot{q} = f(q, \dot{q}); \tag{5.64}$$

so we have:

$$f(q, \dot{q}) = m\ddot{q} = m[\ddot{M}_t^*Q(t) + 2\dot{M}_t^*\dot{Q}(t) + M_t{}^*\ddot{Q}(t) + \ddot{r}(t)]. \tag{5.65}$$

The special case in which M is translational ($M_t^* = M_{t_o}^* =$ constant) implies that

$$mM_{t_o}^*\ddot{Q}(t) = m(\ddot{q} - \ddot{r}) = f(q, \dot{q}) - m\ddot{r}(t)$$

or

$$m\ddot{Q}(t) = (M_{t_o}{}^*)^{-1}f(q, \dot{q}) - (M_{t_o}{}^*)^{-1}m\ddot{r}(t).$$

The case in which M is a rotation ($r(t) = 0$ for all t) gives from (5.65):

$$m\ddot{Q}(t) = (M_t{}^*)^{-1}[f(q, \dot{q}) - m\ddot{M}_t^*Q(t) - 2m\dot{M}_t^*\dot{Q}(t)],$$

so

$$m\ddot{Q}(t) = (M_t{}^*)^{-1}f(q, \dot{q}) - 2m\Omega(t) \times \dot{Q}(t) - m(M_t{}^*)^{-1}\ddot{M}_t^*Q(t). \tag{5.66}$$

From the definition of $\Omega(t)$ we have

$$(M_t{}^*)^{-1}\dot{M}_t^* Y = \Omega(t) \times Y \qquad \text{or}$$

$$\dot{M}_t^* Y = M_t{}^*(\Omega(t) \times Y) \quad \text{for all} \quad Y \in K; \tag{5.67}$$

The derivative of (5.67) gives

$$\ddot{M}_t^* Y = \dot{M}_t^*(\Omega(t) \times Y) + M_t{}^*(\dot{\Omega}(t) \times Y)$$

and so,

$$(M_t{}^*)^{-1}\ddot{M}_t^* Y = \Omega(t) \times (\Omega(t) \times Y) + \dot{\Omega}(t) \times Y$$

for all $Y \in K$ and, in particular, for $Y = Q(t)$, that is,

$$(M_t{}^*)^{-1}\ddot{M}_t^* Q(t) = \Omega(t) \times (\Omega(t) \times Q(t)) + \dot{\Omega}(t) \times Q(t)$$

and this last equality can be introduced in (5.66) giving, after setting $(M_t{}^*)^{-1} f(q, \dot{q}) = F(t, q, \dot{q})$:

$$\begin{aligned} m\ddot{Q}(t) = &- m\Omega(t) \times (\Omega(t) \times Q(t)) - 2m\Omega(t) \times \dot{Q}(t) \\ &- m\dot{\Omega}(t) \times Q(t) + F(t, q, \dot{q}) \end{aligned}$$

where one calls
$F_1 = -m\dot{\Omega}(t) \times Q(t)$: **the inertial force of rotation**,
$F_2 = -2m\Omega(t) \times \dot{Q}(t)$: **the Coriolis force**,
$F_3 = -m\Omega(t) \times (\Omega(t) \times Q(t))$: **the centrifugal force**.

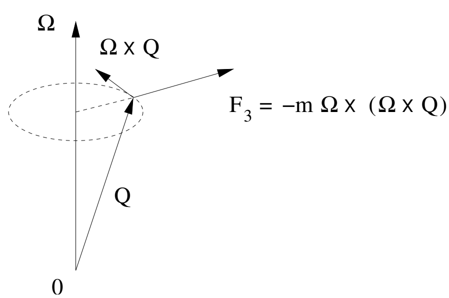

Fig. 5.3. Centrifugal force.

Thus one can state the following:

Proposition 5.6.7. *The motion in a (non inertial) rotating coordinate system takes place as if three additional inertial forces (the inertial force of rotation F_1, the Coriolis force F_2 and the centrifugal force F_3) together with the external force $F(t, q, \dot{q}) = (M_t{}^*)^{-1} f(q, \dot{q})$ acted on every moving point $Q(t)$ of mass m.*

For the purposes of giving a mathematical definition of a rigid body, we start by saying that a **body** is a bounded borelian set $S \subset K$, and a **rigid body** $S \subset K$ is a bounded connected Borel set $S \subset K$ such that during the action of any rigid motion $M : t \mapsto M_t$ of K relative to k, the points $\xi \in S$ do not move, that is

$$Q(t, \xi) = \xi \quad \text{for any} \quad t \quad \text{and any} \quad \xi \in S. \tag{5.68}$$

The distribution of the masses on S will be considered in the sequel. Without loss of generality one assumes, from now on, that the origin O of K belongs to S.

A rigid motion M of K relative to k induces, by restriction, a motion of S relative to k, and, when S is a rigid body, we have from (5.49) and (5.68):

$$q(t, \xi) = M_t(Q(t, \xi)) = M_t(\xi) = {M_t}^* \xi + r(t) \tag{5.69}$$

for any t and any $\xi \in S$.

If a rigid motion is a rotation ($r(t) \equiv 0$), its action on the rigid body S is given, from (5.69), by the equation

$$q(t, \xi) = {M_t}^* \xi, \quad \text{for all} \quad \xi \in S, \tag{5.70}$$

that is, by a transferred rotation of each $\xi \in S$; so, a rotation acting on a rigid body S is said to be a **motion of** S **with a fixed point**, the origin $0 \in K$, since $r(t) = M_t(0) = 0$. At each instant t, either the image $M_t(S)$ of S has an instantaneous axis of rotation passing through $0 \in k$, the points $q(t, \xi) \in M_t(S)$ with velocities $\omega(t) \times q(t, \xi)$, or all the points of $M_t(S)$ have zero velocity, according what states Proposition 5.6.6 above.

If M is translational (${M_t}^* = {M_{t_o}}^*$ for all t), its action on a rigid body S is given, from (5.69) by the equation

$$q(t, \xi) = {M_{t_o}}^* \xi + r(t) = {M_{t_o}}^* \xi + M_t(0)$$

so, $\dot{q}(t, \xi) = \dot{r}(t)$, that is, the velocity of any point of $M_t(S)$ is equal to the velocity $\dot{r}(t)$ of $M_t(0)$.

We will introduce now the notions of **mass, center of mass, kinetic energy** and **kinetic or angular momentum** of a rigid body S.

A **distribution of mass** on a rigid body S is defined through a **positive scalar measure** m on K; the following hypothesis is often used:

$$m(U) > 0 \textit{ for all nonempty open subset } U \textit{ of } S. \tag{5.71}$$

(Here we are considering the induced topology; in particular $m(S) > 0$ if $S \neq \emptyset$).

The **center of mass of** S corresponding to a distribution of mass m is the point $G \in K$ given by

$$G = \frac{1}{m(S)} \int_S \xi dm(\xi) \tag{5.72}$$

where $m(S)$ is the **total mass** of the rigid body S which is a positive number (see the fundamental hypothesis).

Under the action of a rigid motion $t \to M_t$, the center of mass describes a curve in k given by:

$$g(t) \stackrel{def}{=} M_t(G) = \frac{1}{m(S)} \int_S M_t \xi dm(\xi) = \frac{1}{m(S)} \int_S q(t, \xi) dm(\xi) \tag{5.73}$$

Proposition 5.6.8. *The velocity $\dot{q}(t, \xi)$ of a point ξ of a given rigid body S under the action of a rigid motion $t \to M_t$ is given by*

$$\dot{q}(t, \xi) = \dot{g}(t) + \omega(t) \times [q(t, \xi) - g(t)]$$

where $\omega(t) \times = \dot{M}_t^* ({M_t}^*)^{-1}$.

Proof: By (5.68) and (5.69) we have for all $\xi \in K$:

$$q(t, \xi) = {M_t}^* \xi + r(t) \quad \text{and} \quad \xi = ({M_t}^*)^{-1} [q(t, \xi) - r(t)];$$

so, by derivative one obtains:

$$\begin{aligned} &\dot{q}(t, \xi) = \dot{M}_t^* \xi + \dot{r}(t) = \dot{M}_t^* ({M_t}^*)^{-1} [q(t, \xi) - r(t)] + \dot{r}(t) \quad \text{or} \\ &\dot{q}(t, \xi) = \omega(t) \times [q(t, \xi) - r(t)] + \dot{r}(t), \quad \text{for all} \quad \xi \in K. \end{aligned} \tag{5.74}$$

Choosing $\xi = G$ we get

$$\dot{g}(t) = \omega(t) \times [g(t) - r(t)] + \dot{r}(t); \tag{5.75}$$

then (5.74) and (5.75) prove the result. ∎

The **kinetic energy** of the motion of a rigid body S at a certain time t is, by definition,

$$K^c(t) = \frac{1}{2} \int_S \mid \dot{q}(t, \xi) \mid^2 dm(\xi) \tag{5.76}$$

The vectors $\omega(t)$ and $\Omega(t) = (M_t^*)^{-1} \omega(t)$ characterized by the equalities $\dot{M}_t^* (M_t^*)^{-1} = \omega(t) \times$ and $(M_t^*)^{-1} \dot{M}_t^* = \Omega(t) \times$ are called the **instantaneous angular velocities relative to** k and K, respectively.

The **angular momentum relative to** k of the motion of S at a certain time t is the vector

$$p(t) = \int_S [q(t, \xi) \times \dot{q}(t, \xi)] dm(\xi) \tag{5.77}$$

and the **angular momentum relative to the body** is

$$P(t) = (M_t{}^*)^{-1} p(t) \tag{5.78}$$

Special case: rigid body with a fixed point.
In this case $r(t) = 0$ for all t and then:

$$q(t,\xi) = M_t^* \xi, \qquad \dot{q}(t,\xi) = \omega(t) \times q(t,\xi);$$

$$K^c(t) = \frac{1}{2}\int_S |\, \omega(t) \times q(t,\xi)|^2 dm(\xi) = \frac{1}{2}\int_S |\Omega(t) \times \xi|^2 dm(\xi)$$

$$p(t) = \int_S [M_t^* \xi \times (\omega(t) \times M_t^* \xi)] dm(\xi);$$

$$P(t) = \int_S [\xi \times (\Omega(t) \times \xi)] dm(\xi). \tag{5.79}$$

The last expression (5.79) suggests how to give a definition for the **inertia operator of a rigid body** S:

$$A : X \in K \longmapsto [\int_S \xi \times (X \times \xi) dm(\xi)] \in K. \tag{5.80}$$

Proposition 5.6.9. *The inertia operator A of a rigid body $S \subset K$ is symmetric and positive with respect to the inner product of K. If, moreover, S has at least two points whose radii vectors are linearly independent and the distribution of mass satisfies (5.71), then A is positive definite.*

Proof:

$$(AX, Y) = (Y, \int_S \xi \times (X \times \xi) dm(\xi)) = \int_S (Y, \xi \times (X \times \xi)) dm(\xi)$$

and then

$$(AX, Y) = \int_S (X \times \xi, Y \times \xi) dm(\xi) = (X, AY), \tag{5.81}$$

so A is symmetric. Assume now that $(AY, Y) = \int_S |Y \times \xi|^2 dm(\xi) = 0$. This implies that the set $E = \{\xi \in S | |Y \times \xi| \neq 0\}$ has measure $m(E) = 0$. On the other hand, if there exist $a, b \in S$ linearly independent then there exist neighborhoods U_a, U_b in K of a and b, such that v_1, v_2 are linearly independent for all $v_1 \in U_a$ and $v_2 \in U_b$. From the hypothesis on the measure m we have $m(U_a \cap S) > 0$ and $m(U_b \cap S) > 0$; so, there exist $u \in U_a \cap S$ and $v \in U_b \cap S$ such that $u, v \notin E$, that is, $|Y \times u| = |Y \times v| = 0$; since u and v are linearly independent, $Y = 0$, that is, A is positive definite. ∎

If we come back to the special case of the motion of a rigid body S with a fixed point $O \in K$, we have from (5.79):

$$P(t) = A\Omega(t)$$
$$K^c(t) = \frac{1}{2}(A\Omega(t), \Omega(t)). \tag{5.82}$$

In fact,

$$\begin{aligned}
K^c(t) &= \frac{1}{2}\int_S |\Omega(t) \times \xi|^2 dm(\xi) = \frac{1}{2}\int_S (\Omega(t), \xi \times (\Omega(t) \times \xi))dm(\xi) \\
&= \frac{1}{2}(\Omega(t), \int_S \xi \times (\Omega(t) \times \xi)dm(\xi)) \\
&= \frac{1}{2}(\Omega(t), A\Omega(t)).
\end{aligned}$$

Another remark on the inertia operator A is the following: since A is linear and symmetric, there exists an orthonormal basis (E_1, E_2, E_3) in K where E_i is an eigenvector of a (real) eigenvalue I_i of A; since A is positive, $I_i \geq 0, i = 1, 2, 3$. If $\Omega(t) = \Omega_1(t)E_1 + \Omega_2(t)E_2 + \Omega_3(t)E_3$ we have

$$K^c(t) = \frac{1}{2}(I_1{\Omega_1}^2(t) + I_2{\Omega_2}^2(t) + I_3{\Omega_3}^2(t)).$$

Since $AE_i = I_iE_i, i = 1, 2, 3$, and because we had assumed, without loss of generality, that the fixed point 0 belongs to S, the three lines: $0 + \lambda E_i, \lambda \in \mathbb{R}, i = 1, 2, 3$, are mutually orthogonal, and are called the **principal axis** of S at the point 0.

The set $\{\Omega \in K | (A\Omega, \Omega) = 1\}$ is called the **inertia ellipsoid of the rigid body** S at the point 0. The equation of such ellipsoid, with respect to the reference frame $(0, E_1, E_2, E_3)$, is

$$I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = 1$$

where $\Omega = \Omega_1E_1 + \Omega_2E_2 + \Omega_3E_3$.

Special case: motion of a rigid body with a fixed axis.

If $S \subset K$ is a rigid body with a fixed point ($r(t) = M_t(0) = 0$ for all t) and if $\omega(t) = \omega \neq 0$ is constant, we say that S rotates around the **axis** $e = \frac{\omega}{|\omega|} \in k$ with constant **angular velocity** ω. In this case, the motions $q(t, \xi)$ of S satisfy:

$$\dot{q}(t, \xi) = \omega \times q(t, \xi)$$
$$q(0, \xi) = M_o^*\xi.$$

The solution of that ordinary differential equation, with the initial condition above, can be easily found. In fact let $\bar{\omega} = \omega\times$ be the skew symmetric operator corresponding to the vector $\omega \neq 0$; the solution is

$$q(t,\xi) = exp(t\bar{\omega})M_o^*\xi$$

Since in the present case $q(t,\xi) = M_t^*\xi$ one has:

$$M_t{}^* = exp(t\bar{\omega})M_o{}^*$$

Exercise 5.6.10. Assume that S rotates around the axis $e = \frac{\omega}{|\omega|}$ with constant angular velocity; then show that:

1) The distance $\rho(\xi)$ between $q(t,\xi)$ and the axis $\{\lambda e | \lambda \in \mathbb{R}\}$ does not depend on t.

2) The kinetic energy is given by

$$K^c(t) = \frac{1}{2} I_e |\omega|^2, \quad \text{where} \quad I_e = \int_S \rho^2(\xi) dm(\xi)$$

is called the moment of inertia of the rigid body with respect to the axis $\{\lambda e | \lambda \in \mathbb{R}\}$.

3) $\Omega(t) = (M_t{}^*)^{-1}\omega = \Omega$ is constant and

$$K^c(t) = \frac{1}{2} I_\Omega |\Omega|^2, \qquad \text{where}$$

$$I_\Omega = \int_S |E \times \xi|^2 dm(\xi)$$

is the moment of inertia of the rigid body with respect to the axis $\{\lambda E | \lambda \in \mathbb{R}\}, E = \frac{\Omega}{|\Omega|}$.

4) The eigenvalues I_1, I_2 and I_3 of the inertia operator A are the momenta of inertia of the rigid body with respect to the principal axis of S.

Exercise 5.6.11. (Steiner's theorem) The moment of inertia of the rigid body with respect to an axis is equal to the sum of the moment of inertia with respect to another axis through the center of mass and parallel to the first one plus $m(S)d^2$ where d is the distance between the two axes.

The dynamics of a rigid body S is introduced for bodies S that have at least three non-colinear points. Let us fix, from now on, a proper linear isometry $B : K \to k$. The Lie group $SO(k;3)$ of all proper (linear) orthogonal operators of k is a compact manifold with dimension three. The **configuration space** of a rigid body is a six-dimensional manifold, namely $k \times SO(k;3)$.

Proposition 5.6.12. *The set of all proper isometries M of K onto k is diffeomorphic to the six-dimensional manifold $k \times SO(k;3)$.*

Proof: Let us consider the map

$$\Phi_B : M \longmapsto (M(0), M^* B^{-1}) \tag{5.83}$$

where B is the linear isometry fixed above and M^* is the linear map associated to M, that is,

$$M^*(X) = M(X) - M(0) \quad \text{for all} \quad X \in K.$$

It is easy to see that Φ_B is differentiable, injective and has a differentiable inverse Ψ_B given by

$$\Psi_B : (r, h) \in k \times SO(k; 3) \longmapsto N$$

where N is the proper isometry defined by $N(X) = r + hB(X)$.

By (5.69) the motion of S is given by

$$q(t, \xi) = M_t{}^*(\xi) + r(t), \qquad r(t) = M_t(0);$$

taking into account the map Φ_B (see (5.69)), to the proper isometry M_t there corresponds a pair $(r(t), h(t)) \in k \times SO(k; 3)$ that is:

$$\Phi_B(M_t) = (r(t), h(t) = M_t{}^* B^{-1}). \tag{5.84}$$

So, we can write:

$$q(t, \xi) = r(t) + M_t{}^*(\xi) = r(t) + h(t)B\xi. \tag{5.85}$$

■

Let us denote by β the **σ-algebra of all Borel sets of** K, by λ a **real-valued measure on** (K, β) and let $f : K \to \mathbb{R}$ be a **(real-valued) λ-measurable function**. The correspondence

$$\nu : E \in \beta \longmapsto \int_E f(\xi) d\lambda(\xi) \tag{5.86}$$

is a real-valued measure on (K, β). Moreover, for any λ-measurable function $g : K \to \mathbb{R}$, one has

$$\int_E g(\xi) d\nu(\xi) \stackrel{def}{=} \int_E g(\xi) f(\xi) d\lambda(\xi). \tag{5.87}$$

Given a **vector-valued λ-measurable function** $G : K \to k$, one obtains (taking in k a positive orthonormal basis) its components $g_i, i = 1, 2, 3$, that are (real-valued) λ-measurable functions. So, the vector $\nu(E) = \int_E G(\xi) d\lambda(\xi)$ has three components:

$$\nu_i(E) = \int_E g_i(\xi) d\lambda(\xi), \qquad i = 1, 2, 3. \tag{5.88}$$

It can be also introduced the notion of **vector-valued measure on** (K, β) or **measure on** (K, β) **with values on** k, through the utilization of its three components. In fact if Φ is a measure on (K, β) with values on k and Φ_1, Φ_2, Φ_3

its components in a positive orthonormal basis of k, and given a Φ-measurable (real-valued) function $f : K \to \mathbb{R}$, one denotes by $\int_E f(\xi)d\Phi(\xi)$ the vector in k with components $\int_E f(\xi)d\Phi_i(\xi), i = 1, 2, 3$. Given a Φ-measurable vector-valued function $v : K \to k$, we have that $\int_E v(\xi).d\Phi(\xi)$ is the number given by $\sum_i(\int_E v_i(\xi)d\Phi_i(\xi))$ and $\int_E v(\xi) \times d\Phi(\xi)$ is the vector in k with components:

$$\int_E v_2(\xi)d\Phi_3(\xi) - \int_E v_3(\xi)d\Phi_2(\xi);$$
$$\int_E v_3(\xi)d\Phi_1(\xi) - \int_E v_1(\xi)d\Phi_3(\xi);$$
$$\int_E v_1(\xi)d\Phi_2(\xi) - \int_E v_2(\xi)d\Phi_1(\xi).$$

eq If ν is the vector-valued measure introduced by (5.88) depending on a λ-measurable function $G : K \to k$ with components $g_i : K \to \mathbb{R}$, we have

$$\int_E (v(\xi), d\nu(\xi)) = \int_E (v(\xi), G(\xi))d\lambda(\xi) \quad \text{and}$$
$$\int_E v(\xi) \times d\nu(\xi) = \int_E [v(\xi) \times G(\xi)]d\lambda(\xi).$$

We want to consider now the notion of **(physical) fields of forces** acting on a rigid body S. If S is under the action of the gravitational acceleration $\mathbf{g} \in k, |\mathbf{g}| = g$, one understands that each m-measurable subset $E \subset S$ with mass $m(E)$, is subjected to an external force $m(E)\mathbf{g}$. So, one can define the weight field of forces as a vector-valued measure on S:

$$E \subset S \longmapsto m(E)\mathbf{g} = \int_E \mathbf{g}dm(\xi). \tag{5.89}$$

In general, a **field of forces acting on** $S \subset K$ is a law

$$w \in T(k \times SO(k; 3)) \longrightarrow f_w$$

where f_w is a vector-valued measure on S with values on k.

Since $q(t, \xi) = r(t) + h(t)B\xi$ (see (5.85) and so:

$$\dot{q}(t, \xi) = \dot{r}(t) + \dot{h}B\xi, \tag{5.90}$$

we see that to each $w = (u, s) \in T_{(r,h)}(k \times SO(k; 3))$ there correspond the maps $q, v : K \to k$ defined by

$$q(\xi) = r + hB\xi, \qquad v(\xi) = u + sB\xi. \tag{5.91}$$

It is usual, in Physics, to consider surface forces, volume forces, etc., in the following way: one defines on S a (real-valued) measure σ and a bounded function $\alpha : k \times k \to k$ such that the vector-valued measure on S, with values on k, given by:

$$f_w(E) = \int_E \alpha(q(\xi), v(\xi)) d\sigma(\xi) \tag{5.92}$$

for any Borel subset $E \subset S$, is well defined.

As in the case of a finite system of mass points, it is usual to consider the **field of external forces** ${f_w}^{ext}$ and the **field of internal forces** ${f_w}^{int}$. Given a rigid motion $M : t \mapsto M_t$ of K with respect to k, from (5.85) and (5.90) each proper isometry M_t is represented by the pair $(r(t), h(t)) \in k \times SO(k, 3)$ and, at this point, the tangent vector $w(t) = (\dot{r}(t), \dot{h}(t))$ determines the measures

$${f_t}^{ext} = {f_{w(t)}}^{ext} \quad \text{and} \quad {f_t}^{int} = {f_{w(t)}}^{int}, \quad \text{for each } t.$$

We say that two fields of forces f_w and g_w, acting on a rigid body $S \subset K$, are said to be **equivalent with respect to** M_t if

$$\int_S df_t(\xi) = \int_S dg_t(\xi) \qquad \text{and}$$

$$\int_S M_t\xi \times df_t(\xi) = \int_S M_t\xi \times dg_t(\xi) \tag{5.93}$$

As in the case of a finite number of mass points, the fundamental laws, in classical mechanics, relative to the motions of a rigid body S, are:

I - **Newton law**

"The sum of the internal and external fields of forces is, at each time t, equal to the kinematical distribution D_t (assumed to be well defined)", that is:

$$D_t(E) \overset{def}{=} \int_E \ddot{q}(t, \xi) dm(\xi) = \int_E d{f_t}^{ext}(\xi) + \int_E d{f_t}^{int}(\xi),$$

for all Borel subsets E of S.

II - **Action and reaction principle:**

"The field of internal forces ${f_w}^{int}$ is equivalent to zero with respect to any proper isometry M_t of an arbitrary rigid motion M of K relative to k."

The general equations for the motion of a rigid body S are the equations EG_1) and EG_2) below that follow from I and II:

EG_1)

$$\int_S \ddot{q}(t, \xi) dm(\xi) = \int_S d{f_t}^{ext}(\xi) \overset{def}{=} {F_t}^{ext} \tag{5.94}$$

EG_2)

$$\int_S [(q(t, \xi) - c) \times \ddot{q}(t, \xi)] dm(\xi) = \int_S (q(t, \xi) - c) \times d{f_t}^{ext}(\xi) \overset{def}{=} {P_{t,c}}^{ext} \quad \text{for all } c \in k. \tag{5.95}$$

Exercise 5.6.13. Prove the following formula that gives the variation of the kinetic energy $K^c(t)$ (see (5.76)):

$$\frac{dK^c(t)}{dt} = \int_S (\dot{q}(t,\xi), d{f_t}^{ext}(\xi)) = (\dot{g}(t), {F_t}^{ext}) + (\omega(t), {P_{t,g(t)}}^{ext}),$$

where ${F_t}^{ext}$ and ${P_{t,c}}^{ext}$ (for $c = g(t)$) appear in EG_1 and EG_2.

A rigid body S is said to be **free** under the action of a rigid motion $M : t \mapsto M_t$ of K relatively to k if ${f_t}^{ext}$ is equivalent to zero with respect to M_t for all t. In particular, if ${f_w}^{ext} = 0$ that is, in the absence of external forces, the rigid body is said to be **isolated**; for an (approximate) example we can think about the rolling of a spaceship.

If G is the center of mass of S, that is, $G = \frac{1}{m(S)} \int_S \xi dm(\xi)$, then $g(t) = M_t G = \frac{1}{m(S)} \int_S M_t \xi dm(\xi) = \frac{1}{m(S)} \int_S q(t,\xi) dm(\xi)$.

Differentiating twice with respect to time one has:

$$m(S)\ddot{g}(t) = \int_S \ddot{q}(t,\xi) dm(\xi);$$

by EG_1) and assuming that S is free, one obtains $\ddot{g}(t) = 0$ for all t:

Proposition 5.6.14. *If a rigid body S is free under the action of $M : t \mapsto M_t$, its center of mass moves uniformly and linearly. Moreover, the kinetic momentum and the kinetic energy are constants of motion.*

Proof: From (5.77) one obtains

$$\dot{p}(t) = \int_S [q(t,\xi) \times \ddot{q}(t,\xi)] dm(\xi)$$

and EG_2) (with $c = 0$) implies:

$$\dot{p}(t) = \int_S q(t,\xi) \times d{f_t}^{ext}(\xi) = \int_S M_t(\xi) \times d{f_t}^{ext}(\xi);$$

but the fact that S is free under the action of $M : t \mapsto M_t$, together with (5.93), yields $\dot{p}(t) = 0$. By an analogous argument with the expression of $\frac{dK^c(t)}{dt}$ given by the result of Exercise 5.6.13 we see that $\frac{dK^c(t)}{dt} = 0$; so, $p(t)$ and $K^c(t)$ are constants of motion. More precisely, since $p(t)$ is a vector-valued constant of motion, one obtains four (scalar valued) constants of motion for any rigid body S free under the action of M. ∎

Assume we are looking at an inertial coordinate system where the center of mass is stationary. Then

Proposition 5.6.15. *A free rigid body rotates around its center of mass as if the center of mass were fixed.*

Let us consider the motion of a rigid body around a stationary point, in the absence of external forces. In this case, there exist four real valued constants of motion given by Proposition 13.5. One can also consider the induced functions

$$K^c : T(SO(k;3)) \longrightarrow \mathbb{R} \qquad p : T(SO(k;3)) \longrightarrow k, \tag{5.96}$$

defined by

$$\begin{aligned} s_h \in T(SO(k;3)) &\longmapsto K^c(s_h) = \frac{1}{2}\int_S |sB\xi|^2 dm(\xi), \\ s_h \in T(SO(k;3)) &\longmapsto p(s_h) = \int_S (hB\xi \times sB\xi) dm(\xi), \end{aligned} \tag{5.97}$$

respectively. In general (if the rigid body does not have any particular symmetry) the four scalar-valued maps (K^c and the components p_i of p in a basis of k) defined on the six-dimensional manifold $T(SO(k,3))$ are independent in the sense that they do not have critical points, that is, the inverse image of any value (K_o, p_o) (if non empty) is a two dimensional orientable compact invariant manifold, provided that the value K_o of $K^c(s_h)$ is positive. Moreover, $K_o > 0$ implies that the vector field induced on the inverse image of (K_o, p_o) by (K^c, p) has no singular points, that is, each connected component $(K^c, p)^{-1}$ (K_o, p_o) is a bi-dimensional torus.

Proposition 5.6.16. *The angular momentum $P(t)$ relative to a rigid body S that is free under the action of $M : t \mapsto M_t$, satisfies the* **Euler equation***: $\dot{P}(t) = P(t) \times \Omega(t)$. Moreover, $\Omega(t)$ is given by the relation, $A\dot{\Omega}(t) = [A\Omega(t)] \times \Omega(t)$, A being the inertia operator.*

Proof: In fact, $p(t) = {M_t}^* P(t)$, so by Proposition 5.6.14 we have

$$\dot{p}(t) = \dot{M}_t^* P(t) + M_t^* \dot{P}(t) = 0, \qquad \text{and so}$$

$$\dot{P}(t) = -({M_t}^*)^{-1} \dot{M}_t^* P(t) = -\Omega(t) \times P(t) = P(t) \times \Omega(t).$$

But, since $P(t) = A\Omega(t)$, we also have $A\dot{\Omega}(t) = [A\Omega(t)] \times \Omega(t)$. ∎

Proposition 5.6.17. *In the motion of a rigid body S with a fixed point, subjected to a field of external forces, the kinetic momenta $p(t)$ and $P(t)$ satisfy the equations*

$$\dot{p}(t) = \int_S ({M_t}^* \xi) \times d{f_t}^{ext}(\xi),$$

$$\dot{P}(t) = P(t) \times \Omega(t) + \int_S \xi \times [({M_t}^*)^{-1} d{f_t}^{ext}(\xi)].$$

Proof: From (5.77) one obtains $\dot{p}(t) = \int_S [q(t,\xi) \times \ddot{q}(t,\xi)]dm(\xi)$ and since there is a fixed point we can write $q(t,\xi) = M_t{}^*\xi$; using EG_2) with $c = 0$ we have the equation for $\dot{p}(t)$. Since $P(t) = (M_t{}^*)^{-1}p(t)$ and using again (5.77) one can write by differentiating:

$$\dot{P}(t) = (\dot{M_t^*})^{-1} \int_S [q(t,\xi) \times \dot{q}(t,\xi)]dm(\xi) + (M_t{}^*)^{-1}\dot{p}(t);$$

but $M_t{}^*(M_t{}^*)^{-1} = Id$ implies, by differentiating, that

$$(\dot{M_t^*})^{-1} = -(M_t{}^*)^{-1}\dot{M_t^*}(M_t{}^*)^{-1};$$

so,

$$\begin{aligned}\dot{P}(t) = &\int_S [\xi \times (M_t{}^*)^{-1} df_t{}^{ext}(\xi)] \\ &- \Omega(t) \times [(M_t{}^*)^{-1} \int_S [q(t,\xi) \times \dot{q}(t,\xi)]dm(\xi)]\end{aligned}$$

and finally,

$$\dot{P}(t) = P(t) \times \Omega(t) + \int_S \xi \times [(M_t{}^*)^{-1} df_t{}^{ext}(\xi)]$$

■

In order to relate the properties EG_1) and EG_2) with the abstract Newton law, we start by defining the metric $\langle , \rangle$ on $k \times SO(k;3)$. This metric is induced by the kinetic energy. Since (see (5.90))

$$\begin{aligned} q(t,\xi) &= r(t) + h(t)B\xi \qquad \text{and} \\ \dot{q}(t,\xi) &= \dot{r}(t) + \dot{h}(t)B\xi, \end{aligned}$$

we have

$$K^c(t) = \frac{1}{2}\int_S |\dot{r}(t) + \dot{h}B\xi|^2 dm(\xi); \tag{5.98}$$

We will assume that the origin $0 \in K$ coincides with the center of mass $G = \frac{1}{m(S)}\int_S \xi dm(\xi)$; so, we have $\int_S \xi dm(\xi) = 0$, which implies

$$K^c(t) = \frac{1}{2}m(S)|\dot{r}(t)|^2 + \frac{1}{2}\int_S |\dot{h}(t)B\xi|^2 dm(\xi).$$

The last expression suggests the introduction of a metric on $k \times SO(k;3)$; in fact, given two tangent vectors $(u,s), (\bar{u},\bar{s})$ at the point $(r,h) \in k \times SO(k;3)$ one defines

$$\langle (u,s), (\bar{u},\bar{s}) \rangle_{(r,h)} \stackrel{\text{def}}{=} m(S)(u,\bar{u}) + \int_S (sB\xi, \bar{s}B\xi)dm(\xi)$$

in which the right hand side defines two inner products,

$$\langle u, \bar{u}\rangle_r = m(S)(u, \bar{u}) \qquad \text{and} \qquad \langle s, \bar{s}\rangle_h = \int_S (sB\xi, \bar{s}B\xi)dm(\xi), \quad (5.99)$$

on k and $SO(k;3)$, respectively. Recall that s and $\bar{s}$ are tangent vectors at $h \in SO(k;3)$. So, we have defined on $SO(k;3)$ a Riemannian metric which is left invariant, that is, the left translations are isometries. In fact, given $g \in SO(k;3)$, the left translation L_g is defined by the expression $L_g(x) = gx$, for all $x \in SO(k;3)$ and, since g is a linear transformation acting on k, its derivative satisfies $dL_g(x) = L_g$; so one obtains

$$\begin{aligned} \langle dL_g(h)s, dL_g(h)\bar{s}\rangle_{gh} &= \langle gs, g\bar{s}\rangle_{gh} \\ = \int_S (gsB\xi, g\bar{s}B\xi)dm(\xi) &= \int_S (sB\xi, \bar{s}B\xi)dm(\xi) \\ = \langle s, \bar{s}\rangle_h. & \end{aligned}$$

The acceleration, in the product metric, corresponding to a vector $\dot{q} = (\dot{r}, \dot{h})$ tangent to $k \times SO(k;3)$ at the point (r,h), is equal to

$$\frac{D\dot{q}}{dt} = \frac{D}{dt}(\dot{r}, \dot{h}) = (\ddot{r}, \frac{D\dot{h}}{dt}).$$

The mass operator in the product metric acts on $\frac{D\dot{q}}{dt}$ as

$$\mu(\frac{D\dot{q}}{dt})(u, s) = \langle \ddot{r}, u\rangle_r + \langle \frac{D\dot{h}}{dt}, s\rangle_h.$$

Let us introduce now an abstract field of forces $\mathcal{F} : T(k \times SO(k;3)) \longrightarrow T^*(k \times SO(k;3))$ in a suitable way such that the generalized Newton law

$$\mu(\frac{D\dot{q}}{dt}) = \mathcal{F}(\dot{q})$$

becomes equivalent to the general equations EG$_1$) and EG$_2$), for the motion of a rigid body. The way we define $\mathcal{F}$ is the following: for $(\bar{u}, \bar{s})$ and $w = (u, s)$ in $T_{r,h}(k \times SO(k;3))$ we set:

$$(\mathcal{F}(u, s))(\bar{u}, \bar{s}) = \int_S (\bar{u}, df_w^{ext}(\xi)) + \int_S (\bar{s}B\xi, df_w^{ext}(\xi)). \qquad (5.100)$$

Recall (see (5.94), (5.95)) the general equations:

$$\text{EG}_1) \qquad \int_S \ddot{q}(t,\xi)dm(\xi) = \int_S df_t^{ext}(\xi) = F_t^{ext}$$

$$\text{EG}_2) \qquad \int_S (q(t,\xi) - c) \times \ddot{q}(t,\xi)dm(\xi) = \int_S (q(t,\xi) - c) \times df_t^{ext}(\xi) = P_{t,c}^{ext},$$

for all $c \in k$.

It is a simple matter to see that EG_1) and EG_2) are equivalent to EG_1) and EG'$_2$), where

$$\text{EG}_2^{'}) : \quad \int_S (q(t,\xi) - g(t)) \times \ddot{q}(t,\xi) dm(\xi) = \int_S (q(t,\xi) - g(t)) \times df_t^{ext}(\xi) \\ = P_{t,g(t)}^{ext}$$

with

$$g(t) = M_t G = M_t \left[\frac{1}{m(S)} \int_S \xi dm(\xi) \right] = \frac{1}{m(S)} \int_S q(t,\xi) dm(\xi),$$

G being the center of mass of S, which we already set equal to the origin 0 of K. Thus we can write:

$$\int_S \xi dm(\xi) = 0. \tag{5.101}$$

The expression of $q(t,\xi) = M_t(\xi)$ is, in this case, $q(t,\xi) = M_t(0) + M_t^* \xi = g(t) + h(t)B\xi$, with $M_t^* = h(t)B$. So we have

$$\dot{q}(t,\xi) = \dot{g}(t) + \dot{M}_t^* \xi \qquad \text{and} \qquad \ddot{q}(t,\xi) = \ddot{g}(t) + \ddot{M}_t^* \xi,$$

then EG_1) becomes equivalent to

$$\int_S \ddot{g}(t) dm(\xi) + \ddot{M}_t^* \int_S \xi dm(\xi) = F_t^{ext},$$

and, by (5.101), we have EG_1) equivalent to

$$m(S)(\ddot{g}(t), \bar{u}) = (F_t^{ext}, \bar{u}), \qquad \text{for all} \qquad \bar{u} \in k. \tag{5.102}$$

On the other hand $\text{EG}_2)^{'}$ is equivalent to

$$\begin{aligned} P_{t,g(t)}^{ext} &= \int_S M_t^* \xi \times (\ddot{g}(t) + \ddot{M}_t^* \xi) dm(\xi) \\ &= \left(\int_S M_t^* \xi dm(\xi) \right) \times \ddot{g}(t) + \int_S \frac{d}{dt} (M_t^* \xi \times \dot{M}_t^* \xi) dm(\xi) = \\ &= M_t^* \left(\int_S \xi dm(\xi) \right) \times \ddot{g}(t) + \frac{d}{dt} \int_S (M_t^* \xi \times \dot{M}_t^* \xi) dm(\xi); \end{aligned}$$

again by (5.101) $\text{EG}_2)^{'}$ is equivalent to

$$\left(\frac{d}{dt} \int_S (M_t^* \xi \times \dot{M}_t^* \xi) dm(\xi), \quad \bar{u} \right) = (P_{t,g(t)}^{ext}, \quad \bar{u}) \qquad \text{for all} \qquad \bar{u} \in k. \tag{5.103}$$

From what is said in Exercise 5.6.4 there is a linear isomorphism Φ between k and the space $s(k)$ of all linear skew-symmetric operators of k. In fact, for any $A \in s(k)$, $\Phi(A)$ is the unique vector in k such that $Av = \Phi(A) \times v$ for all $v \in k$. With that notation, $\mathrm{EG}_2)'$ being equivalent to (5.103) means being equivalent to

$$(P^{ext}_{t,g(t)}, \Phi(A)) = \frac{d}{dt}\int_S (M_t^*\xi \times \dot{M}_t^*\xi, \Phi(A))dm(\xi)$$
$$= \frac{d}{dt}\int_S (\Phi(A) \times M_t^*\xi, \dot{M}_t^*\xi)dm(\xi);$$

thus $\mathrm{EG}_2)'$ is equivalent to

$$(P^{ext}_{t,g(t)}, \Phi(A)) = \frac{d}{dt}\int_S (AM_t^*\xi, \dot{M}_t^*\xi)dm(\xi), \quad \text{for all} \quad A \in s(k). \tag{5.104}$$

There is also a linear isomorphism between the tangent space $T_hSO(k;3)$ and $s(k)$ (see Exercise 5.6.18 below) through the map

$$\dot{\bar{h}} \in T_hSO(k,3) \longmapsto \dot{\bar{h}}\ \ h^{-1} \in s(k) \tag{5.105}$$

(which is the derivative of the right translation $R_{h^{-1}}$ defined as $R_{h^{-1}}(x) = xh^{-1}$, for all $x \in SO(k;3)$).

Exercise 5.6.18. Prove that $\dot{\bar{h}}\ \ h^{-1} \in s(k)$ in (5.104) and that the map above is a linear isomorphism.

We recall that $M_t^* = h(t)B$, so (5.104) and (5.105) imply that $\mathrm{EG}_2)'$ is equivalent to

$$(P^{ext}_{t,g(t)}, \Phi(\dot{\bar{h}}\ \ h^{-1}(t)) = \frac{d}{dt}\int_S (\dot{\bar{h}}\ \ h^{-1}(t)h(t)B\xi, \dot{h}(t)B\xi)dm(\xi)$$
$$= \frac{d}{dt}\int_S (\dot{\bar{h}}B\xi, \dot{h}(t)B\xi)dm(\xi)$$

for all $\dot{\bar{h}} \in T_{h(t)}SO(k;3)$.

From (5.99), (5.102) and the last expression, one can say that $\mathrm{EG}_1)$ and $\mathrm{EG}_2)'$ are equivalent to

$$(F_t^{ext}, \bar{u}) + (P^{ext}_{t,g(t)}, \Phi(\dot{\bar{h}}\ \ h^{-1}(t)) =$$
$$= \frac{d}{dt}\left[m(S)(\dot{g}(t), \bar{u}) + \int_S (\dot{\bar{h}}B\xi, \dot{h}(t)B\xi)dm(\xi)\right]$$
$$= \frac{d}{dt}\langle(\dot{g}(t), \dot{h}(t)), (\bar{u}, \dot{\bar{h}})\rangle$$
$$\text{for all} \quad (\bar{u}, \dot{\bar{h}}) \in T_{(g(t),h(t))}\ \ k \times SO(k;3). \tag{5.106}$$

Notice that if we extend, by parallel transport, the vector $(\bar{u}, \dot{\tilde{h}})$ along the motion $q(t) = (g(t), h(t))$, one obtains a vector field along $q(t)$ still denoted by $(\bar{u}, \dot{\tilde{h}})$ so that $\frac{D}{dt}(\bar{u}, \dot{\tilde{h}}) = 0$ and then the right-hand side of (5.106) can be written as

$$\frac{d}{dt}\langle \dot{q}, (\bar{u}, \dot{\tilde{h}})\rangle = \langle \frac{D\dot{q}}{dt}, (\bar{u}, \dot{\tilde{h}})\rangle + \langle \dot{q}, \frac{D}{dt}(\bar{u}, \dot{\tilde{h}})\rangle = \langle \frac{D\dot{q}}{dt}, (\bar{u}, \dot{\tilde{h}})\rangle. \tag{5.107}$$

Let us recall the field of forces

$$\mathcal{F} : T(k \times SO(k;3)) \longrightarrow T^*(k \times SO(k;3))$$

given in the following way: if $(u, s) \in T_{(r,h)}(k \times SO(k;3))$ then we have $\mathcal{F}(u, s) \in T^*_{(r,h)}(k \times SO(k,3))$ if, and only if (5.100) holds, that is, for $(u, s) = \dot{q}$:

$$(\mathcal{F}(\dot{q}))(\bar{u}, \dot{\tilde{h}}) = (F_t^{ext}, \bar{u}) + (P_{t,g(t)}^{ext}, \Phi(\dot{\tilde{h}}h^{-1}(t))). \tag{5.108}$$

The constructions of $h^{-1}(t)$, F_t^{ext} and $P_{t,g(t)}^{ext}$ are possible because given $(r, h) \in k \times SO(k;3)$ and $(u, s) \in T_{(r,h)}(k \times SO(k;3))$ we are able to find $q(t, \xi)$ and so $\dot{q}(t, \xi)$ that determine $h^{-1}(t), F_t^{ext}$ and $P_{t,g(t)}^{ext}$. The conclusion is then the following result:

Proposition 5.6.19. *The general equations* EG_1) *and* EG_2) *that govern the motions of a rigid body S (see* (5.94) *and* (5.95)*) are equivalent to the generalized Newton law $\mu(\frac{D\dot{q}}{dt}) = \mathcal{F}(\dot{q})$ on the manifold $k \times SO(3)$ with the Riemannian metric given by equations* (5.99) *and the field of forces $\mathcal{F}$ characterized by* (5.100).

Proof: As we saw, the equations EG_1) and EG_2) are equivalent to (5.106); using (5.106) and (5.107) we see that

$$\langle \frac{D\dot{q}}{dt}, v\rangle = [\mathcal{F}(\dot{q})]v \qquad \text{for all} \qquad v \in T_{q(t)}[k \times SO(k;3)],$$

and so

$$\mu(\frac{D\dot{q}}{dt}) = \mathcal{F}(\dot{q}).$$

■

We intend, now, to derive the **Lagrange equations for the motion of a rigid body** S. We take a positive orthonormal basis $\{e_1, e_2, e_3\}$ for the vector space k and denote by (r_1, r_2, r_3) the coordinates of a vector $r \in k$. Let (h_1, h_2, h_3) be a local system of coordinates for $SO(k;3)$. So if $(\bar{u}, \bar{s}) \in T_{(r,h)}(k \times SO(k;3))$ we have $\bar{u} = \Sigma_{i=1}^3 \bar{u}_i e_i$ and $\bar{s} = \Sigma_{i=1}^3 \bar{s}_i \frac{\partial}{\partial h_i}(h)$. The force $\mathcal{F}$ defined in (5.100) has the following expression in those local coordinates

$$\mathcal{F}(r,h))(\bar{u},\bar{s}) = \int_S (\bar{u}, df_w^{ext}(\xi)) + \int_S (\bar{s}B\xi, df_w^{ext}(\xi)) =$$

$$= \sum_{i=1}^{3} \bar{u}_i(e_i, \int_S df_w^{ext}(\xi)) + \sum_{i=1}^{3} \bar{s}_i \int_S (\frac{\partial}{\partial h_i}(h)B\xi, df_w^{ext}(\xi)) =$$

$$= \sum_{i=1}^{3} (\int_S df_w^{ext}(\xi))_i, dr_i(\bar{u}) + \sum_{i=1}^{3} (\int_S (\frac{\partial}{\partial h_i}(h)B\xi, df_w^{ext}(\xi)))dh_i(\bar{s}) \tag{5.109}$$

Then if $t \to (r(t), h(t)) \in k \times SO(k;3)$ is a motion of S under the external forces f^{ext} and being $K^c(t)$ the kinetic energy along this motion, the Newton law gives

$$\frac{d}{dt}\frac{\partial K^c}{\partial \dot{r}_i} - \frac{\partial K^c}{\partial r_i} = (\int_S df_t^{ext}(\xi))_i, \quad i = 1,2,3, \tag{5.110}$$

$$\frac{d}{dt}\frac{\partial K^c}{\partial \dot{h}_i} - \frac{\partial K^c}{\partial h_i} = \int_S (\frac{\partial}{\partial h_i}(h)B\xi, df_t^{ext}(\xi)), \quad i = 1,2,3. \tag{5.111}$$

We will relate the right hand sides of equations (5.110) and (5.111) above, with the physical notions of total force and momentum of external forces with respect to a point.

Since $\frac{\partial}{\partial h_i}(h)h^{-1}(t) \in T_e(SO(k,3))$, it follows that, for each t, there exist vectors $\omega_i(t) \in k$ such that

$$\omega_i(t)\times = \frac{\partial}{\partial h_i}(h)h^{-1}(t), \qquad i = 1,2,3. \tag{5.112}$$

This implies

$$\begin{aligned}\frac{d}{dt}\frac{\partial K^c}{\partial \dot{h}_i} - \frac{\partial K^c}{\partial h_i} &= \int_S (\omega_i(t) \times h(t)B\xi, df_t^{ext}(\xi)) = \\ &= (\omega_i(t), \int_S hB\xi \times df_t^{ext}(\xi)), \quad i = 1,2,3.\end{aligned} \tag{5.113}$$

Introducing the usual notation $F_t^{ext} = \int_S df_t^{ext}(\xi)$ (total force at t) and $P_t^{ext} = P_{t,r(t)}^{ext} = \int_S (q(t,\xi) - r(t)) \times df_t^{ext}(\xi) = \int_S hB\xi \times df_t^{ext}(\xi)$ (the momentum of external forces with respect to $r(t)$ at the time t) we obtain the **Lagrange equations for the motions of a rigid body** S:

$$\frac{d}{dt}\frac{\partial K^c}{\partial \dot{r}_i} - \frac{\partial K^c}{\partial r_i} = (F_t^{ext})_i, \qquad i = 1,2,3 \tag{5.114}$$

$$\frac{d}{dt}\frac{\partial K^c}{\partial \dot{h}_i} - \frac{\partial K^c}{\partial h_i} = (\omega_i(t), P_t^{ext}), \quad i = 1,2,3. \tag{5.115}$$

Since $K^c(t) = \frac{1}{2}m(S)|\dot{r}|^2 + \frac{1}{2}\int_S |\dot{h}B\xi|^2 dm(\xi)$ the first Lagrange equation gives us

$$m(S)\ddot{r}(t) = F_t^{ext},$$

and the hypothesis $G = 0$ implies $r(t) = g(t)$ so we obtain the classical Newton law for the motion of G. If the rigid body moves with a fixed point, the second of the Lagrange equations are the only ones to be considered.

Exercise 5.6.20. Let $S \subset K$ be a rigid body with fixed point $O \in S$. Assume $K = k$, $B = id$, $(O,\ e_x, e_y, e_z)$ and (O, e_1, e_2, e_3) orthogonal positively oriented frames fixed in k and in S, respectively. If $e_z \times e_3 \neq 0$, let $e_N = \frac{e_z \times e_3}{|e_z \times e_3|}$. The **nodal line** passes through O and has direction e_N. The **Euler angles** (φ, θ, ψ) are defined as follows: φ is the angle of rotation along the axis $(0, e_z)$ which sends e_x to e_N; θ is the angle of rotation along $(0, e_N)$ which sends e_z to e_3; ψ is the rotation along $(0, e_3)$ which sends e_N to e_1. Show that to each (φ, θ, ψ) satisfying $0 < \varphi < 2\pi$, $0 < \psi < 2\pi$, $0 < \theta < \pi$, corresponds a rotation $R(\varphi, \theta, \psi)$ defining local coordinates for $SO(k; 3)$.Denote by I_1, I_2, I_3 the moments of inertia of S relative to (e_1, e_2, e_3) and prove that $\Omega = Ae_1 + Be_2 + Ce_3$, $\omega = \bar{A}e_x + \bar{B}e_y + \bar{C}e_z$, $K^c = \frac{1}{2}(I_1A^2 + I_2B^2 + I_3C^2)$ where $A = \dot{\varphi}\sin(\psi)\sin(\theta) + \dot{\varphi}\cos(\psi)$, $B = \dot{\varphi}\cos(\psi)\sin(\theta) - \dot{\theta}\sin(\psi)$ and $C = \dot{\varphi}\cos(\theta) + \dot{\psi}$. Compute $\bar{A}$, $\bar{B}$ and $\bar{C}$.

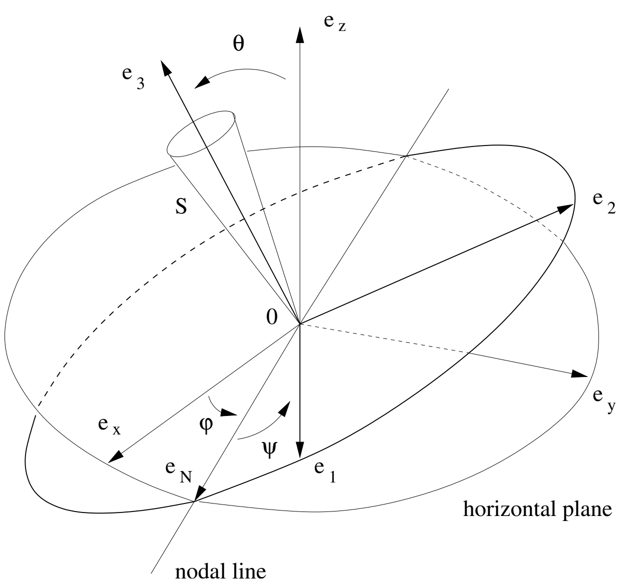

Fig. 5.4. Euler angles.

5.7 Dynamics of pseudo-rigid bodies

The present section corresponds to Dirichlet–Riemann formulation of ellipsoidal motions for fluid masses (also called pseudo-rigid bodies).

As in the previous section, k and K are two 3-dimensional Euclidean vector spaces considered as affine spaces; they represent the fixed (inertial) space and the moving space respectively.

A motion $t \mapsto M_t$ is a smooth map where each $M_t : K \to k$ is an orientation preserving affine transformation (bijection) such that takes the zero vector $O \in K$ (corresponding to the center of mass) into the zero vector $0 \in k$.

If we fix a ball $\mathcal{B}_r \subset K$ of radius r and centered in O, a motion of a pseudo-rigid body is the motion

$$t \mapsto M_t(\mathcal{B}_r) \subset k$$

of a solid ellipsoid.

Given M_t, we call $B = M_{t=0}$ and set $Q_t = M_t \circ B^{-1} : k \to k$, so $Q_t \in GL^+(k,3)$. The derivative $\dot{Q}_t = \dot{M}_t \circ B^{-1}$ represents the tangent vector at the point $Q_t \in GL^+(k,3)$ to the curve $t \mapsto Q_t$. Take a point $X \in \mathcal{B}_r$; then $q(t,X) = M_t X$ is a curve in k with velocity $\dot{q}(t,X) = \dot{M}_t X$.

The **kinetic energy** of the motion of the solid ellipsoid is

$$K^c(t) = \frac{1}{2}\int_{\mathcal{B}_r} |\dot{q}(t,X)|^2 \, dm(X)$$

where the positive measure m on K is the distribution of mass. So

$$K^c(t) = \frac{1}{2}\int_{\mathcal{B}_r} |\dot{Q}_t \circ BX|^2 \, dm(X) = \frac{1}{2}\int_{\mathcal{B}_r} |\dot{Q}_t \circ BX|^2 \, \rho dV(X)$$

where ρ is the density and V is the Lebesgue volume. When $\rho =$ constant,

$$K^c(t) = \frac{\rho}{2}\int_{\mathcal{B}_r} |\dot{Q}_t \circ BX|^2 \, dV(X).$$

In order to work with matrices, we fix two positive orthonormal bases (e_1, e_2, e_3) and (E_1, E_2, E_3) in k and K, respectively. For simplicity, we consider the particular case in which the matrix of B is Id, the identity matrix. We shall denote by Q_t and X the corresponding matrices of Q_t and X with respect to the fixed bases. Then

$$K^c(t) = \frac{\rho}{2}\int_{\mathcal{B}_r} |\dot{Q}_t X|^2 \, dV(X). \tag{5.116}$$

Proposition 5.7.1. *Any real $n \times n$ matrix G has a (non unique) bipolar decomposition $G = LDR$, that is L, R are orthogonal matrices and $D =$* diag $(\sqrt{\sigma_1}, \ldots, \sqrt{\sigma_n})$. *Moreover $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ are the non negative eigenvalues of $G^T G$ (G^T is the transpose of G).*

Proposition 5.7.2. *The matrix* $\mathcal{E}_0 = \rho \int_{\mathcal{B}_r} XX^T dV(X)$ *is given by* $\mathcal{E}_0 = \frac{4\rho\pi r^5}{15} Id = \bar{m} Id$. *(Note carefully that* XX^T *is a* 3×3 *matrix).*

Proposition 5.7.3. *The kinetic energy ((5.116)) is given by*

$$K^c(t) = \frac{1}{2}\mathrm{tr}(\dot{Q}_t\, \mathcal{E}_0\, \dot{Q}_t^T).$$

(Here $\mathrm{tr}\, A$ *denotes the trace of the matrix* A*).*

From the propositions above it follows that

$$K^c(t) = \frac{1}{2}\bar{m}\, \mathrm{tr}\,(\dot{Q}_t\, \dot{Q}_t^T). \tag{5.117}$$

Exercise 5.7.4. Prove the three last propositions.

Let us assume, from now on, that $\bar{m} = 1$.

Remark 5.7.5. The expression ((5.117)) suggests the following Riemannian metric for the group $GL^+(3)$ of all 3×3 matrices of positive determinant:

$$\langle A\,,\, B\,\rangle_Q := \mathrm{tr}\,(AB^T), \tag{5.118}$$

for all $Q \in GL^+(3)$ and all $A, B \in T_Q GL^+(3)$.

Assume that a smooth motion has a (not necessarily unique) smooth bipolar decomposition $Q_t = T_t^T\, A_t\, S_t$ (i.e. three smooth paths: A_t diagonal, and T_t, S_t orthogonal paths).

In the case when Q_t is analytic, this is always possible; also, if the eigenvalues of $Q_t Q_t^T$ are distinct and Q_t is not analytic, the smooth decomposition is still possible. However, there are examples of C^∞ paths Q_t for which there is no continuous bipolar decomposition (see Montaldi [50], Kato [34] and Roberts - S. Dias [57]). We have:

Proposition 5.7.6. *From the equation of continuity in hydrodynamics and* $\rho =$ *constant, it follows that a smooth path* $Q_t = M_t \circ B^{-1}$ *corresponding to an ellipsoidal motion satisfies* $\det Q_t = 1$*, that is,* Q_t *is a curve in the Lie group* $SL(3)$.

Proof: Assume $Q_t = T_t^T\, A_t S_t$ and call

$$x = T_t q(t, X) = T_t M_t X = T_t Q_t B X$$

where $T_t = (T_{ki})$ means a rotation that takes (e_1, e_2, e_3) to the orthonormal basis $(\bar{e}_1(t), \bar{e}_2(t), \bar{e}_3(t))$, that is $\bar{e}_i(t) = \sum_{k=1}^{3} T_{ki} e_k$, $i = 1, 2, 3$.

Then $u := \dot{x} = \left(\dot{T}_t Q_t + T_t \dot{Q}_t\right)\, BX$ and $BX = Q_t^{-1}\, T_t^T x$ so,

$$u = \left(\dot{T}_t\, T_t^T + T_t \dot{Q}_t Q_t^{-1} T_t^T\right)\, x$$

and $div\, u = \sum_k \dfrac{\partial u_k}{\partial x_k} = \mathrm{tr}\,(\dot{Q}_t Q_t^{-1}) = \dfrac{1}{\det Q_t}\, \dfrac{d}{dt}\,(\det Q_t)$. Finally $div\, u = 0$ if and only if $\dfrac{d}{dt}\,(\det Q_t) = 0$ if and only if $\det Q_t =$ constant. Thus $\det Q_t = 1$ because for $t = 0$ we have $\det Q_0 = \det\,(BB^{-1}) = 1$. ■

From Dirichlet–Riemann formulation (see Chandrasekhar [15] and Montaldi [50]) the motions of pseudo-rigid bodies are given by a generalized Newton law describing a mechanical system on the configuration space $GL^+(3)$ with a holonomic constraint defined by the submanifold $SL(3)$ of $GL^+(3)$, that is:

$$\mu \frac{D\dot{Q}}{dt} = -\, dV + \lambda df, \qquad Q \in SL(3). \tag{5.119}$$

Here $f : GL(3) \to \mathbb{R}$ is the determinant function and $\lambda : TSL(3) \to \mathbb{R}$ is the so-called Lagrange multiplier; also, $SL(3) = f^{-1}(1) \subset GL^+(3)$ is an analytic 8-dimensional orientable submanifold of $GL^+(3)$,

$$\mu : TGL^+(3) \to T^*GL^+(3)$$

is the mass operator (Legendre transformation) relative to the trace metric, $\mu(v)(\cdot) := \langle v\,,\,\cdot\rangle$ (see (5.118)), and $\frac{D\dot{Q}}{dt}$ is the covariant derivative of $\dot{Q}(t)$ (acceleration) along $Q(t)$ in that metric. The map $df : TGL^+(3) \to T^*GL^+(3)$ is given by

$$v \mapsto df(\pi v)$$

where $\pi : TGL^+(3) \to GL^+(3)$ is the canonical bundle projection. We still denote by df its restriction to $TSL(3)$. We will show that $\mu^{-1} df : TSL(3) \to TGL^+(3)$ satisfies **d'Alembert principle**. In fact for any $A \in TSL(3)$ we have

$$(\mu^{-1} df)\, A = w \in T_{\pi(A)} GL^+(3)$$

where w is such that $\langle w\,,\,\cdot\rangle = \left[df_{\pi(A)}\right](\cdot)$, so w is orthogonal to $T_{\pi(A)}SL(3)$.

Then there exists a unique Lagrange multiplier $\lambda : TSL(3) \to \mathbb{R}$, yielding the reaction force . The function

$$V : GL^+(3) \to \mathbb{R}$$

is the **potential energy** and corresponds to the gravitational potential (see examples below).

Proposition 5.7.7. *The generalized Newton law ((5.119)) is equivalent to the system*

$$\ddot{Q} = -\frac{\partial V}{\partial Q} + \lambda \frac{\partial f}{\partial Q}, \qquad \det Q = 1. \tag{5.120}$$

Proof: Here $Q, \ddot{Q}, \frac{\partial V}{\partial Q}, \frac{\partial f}{\partial Q}$ are 3×3 matrices: $Q = (q_{ij}), \ddot{Q} = (\ddot{q}_{ij}), \frac{\partial V}{\partial Q} = (\frac{\partial V}{\partial q_{ij}})$ and $\frac{\partial f}{\partial Q} = (\frac{\partial f}{\partial q_{ij}})$, respectively. We also have that (see Exercise 5.1.1):

$$\mu\left(\frac{D\dot{Q}}{dt}\right) = \sum_{i,j}\left[\frac{d}{dt}\,\frac{dK^c}{d\dot{q}_{ij}} - \frac{dK^c}{dq_{ij}}\right] dq_{ij},$$

where

$$K^c = \frac{1}{2}\langle\dot{Q},\dot{Q}\rangle = \frac{1}{2}\ \left[\dot{q}_{11}^2 + \dot{q}_{12}^2 + \cdots + \dot{q}_{33}^2\right].$$

Then

$$\mu\left(\frac{D\dot{Q}}{dt}\right) = -dV + \lambda df \longleftrightarrow \sum_{ij}\ddot{q}_{ij}\,dq_{ij} = \sum_{ij}\left(-\frac{\partial V}{\partial q_{ij}} + \lambda\frac{\partial f}{\partial q_{ij}}\right) dq_{ij}$$

and the proof is complete. ∎

For the Dirichlet–Riemann formulation (see [15]) one considers, from the smooth bi-polar decomposition $Q_t = T_t^T\,A_t\,S_t$, the new variables

$$\Omega^* := \dot{T}T^T \qquad\qquad \Lambda^* := \dot{S}S^T$$

which are skew symmetric paths because differentiation of $TT^T = SS^T = I$ gives

$$\dot{T}T^T + T\dot{T}^T = 0 = \dot{S}S^T + S\dot{S}^T.$$

Thus we obtain:

$$\dot{Q} = T^T\left({\Omega^*}^T A + \dot{A} + A\Lambda^*\right) S$$

and also, from last Proposition 5.7.7:

$$\ddot{Q} = \dot{T}^T\left({\Omega^*}^T A + \dot{A} + A\Lambda^*\right) S + T^T\left({\Omega^*}^T A + \dot{A} + A\Lambda^*\right)\dot{S} +$$

$$+ T^T\ddot{A}S + T^T\left[\tfrac{d}{dt}\left(A\Lambda^* - \Omega^* A\right)\right] S =$$

$$= \left[-\tfrac{\partial V}{\partial Q} + \lambda\tfrac{\partial(\det Q)}{\partial Q}\right]_{Q=T^T AS}.$$

So, one obtains the **equation of motion**:

$$\ddot{A} + \Omega^*\left(\Omega^* A - \dot{A} - A\Lambda^*\right) + \left(-\Omega^* A + \dot{A} + A\Lambda^*\right)\Lambda^* + \tfrac{d}{dt}\left(A\Lambda^* - \Omega^* A\right)$$

$$= \left[-T\left(\tfrac{\partial V}{\partial Q}\right)_{Q=T^T AS} S^T + \lambda T\left(\tfrac{\partial(\det Q)}{\partial Q}\right)_{Q=T^T AS} S^T\right]. \tag{5.121}$$

Exercise 5.7.8. Show that

I. If $f = \det Q$, $Q \in GL^+(3)$, then $df_Q(B) = (\det Q)\,\mathrm{tr}(Q^{-1}B)$ for any 3×3 real matrix B.
II. For any function $\phi : GL^+(3) \to \mathbb{R}$ then $\frac{\partial \phi}{\partial Q} = [\, d\phi_Q(B_{ij})]$ where B_{ij} is the matrix with 1 at the $(ij)-$entry and zero otherwise.
III. $T\,\frac{\partial(\det Q)}{\partial Q} S^T = A^{-1}(\det A)$ for any $Q \in GL^+(3)$.

IV. If for any $Q \in GL^+(3)$, $V(Q) = V(T^T A S) = \bar{V}(A)$ depends only on $A = diag(a_1, a_2, a_3)$, $0 < a_1 < a_2 < a_3$, then

$$T(\frac{\partial V}{\partial Q})_{Q=T^T AS} S^T = \frac{\partial \bar{V}}{\partial A}.$$

Example 5.7.9. (**Examples of potentials**)
Assume that $V : GL^+(3) \to \mathbb{R}$ is of the form:

$$V(Q) = \bar{V}\left(\mathrm{I}(C), \mathrm{II}(C), \mathrm{III}(C)\right)$$

where $C = QQ^T$ and $\mathrm{I}(C) = \mathrm{tr}\, C$, $\mathrm{II}(C) = \frac{1}{2}\left[(\mathrm{tr}\, C)^2 - \mathrm{tr}\,(C^2)\right]$, $\mathrm{III}(C) = \det C$.

1. **Gravitational potential**

$$\bar{V} = -2\pi G\rho \int_0^\infty \frac{ds}{\left[(s^3 + \mathrm{I}(C)\, s^2 + \mathrm{II}(C)\, s + \mathrm{III}(C)\right]^{1/2}}.$$

2. **Ciarlet-Geymonat material** (see [42])

$$\bar{V} = \frac{1}{2}\,\lambda\left(\mathrm{III}(C) - 1 - \ln \mathrm{III}(C)\right) + \frac{1}{2}\,\mu\left(\mathrm{I}(C) - 3 - \ln \mathrm{III}(C)\right).$$

3. **Saint Venant-Kirchhoff material** (see [42])

$$\bar{V} = \frac{1}{2}\,\lambda\left(\mathrm{tr}\,(C - Id)\right)^2 + \mu\,\left(\mathrm{tr}\,(C - Id)^2\right).$$

Remark 5.7.10. For general purposes we write:

$$\frac{\partial V}{\partial Q} = \frac{\partial \bar{V}}{\partial \mathrm{I}}\frac{\partial \mathrm{I}\,(C)}{\partial Q} + \frac{\partial \bar{V}}{\partial \mathrm{II}}\frac{\partial \mathrm{II}\,(C)}{\partial Q} + \frac{\partial \bar{V}}{\partial \mathrm{III}}\frac{\partial \mathrm{III}\,(C)}{\partial Q}.$$

Proposition 5.7.11. *(see [58])*

$$\frac{\partial \mathrm{I}\,(C)}{\partial Q} = 2Q$$

$$\frac{\partial \mathrm{II}\,(C)}{\partial Q} = 2\left[Id\,\mathrm{tr}\,(QQ^T) - QQ^T\right]\,Q$$

$$\frac{\partial \mathrm{III}\,(C)}{\partial Q} = 2\det\,(QQ^T)\,(Q^{-1})^T.$$

Remark 5.7.12. Using the expression of the gravitational potential and the results III and IV of Exercise 5.7.8, we see that equation (5.121) is precisely the so-called **Dirichlet–Riemann** equation (see [15] p.71, eq(57)), provided that $detA = 1$ and $\lambda = \frac{2p_c}{\rho}$.

5.8 Dissipative mechanical systems

The results we will present in this section have their proofs in the article "Dissipative Mechanical Systems", by I. Kupka and W.M. Oliva, appeared in *Resenhas* IME-USP 1993, vol. 1, no. 1, 69-115 (see [38]).

A mechanical system $(Q, \langle , \rangle, \mathcal{F})$, is said to be **dissipative** if the field of external forces $\mathcal{F} : TQ \to T^*Q$ is given by

$$\mathcal{F}(v) = -dV(p) + \tilde{D}(v) \quad \text{for all} \quad v \in T_pQ;$$

where $V : Q \to \mathbb{R}$ is a $C^{r+1}(r \geq 1)$ potential energy and $\tilde{D} \in C^1$ verifies $(\tilde{D}(v))v < 0$ for all $0 \neq v \in TQ$. $\tilde{D}$ is called a **dissipative** external field of forces (or simply a **dissipative** force) and $(-dV)$ is said to be the **conservative** force.

Remark 5.8.1. $\tilde{D}(0_p) = 0$ $\forall p \in Q$ (0_p is the zero vector of T_pQ). In fact, continuity of $\tilde{D}$ shows that $(\tilde{D}(0_p))v = \lim_{\lambda\to 0} \frac{1}{\lambda}(\tilde{D}(\lambda v))\lambda v \leq 0$ for $\lambda > 0$ and $0 \neq v \in T_pQ$ implies $(\tilde{D}(0_p))v = 0$ (otherwise $(\tilde{D}(\epsilon v))v < 0$ for small $\epsilon < 0$ and then $(\tilde{D}(\epsilon v))(\epsilon v) > 0$ which is a contradiction).

Remark 5.8.2. The mass operator $\mu : TQ \to T^*Q$ defines $D = \mu^{-1}\tilde{D} : TQ \to TQ$ and $(\tilde{D}(v))v < 0$ is equivalent to $\langle D(v), v\rangle < 0$ for all $0 \neq v \in TQ$.

It is usual to say that D is a dissipative force when $\tilde{D} = \mu D$ is a dissipative force.

Let us denote by DMS the set of all vector fields $X \in C^r(TQ, TTQ)$ such that X is defined by a dissipative mechanical system, that is, by a pair (V, D) as above. If z is a trajectory of (V, D) and q its projection on Q, then $z = \frac{dq}{dt} = \dot{q}$ and the motion $q = q(t)$ satisfies the generalized Newton law

$$\frac{D\dot{q}}{dt} = -(grad \;\; V)(q) + D(\dot{q}). \tag{5.122}$$

It is useful to remark that the mechanical energy E_m decreases along non trivial integral curves of any mechanical system (V, D). In fact, we have:

$$\dot{E}_m = \frac{d}{dt}(\frac{1}{2}\langle\dot{q}, \dot{q}\rangle + V(q(t))) = \langle D\dot{q}, \dot{q}\rangle$$

which shows that E_m decreases on all integral curves not reduced to a singular point. The singular points of X lie on the zero section $O(Q)$; moreover $0_p \in O(Q)$ is a singular point if and only if p is critical for V.

A function $V \in C^{r+1}(Q, \mathbb{R})$ is said to be a **Morse function** if the Hessian of V at each critical point is a non-degenerate quadratic form. It is well known that the set of all Morse functions is an open dense subset of $C^{r+1}(Q, \mathbb{R})$ with the standard C^{r+1} topology.

A dissipative mechanical system (V, D) is said to be **strongly dissipative** if V is a Morse function and D comes from a **strongly dissipative force** that is, satisfies the following additional condition: for all $p \in Q$ and all $\omega \neq 0$, $\omega \in T_pQ$, one has $(\langle d_v D(0_p)\omega, \omega\rangle) < 0$ where $d_v D$ denotes the vertical differential of D.

From now on let us denote by SDMS the set of all $X \in$ DMS such that $X = (V, D)$ is strongly dissipative and by $\mathcal{D}$ the set of all strongly dissipative forces D.

Proposition 5.8.3. *Let (V, D) be a strongly dissipative mechanical system. Then the following properties hold:*

i) *The singular points of (V, D) are hyperbolic.*
ii) *The stable and unstable manifolds $W^s(0)$ and $W^u(0)$ of a singular point 0 are properly embedded.*
iii) $\dim W^u(0)$ *is the Morse index of V at $\tau(0) \in Q$.*
iv) $\dim W^u(0) \leq \dim Q \leq \dim W^s(0)$.

Exercise 5.8.4. Exercise 11.5 Prove property (ii) in the last proposition.

Two submanifolds S_1 and S_2 of a manifold M are said to be in **general position** or **transversal** if either $S_1 \cap S_2$ is empty or at each point $x \in S_1 \cap S_2$ the tangent spaces T_xS_1 and T_xS_2 span the tangent space T_xM.

Let us denote by $SDMS(D)$ the set of all C^r strongly dissipative mechanical systems $X = (V, D)$ with a fixed D. Analogously we introduce the set $SDMS(V)$.

All the subsets of DMS are endowed with the topology induced by the C^r-Whitney topology of $C^r(TQ, TTQ)$.

This topology possesses the Baire property.

Proposition 5.8.5. *The set of all systems X in $SDMS$ such that their stable and unstable manifolds are pairwise transversal is open in $SDMS$.*

Proposition 5.8.6. *Assume* $\dim Q > 1$, $r > 3(1 + \dim Q)$ *and let $\mathcal{G}$ be the subset of $SDMS(D)$ (resp. $SDMS(V)$) of all systems X such that their invariant manifolds are pairwise transversal. Then $\mathcal{G}$ is open dense in $SDMS(D)$ (resp. $SDMS(V)$).*

As usual, we say that $X \in SDMS$ is **structurally stable** if there exists a neighborhood W of X (in the Whitney C^r-topology) and a continuous map h from W into the set of all homeomorphisms of TQ (with the compact open topology), such that:

1) $h(X)$ is the identity map;

2) $h(Y)$ takes orbits of X into orbits of Y, for all $Y \in W$, that is, $h(Y)$ is a **topological equivalence** between X and Y.
   If the topological equivalence $h(Y)$ preserves time, that is, if X_t (resp. Y_t) is the flow map of X (resp Y) and $h(Y) \circ X_t = Y_t \circ h(Y)$ for all $t \in \mathbb{R}$, then we say that $h(Y)$ is a **conjugacy** between X and Y.

Recall that the subset of all complete C^r vector fields X on a manifold M (the flow map X_t of X is defined for all $t \in \mathbb{R}$) is open in the set of all C^r-vector fields with the Whitney C^r-topology.

Proposition 5.8.7. *Any complete strongly dissipative mechanical system where all the stable and unstable manifolds of singular points are in general position is structurally stable and the topological equivalence is a conjugacy.*

If in the last proposition we do not assume the mechanical system to be complete, the same arguments used in the proof also shows that the corresponding time-one map flow is a Morse–Smale map in the sense presented in [29], then stable with respect to the attractor $\mathcal{A}$ (V, D), which in this case is the union of the unstable manifolds of all singular points of (V, D).

Example 5.8.8. Let us consider an example of a strongly dissipative mechanical system which does not satisfy the conclusions of Proposition 5.8.6 in the sense that it does not belong to $\mathcal{G}$; it is the system which describes the motions of a particle (unit mass) constrained to move on the surface Q of a symmetric vertical solid torus of $\mathbb{R}^3$ obtained by the rotation around the x-axis, of a circle defined by the equations $y = 0$ and $x^2 + (z - 3)^2 = 1$. The potential is proportional to the height function of Q and the dissipative force D is given by $D(v) = -cv$, $c > 0$, for all $v \in TQ$. These data define a strongly dissipative mechanical system with Q as the configuration space. The metric of Q is the one induced by the usual inner product of $\mathbb{R}^3$ and the potential is a well known Morse function with four critical points. The symmetry of the problem shows that the unstable manifold of dimension one of a saddle is contained in the stable manifold of dimension 3 of the other saddle hence they are not in general position since $\dim TQ = 4$.

A dissipative force D is said to be **complete** if, for any Morse function V, the vector field associated to (V, D) is complete, that is, all of its integral curves are defined for all time.

Example 5.8.9. Let us consider a linear dissipative field of forces, that is, a function D defined by

$$D(v) = -c(\tau(v))v, \quad \text{for all} \quad v \in TQ$$

where $c : Q \to \mathbb{R}$ is a strictly positive C^r function and Q is compact. It is a simple matter to show that D is a strongly dissipative force. We will show that D is complete. If it were not the case, there would exist a smooth function

$V : Q \to \mathbb{R}$ and a motion $t \to q(t)$ of (V, D) whose maximal interval of a existence is $]\alpha, +\infty[$ with $-\infty < \alpha < 0$. We know that $\frac{d}{dt}(E_m(\dot{q})) = \langle D(\dot{q}), \dot{q}\rangle$ is negative and also that

$$0 < |\langle D(\dot{q}), \dot{q}\rangle| \leq \mu|\dot{q}|^2 \leq 2\mu(E_m(\dot{q}) + k)$$

where $\mu > 0$ is the maximum of c on Q and $k = |\nu|$, ν being the minimum of V on Q. For all t, $\alpha < t < 0$, we may write

$$-2\mu(E_m(\dot{q}) + k) \leq \dot{E}_m(\dot{q}) \leq \frac{d}{dt}(E_m(\dot{q}) + k) < 0$$

or

$$\frac{d(E_m(\dot{q}) + k)}{E_m(\dot{q}) + k} \geq -2\mu dt$$

which implies

$$E_m(\dot{q}) + k \leq (E_m(\dot{q}(0)) + k)e^{-2\mu t}$$

and then $E_m(\dot{q}(t))$ is bounded and strictly decreasing, so there exists

$$\lim_{t \to \alpha_-} E_m(\dot{q}(t)) = L < +\infty.$$

This shows that $|\dot{q}|^2 = 2(E_m(\dot{q}) - V(q(t))$ is also bounded, because V is bounded; now it is immediate that we have a contradiction.