Volume Requirements of 3D Upward Drawings*

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Abstract. This paper studies the problem of drawing directed acyclic graphs in three dimensions in the straight-line grid model, and so that all directed edges are oriented in a common (upward) direction. We show that there exists a family of outerplanar directed acyclic graphs whose volume requirement is super-linear. We also prove that for the special case of rooted trees a linear volume upper bound is achievable.

1 Introduction

The problem of computing 3D grid drawings of graphs so that the vertices are represented at integer grid-points, the edges are crossing-free straight-line segments, and the volume is small, has received a lot of attention in the graph drawing literature (e.g., [4, 5, 7, 8, 9, 12, 13]). While the interested reader is referred to the exhaustive introduction and list of references of [9] for reasons of space, we recall in this extended abstract some of the more recent results on the subject. In what follows, n denotes the number of vertices, and m the number of edges of a graph.

Dujmović and Wood [12] proved that drawings on an integer grid with an $O(n^{1.5})$ volume can be obtained for planar graphs, graphs with bounded degree, graphs with bounded genus, and graphs with no K_h (h constant) as a minor. Bose et al. [3] proved that the maximum number of edges in a grid drawing of dimensions $X \times Y \times Z$ is $(2X-1)(2Y-1)(2Z-1)-XYZ$, which implies a lower bound of $\frac{m+n}{8}$ on the volume of a 3D grid drawing of any graph. Felsner et al. [13] initiated the study of restricted integer grids, where all vertices are drawn on a small set of parallel grid lines, called tracks and proved that outerplanar graphs can be drawn by using three tracks on an integer grid of size $O(1) \times O(1) \times O(n)$. Dujmović, Morin, and Wood [9] showed that a graph G admits a drawing on an integer grid of size $O(1) \times O(1) \times O(n)$ if and only if G admits a drawing

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on an integer grid consisting of a constant number of tracks. Dujmović, Morin, and Wood used this result to show in [9] that graphs of bounded tree-width (including, for example, series-parallel graphs and k-outerplanar graphs with constant k) have 3D straight-line grid drawings of $O(n)$ volume. Some of the constant factors in the volume bounds of [9] are improved in [8]. As far as we know, the question of whether all planar graphs admit a 3D straight-line grid drawing of $O(n)$ volume remains a fascinating open problem.

This paper studies the problem of computing 3D straight-line grid drawings of directed acyclic graphs so that all edges are drawn oriented in a common direction; such drawings are called *3D upward drawings* in the remainder of the paper. Recall that 2D straight-line grid drawings of directed acyclic graphs such that all edges are drawn upward are a classical subject of investigation in the graph drawing literature (see, e.g. [1, 2, 14, 18]). Little is known about volume requirements of 3D upward drawings. Poranen [19] presented an algorithm to compute a 3D upward drawing of an arbitrary series-parallel digraph in $O(n^3)$ volume. This bound can be improved to $O(n^2)$ and $O(n)$ if the series-parallel digraph has some additional properties. The major contributions of the present paper can be listed as follows.

- We introduce and study the notion of *upward track layout*, which extends a similar concept studied by Dujmović, Morin, and Wood (see, e.g. [9, 10, 11, 12]). We relate upward track layouts to upward queue layouts and use this relationship to prove some of our volume bounds.
- We show that there exist outerplanar directed acyclic graphs which have a $\Omega(n^{1.5})$ volume lower bound. This result could be regarded as the 3D counterpart of a theorem in [6], which proves that upward grid drawings in 2D can require area exponential in the number of vertices. Note however that the class of graphs that we use for our lower bound has an $O(n^2)$ upward drawing in 2D. Also note that undirected outerplanar graphs admit a 3D grid drawing in optimal $O(n)$ volume [13].
- Motivated by the above super-linear lower bound, we investigate families of outerplanar graphs which admit upward 3D drawings of linear volume. In particular, we show that every tree has an upward 3D drawing on a grid of size $O(1) \times O(1) \times O(n)$.

The remainder of this paper is organized as follows. Preliminaries can be found in Section 2. The definition of upward track layout, and the volume lower bound for 3D upward drawings of outerplanar graphs are in Section 3. How to compute linear-volume 3D upward drawings of trees is the subject of Section 4. Other families of graphs and gaps on the volume are discussed in Section 5. Some proofs are sketched or omitted for reasons of space.

2 Preliminaries

Let G be a directed acyclic graph (DAG). The *underlying undirected graph* $\widehat{G}$ of G is the undirected graph obtained by ignoring the directions of the edges of G.

A *3D straight-line grid drawing* Γ of an *undirected* graph G maps each vertex of G to a distinct point of $\mathbb{Z}^3$ and each edge of G to the straight-line segment between its vertices. we denote the x-, y- and z-coordinates of p by $x(v)$, $y(v)$ and $z(v)$. A *crossing-free straight-line grid drawing* is a straight-line grid drawing such that edges intersect only at shared end-vertices and an edge only intersect a vertex that is an end-vertex of that edge.

A (crossing-free) straight-line grid drawing of a *DAG* G is a (crossing-free) straight-line grid drawing of the underlying undirected graph $\widehat{G}$ of G. A 3D straight-line grid drawing of G is *upward* if for each directed edge $(u, v) \in G$ we have $z(u) < z(v)$.

The *bounding box* of a straight-line grid drawing Γ of a graph G is the minimum axis-aligned box containing Γ. If the sides of the bounding box of a 3D straight-line grid drawing Γ parallel to the x-, y-, and z-axis have lengths $W-1$, $D-1$ and $H-1$, respectively, we say that Γ has *width* W, *depth* D and *height* H. We also say that Γ has *size* $W \times D \times H$ and *volume* $W \cdot D \cdot H$.

3 Volume Requirements of 3D Upward Drawings

In this section we present a super-linear lower bound on the volume of 3D upward drawings of outerplanar DAGs. In order to do this, we start by introducing and studying the concept of an *upward track layout*, which extends the (undirected) notion of an improper track layout as defined by Dujmović et al. [10].

3.1 Upward Track Layouts

Let $G = (V, E)$ be an undirected graph. A *t-track assignment* γ of G consists of a partition of V into t sets $V_0, V_1, \ldots, V_{t-1}$ and a total order $\leq_i$ for each set V_i. We write $u <_i w$ if $u \leq_i w$ and $u \neq w$. There is an *overlap* if there exist three vertices u, v, w such that $u, v, w \in V_i$, $(u, w) \in E$ and $u <_i v <_i w$. There is an *X-crossing* if there exist two edges (u, v) and (w, z) such that $u, w \in V_i$, $v, z \in V_j$, with $i \neq j$, and $u <_i w$ and $z <_j v$. A *t-track layout* of G is a t-track assignment of G without overlaps and X-crossings. The minimum value of t such that G has an t-track layout is called the *track number* of G and is denoted as $\mathsf{tn}(G)$.

Definition 1. *Let $G = (V, E)$ be a DAG. An upward t-track layout of G is a partition of V into t sets $V_0, V_1, \ldots, V_{t-1}$, called* tracks*, a total order $\leq_i$ for each track V_i and a partial order $\preccurlyeq$ on V such that there is no overlap, there is no X-crossing, if $(u, v) \in E$ then $u \preccurlyeq v$ and if $u \leq_i v$ for some i then $u \preccurlyeq v$.*

The minimum value of t such that G has an upward t-track layout is called the *upward track number* of G and is denoted as $\mathsf{utn}(G)$.

We complete this section by studying the relationship between upward track layout and another well-known graph parameter, namely the upward queue-number [15, 16, 17].

Let $G = (V, E)$ be an undirected graph. A *q-queue layout* of G consists of a total ordering $\leq_\sigma$ of V and a partition of E into q sets, called *queues*, such that there are no two edges (u,v) and (w,z) in the same queue such that $u <_\sigma w <_\sigma z <_\sigma v$, where $u <_\sigma w$ means $u \leq_\sigma w$ and $u \neq w$. The minimum value of q such that G has a q-queue layout is called the *queue number* of G and is denoted as $\mathsf{qn}(G)$.

Let $G=(V,E)$ be a DAG. An *upward q-queue layout* of G consists of a total ordering $\leq_\sigma$ of V and a partition of E into q sets, called *queues*, such that it is a q-queue layout for the undirected underlying graph $\widehat{G}$ of G and for each edge $(u,v) \in E$, we have $u <_\sigma v$. The minimum value of q such that G has an upward q-queue layout is called the *upward queue number* of G and is denoted as $\mathsf{uqn}(G)$.

Lemma 1. *Let G be a DAG. Then*

$$\mathsf{uqn}(G) \leq \binom{\mathsf{utn}(G)}{2} + \mathsf{utn}(G).$$

Sketch of Proof. The total ordering σ of the queue layout is a total order that respects the partial order $\preccurlyeq$ of the track layout. All edges between any pair of tracks can be put in a queue. All edges on a track can be put in a queue. $\square$

Note that in the undirected case Dujmović et al. [10] proved that $\mathsf{qn}(G) \leq \mathsf{tn}(G)$, for every graph G. As the following lemma shows, the relationship stated by Lemma 1 can be asimptotically tight for DAGs.

Lemma 2. *For all n there exists a DAG G with at least n vertices such that $\mathsf{uqn}(G) \geq (\mathsf{utn}(G) - 2)^2/2$.*

Proof. Let k be the smallest integer such that there is a value t for which $t(t+1) = 2k \geq n$. Consider the graph $G=(V,E)$. The set V is $V_u \cup V_v$ where $V_u = \{u_0, u_1, \dots, u_{k-1}\}$ and $V_v = \{v_0, v_1, \dots, v_{k-1}\}$. The set of edges E is $E_u \cup E_v \cup E_{uv}$ where $E_u = \{(u_i, u_{i+1}) \mid 0 \leq i < k-1\}$, $E_v = \{(v_i, v_{i+1}) \mid 0 \leq i < k-1\}$ and $E_{uv} = \{(u_i, v_{k-1-i}) \mid 0 \leq i < k\}$. The graph G contains the Hamiltonian path consisting of $E_u \cup E_v$ plus the edge (u_{k-1}, v_0). The order of the vertices of G in this Hamiltonian path is the unique topological sort of G and therefore it must be the total order for the upward queue layout. Since no two edges from E_{uv} can belong to the same queue, it follows that $\mathsf{uqn}(G) \geq k$. It is not hard to see that in fact $\mathsf{uqn}(G) = k$.

Consider the following upward layout of G on $t+2$ tracks. Place all vertices of V on the tracks in the order given below. For an illustration see Figure 1, where $k=10$ and $t=4$. Place vertices $u_0, u_1, \dots, u_{t-1}$ on track 0. Then place the next $t-1$ vertices of V_u on track 1, the next $t-2$ vertices of V_u on track 2, etc. So track $t-1$ contains the vertex u_{k-1}. Place v_0 on track $t+1$. Place v_1 and v_2 on tracks $t+1$ and t respectively. Then place the next three vertices of V_v on tracks $t+1$, t and $t-1$, etc. So the last group of vertices placed is $\{v_{k-t}, v_{k-t-1}, \dots, v_{k-1}\}$, and they lie on tracks $t+1, t, \dots, 2$. It can easily be verified that the edges of E do not form an X-crossing. So $\mathsf{utn}(G) \leq t+2$. We have $\mathsf{uqn}(G) = k = t(t+1)/2 \geq (\mathsf{utn}(G) - 2)^2/2$, so the lemma holds. $\square$

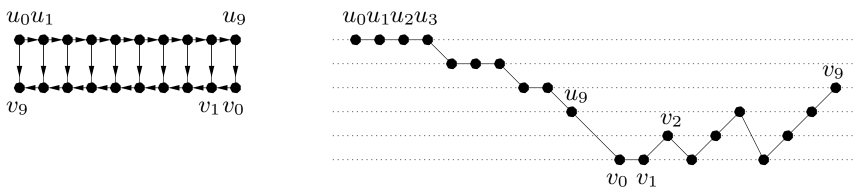

Fig. 1. A graph G and a partial 5-track layout of G

3.2 Volume Requirement

We next show a super-linear volume lower bound by using the results in the previous subsection and the following lemma.

Lemma 3. *Let G be a DAG and let Γ be a 3D straight-line upward grid drawing of G such that the sides of the bounding box of Γ parallel to the x-, y-, and z-axis have length W, D, and H, respectively. Then* $\mathsf{utn}(G) \leq W \cdot D$.

Sketch of Proof. All lines in Γ parallel to the z-axis are the tracks of the track layout. The total ordering $\leq_i$ on each track V_i and the partial order $\preccurlyeq$ for the track layout can be defined according to the z-coordinates of the vertices in Γ. □

Theorem 1. *There exists an outerplanar DAG G with n vertices such that any crossing-free 3D straight-line upward grid drawing of G requires $\Omega(n^{1.5})$ volume.*

Proof. Consider the DAG $G = (V, E)$ with $m = 3n/2 - 2$ edges as defined in the proof of Lemma 2 and illustrated in Figure 1 with $n = 20$.

As we saw in the proof of Lemma 2, $\mathsf{uqn}(G) = k = n/2$. Assume for contradiction that there exists a 3D straight-line upward grid drawing Γ of G with volume $o(n^{1.5})$. Let W, D, and H be the width, depth, and height of Γ. Since Γ is upward, we have $z(u_0) < z(u_1) < \cdots < z(u_{k-1}) < z(v_0) < z(v_1) < \cdots < z(v_{k-1})$. This implies that $H \geq n$. In order to have a volume of $o(n^{1.5})$ it must be that $W \cdot D = o(n^{\frac{1}{2}})$. By Lemma 3 this would imply $\mathsf{utn}(G) = o(n^{\frac{1}{2}})$. By Lemma 1, we have $\mathsf{uqn}(G) = O(\mathsf{utn}(G)^2)$ and therefore it would be $\mathsf{uqn}(G) = o(n)$, but this is impossible because we proved that $\mathsf{uqn}(G) = \Omega(n)$. □

Note that in contrast to Theorem 1, undirected outerplanar graphs admit a crossing-free 3D straight-line upward grid drawing in optimal $O(n)$ volume [13]. Theorem 1 can be regarded as the three-dimensional counterpart of well-known results which show that in two-dimensions, undirected and directed planar graphs have different area requirements [6].

4 Compact 3D Upward Drawings of Trees

Based on the result of Theorem 1 we next investigate whether there exist meaningful families of outerplanar DAGs with $o(n^{1.5})$ volume upper bounds. In this section we study compact 3D upward drawings of trees and paths. We recall that Heath et al. [17] proved that every tree DAG has an upward 2-queue layout and that every path DAG has an upward 1-queue layout.

Definition 2. *Let $G = (V, E)$ be a DAG. A 3D upward straight-line grid drawing Γ of G on t lines is a drawing of G with vertices placed on t lines parallel to the z-axis such that the drawing induced by the vertices on two of the t lines is crossing-free.*

Recall that in an upward grid drawing, we also have $z(u) < z(v)$ for all edges (u, v).

Lemma 4. *If DAG G has a 3D upward straight-line grid drawing on t-lines of height H, then G has a 3D crossing-free straight-line upward grid drawing of size $t \times p \times p \cdot H$ and volume $O(t^3 \cdot H)$, where p is the smallest prime number such that $p \geq t$.*

The lemma follows directly from a similar result in [9].

Corollary 1. *Let G be a DAG with n vertices. G has a 3D crossing-free straight-line upward grid drawing of size $\mathsf{utn}(G) \times p \times p \cdot n$ and volume $O(\mathsf{utn}(G)^3 \cdot n)$, where p is the smallest prime number such that $p \geq \mathsf{utn}(G)$.*

Lemma 5. *Let T be a directed tree with n vertices. Then T admits an upward straight-line grid drawing on 7 lines.*

Sketch of Proof. Let v be a vertex of T. The set of edges of T is E. We use $T^+(v)$ to denote the subtree of T induced by all vertices w for which there is a directed path of length ≥ 0 from v to w. Similarly, $T^-(v)$ is the subtree of T induced by all vertices w for which there is a directed path of length ≥ 0 from w to v.

Let r be a vertex of T that has no incoming edges, i.e. there are no edges (v, r) in E. Let F_0 be $T^+(r)$. Let $F_1 = \{T^-(w) \mid v \in F_0, w \notin F_0, (w, v) \in E\}$. In other words F_1 is a forest of trees $T^-(w)$ for all nodes w for which there is an edge (w, v) in E with $v \in F_0$ and $w \notin F_0$. Similarly, let $F_2 = \{T^+(w) \mid v \in F_1, w \notin F_0 \cup F_1, (v, w) \in E\}$, $F_3 = \{T^-(w) \mid v \in F_2, w \notin F_1 \cup F_2, (w, v) \in E\}$, etc. Since T is connected, it follows that each vertex v of T belongs to some F_i.

We first draw the single tree of F_0, i.e. the tree $T^+(r)$, on tracks 0, 1 and 2 using the wrap-around algorithm described in [13]. We then place the roots

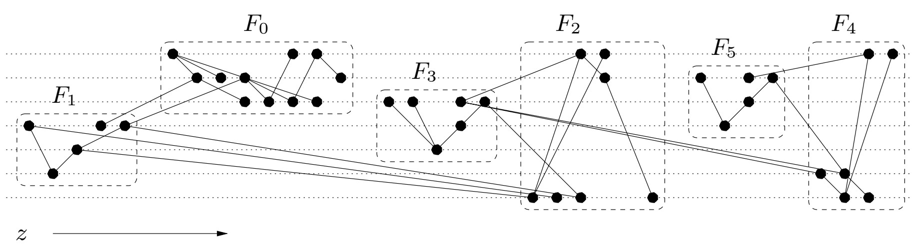

Fig. 2. Drawing of a tree decomposed into 6 forests

of the trees in F_1 on track 3. We then use the algorithm of [13] again to place the remaining vertices of F_1 on tracks 3, 4 and 5, but now wrapping the trees from high z values to smaller z values. Suppose we have placed F_i on tracks $j, j+1, j+2$. If i is odd we place the roots of the forest F_{i+1} on track $j+3$, sufficiently far above F_{i-1} to leave room for F_{i+2}. We then use the wrap-around algorithm to place the remaining vertices of F_{i+1} on tracks $j+3, j+4, j+5$. If i is even we place the roots of the forest F_{i+1} on track $j+3$, below all vertices of F_i, but above F_{i-2}. We then use the wrap-around algorithm to place the remaining vertices of F_{i+1} on tracks $j+3, j+4, j+5$, so that all vertices of F_{i+1} are above the vertices of F_{i-2}. It can be shown that the resulting drawing has no overlaps and no X-crossings. For an illustration, see Figure 2. □

Theorem 2. *Every directed tree T with n vertices admits a 3D crossing-free straight-line upward grid drawing of size $7 \times 7 \times 7 \cdot n$ and volume $O(n)$.*

An immediate consequence of Lemma 5 is that for every tree T $\mathsf{utn}(T) \leq 7$. It is possible to prove that there exists a directed tree T such that $\mathsf{utn}(T) \geq 4$. Therefore the following theorem holds.

Theorem 3. *Let T be a directed tree. Then $4 \leq \mathsf{utn}(T) \leq 7$.*

For the special case of a path, the result of Theorem 2 can be further improved as shown in the following.

Theorem 4. *Every directed path P with n vertices admits a 3D crossing-free straight-line upward grid drawing of size $2 \times 2 \times n$ and volume $O(n)$.*

Sketch of Proof. Let P be a directed path with vertices $v_0, \ldots, v_{n-1}$. We assume without loss of generality that the first edge from v_0 to v_1 is directed in the direction from v_0 to v_1. Decompose the path into k chains of consecutive edges that are similarly directed. We refer to the vertices where the path changes direction as $w_0 = v_0, w_1, w_2, ..., w_k = v_{n-1}$, and the directed chains as $W_0 = w_0 \rightarrow w_1$, $W_1 = w_1 \leftarrow w_2$, $W_2 = w_2 \rightarrow w_3$, etcetera.

These chains alternate in direction and our goal is to draw them on three tracks in the order $0, 1, 2, 0, 1, \ldots$. The algorithm to layout the chains is straightforward except that some care is required if there is a long down chain that

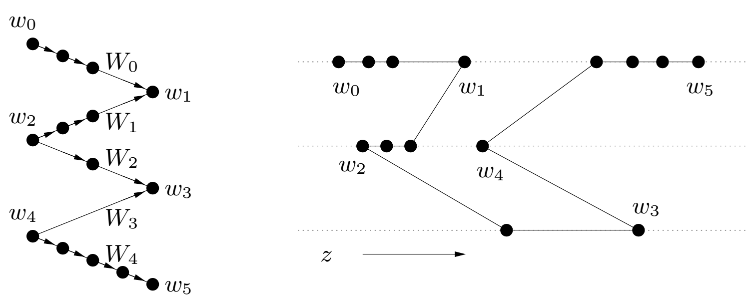

Fig. 3. Paths on 3 tracks

might interfere with previously placed vertices. To avoid this, we maintain an invariant that places vertices of the form w_{2i+1} sufficiently high. See Figure 3 for the general technique. Finally, the tracks can be drawn in 3D (non-coplanarly) in an upward manner and achieving the stated size and volume. □

5 Extensions to Other Families of DAGs

Let $G = (V, E)$ be a DAG. A *vertex c-colouring* of G is a partition $\{V_i : 1 \leq i \leq c\}$, such that for every edge $(u, v) \in E$, if $u \in V_i$ and $v \in V_j$, then $i \neq j$. The minimum value of c such that G has a vertex c-colouring is called the *chromatic number* and is denoted by $\chi(G)$. A *strong star colouring* of a graph G is a vertex colouring of G such that each bichromatic subgraph consists of a star and possibly some isolated vertices. The minimum value of c such that G has a strong star colouring with c colours is called the *strong star chromatic number* and is denoted by $\chi_{sst}(G)$. Th definition of strong star chromatic number is due to Dujmović and Wood [12] who observed that track number is at most strong star chromatic number, i.e. $\mathsf{tn}(G) \leq \chi_{sst}(G)$. It is easy to prove that also $\mathsf{utn}(G) \leq \chi_{sst}(G)$.

In [12] it has been proven that every graph G with m edges and maximum degree $\Delta \geq 1$ has strong star chromatic number $\chi_{sst}(G) < 14\sqrt{\Delta m}$ and $\chi_{sst}(G) < 15m^{2/3}$. Consequences of these results are that every planar graph has upward track number $O(n^{2/3})$ and that this bound reduces to $O(\sqrt{n})$ if the planar graph has bounded degree. This allows us to find upper bounds on the volume of a 3D crossing-free straight-line upward grid drawing of several families of graphs. In particular, outerplanar graphs and Halin Graphs as special cases of planar graphs with unbounded degree, have upward track number $O(n^{2/3})$ and by Corollary 1, volume $O(n^3)$. On the other hand k-planar graphs (i.e. planar graphs with maximum vertex degree at most k) and X-trees as examples of planar graphs with bounded degree have upward track number $O(\sqrt{n})$ and by Corollary 1, volume $O(n^{2.5})$. It is easy to construct an X-tree and a Halin graph that contains the graph of Figure 1, which is outerplanar, planar and k-planar

Table 1. Upper and Lower Bounds on the Volume of a 3D crossing-free straight-line upward grid drawing of different families of graphs

Family of DAGs	Volume Upper Bound	Volume Lower Bound
Trees	$O(n)$ $(7 \times 7 \times 7 \cdot n)$	$\Omega(n)$
Paths	$O(n)$ $(2 \times 2 \times n)$	$\Omega(n)$
X-trees	$O(n^{2.5})$	$\Omega(n^{1.5})$
Halin	$O(n^3)$	$\Omega(n^{1.5})$
Outerplanar	$O(n^3)$	$\Omega(n^{1.5})$
Planar	$O(n^3)$	$\Omega(n^{1.5})$
k-planar	$O(n^{2.5})$	$\Omega(n^{1.5})$
arbitrary	$O(n^4)$	$\Omega(n^{1.5})$

for each $k \geq 3$. It follows that a lower bound of $\Omega(n^{1.5})$ on the volume of a 3D crossing-free straight-line upward grid drawing can be established for all these families of graphs. We conclude by observing that a trivial upper bound on the upward track number of an arbitrary graph G is $O(n)$ and hence by Corollary 1 a trivial upper bound on the volume is $O(n^4)$. Table 1 summarizes these upper and lower bounds on the volume.

References

1. P. Bertolazzi, G. Di Battista, G. Liotta, and C. Mannino. Upward drawings of triconnected digraphs. *Algorithmica*, 6(12):476–497, 1994.
2. P. Bertolazzi, G. Di Battista, C. Mannino, and R. Tamassia. Optimal upward planarity testing of single-source digraphs. *SIAM J. Computing*, 27(1):132–169, 1998.
3. P. Bose, J.Czyzowicz, P. Morin, and D. R. Wood. The maximum number of edges in a three-dimensional grid-drawing. *J. of Graph Algorithms and Applications*, 8(1):21–26, 2004.
4. T. Calamoneri and A. Sterbini. 3D straight-line grid drawing of 4-colorable graphs. *Information Processing Letters*, 63(2):97–102, 1997.
5. R. F. Cohen, P. Eades, T. Lin, and F. Ruskey. Three-dimensional graph drawing. *Algorithmica*, 17(2):199–208, 1997.
6. G. Di Battista, R. Tamassia, and I. G. Tollis. Area requirement and symmetry display of planar upward drawings. *Discrete and Computational Geometry*, 7:381–401, 1992.
7. E. Di Giacomo. Drawing series-parallel graphs on restricted integer 3D grids. In G. Liotta, editor, *Proc. of 11th International Symposium on Graph Drawing, GD 2003*, volume 2912 of *Lecture Notes Computer Science*, pages 238–246. Springer-Verlag, 2004.
8. E. Di Giacomo, G. Liotta, and H. Meijer. Computing straight-line 3D grid drawings of graphs in linear volume. *Computational Geometry.* to appear.
9. V. Dujmović, P. Morin, and D. R. Wood. Layout of graphs with bounded tree-width. *SIAM J. on Computing*, 34(3):553–579, 2005.
10. V. Dujmović, A. Por, and D. R. Wood. Track layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):497–522, 2004.
11. V. Dujmović and D. R. Wood. Layouts of graph subdivisions. In J. Pach, editor, *Proc. 12th International Symposium on Graph Drawing, GD 2004*, volume 3383 of *Lecture Notes in Computer Science*, pages 133–143. Springer, 2004.
12. V. Dujmović and D. R. Wood. Three-dimensional grid drawings with sub-quadratic volume. In J. Pach, editor, *Towards a Theory of Geometric Graphs*, volume 342, pages 55–66. 2004.
13. S. Felsner, G.Liotta, and S. K. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. *J. of Graph Algorithms and Applications*, 7(4):363–398, 2003.
14. A. Garg and R. Tamassia. On the computational complexity of upward and rectilinear planarity testing. *SIAM J. Computing*, 31(2):601–625, 2001.
15. L. S. Heath and S. V. Pemmaraju. Stack and queue layouts of posets. *SIAM J. on Discrete Mathematics*, 10:599–625, 1997.
16. L. S. Heath and S. V. Pemmaraju. Stack and queue layouts of directed acyclic graphs: Part II. *SIAM J. on Computing*, 28:1588–1626, 1999.

17. L. S. Heath, S. V. Pemmaraju, and A. Trenk. Stack and queue layouts of directed acyclic graphs: Part I. *SIAM J. on Computing*, 28:1510–1539, 1999.
18. M. D. Hutton and A. Lubiw. Upward planning of single-source acyclic digraphs. *SIAM J. on Computing*, 25(2):291–311, 1996.
19. T. Poranen. A new algorithm for drawing series-parallel digraphs in 3D. Technical Report A-2000-16, Dept. of Computer and Information Sciences, University of Tampere, Finland, 2000.